Algebraic Geometry 2, 2009/2010
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A GUIDE TO MUMFORD'S BOOK

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The aim of the Algebraic Geometry 2 course is to introduce one of Algebraic Geometry's central ideas, *moduli spaces*, and develop it in one (relatively) simple setting. To this end, we will mostly follow Mumford's classic book [4].

In the course as well as in the book, a fair number of lectures is devoted to a very quick overview of several important topics and techniques in Algebraic Geometry. Most of this material is covered in any advanced course in Algebraic Geometry and can be found in standard textbooks, such as Harthshorne's [2] or Liu's [3]

Algebraic Geometry, in the broad sense, has developed into what is probably the richest, largest and deepest branch of Mathematics, blending Geometry with Algebra, Category Theory, Number Theory, Topology. . . This level of maturity has been attained by developing a formidable arsenal of concepts and techniques that can seem daunting to newcomers. Hartshorne's book is a good illustration: after going through the technical chapters on schemes and cohomology, one is rewarded "only" with the modern treatment of the classical theory of curves and surfaces; fascinating though these may be, they were already well established before WW2.

The intention of the course is to leapfrog much of the technical core to show modern Algebraic Geometry in action. The rewards as well as the dangers of such an approach should be clear.

It should be noted that many of the techniques and definitions we will only skim through have been introduced and developed precisely to study moduli spaces. For instance, moduli spaces do not come naturally as the loci defined by a system of equations in a given projective space: this is the main reason for introducing abstract varieties and schemes.

As a help to the reader who might feel distressed at rushing through so much material, we provide below a link between Mumford's book and Hartshorne's, indicating where to find more details and some recommended exercises. The numbering of the lectures refers to Mumford's book. The numbering of the exercises refers to Hartshorne's.

Lecture 5. Proj schemes: [2] II.2.5. Sheaves of modules, locally free sheaves: II.5. $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$: III, Ex 4.5. Scheme-valued points of $\mathbb{P}^n$: II.7.1. Proj of a sheaf of algebras: II.7 (p. 160). Recommended exercises: II.5.1, II.5.6, II.5.13, II.5.16, II.5.18, II.7.10.

Lecture 6. Coherent and quasi-coherent sheaves of modules: [2] II.5. Subschemes and immersions; finite morphisms: [2] II.3.2. Affine morphisms: [2] exercise II.5.17. Associated primes: [1] chapter 3. Cohomology of the fibres of a flat $\mathcal{O}_X$-module: [2] III.9.3–9.4. Recommended exercises: II.3.4, II.3.5, 3.11.

Lecture 7. Cohomology: we have followed [3] chapter 5. Cohomology of $\mathbb{P}^n$: [2] III.5. Hilbert polynomial: [2] I.7. Characterization of a flat module by the Hilbert polynomial of the fibres: [2] III.9.9. Recommended exercises: I.7.2, III.5.2, III.5.6.

Lecture 8. Nobody has ever improved on Mumford's treatment of flattening stratifications.

Lecture 9. Cartier divisors and invertible sheaves: [2] II.6.11–6.18. Depth & codimension: [1] chapter 18. Normal rings and Serre's criterion: [1] §11.2. Weil divisors: [2] II.6.1–6.6. Recommended exercises: III.6.1, III.6.8.

Lecture 10. Mumford's is still the most accessible account on functoriality for Cartier divisors. [2] only covers the case of curves: IV exercise 2.6.

References

[1] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer GTM 150 (1995).

[2] R. Hartshorne, *Algebraic Geometry*, Springer GTM 52 (1977).

[3] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford GTM 6 (2002).

[4] D. Mumford, *Lectures on Curves on an Algebraic Surface*, Annals of Mathematics Studies 59, Princeton University Press (1966).