Descent theory

Recall that a ring homomorphism $\varphi : R \to R'$ is *faithfully flat* if it is flat and $M \otimes_R R'$ is non-zero for any non-zero R-module M.

Exercise 1. Let $\varphi : R \to R'$ be a flat homomorphism. Show that the following are equivalent:

a) φ is faithfully flat.

b) a sequence $N \to M \to P$ of R-modules is exact if and only if the induced sequence of R'-modules $N \otimes_R R' \to M \otimes_R R' \to P \otimes_R R'$ is exact.

c) the induced morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ is surjective.

d) $\varphi(\mathfrak{m})R' \neq R'$ for every maximal ideal $\mathfrak{m}$ of R.

In particular any flat local homomorphism of local rings is faithfully flat.

Exercise 2. Let $\varphi : R \to R'$ be faithfully flat, M an R-module. Prove that the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{id_M \otimes \varphi} M \otimes_R R' \xrightarrow{d} M \otimes_R R' \otimes_R R' \xrightarrow{\delta} M \otimes_R R' \otimes_R R' \otimes_R R'$$

where $d(m \otimes x) = m \otimes 1 \otimes x - m \otimes x \otimes 1$ and $\delta(m \otimes x \otimes y) = m \otimes 1 \otimes x \otimes y - m \otimes x \otimes 1 \otimes y + m \otimes x \otimes y \otimes 1$. [Hint: assume first that φ has a section, i.e. an homomorphism $\sigma : R' \to R$ such that $\varphi \circ \sigma = id_R$. Show then that there exists a faithfully flat R-algebra S such that $S \to S \otimes_R R'$ admits such a section.]

Exercise 3. Let $\varphi : R \to R'$ be faithfully flat and put $R'' = R' \otimes_R R'$ and $R''' = R' \otimes_R R' \otimes_R R'$. To any R'-module M' we associate two R''-modules using the two R'-algebra structures of R'', namely $M' \otimes_{R'} R''$ (where, for $x, y, z \in R'$ and $m' \in M'$, we have $(x \otimes y)(m' \otimes z) = xm' \otimes yz$) and $R'' \otimes_{R'} M'$ (where $(x \otimes y)(z \otimes m') = xz \otimes ym'$).

a) Let M be an R-module and set $M' = M \otimes_R R'$. Check that

$$\alpha : M' \otimes_{R'} R'' \longrightarrow R'' \otimes_{R'} M' \qquad (m \otimes x) \otimes y \longmapsto x \otimes (m \otimes y)$$

is an isomorphism of R''-modules.

b) Show that the pair $(M \otimes_R R', \alpha)$ determines M.

c) Let (M', α) be any pair consisting of an R'-module and an isomorphism $\alpha : M' \otimes_{R'} R'' \to R'' \otimes_{R'} M'$ of R''-modules. Define

$$\alpha_1 : R' \otimes_R M' \otimes_R R' \longrightarrow R' \otimes_R R' \otimes_R M' \qquad x \otimes m' \otimes y \longmapsto x \otimes \alpha(m' \otimes y);$$

$$\alpha_2 : M' \otimes_R R' \otimes_R R' \longrightarrow R' \otimes_R R' \otimes_R M' \qquad m' \otimes x \otimes y \longmapsto x \otimes \alpha(m' \otimes y);$$

$$\alpha_3 : M' \otimes_R R' \otimes_R R' \longrightarrow R' \otimes_R M' \otimes_R R' \qquad m' \otimes x \otimes y \longmapsto \alpha(m' \otimes x) \otimes y.$$

Show that (M', α) arises from an R-module M (as in a)) if and only if $\alpha_2 = \alpha_1 \circ \alpha_3$.