The Grassmannian

Let R be a ring. Recall that an R-module M is *locally free* if for every prime ideal $\mathfrak{p} \subset R$ the localization $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$ module.

Exercise 1. Let M be an R-module. Show that M is locally free if and only if there is an open covering of $\operatorname{Spec} R$ by standard open subsets $D(f) = \operatorname{Spec} R_f$ such that M_f is a free R_f module.

Quick review of exterior products. If M is an R-module and r is an integer, the r-th *exterior power* $\bigwedge^r M$ of M is the quotient of the r-fold product $M \otimes_R \ldots \otimes_R M$ modulo the submodule generated by the elements $m_1 \otimes \ldots \otimes m_r$ such that $m_i = m_j$ for at least two different indices.
If M is free of rank p with basis $\{m_1, \ldots, m_p\}$ then $\bigwedge^r M$ is also free of rank $\binom{p}{r}$ with basis $\{m_{i_1} \wedge \ldots \wedge m_{i_r} \,|\, 0 \leq i_1 < \ldots < i_r \leq p\}$.
A map of R-modules $\varphi : M \to M'$ induces an R-linear map $\wedge^r \varphi : \bigwedge^r M \to \bigwedge^r M'$. If both modules are free with bases $\{m_1, \ldots, m_p\}$ and $\{m'_1, \ldots, m'_q\}$ and Φ is the matrix of φ in these bases, then the entry in the matrix of $\wedge^r \varphi$ corresponding to the basis elements $m_{i_1} \wedge \ldots \wedge m_{i_r}$ and $m'_{j_1} \wedge \ldots \wedge m'_{j_r}$ is the determinant of the submatrix of Φ corresponding to the columns $i_1, \ldots, i_r$ and the rows $j_1, \ldots, j_r$.

Exercise 2. Let M be a locally free R-module of finite rank and $M^* = Hom_R(M, R)$ its dual.

1. Show that the formula $(\varphi_1 \wedge \ldots \wedge \varphi_r)(m_1 \wedge \ldots \wedge m_r) = \det(\varphi_i(m_j))$ establishes a duality between $\bigwedge^r M^*$ and $\bigwedge^r M$.
2. Let $\psi, \varphi_1, \ldots, \varphi_r \in M^*$. Show that for any $m_1, \ldots, m_{r+1} \in M$ we have:

$$(\psi \wedge \varphi_1 \wedge \ldots \wedge \varphi_r)(m_1 \wedge \ldots \wedge m_{r+1}) =$$
$$\sum_{j=1}^{r+1} (-1)^j \psi(m_j)\, (\varphi_1 \wedge \ldots \wedge \varphi_r)(m_1 \wedge \ldots \wedge \hat{m}_j \wedge \ldots \wedge m_{r+1})$$

where $\hat{m}_j$ means that the term m_j is omitted.

Recall that we have defined the Grassmannian as contavariant functor $Sch^\circ \to Sets$ as follows:

$$\mathbb{G}_{n,r}(S) = \left\{ (\mathcal{E}; s_0, \ldots, s_n) \,\middle|\, \begin{array}{l} \mathcal{E} = \text{locally free } \mathcal{O}_S\text{-module of rank } r \\ s_0, \ldots, s_n \in \Gamma(S, \mathcal{E}) \text{ generating } \mathcal{E} \text{ at every point} \end{array} \right\}_{/\cong}.$$

In particular $\mathbb{G}_{n,1} = \mathbb{P}^n$ is the projective space. Notice that for $S = \operatorname{Spec} R$ this boils down to:

$$\mathbb{G}_{n,r}(R) = \left\{ (E; s_0, \ldots, s_n) \,\middle|\, \begin{array}{l} E = \text{locally free } R\text{-module of rank } r \\ s_0, \ldots, s_n \in E \text{ generating } E/\mathfrak{p}E \text{ at every prime ideal } \mathfrak{p} \end{array} \right\}_{/\cong}.$$

Finally, set $N = \binom{n+1}{r}$ and recall that we have defined the Plücker morphism

$$\pi : \mathbb{G}_{n,r} \longrightarrow \mathbb{P}^{N-1}; \qquad (\mathcal{E}; s_0, \ldots, s_n) \longmapsto \left(\bigwedge^r \mathcal{E}; \{s_{i_1} \wedge \ldots \wedge s_{i_r}\}_{0 \leq i_1 < \ldots < i_r \leq n} \right).$$

Exercise 3. Let R be a ring, M a free R-module and $f : R^{n+1} \to M$ a surjective homomorphism. Let $\{e_0, \dots, e_n\}$ be the standard basis of R^{n+1} and set $s_j = f(e_j)$. Let $\{m_1, \dots, m_r\}$ be a basis for M, write the matrix of f with respect to the chosen bases and consider its maximal minors:

$$\begin{pmatrix} x_{1,0} & x_{1,1} & \dots & x_{1,n} \\ \vdots & \vdots & & \vdots \\ x_{r,0} & x_{r,1} & \dots & x_{r,n} \end{pmatrix}; \qquad P_{i_1,\dots,i_r} = \det \begin{pmatrix} x_{1,i_1} & \dots & x_{1,i_r} \\ \vdots & & \vdots \\ x_{r,i_1} & \dots & x_{r,i_r} \end{pmatrix}$$

1. How do the $P_{i_1,\dots,i_r}$ change if we change the basis $\{m_1, \dots, m_r\}$?
2. Check that the image under the Plücker morphism of the point $(M; s_0 \dots, s_n) \in \mathbb{G}_{n,r}(R)$ is the point of $\mathbb{P}^{N-1}(R) = \mathbb{P}(\bigwedge^r R^{n+1})$ whose projective coordinates are the $P_{i_1,\dots,i_r}$.
3. Fix two subsets $\{s_{i_1}, \dots, s_{i_{r-1}}\}$ and $\{s_{j_1}, \dots, s_{j_{r+1}}\}$. Prove the Plücker relations:

$$\sum_{k=1}^{r+1} (-1)^j \, P_{i_1,\dots,i_{r-1},j_k} \, P_{j_1,\dots,\hat{j}_k,\dots,j_{r+1}} = 0.$$

   [Hint: use exercise 2.2]

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