ALGEBRE MATRICIALI E STRUTTURE DI RANGO: “GEMELLAGGI” E “MATRIMONI”

$$\mathcal{A}_U = \{P = U\Delta U^*, \quad \Delta \text{ diagonale}\}$$

Popolazione in esame: 3 famiglie (allargate?) di algebre note

- ω-circolanti ($\omega \in \mathbb{C}$, $|\omega| = 1$)
- 16 trasformate trigonometriche
- 8 trasformate di Hartley

$\hookrightarrow$ futura estensione a blocchi (in collaborazione con C. Estatico)

Trigonometriche pari ($\eta_0 = \eta_n = \frac{1}{\sqrt{2}}$; $\eta_k = 1$ altrimenti)

$\mathcal{C}_I^e$	$U = \sqrt{\frac{2}{n-1}} \left(\eta_k \eta_{n-1-k} \eta_j \eta_{n-1-j} \cos \frac{kj\pi}{n-1} \right)_{k,j=0}^{n-1}$
$\mathcal{C}_{II}^e$	$U = \sqrt{\frac{2}{n}} \left(\eta_k \cos \frac{k(2j+1)\pi}{2n} \right)_{k,j=0}^{n-1}$
$\mathcal{C}_{III}^e$	$U = \sqrt{\frac{2}{n}} \left(\eta_j \cos \frac{(2k+1)j\pi}{2n} \right)_{k,j=0}^{n-1}$
$\mathcal{C}_{IV}^e$	$U = \sqrt{\frac{2}{n}} \left(\cos \frac{(2k+1)(2j+1)\pi}{4n} \right)_{k,j=0}^{n-1}$
$\mathcal{S}_I^e$	$U = \sqrt{\frac{2}{n+1}} \left(\sin \frac{kj}{n+1}\pi \right)_{k,j=1}^{n}$
$\mathcal{S}_{II}^e$	$U = \sqrt{\frac{2}{n}} \left(\eta_k \sin \frac{k(2j-1)}{2n}\pi \right)_{k,j=1}^{n}$
$\mathcal{S}_{III}^e$	$U = \sqrt{\frac{2}{n}} \left(\eta_j \sin \frac{(2k-1)j}{2n}\pi \right)_{k,j=1}^{n}$
$\mathcal{S}_{IV}^e$	$U = \sqrt{\frac{2}{n}} \left(\sin \frac{(2k-1)(2j-1)}{4n}\pi \right)_{k,j=1}^{n}$

Trigonometriche dispari

$\mathcal{C}_{I}^{o}$	$U = \frac{2}{\sqrt{2n-1}} \left(\eta_i \eta_j \cos \frac{2ij\pi}{2n-1} \right)_{i,j=0}^{n-1}$
$\mathcal{C}_{II}^{o}$	$U = \frac{2}{\sqrt{2n-1}} \left(\eta_i \eta_{n-1-j} \cos \frac{2i(j+\frac{1}{2})\pi}{2n+1} \right)_{i,j=0}^{n-1}$
$\mathcal{C}_{III}^{o}$	$U = \frac{2}{\sqrt{2n-1}} \left(\eta_{n-1-i} \eta_j \cos \frac{2j(i+\frac{1}{2})\pi}{2n-1} \right)_{i,j=0}^{n-1}$
$\mathcal{C}_{IV}^{o}$	$U = \frac{2}{\sqrt{2n+1}} \left(\cos \frac{2(i+\frac{1}{2})(j+\frac{1}{2})\pi}{2n+1} \right)_{i,j=0}^{n-1}$
$\mathcal{S}_{I}^{o}$	$U = \frac{2}{\sqrt{2n+1}} \left(\sin \frac{2ij\pi}{2n+1} \right)_{i,j=1}^{n}$
$\mathcal{S}_{II}^{o}$	$U = \frac{2}{\sqrt{2n+1}} \left(\sin \frac{2i(j-\frac{1}{2})\pi}{2n+1} \right)_{i,j=1}^{n}$
$\mathcal{S}_{III}^{o}$	$U = \frac{2}{\sqrt{2n+1}} \left(\sin \frac{2(i-\frac{1}{2})j\pi}{2n+1} \right)_{i,j=1}^{n}$
$\mathcal{S}_{IV}^{o}$	$U = \frac{2}{\sqrt{2n-1}} \left(\eta_{n-1-i} \sin \frac{2\pi(i-\frac{1}{2})(j-\frac{1}{2})}{2n-1} \right)_{i,j=1}^{n}$

Algebre Hartley $\left(Q = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & & & \\ & I & & J \\ & & \sqrt{2} & \\ & -J & & I \end{pmatrix}\right)$

$\mathcal{H}_I$	$U = \frac{1}{\sqrt{n}}\left(\cos\frac{2ij\pi}{n} + \sin\frac{2ij\pi}{n}\right)_{i,j=0}^{n-1}$
$\mathcal{H}_{II}$	$U = \frac{1}{\sqrt{n}}\left(\cos\frac{2ij\pi}{n} + \sin\frac{2ij\pi}{n}\right)_{i,j=0}^{n-1} Q^t$
$\mathcal{H}_{III}$	$U = \frac{1}{\sqrt{n}}\left(\cos\frac{i(2j+1)\pi}{n} + \sin\frac{i(2j+1)\pi}{n}\right)_{i,j=0}^{n-1}$
$\mathcal{H}_{IV}$	$U = \frac{1}{\sqrt{n}}\left(\cos\frac{i(2j+1)\pi}{n} + \sin\frac{i(2j+1)\pi}{n}\right)_{i,j=0}^{n-1} Q^t$
$\mathcal{H}_{V}$	$U = \frac{1}{\sqrt{n}}\left(\cos\frac{(2i+1)j\pi}{n} + \sin\frac{(2i+1)j\pi}{n}\right)_{i,j=0}^{n-1}$
$\mathcal{H}_{VI}$	$U = \frac{1}{\sqrt{n}}\left(\cos\frac{(2i+1)j\pi}{n} + \sin\left(\frac{(2i+1)j\pi}{n}\right)_{i,j=0}^{n-1} Q^t$
$\mathcal{H}_{VII}$	$U = \frac{1}{\sqrt{n}}\left(\cos\frac{(2i+1)(2j+1)\pi}{2n} + \sin\frac{(2i+1)(2j+1)\pi}{2n}\right)_{i,j=0}^{n-1}$
$\mathcal{H}_{VIII}$	$U = \frac{1}{\sqrt{n}}\left(\cos\frac{(2i+1)(2j+1)\pi}{2n} + \sin\frac{(2i+1)(2j+1)\pi}{2n}\right)_{i,j=0}^{n-1} Q$

MOTIVAZIONE 1

$$\mathcal{A} \text{ algebra } \mapsto \mathcal{P}(A) := \arg\min_{P\in\mathcal{A}} \|A - P\|_F$$

A generica $\Rightarrow$ calcolo $\mathcal{P}(A)$ in $O(n^2 \log n)$

... *casi intermedi*??? ...

T Toeplitz $\Rightarrow$ formule ad hoc in $O(n)$ per $\mathcal{P}(T)$ e $\mathcal{P}(T^tT)$

Riduzione costi $\longleftrightarrow$ "gemellaggi" con strutture di rango.

MOTIVAZIONE 2

Algoritmo $O(n)$ per $\mathcal{P}(T^tT)$ (caso circolante):

$$
\begin{aligned}
T \; &= \; C(\text{circ}) + S(\text{anticirc}) \\
&\Rightarrow \mathcal{P}(T^tT) = \mathcal{P}(C^tC) + \mathcal{P}(S^tS) + \mathcal{P}(C^tS) + \mathcal{P}(S^tC)
\end{aligned}
$$

1) C^tC circolante $\Rightarrow \mathcal{P}(C^tC) = C^tC$

2) S^tS anticircolante $\Rightarrow$ Toeplitz $\Rightarrow$ costo $O(n)$

3–4) $\mathcal{P}(C^tS) = C^t\mathcal{P}(S)$, $\mathcal{P}(S^tC) = \mathcal{P}(S^t)C$, S Toeplitz

Altre algebre: posso sempre scrivere $T = C + S$?

($C \in \mathcal{A}$, $\quad S \in$?? $\qquad$ Costo di $\mathcal{P}(S)$?)

$\longrightarrow$ "matrimoni" tra algebre.

CALCOLO CON MATRICI DI GRAM

Osservazione. $\mathcal{P}(A)$ = proiez. ortogonale di A su $\mathcal{A}$ risp. a

$$\langle X, Y \rangle := \sum_{i,j=1}^{n} X_{ij} Y_{ij}$$

Equazioni normali:

1. Scelgo base di $\mathcal{A} = \mathrm{Span}(P_0, \dots, P_{n-1})$
2. Costruisco *matrice di Gram* $G := (\langle P_k, P_j \rangle)_{k,j=0}^{n-1}$
3. Calcolo termine noto $b := (\langle A, P_j \rangle)_{j=0}^{n-1}$
4. Risolvo il sistema $Gc = b$ (quanto costa?!?)
5. Il precondizionatore $\mathcal{P}(A)$ è $P = c_0 P_0 + \cdots + c_{n-1} P_{n-1}$

Base delle ω-circolanti:

$$W_\omega^k = \begin{pmatrix} & & \omega & & \\ & & & \ddots & \\ & & & & \omega \\ 1 & & & & \\ & \ddots & & & \\ & & 1 & & \end{pmatrix} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} k \qquad k = 0, \ldots, n-1$$

Elemento della Gram: $G_{kh} = \langle W_\omega^k, W_\omega^h \rangle = 0 \quad \forall k \neq h$

$\Rightarrow G$ diagonale (troppo facile...)

Base di $\mathcal{S}_I^e$ (caso $n = 5$): $B_0 = I$,

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Toeplitz+Hankel "pulita" $\Rightarrow$ algebra *non degenere*

Base di $\mathcal{S}^e_{II}$ (caso $n = 5$): $\tilde{B}_0 = I$,

$$\tilde{B}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} & 0 \end{pmatrix}, \qquad \tilde{B}_2 = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \end{pmatrix},$$

$$\tilde{B}_3 = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}_4 = \begin{pmatrix} 0 & 0 & -1 & 0 & \sqrt{2} \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Toeplitz+Hankel “sporca” $\Rightarrow$ algebra *degenere*

Base trigonometrica (caso non degenere): $B_0 = I$; per $k \geq 1$

$$B_k = \begin{pmatrix} & & \alpha & 1 & & \\ & \iddots & & & \ddots & \\ \alpha & & & & & 1 \\ 1 & & & & & \beta \\ & \ddots & & & \iddots & \\ & & 1 & \beta & & \end{pmatrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} k \quad , \quad \alpha, \beta \in \{\pm 1\}$$

$$\left.\begin{array}{l} r = 1^{\text{o}} \text{ indice } k \text{ con antidiag. } \alpha \\ s = 1^{\text{o}} \text{ indice } k \text{ con antidiag. } \beta \end{array}\right\} \Rightarrow \mathcal{A} = \mathcal{A}(\alpha, \beta, r, s)$$

Parametrizzazione trigonometriche non degeneri:

	$\mathcal{S}^e_I$	$\mathcal{S}^e_{III}$	$\mathcal{S}^e_{IV}$	$\mathcal{C}^e_{III}$	$\mathcal{C}^e_{IV}$	$\mathcal{S}^o_I$	$\mathcal{S}^o_{II}$	$\mathcal{S}^o_{III}$	$\mathcal{C}^o_{IV}$
α	-1	-1	-1	1	1	-1	-1	-1	1
β	-1	-1	1	1	-1	-1	-1	1	-1
r	2	1	1	1	1	2	1	2	1
s	2	1	1	1	1	1	2	1	2

Base trigonometrica (caso degenere): $\tilde{B}_0 = I$,

$$\tilde{B}_k = B_k + \begin{pmatrix} & \gamma & & \\ \gamma & & & \\ & & & \delta \\ & & \delta & \end{pmatrix}, \; \tilde{B}_{n-1} = B_{n-1} + \begin{pmatrix} & \eta \\ \eta & \end{pmatrix}$$

$$\Rightarrow \mathcal{A} = \mathcal{A}(\alpha, \beta, r, s, \gamma, \delta, \eta)$$

Parametrizzazione trigonometriche degeneri:

	$\mathcal{S}^e_{II}$	$\mathcal{C}^e_{I}$	$\mathcal{C}^e_{II}$	$\mathcal{S}^o_{IV}$	$\mathcal{C}^o_{I}$	$\mathcal{C}^o_{II}$	$\mathcal{C}^o_{III}$
α	-1	1	1	-1	1	1	1
β	1	1	-1	1	1	1	-1
r	2	0	0	1	0	1	0
s	0	0	2	0	1	0	1
γ	0	$\sqrt{2}-2$	$\sqrt{2}-2$	0	$\sqrt{2}-2$	0	$\sqrt{2}-2$
δ	$-\frac{1}{1+\sqrt{2}}$	$\sqrt{2}-2$	0	$\sqrt{2}-2$	0	$\sqrt{2}-2$	0
η	$-\frac{1}{1+\sqrt{2}}$	-1	$\sqrt{2}-2$	$\sqrt{2}-2$	$\sqrt{2}-2$	$\sqrt{2}-2$	$\sqrt{2}-2$

Teorema. $\forall$ algebra trigonometrica $\mathcal{A}(\alpha, \beta, r, s\ [, \gamma, \delta, \eta]\,)$:

$$
\begin{aligned}
G_{k0} &= \alpha\pi(k,r) + \beta\pi(k,s) && k > 0 \\
G_{k1} &= 2(\alpha + \beta - G_{k0}) && k > 1 \\
G_{kh} &= 2G_{k0} && k > h,\ \ h > 0 \text{ pari} \\
G_{kh} &= G_{k1} && k > h,\ \ h \text{ dispari}
\end{aligned}
$$

dove $\pi(k,m) = \begin{cases} 1 & \text{se } k \equiv m \bmod 2 \\ 0 & \text{altrimenti.} \end{cases}$

Struttura di rango di G (stril$(\cdot)$:= tr. inf. *stretto*)

$$\mathrm{stril}(G) = \mathrm{stril}(\text{rango} \leq 2)$$

Il rango si abbassa:

- se $\alpha = -\beta$ $\quad (G_{k1} = -2G_{k0})$
- se inoltre $r \equiv s \bmod 2$ $\quad (G_{k0} \equiv 0)$
- se $\alpha = \beta$ e $r \not\equiv s \bmod 2$ $\quad (G_{k0} \equiv \alpha,\ G_{k0} \equiv 2\alpha)$

$\Rightarrow G =$ diag. + s.s., quasiseparabile di ordine $p \in \{0, 1, 2\}$

Ordine di separabilità delle trigonometriche:

	$\mathcal{S}^e_I$	$\mathcal{S}^e_{II}$	$\mathcal{S}^e_{III}$	$\mathcal{S}^e_{IV}$	$\mathcal{C}^e_I$	$\mathcal{C}^e_{II}$	$\mathcal{C}^e_{III}$	$\mathcal{C}^e_{IV}$
p	2	0	2	0	2	0	2	0

	$\mathcal{S}^o_I$	$\mathcal{S}^o_{II}$	$\mathcal{S}^o_{III}$	$\mathcal{S}^o_{IV}$	$\mathcal{C}^o_I$	$\mathcal{C}^o_{II}$	$\mathcal{C}^o_{III}$	$\mathcal{C}^o_{IV}$
p	1	1	1	1	1	1	1	1

Base di Hartley: $T_0 = I$;

$$
\begin{matrix} k\left\{\vphantom{\begin{matrix}1\\ \ddots\end{matrix}}\right. \\ \\ T_k = \\ \\ \\ \end{matrix}
\begin{pmatrix}
 & & 1 & & \sigma & & \\
 & & & \ddots & & \ddots & \\
 & & & & \ddots & & \\
1 & & & & & & \sigma \\
 & \ddots & & & & \ddots & \\
\sigma & & \ddots & & & & 1 \\
 & \ddots & & \ddots & & & \\
 & & \sigma & & 1 & &
\end{pmatrix}
\begin{matrix} \left.\vphantom{\begin{matrix}1\\ \ddots\end{matrix}}\right\}k \\ \\ \\ \\ \\ \end{matrix}
$$

per $k = 1, \ldots, \lfloor n/2 \rfloor - 1$ (anche $\lfloor n/2 \rfloor$ se n è dispari)...

$$
\ldots\ H_k = \begin{matrix} k-r\left\{\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right. \\ \\ k-r\left\{\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right. \end{matrix}
\begin{pmatrix}
 & & \tau & & 1 & \\
 & \iddots & & \iddots & & \\
\tau & & \iddots & & & \sigma\tau \\
 & \iddots & & & \iddots & \\
1 & & & \iddots & & \sigma \\
 & & \iddots & & \iddots & \\
 & & \sigma\tau & & \sigma &
\end{pmatrix}
\begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} k-r+1 \\ \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} k-r-1 \end{matrix}
$$

$+$ eventualmente H_0 oppure $T_{\lfloor n/2 \rfloor}$ oppure $H_{\lfloor n/2 \rfloor}$ (secondo $\mathcal{A}$)

Parametrizzazione (e base) delle Hartley:

	$\mathcal{H}_I$	$\mathcal{H}_{II}$	$\mathcal{H}_{III}$	$\mathcal{H}_{IV}$	$\mathcal{H}_V$	$\mathcal{H}_{VI}$	$\mathcal{H}_{VII}$	$\mathcal{H}_{VIII}$
σ	1	1	1	1	-1	-1	-1	-1
τ	-1	1	-1	1	1	-1	1	-1
r	0	0	1	1	0	0	1	1
base	$T_{\lfloor n/2 \rfloor}$	$T_{\lfloor n/2 \rfloor}$	$T_{\lfloor n/2 \rfloor}$	$T_{\lfloor n/2 \rfloor}$	$H_{\lfloor n/2 \rfloor}$	H_0	$H_{\lfloor n/2 \rfloor}$	H_0

Teorema. $\forall$ algebra di Hartley $\mathcal{A}(\sigma,\tau,r)$: $G = \begin{pmatrix} D_T & M^t \\ M & D_H \end{pmatrix}$,
$D_T = (\langle T_k, T_h\rangle), D_H = (\langle H_k, H_h\rangle)$ diagonali, $M = (\langle H_k, T_h\rangle)$

$$M_{k0} = (\sigma+\tau)\pi(k,r) + (1+\sigma\tau)\pi(k,n+r)$$

$$M_{k1} = 2[(1+\sigma)(1+\tau) - M_{k0}]$$

$$M_{kh} = 2M_{k0} \quad \text{se } h \text{ pari e } 0 < h < \lfloor n/2 \rfloor$$

$$M_{kh} = M_{k1} \quad \text{se } h < \lfloor n/2 \rfloor \text{ dispari}$$
$$M_{k,\lfloor n/2\rfloor} = M_{k,\lfloor n/2\rfloor-2} \text{ se } n \text{ dispari}, \quad \frac{1}{2}M_{k,\lfloor n/2\rfloor-2} \text{ se } n \text{ pari.}$$

Struttura di rango di $M \Rightarrow$ struttura di G + semplificazioni...

Ordine di separabilità delle Hartley:

	$\mathcal{H}_I$	$\mathcal{H}_{II}$		$\mathcal{H}_{III}$	$\mathcal{H}_{IV}$		$\mathcal{H}_V$	$\mathcal{H}_{VI}$		$\mathcal{H}_{VII}$	$\mathcal{H}_{VIII}$	
n		pari	disp.		pari	disp.		pari	disp.		pari	disp.
p	0	2	1	0	2	1	0	0	1	0	0	1

Il termine noto...

G quasiseparabile $\Rightarrow$ costo per $\mathcal{P}(A) = O(n) +$ calcolo $(\langle A, B_k \rangle)_k$

<u>Caso di A Toeplitz</u>: $B_k = T_k + H_k + E_k$

- calcolo $\langle A, T_k \rangle$ immediato (el. di $A \times$ lungh. diagonali)
- calcolo $\langle A, E_k \rangle$ immediato (somma pesata di $2 \div 4$ el. di A)
- calcolo $\langle A, H_k \rangle \mapsto H_k = \alpha H'_k + \beta H''_k + \cdots$ (separo antidiag.)
- $\langle A, H'_k \rangle = \langle A, H'_{k\pm 2} \rangle + 2$ el. di A
- linearità $\Rightarrow O(n)$

<u>Caso di A Hankel</u>: $\langle A, B_k \rangle = \langle JA, JB_k \rangle$ (JA Toeplitz)

- $JB_k = JT_k$ (Hankel) $+JH_k$ (Toeplitz) $+JE_k \Rightarrow$ come sopra!

Congettura: $A = \sum_i A_i$ con $(\langle A_i, B_k \rangle)_{i,k}$ quasiseparabile

MATRIMONI I — Fatte l'una per l'altra:

$$\mathcal{C}^e_{IV} = \left\{ T(c_1, \ldots, c_n) + \begin{pmatrix} c_2 & \cdots & c_n & 0 \\ \vdots & \iddots & & -c_n \\ c_n & & \iddots & \vdots \\ 0 & -c_n & \cdots & -c_2 \end{pmatrix} \right\},$$

$$\mathcal{S}^e_{IV} = \left\{ T(s_1, \ldots, s_n) + \begin{pmatrix} -s_2 & \cdots & -s_n & 0 \\ \vdots & \iddots & & s_n \\ -s_n & & \iddots & \vdots \\ 0 & s_n & \cdots & s_2 \end{pmatrix} \right\}$$

$T(a_1, \ldots, a_n) = C + S$ con $C \in \mathcal{C}^e_{IV}$, $S \in \mathcal{S}^e_{IV}$:

$$c_j = s_j = \frac{a_j}{2}$$

MATRIMONI II — Algebre... di facili costumi:

$$\mathcal{S}^e_{III} = \left\{ T(c_1, \ldots, c_n) - \begin{pmatrix} c_2 & \cdots & c_n & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & & c_n \\ c_n & & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & c_n & \cdots & c_2 \end{pmatrix} \right\},$$

$$\mathcal{C}^e_{III} = \left\{ T(s_1, \ldots, s_n) + \begin{pmatrix} s_2 & \cdots & s_n & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & & s_n \\ s_n & & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & s_n & \cdots & s_2 \end{pmatrix} \right\} \ldots \text{ma non solo!}$$

$$\mathcal{S}_I^e = \left\{ T(s_1, \ldots, s_n) - \begin{pmatrix} s_3 & \cdots & s_n & 0 & 0 \\ \vdots & \iddots & & & 0 \\ s_n & & 0 & & s_n \\ 0 & & & \iddots & \vdots \\ 0 & 0 & s_n & \cdots & s_3 \end{pmatrix} \right\} \quad \text{OK ponendo}$$

$$c_j = \begin{cases} 0 & j = 1, n \\ -\sum_{i=j+1}^{n} a_i & j \neq 1, n \end{cases}, \quad s_j = \begin{cases} a_1 & j = 1 \\ \sum_{i=j}^{n} a_i & j \geq 2; \end{cases}$$

$$\mathcal{C}_I^e = \left\{ T(s_1, \ldots, s_n) + \begin{pmatrix} s_1 & \cdots & s_n \\ \vdots & \iddots & \vdots \\ s_n & \cdots & s_1 \end{pmatrix} + \begin{pmatrix} * & \cdots & * \\ \vdots & 0 & \vdots \\ * & \cdots & * \end{pmatrix} \right\} \text{OK}$$

$$\text{con } c_j = \sum_{i=j}^{n} (-1)^{j-i} a_i \,,\; s_j = \sum_{i=j+1}^{n} (-1)^{j-i+1} a_i \Rightarrow T = C + S + E$$

MATRIMONI III — Single impenitenti?

$$\mathcal{S}^e_{II} = \left\{ T(c_1,\dots,c_n) + \begin{pmatrix} -c_3 & \cdots & -c_n & 0 & c_n \\ \vdots & \iddots & & \iddots & \vdots \\ -c_n & & \iddots & & c_3 \\ 0 & \iddots & & c_3 & c_2 \\ c_n & \cdots & c_3 & c_2 & c_1 \end{pmatrix} + \begin{pmatrix} 0 & & * \\ & & \vdots \\ * & \cdots & * \end{pmatrix} \right\}$$

$$\textbf{Trucco:} \quad \mathcal{A} = \left\{ \begin{pmatrix} \hat{S} & 0 \\ 0 & \sigma \end{pmatrix} : \ \hat{S} \in \mathcal{S}^e_{IV},\ \sigma \in \mathbb{R} \right\},$$

$$\hat{S} = T(s_1,\dots,s_n) + \begin{pmatrix} -s_2 & \cdots & -s_n & 0 \\ \vdots & \iddots & & s_n \\ -s_n & & \iddots & \vdots \\ 0 & s_n & \cdots & s_2 \end{pmatrix}$$

$\Rightarrow c_j = \sum_{i=j}^n a_i,\ s_j = -\sum_{i=j+1}^n a_i$ e ancora

$$T(a_1,\dots,a_n) = C\ (\in \mathcal{S}^e_{II})\ + S\ (\in \mathcal{A})\ +\ E$$

Trasformate dispari:

$$\mathcal{S}_I^o \longleftrightarrow \mathcal{C}_{II}^o,\ \mathcal{S}_{II}^o \longleftrightarrow \mathcal{C}_I^o,\ \mathcal{S}_{III}^o \longleftrightarrow \mathcal{S}_{IV}^o,\ \mathcal{C}_{III}^o \longleftrightarrow \mathcal{C}_{IV}^o$$

Algebre Hartley:

$$\mathcal{H}_{I,II,III,IV} \longleftrightarrow \mathcal{H}_{V,VI,VII,VIII}$$

(basta scegliere C circolante, S anticircolante)