Cohomology of complete intersections

Exercise 1. This exercise is related to exercise III.5.5 in Hartshorne [1], which you should do before (though this one might inspire you for solving that).

We work over a field k: let $\mathbb{P}^n_k = \operatorname{Proj} S$, where $S = k[T_0, \ldots, T_n]$. Let $i : X \hookrightarrow \mathbb{P}^n_k$ be a *complete intersection*: a closed subscheme whose ideal $I = (f_1, \ldots, f_{n-d}) \subset S$ is generated by $n-d$ polynomials where $d = \dim X$. Let $m_i = \deg f_i$.

Let K_1 be the direct sum of $n-d$ copies of S; make it into a graded S-module by declaring that the i-th basis element e_i has degree m_i. Let $K_r = \bigwedge^r K_1$: it is a graded S-algebra where $\deg(e_{i_1} \wedge \ldots \wedge e_{i_r}) = \sum_{j=1}^r m_{i_j}$. Let $K_0 = S$ and define

$$\delta : K_r \longrightarrow K_{r-1} : \qquad \delta(e_{i_1} \wedge \ldots \wedge e_{i_r}) = \sum_{j=1}^{r} (-1)^j f_{i_j}\, e_{i_1} \wedge \ldots \wedge \hat{e}_{i_j} \wedge \ldots \wedge e_{i_r}$$

a) Check that the following sequence is exact:

$$0 \longrightarrow K_{n-d} \xrightarrow{\delta} \ldots \xrightarrow{\delta} K_1 \xrightarrow{\delta} K_0 \longrightarrow S/I \longrightarrow 0$$

where the last nonzero map sends e_i to f_i.

Recall (or accept as a definition) that for any S-module M the group $\operatorname{Ext}^r_S(S/I, M)$ is the i-th cohomology group of the complex

$$\operatorname{Hom}_S(K_\bullet, M) : \qquad 0 \longrightarrow \operatorname{Hom}_S(K_0, M) \longrightarrow \operatorname{Hom}_S(K_1, M) \longrightarrow \cdots$$

Take $M = S(-n-1)$ (free module with basis one element of degree $n+1$).

b) Recall that that $H^i(X, \mathcal{F}) = H^i(\mathbb{P}^n_k, i_*\mathcal{F})$ for any quasi-coherent sheaf $\mathcal{F}$ on X. Use Serre's duality theorem ([1] III theorem 7.1) and exercise III.5.5 to conclude that $\operatorname{Ext}^i_S(S/I, S(-n-1)) = 0$ for $n-d < i < n$.

c) Compute $\operatorname{Ext}^{n-d}_S(S/I, S(-n-1))$ as the k-subspace of elements of S/I that are homogeneous of a suitable degree.

d) Compute the dimension of $H^d(X, \mathcal{O}_X)$ in terms of the degrees $m_1, \ldots, m_{n-d}$.

Exercise 2. Let A be a Dedekind domain and denote its fraction field by K.

a) Show that the canonical map $\mathbb{P}^n(A) \to \mathbb{P}^n(K)$ (induced by $A \subset K$) is a bijection.

b) More generally, let $X \subseteq \mathbb{P}^n_A$ be a closed A-subscheme. Show that the canonical map $X(A) \to X(K)$ is bijective.

c) Let X be as in b) and put $X_K = X \times_{\operatorname{Spec} A} \operatorname{Spec} K$. Show that any effective Cartier divisor on X_K extends uniquely to a relative Cartier divisor on X.

May 5, 2010

References

[1] R. Hartshorne, *Algebraic Geometry*, Springer GTM 52 (1977).