VARIATION OF ANTICYCLOTOMIC IWASAWA INVARIANTS IN HIDA FAMILIES

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Abstract. In this paper, using the construction of big Heegner points [LV11] in the definite quaternionic setting and their relation to special values of $L$-functions [CL14], we obtain anticyclotomic analogs of the results of Emerton–Pollack–Weston [EPW06] on the variation of Iwasawa invariants in Hida families.

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Introduction

The purpose of this paper is the study of anticyclotomic analogs of the results of [EPW06] on the variation of Iwasawa invariants in Hida families. Let $\bar{\rho}: G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{F})$ be an odd and absolutely irreducible Galois representation over a finite field $\mathbb{F}$ of characteristic $p$. After the celebrated proof of Serre’s conjecture [KW09], we know that $\bar{\rho}$ is modular. Let $\mathcal{H}(\bar{\rho})$ denote the set of all $p$-ordinary and $p$-stabilized newforms with mod $p$ Galois representation isomorphic to $\bar{\rho}$.

Let $K$ be an imaginary quadratic field of discriminant prime to $p$. Let $N^-$ be a square-free product of an odd number of primes, each inert in $K$, containing all such primes at which $\bar{\rho}$ is ramified. As in [PW11], we say that $(\bar{\rho}, N^-)$ satisfies condition (CR) if the following hold:

Assumption (CR).

1. $\bar{\rho}$ is irreducible, and surjective if $\mathbb{F} = \mathbb{F}_5$.
2. If $\ell|N^-$ and $\ell \equiv \pm 1 \pmod{p}$, then $\bar{\rho}$ is ramified at $\ell$.

Let $\Gamma$ be the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension $K_{\infty}/K$. Associated with each $f \in \mathcal{H}(\bar{\rho})$ there is a $p$-adic $L$-function $L_p(f/K) \in \Lambda := \mathcal{O}[[\Gamma]]$, where $\mathcal{O}$ is the ring of integers of a finite extension $F$ of $\mathbb{Q}_p$ over which $f$ is defined, characterised by an interpolation property of the form

$$
\chi(L_p(f/K)) = C_p(f, \chi) \cdot E_p(f, \chi) \cdot \frac{L(f, \chi, k/2)}{\Omega_{f,N^-}}
$$

as $\chi$ runs over the $p$-adic characters of $\Gamma$ corresponding to certain algebraic Hecke characters of $K$, where $C_p(f, \chi)$ is an explicit nonzero constant, $E_p(f, \chi)$ is a $p$-adic multiplier, and $\Omega_{f,N^-}$ is a complex period (specified up to a $p$-adic unit) making the above ratio algebraic.

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The anticyclotomic Iwasawa main conjecture gives an arithmetic interpretation of $L_p(f/K)$. More precisely, let
\[ \rho_f : G_Q \to \text{Aut}_F(V_f) \simeq \mathbb{GL}_2(F) \]
be a self-dual twist of the $p$-adic Galois representation associated to $f$, fix an $\mathcal{O}$-stable lattice $T_f \subset V_f$, and set $A_f = V_f/T_f$. Since $f$ is $p$-ordinary, there is a unique one-dimensional $G_{Q_p}$-invariant subspace $F_p^+ V_f \subset V_f$ where the inertia group at $p$ acts via $\varepsilon_p^{1/2}$, where $\varepsilon_p$ is the $p$-adic cyclotomic character and $\psi$ is of finite order. Let $F_p^+ A_f$ be the image of $F_p^+ V_f$ in $A_f$ and set $F_p^- A_f = A_f/F_p^+ A_f$. Define the minimal Selmer group of $f$ by
\[ \text{Sel}(K_{\infty}, f) := \ker \left\{ H^1(K_{\infty}, A_f) \to \prod_{w \mid p} H^1(K_{\infty,w}, A_f) \times \prod_{w \mid p} H^1(K_{\infty,w}, F_p^- A_f) \right\} \]
where $w$ runs over the places of $K_{\infty}$. By standard arguments (see [Gre89], for example), one knows that the Pontryagin dual of $\text{Sel}(K_{\infty}, f)$ is finitely generated over $\Lambda$. The anticyclotomic main conjecture is then the following.

**Conjecture 1.** $\text{Sel}(K_{\infty}, f)^\vee$ is $\Lambda$-torsion, and
\[ \text{Ch}_\Lambda(\text{Sel}(K_{\infty}, f)^\vee) = (L_p(f/K)). \]

For $f$ corresponding to $p$-ordinary elliptic curves, and under rather stringent assumptions on $\bar{\rho}_f$ which were later relaxed by Pollack–Weston [PW11], one of the divisibilities predicted by Conjecture 1 was obtained by Bertolini–Darmon [BD05] using Heegner points and Kolyvagin’s method of Euler systems. More recently, after the work of Chida–Hsieh [CH15] the divisibility
\[ \text{Ch}_\Lambda(\text{Sel}(K_{\infty}, f)^\vee) \supseteq (L_p(f/K)) \]
is known for all newforms $f \in \mathcal{S}(\bar{\rho})$ of weight $k \leq p - 2$ and trivial nebentypus, provided the pair $(\bar{\rho}, N_f^-)$ satisfies a mild strengthening of condition (CR). Here, $N_f^-$ denotes as usual the product of the prime factors of $N_f$ which are inert in $K$.

The restriction to weights $k \leq p - 2$ in [CH15] comes from the use of the version of Ihara’s lemma proved in [DT94]. While it seems difficult to directly extend their arguments to higher weights, it might be possible to obtain the above divisibility for all weights by adapting the strategy of Bertolini–Darmon [BD05] to the setting of Heegner points in Hida families [LV11]. In fact, the results of this paper complete the proof of many new cases of Conjecture 1 using big Heegner points, but by a rather different approach, as we now explain.

Associated with every $f \in \mathcal{S}(\bar{\rho})$ there are anticyclotomic Iwasawa invariants $\mu^\text{an}(K_{\infty}, f)$, $\lambda^\text{an}(K_{\infty}, f)$, $\mu^\text{alg}(K_{\infty}, f)$, and $\lambda^\text{alg}(K_{\infty}, f)$. The analytic (resp. algebraic) $\lambda$-invariants are the number of zeros of $L_p(f/K)$ (resp. of a generator of the characteristic ideal of $\text{Sel}(K_{\infty}, f)^\vee$), while the $\mu$-invariants are defined as the exponent of the highest power of $p$ dividing the same objects. In this paper we study the behavior of these invariants as $f$ varies over the subset $\mathcal{H}(\bar{\rho})$ of $\mathcal{S}(\bar{\rho})$ consisting of newforms with $N_f^- = N^-$. Our main result is then the following.

**Theorem 2.** Suppose that $\bar{\rho}$ is $p$-ordinary, $p$-distinguished, and ramified at all $\ell | N^-$, and fix $* \in \{\text{alg, an}\}$.

1. For all $f \in \mathcal{H}(\bar{\rho})$, we have
\[ \mu^*(K_{\infty}, f) = 0. \]

2. Let $f_1, f_2 \in \mathcal{H}(\bar{\rho})$ lie on the branches $T(a_1)$, $T(a_2)$, respectively. Then
\[ \lambda^*(K_{\infty}, f_1) - \lambda^*(K_{\infty}, f_2) = \sum_{\ell | N^+_f \cap N^-_f} e_\ell(a_2) - e_\ell(a_1) \]
where the sum is over the split primes in $K$ which divide the tame level of $f_1$ or $f_2$, and $e_\ell(a_j)$ is an explicit non-negative invariant of the branch $T(a_j)$ and the prime $\ell$.  

Provided \( p \) splits in \( K \), and under the same assumptions on \( \bar{\rho} \) as in Theorem 2, the deep work of Skinner–Urban [SU14] establishes one of the divisibilities in a related “three-variable” Iwasawa main conjecture. Combining their work with the main result of this paper, we deduce the following.

**Corollary 3.** Let \( \bar{\rho} \) be as in Theorem 2 and suppose that \( p \) splits in \( K \). If the anticyclotomic main conjecture holds for some newform \( f_0 \in \mathcal{H}(\bar{\rho}) \) of weight \( k_0 \equiv 2 \pmod{p-1} \) and trivial nebentypus, then it holds for all newforms \( f \in \mathcal{H}(\bar{\rho}) \) of weight \( k \equiv 2 \pmod{p-1} \) and trivial nebentypus.

As hinted at above, the proof of our main results closely follows the methods of [EPW06]. In fact, on the algebraic side the arguments of *loc.cit.* apply in our context almost verbatim, and the main contribution of this paper is the development of anticyclotomic analogs of their results on the analytic side. Indeed, the proof of the analytic parts of [EPW06] is based on the study of certain variants of the two-variable \( p \)-adic \( L \)-functions of Mazur–Kitagawa, whose construction relies on the theory of modular symbols on classical modular curves. In contrast, by our assumptions on \( N^- \), we are led to work on a family of Shimura curves associated with a (definite) quaternion algebra over \( \mathbb{Q} \) of discriminant \( N^- > 1 \), and these curves are well-known to have no cusps.

In the cyclotomic case, modular symbols are useful two ways: They yield a concrete realization of the degree one compactly supported cohomology of open modular curves, and provide a powerful tool for studying the arithmetic properties of Hecke \( L \)-functions. Our basic observation is that in the present anticyclotomic setting, Heegner points on definite Shimura curves provide a similarly convenient way of describing the central critical values of the Rankin \( L \)-series \( L(f/K, \chi, s) \).

Also fundamental for the method of [EPW06] is the possibility to “deform” modular symbols in Hida families. In our anticyclotomic context, the construction of big Heegner points in Hida families was obtained in the work [LV11] of the third named author in collaboration with Vigni, following an original construction due to Howard [How07], while the relation between these points and classical \( L \)-values was established in the work [CL14] by the first and third named authors. With these key results at hand, and working over appropriate quotients of the Hecke algebras considered in [EPW06] via the Jaquet–Langlands correspondence, we are then able to adapt the arguments of *loc.cit.* to our setting, making use of the ramification hypotheses on \( \bar{\rho} \) to ensure a multiplicity one property of certain Hecke modules (among other uses).

We conclude this introduction with the following overview of the contents of this paper. In the next section, we briefly recall the Hida theory used in this paper, following the exposition in [EPW06, §1] for the most part. In Section 2, we describe an extension of the construction of big Heegner points to “imprimitive” branches of the Hida family, an extension necessary for the purposes of this paper. In Section 3, we construct two-variable \( p \)-adic \( L \)-functions attached to a Hida family and to each of its irreducible components (or branches), and prove Theorem 3.9 relating the two. This theorem is the key technical result of this paper, and the analytic part of Theorem 2 follows easily from this. In Section 4, we deduce the algebraic part of Theorem 2 using the residual Selmer groups studied in [PW11, §3.2]. Finally, in Section 5 we give the applications of our results to the anticyclotomic main conjecture.

### 1. Hida theory

#### 1.1. Hecke algebras

Fix a positive integer \( N \) admitting a factorization \( N = N^+N^- \) with \( (N^+, N^-) = 1 \) and \( N^- \) square-free, and fix a prime \( p \nmid N \).

For each integer \( k \geq 2 \), denote by \( \mathfrak{h}_{N,r,k} \) the \( \mathbb{Z}_p \)-algebra generated by the Hecke operators \( T_\ell \) for \( \ell \nmid Np \), the operators \( U_\ell \) for \( \ell \mid Np \), and the diamond operators \( \langle a \rangle \) for \( a \in \langle \mathbb{Z}/p^r \mathbb{Z} \rangle^\times \), acting on the space \( S_k(\Gamma_0,1(N,p^r), \mathfrak{Q}_p) \) of cusp forms of weight \( k \) on \( \Gamma_{0,1}(N,p^r) := \Gamma_0(N) \cap \Gamma_1(p^r) \). For \( k = 2 \), we abbreviate \( \mathfrak{h}_{N,r} := \mathfrak{h}_{N,r,2} \).
Let \( e^{\text{ord}} := \lim_{n \to \infty} U_{p}^{n} \) be Hida’s ordinary projector, and define
\[
\mathfrak{h}_{N,r,k}^{\text{ord}} := e^{\text{ord}} \mathfrak{h}_{N,r,k}, \quad \mathfrak{h}_{N,r}^{\text{ord}} := e^{\text{ord}} \mathfrak{h}_{N,r}, \quad \mathfrak{h}_{N}^{\text{ord}} := \lim_{r \to \infty} \mathfrak{h}_{N,r}^{\text{ord}}
\]
where the limit is over the projections induced by the natural restriction maps.

Let \( T_{N,r,k}^{\text{ord}} \) be the quotient of \( \mathfrak{h}_{N,r,k}^{\text{ord}} \) acting faithfully on the subspace of \( e^{\text{ord}} S_{k}(\Gamma_{0,1}(N, p^{r}), \overline{\mathbb{Q}}_{p}) \) consisting of forms which are new at all primes dividing \( N^{-} \). Set \( T_{N,r}^{\text{ord}} := T_{N,r,2}^{\text{ord}} \) and define
\[
T_{N}^{\text{ord}} := \lim_{r \to \infty} T_{N,r}^{\text{ord}}.
\]

Each of these Hecke algebras are equipped with natural \( \mathbb{Z}_{p}[\mathbb{Z}_{p}^{\text{ord}}] \)-algebra structures via the diamond operators, and by a well-known result of Hida, \( \mathfrak{h}_{N}^{\text{ord}} \) is finite and flat over \( \mathbb{Z}_{p}[1+p\mathbb{Z}_{p}] \).

1.2. Galois representations on Hecke algebras. For each positive integer \( M \mid N \) we may consider the new quotient \( T_{M}^{\text{new}} \) of \( \mathfrak{h}_{M}^{\text{ord}} \), and the Galois representation
\[
\rho_{M} : G_{Q} \to \GL_{2}(T_{M}^{\text{new}} \otimes \mathcal{L})
\]
described in [EPW06, Thm. 2.2.1], where \( \mathcal{L} \) denotes the fraction field of \( \mathbb{Z}_{p}[1+p\mathbb{Z}_{p}] \).

Let \( T_{N}^{\text{new}} \) be the quotient of \( \mathfrak{h}_{N}^{\text{ord}} \) generated by the image under the natural projection \( \mathfrak{h}_{N}^{\text{ord}} \to T_{N}^{\text{ord}} \) of the Hecke operators of level prime to \( N \). As in [EPW06, Prop. 2.3.2], one can show that the canonical map
\[
T_{N}^{\text{new}} \to \prod_{M} T_{M}^{\text{new}}
\]
where the product is over all integers \( M \geq 1 \) with \( N^{-} \mid M \mid N \), becomes an isomorphism after tensoring with \( \mathcal{L} \). Taking the product of the Galois representations \( \rho_{M} \) we thus obtain
\[
\rho : G_{Q} \to \GL_{2}(T_{N}^{\text{new}} \otimes \mathcal{L}).
\]

For any maximal ideal \( \mathfrak{m} \) of \( T_{N} \), let \( (T_{N})_{\mathfrak{m}} \) denote the localization of \( T_{N} \) at \( \mathfrak{m} \) and let
\[
\rho_{\mathfrak{m}} : G_{Q} \to \GL_{2}((T_{N})_{\mathfrak{m}} \otimes \mathcal{L})
\]
be the resulting Galois representation. If the residual representation \( \overline{\rho}_{\mathfrak{m}} \) is irreducible, then \( \rho_{\mathfrak{m}} \) admits an integral model (still denoted in the same manner)
\[
\rho_{\mathfrak{m}} : G_{Q} \to \GL_{2}((T_{N})_{\mathfrak{m}})
\]
which is unique up to isomorphism.

1.3. Residual representations. Let \( \overline{\rho} : G_{Q} \to \GL_{2}(\mathbb{F}) \) be an odd irreducible Galois representation defined over a finite field \( \mathbb{F} \) of characteristic \( p \). By [KW09], \( \overline{\rho} \) is modular, meaning that it arises as the residual representation associated with a modular form of some weight and level defined over \( \overline{\mathbb{Q}}_{p} \). Consider three more conditions we may impose on \( \overline{\rho} \), where \( N^{-} \) is a fixed square-free product of an odd number of primes.

Assumption (SU). (1) \( \overline{\rho} \) is \( p \)-ordinary: the restriction of \( \overline{\rho} \) to a decomposition group
\[
G_{p} \subset G_{Q} \text{ at } p \text{ has a one-dimensional unramified quotient over } \mathbb{F}.
\]
(2) \( \overline{\rho} \) is \( p \)-distinguished: \( \overline{\rho}|_{G_{p}} \sim (\overline{\chi}, \overline{\delta}) \) with \( \overline{\epsilon} \neq \overline{\delta} \).
(3) \( \overline{\rho} \) is ramified at all primes \( \ell \mid N^{-} \).

Fix once and for all a representation \( \overline{\rho} \) as above satisfying Assumption (SU), together with a \( p \)-stabilization of \( \overline{\rho} \) in the sense of [EPW06, Def. 2.2.10]. Let \( \nabla \) be a two-dimensional \( \mathbb{F} \)-vector space which affords \( \overline{\rho} \), and for any finite set of primes \( \Sigma \) that does not contain \( p \), define
\[
N(\Sigma) := N(\overline{\rho}) \prod_{\ell \in \Sigma} \ell^{m_{\ell}}
\]
where $N(\bar{\rho})$ is the tame conductor of $\bar{\rho}$, and $m_\ell := \dim_{\F} V_{I_\ell}$.

Combining [EPW06, Thm. 2.4.1] and [EPW06, Prop. 2.4.2] with the fact that $\bar{\rho}$ is ramified at the primes dividing $N^-$, one can see that there exist unique maximal ideals $\mathfrak{n}$ and $\mathfrak{m}$ of $T'_N(\Sigma)$ and $T'_N(\Sigma')$, respectively, such that $\mathfrak{n}$ lifts $\mathfrak{m}$, $(T'_N(\Sigma))_\mathfrak{m} \simeq (T'_N(\Sigma))_\mathfrak{n}$, and $\bar{\rho}_\mathfrak{m} \simeq \bar{\rho}$. Define the ordinary Hecke algebra $T_\Sigma$ attached to $\bar{\rho}$ and $\Sigma$ by

$$T_\Sigma := (T'_N(\Sigma))_\mathfrak{m}.$$ 

Thus $T_\Sigma$ is a local factor of $T'_N(\Sigma)$, and we let

$$\rho_{\Sigma} : G_{\Q} \to \GL_2(T_\Sigma)$$

denote the Galois representation deduced from $\rho_m$.

Following the terminology of [EPW06, §2.4], we shall refer to $\Spec(T_\Sigma)$ as “the Hida family” $\mathcal{H}(\bar{\rho})$ attached to $\bar{\rho}$ (and our chosen $p$-stabilization) that is minimally ramified outside $\Sigma$.

1.4. Branches of the Hida family. If $\mathfrak{a}$ is a minimal prime of $T_\Sigma$ (for a finite set of primes $\Sigma$ as above), we put $T(\mathfrak{a}) := T_\Sigma/\mathfrak{a}$ and let

$$\rho(\mathfrak{a}) : G_{\Q} \to \GL_2(T(\mathfrak{a}))$$

be the Galois representation induced by $\rho_{\Sigma}$. As in [EPW06, Prop. 2.5.2], one can show that there is a unique divisor $N(\mathfrak{a})$ of $N(\Sigma)$ and a unique minimal prime $\mathfrak{a}' \subset T_{N(\mathfrak{a})}$ above $\mathfrak{a}$ such that the diagram

$$\begin{array}{ccc}
T_\Sigma & \longrightarrow & T'_N(\Sigma) \\
\downarrow & & \downarrow \\
T_\Sigma/\mathfrak{a} & \longrightarrow & T(\mathfrak{a}) \\
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\downarrow & & \downarrow \\
\prod_{N^{-}|M| N(\Sigma)} T_{M}^\text{new} & \longrightarrow & T_{N(\mathfrak{a})}^\text{new}/\mathfrak{a}' \\
\end{array}$$

commutes. We call $N(\mathfrak{a})$ the tame conductor of $\mathfrak{a}$ and set

$$T(\mathfrak{a})^\circ := T_{N(\mathfrak{a})}^\text{new}/\mathfrak{a}' .$$

In particular, note that $N^{-}|N(\mathfrak{a})$ by construction, and that the natural map $T(\mathfrak{a}) \to T(\mathfrak{a})^\circ$ is an embedding of local domains.

1.5. Arithmetic specializations. For any finite $\Z_p[[1+p\Z_p]]$-algebra $T$, we say that a height one prime $\varphi$ of $T$ is an arithmetic prime of $T$ if $\varphi$ is the kernel of a $\Z_p$-algebra homomorphism $T \to \Q_p$ such that the composite map

$$1 + p\Z_p \to \Z_p[[1+p\Z_p]]^\times \to T^\times \to \Q_p^\times$$

is given by $\gamma \mapsto \gamma^{k-2}$ on some open subgroup of $1 + p\Z_p$, for some integer $k \geq 2$. We then say that $\varphi$ has weight $k$.

Let $\mathfrak{a} \subset T_\Sigma$ be a minimal prime as above. For each $n \geq 1$, let $\mathfrak{a}_n \in T(\mathfrak{a})^\circ$ be the image of $T_n$ under the natural projection $\mathfrak{h}_{N(\Sigma)}^{\text{ord}} \to T(\mathfrak{a})^\circ$, and form the $q$-expansion

$$f(\mathfrak{a}) = \sum_{n \geq 1} \mathfrak{a}_n q^n \in T(\mathfrak{a})^\circ[[q]].$$

By [Hid86, Thm. 1.2], if $\varphi$ is an arithmetic prime of $T(\mathfrak{a})$ of weight $k$, then there is a unique height one prime $\varphi'$ of $T(\mathfrak{a})^\circ$ such that

$$f_{\varphi}(\mathfrak{a}) := \sum_{n \geq 1} (\mathfrak{a}_n \mod \varphi') q^n \in \mathcal{O}_{\varphi'}[[q]]$$

is the $q$-expansion a $p$-ordinary eigenform $f_{\varphi}$ of weight $k$ and tame level $N(\mathfrak{a})$, where $\mathcal{O}_{\varphi'} := T(\mathfrak{a})^\circ/\varphi'$ (see [EPW06, Prop. 2.5.6]).
2. Big Heegner points

Fix an integer $N \geq 1$ admitting a factorization $N = N^+ N^-$ with $(N^+, N^-) = 1$ and $N^-$ equal to the square-free product of an odd number of primes, and fix a prime $p \nmid 6N$. Also, let $K$ be an imaginary quadratic field of discriminant $-D_K < 0$ prime to $Np$ and such that every prime factor of $N^+$ (resp. $N^-$) splits (resp. is inert) in $K$.

In this section we describe a mild extension of the construction in [LV11] (following [How07]) of big Heegner points attached to $K$. Indeed, using the results from the preceding section, we can extend the constructions of loc.cit. to branches of the Hida family which are not necessarily primitive (in the sense of [Hid86, §1]). The availability of such extension is fundamental for the purposes of this paper.

2.1. Definite Shimura curves. Let $B$ be the definite quaternion algebra over $\mathbb{Q}$ of discriminant $N^-$. We fix once and for all an embedding of $\mathbb{Q}$-algebras $K \hookrightarrow B$, and use it to identify $K$ with a subalgebra of $B$. Denote by $z \mapsto \overline{z}$ the nontrivial automorphism of $K$, and choose a basis $\{1, j\}$ of $B$ over $K$ with

- $j^2 = \beta \in \mathbb{Q}^\times$ with $\beta < 0$, 
- $jt = i\bar{t}$ for all $t \in K$, 
- $\beta \in (\mathbb{Z}_q)^2$ for $q \mid pN^+$, and $\beta \in \mathbb{Z}_q^\times$ for $q \mid D_K$.

Fix a square-root $\delta_K = \sqrt{-D_K}$, and define $\theta \in K$ by

$$\theta := \frac{D' + \delta_K}{2}, \text{ where } D' := \begin{cases} D_K & \text{if } 2 \nmid D_K, \\ D_K/2 & \text{if } 2 \mid D_K. \end{cases}$$

For each prime $q \mid pN^+$, define $i_q : B_q := B \otimes \mathbb{Q} \cong M_2(\mathbb{Q}_q)$ by

$$i_q(\theta) = \begin{pmatrix} \text{Tr}(\theta) & -\text{Nm}(\theta) \\ 1 & 0 \end{pmatrix}, \quad i_q(j) = \sqrt{\beta} \begin{pmatrix} -1 & \text{Tr}(\theta) \\ 0 & 1 \end{pmatrix},$$

where $\text{Tr}$ and $\text{Nm}$ are the reduced trace and reduced norm maps on $B$, respectively. On the other hand, for each prime $q \nmid Np$ we fix any isomorphism $i_q : B_q \cong M_2(\mathbb{Q}_q)$ with the property that $i_q(\mathcal{O}_K \otimes \mathbb{Z} \mathcal{Z}_q) \subset M_2(\mathbb{Z}_q)$.

For each $r \geq 0$, let $R_{N+r}$ be the Eichler order of $B$ of level $N^+ p^r$ with respect to the above isomorphisms $\{i_q : B_q \cong M_2(\mathbb{Q}_q)\}_{q \mid N^+}$, and let $U_{N+r}$ be the compact open subgroup of $\hat{R}_{N+r}^\times$ defined by

$$U_{N+r} := \left\{ (x_r)_{q \mid N} \in \hat{R}_{N+r}^\times \mid i_p(x_p) \equiv \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix} \pmod{p^r} \right\}.$$

Consider the double coset spaces

$$\tilde{X}_{N+r} = B^\times \backslash \left( \text{Hom}_\mathbb{Q}(K, B) \times \hat{B}^\times \right) / U_{N+r},$$

where $b \in B^\times$ acts on $(\Psi, g) \in \text{Hom}_\mathbb{Q}(K, B) \times \hat{B}^\times$ by

$$b \cdot (\Psi, g) = (b\Psi b^{-1}, bg)$$

and $U_{N+r}$ acts on $\hat{B}^\times$ by right multiplication.

As explained in [LV11, §2.1], $\tilde{X}_{N+r}$ is naturally identified with the set of $K$-rational points of certain genus zero curves defined over $\mathbb{Q}$. Nonetheless, there is a nontrivial Galois action on $\tilde{X}_{N+r}$ defined as follows: If $\sigma \in \text{Gal}(K^{ab}/K)$ and $P \in \tilde{X}_{N+r}$ is the class of a pair $(\Psi, g)$, then

$$\sigma P := [(\Psi, \Psi(a)g)]$$

where $a \in K^\times \backslash \hat{K}^\times$ is chosen so that $\text{rec}_K(a) = \sigma$. It will be convenient to extend this action to an action of $G_K := \text{Gal}(\overline{\mathbb{Q}}/K)$ by letting $\sigma \in G_K$ act on $\tilde{X}_{N+r}$ as $\sigma|_{K^{ab}}$. Since $\text{Gal}(K^{ab}/K)$ is obviously abelian, we will set $P^\sigma := \sigma P$ for the ease of notation.
Finally, we note that $\tilde{X}_{N,+}$ is also equipped with standard actions of $U_p$, Hecke operators $T_\ell$ for $\ell \nmid Np$, and diamond operators $\langle d \rangle$ for $d \in (\mathbb{Z}/p^r\mathbb{Z})^\times$ (see [LV11, §2.4], for example).

2.2. Compatible systems of Heegner Points. For each integer $c \geq 1$, let $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$ be the order of $K$ of conductor $c$.

Definition 2.1. We say that $P \in \tilde{X}_{N,+}$ is a Heegner point of conductor $c$ if $P$ is the class of a pair $(\Psi, g)$ with

$$\Psi(\mathcal{O}_c) = \Psi(K) \cap (B \cap g\tilde{R}_{N,+}g^{-1})$$

and

$$\Psi_p((\mathcal{O}_c \otimes \mathbb{Z}_p)^\times) \cap (1 + p^r\mathcal{O}_K \otimes \mathbb{Z}_p)^\times = \Psi_p((\mathcal{O}_c \otimes \mathbb{Z}_p)^\times) \cap g_pU_{N,+}g_p^{-1}$$

where $U_{N,+}p$ denotes the $p$-component of $U_{N,+}$.

Fix a decomposition $N^+\mathcal{O}_K = \mathfrak{m}^+\overline{\mathfrak{m}}^+$, and define, for each prime $q \nmid p$,

- $\varsigma_q = 1$, if $q \nmid N^+$,
- $\varsigma_q = \delta^{-1}_K\left(\frac{\theta}{1} \begin{pmatrix} p^s & 0 \\ 0 & 1 \end{pmatrix} \right) \in \text{GL}_2(K_q) = \text{GL}_2(\mathbb{Q}_q)$, if $q = q\overline{q}$ splits with $q|\mathfrak{m}^+$,

and for each $s \geq 0$,

- $\varsigma^{(s)}_\mathfrak{p} = \left(\begin{array}{cc} \theta & -1 \\ 1 & 0 \end{array} \right)$ if $p = q\overline{q}$ splits in $K$,
- $\varsigma^{(s)}_\mathfrak{p} = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ if $p$ is inert in $K$.

Set $\varsigma^{(s)} = \varsigma^{(s)}_\mathfrak{p} \prod_{q \neq \mathfrak{p}} \varsigma_q \in \hat{B}^\times$, and let $i_K : K \hookrightarrow B$ be the inclusion. For all $n, r \geq 0$, it is easy to see that the point

$$\tilde{P}_{p^n,r} := [(i_K, \varsigma^{(n+r)})] \in \tilde{X}_{N,+}$$

is a Heegner point of conductor $p^{n+r}$. Moreover, one can show that the points $\tilde{P}_{p^n,r}$ enjoy the following properties:

- **Field of definition**: $\tilde{P}_{p^n,r} \in H^0(L_{p^n,r}, \tilde{X}_{N,+})$, where $L_{p^n,r} := H_{p^{n+r}}(\mu_{p^n})$ and $H_c$ is the ring class field of $K$ of conductor $c$.
- **Galois equivariance**: For all $\sigma \in \text{Gal}(L_{p^n,r}/H_{p^{n+r}})$,

$$\tilde{P}^{\sigma}_{p^n,r} = \langle \vartheta(\sigma) \rangle \cdot \tilde{P}_{p^n,r}$$

where $\vartheta : \text{Gal}(L_{p^n,r}/H_{p^{n+r}}) \to \mathbb{Z}_p^\times / \{\pm 1\}$ is such that $\vartheta^2 = \varepsilon_{\text{cyc}}$.
- **Horizontal compatibility**: If $r > 1$, then

$$\sum_{\sigma \in \text{Gal}(L_{p^n,r}/L_{p^{n-1},r})} \tilde{\alpha}_r(\tilde{P}^{\sigma}_{p^n,r}) = U_p \cdot \tilde{P}_{p^{n-1},r}$$

where $\tilde{\alpha}_r : \tilde{X}_{N,+} \to \tilde{X}_{N,+}^{r-1}$ is the map induced by the inclusion $U_{N,+} \subset U_{N,+}^{r-1}$.
- **Vertical Compatibility**: If $n > 0$, then

$$\sum_{\sigma \in \text{Gal}(L_{p^n,r}/L_{p^{n-1},r})} \tilde{P}^{\sigma}_{p^n,r} = U_p \cdot \tilde{P}_{p^{n-1},r}.$$
2.3. Critical character. Factor the $p$-adic cyclotomic character as
\[ \varepsilon_{\text{cyc}} = \varepsilon_{\text{tame}} \cdot \varepsilon_{\text{wild}} : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p) \]
and define the critical character $\Theta : G_{\mathbb{Q}} \to \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]^\times$ by
\[ \Theta(\sigma) = [\varepsilon_{\text{wild}}^{1/2}(\sigma)] \]
where $\varepsilon_{\text{wild}}^{1/2}$ is the unique square-root of $\varepsilon_{\text{wild}}$ taking values in $1 + p\mathbb{Z}_p$, and $[\cdot] : 1 + p\mathbb{Z}_p \to \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]^\times$ is the map given by the inclusion as group-like elements.

2.4. Big Heegner points. Recall the Shimura curves $\tilde{X}_{N^+,p^r}$ from Section 2.1, and set
\[ \mathfrak{D}_{N^+,r} := e^{\text{ord}}(\text{Div}(\tilde{X}_{N^+,r}) \otimes \mathbb{Z}_p); \]
by the Jacquet–Langlands correspondence, $\mathfrak{D}_{N^+,r}$ is naturally endowed with an action of the Hecke algebra $\mathbb{T}_{N^+,r}$. Let $(\mathbb{T}_{N^+,r})^\dagger$ be the free $\mathbb{T}_{N^+,r}$-module of rank one equipped with the Galois action via the inverse of the critical character $\Theta$, and set
\[ \mathfrak{D}_{N^+,r}^\dagger := \mathfrak{D}_{N^+,r} \otimes_{\mathbb{T}_{N^+,r}} (\mathbb{T}_{N^+,r})^\dagger. \]

Let $\tilde{P}_{p^s,r} \in \tilde{X}_{N^+,r}$ be the system of Heegner points of Section 2.2, and denote by $P_{p^s,r}$ the image of $e^{\text{ord}}\tilde{P}_{p^s,r}$ in $\mathfrak{D}_{N^+,r}$. By the Galois equivariance of $\tilde{P}_{p^s,r}$ (see [LV11, §7.1]), we have
\[ P_{p^s,r} := \Theta(\sigma) \cdot P_{p^s,r} \]
for all $\sigma \in \text{Gal}(L_{p^s,r}/H_{p^{s+r}})$, and hence $P_{p^s,r}$ defines an element
\[ P_{p^s,r} \otimes \zeta_r \in H^0(H_{p^{s+r}}, \mathfrak{D}_{N^+,r}^\dagger). \]

In the next section we shall see how this system of points, for varying $n$ and $r$, can be used to construct various anticyclotomic $p$-adic $L$-functions.

3. Anticyclotomic $p$-adic $L$-functions

3.1. Multiplicity one. Keep the notations introduced in Section 2. For each integer $k \geq 2$, denote by $\mathcal{L}_k(R)$ the set of polynomials of degree less than or equal to $k - 2$ with coefficients in a ring $R$, and define
\[ \mathfrak{J}_{N^+,r,k} := e^{\text{ord}}H_0(\tilde{X}_{N^+,r}, \mathcal{L}_k(\mathbb{Z}_p)) \]
where $\mathcal{L}_k(\mathbb{Z}_p)$ is the local system on $\tilde{X}_{N^+,r}$ associated with $L_k(\mathbb{Z}_p)$. Note that $\mathfrak{J}_{N^+,r,k}$ is naturally a module over the Hecke algebra $\mathbb{T}_{N^+,r,k}$.

**Theorem 3.1.** Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_{N^+,r,k}$ whose residual representation is irreducible and satisfies Assumption (SU). Then $(\mathfrak{J}_{N^+,r,k})_\mathfrak{m}$ is free of rank one over $(\mathbb{T}_{N^+,r,k})_\mathfrak{m}$. In particular, there is a $(\mathbb{T}_{N^+,r,k})_\mathfrak{m}$-module isomorphism
\[ (\mathfrak{J}_{N^+,r,k})_\mathfrak{m} \cong (\mathbb{T}_{N^+,r,k})_\mathfrak{m}. \]

**Proof.** If $k = 2$ and $r = 1$, this follows by combining [PW11, Thm. 6.2] and [loc.cit., Prop. 6.5]. The general case will be deduced from this case in Section 3.3 using Hida theory.

Associated with any $N^-$-new eigenform $f \in S_k(\Gamma_{0,1}(N,p^r))$ whose associated maximal ideal in $\mathbb{T}_{N^+,r,k}$ is $\mathfrak{m}$, there is a $\mathbb{Z}_p$-algebra homomorphism $(\mathbb{T}_{N^+,r,k})_\mathfrak{m} \to \mathcal{O}$, where $\mathcal{O}$ is the ring of integers of a finite extension $F/\mathbb{Q}_p$ generated by the Hecke eigenvalues of $f$. Composing with a fixed isomorphism $\alpha_{N^+,r,k}$ as in Theorem 3.1, we thus obtain the functional
\[ \delta_f : (\mathfrak{J}_{N^+,r,k})_\mathfrak{m} \to \mathcal{O}. \]
On the other hand, if $\phi_f \in S_k(\tilde{X}_{N+r})$ is a $p$-adically normalised Jacquet–Langlands transfer (in the sense of [CH13, §4.1]) of $f$, then by evaluation $\phi_f$ defines another $\mathcal{O}$-valued functional
\[
\phi_f : (\mathfrak{J}_{N^+,r,k})_m \rightarrow \mathcal{O}.
\]

By the multiplicity one theorem, $\phi_f$ and $\delta_f$ differ by a nonzero constant $\lambda_f \in F^\times$ which is easily seen to be necessarily a $p$-adic unit. Since both $\phi_f$ and $\delta_f$ are themselves defined up to a $p$-adic unit, we may assume $\phi_f = \delta_f$, as we shall do in the following.

If $f$ is in fact a newform, following [PW11, §2.1] and [CH13, §4.1] we define Gross period
\[
\Omega_{f,N^-} := (f,f)_{\Gamma_0(N)} / \xi_f(N^+,N^-)
\]
where $\xi_f(N^+,N^-)$ is the self-product of $\phi_f$ with respect to a certain “intersection” pairing (see [CH13, Eq.(3.9)]). In [loc.cit., §5.4], it is shown that a certain $p$-adic $L$-function $L_p(f/K)$ normalized by the complex period $\Omega_{f,N^-}$ has vanishing $\mu$-invariant. The preceding description of $\phi_f$ in terms of $\delta_f$ will thus allow us to show that this property is preserved over the Hida family.

3.2. One-variable $p$-adic $L$-functions. Denote by $\Gamma$ the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension $K_{\infty}/K$. For each $n$, let $K_n \subset K_\infty$ be defined by $\text{Gal}(K_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$ and let $\Gamma_n$ be the subgroup of $\Gamma$ such that $\Gamma / \Gamma_n \simeq \text{Gal}(K_n/K)$.

Let $\mathcal{P}_{p^{n+1},r} \otimes \zeta_r \in H^0(H_{p^{n+1},r}, \mathcal{D}_{N^+,r}^\dagger)$ be the Heegner point of conductor $p^{n+1}$, and define
\[
\mathcal{Q}_{n,r} := \text{Cor}_{H_{p^{n+1},K_n} / H_{p^n+1,r}}(\mathcal{P}_{p^{n+1},r} \otimes \zeta_r) \in H^0(K_n, \mathcal{D}_{N^+,r}^\dagger);
\]
with a slight abuse of notation, we will still denote by $\mathcal{Q}_{n,r}$ the image under the natural map $H^0(K_n, \mathcal{D}_{N^+,r}^\dagger) \subset \mathcal{J}_{N^+,r}$ composed with localization at $m$, where $\mathcal{J}_{N^+,r} := \mathcal{J}_{N^+,r,2}$.

Definition 3.2. For any open subset $\sigma \Gamma_n$ of $\Gamma$, define
\[
\mu_r(\sigma \Gamma_n) := U_p^{-n} \cdot \mathcal{Q}_{n,r} \in (\mathcal{J}_{N^+,r})_m.
\]

Proposition 3.3. The rule $\mu_r$ is a measure on $\Gamma$.

Proof. This follows immediately from the “horizontal compatibility” of Heegner points. \hfill \Box

3.3. Gross periods in Hida families. Keep the notations of Section 3.1, and let
\[
(\mathcal{J}_{N^+})_m := \lim_{r} (\mathfrak{J}_{N^+,r})_m
\]
which is naturally equipped with an action of the big Hecke algebra $T_{N^-}^N = \lim_{r} T_{N,r}^N$.

Theorem 3.4. Let $m$ be a maximal ideal of $T_{N}^{N^-}$ whose residual representation is irreducible and satisfies Assumption (SU). Then $(\mathcal{J}_{N^+})_m$ is free of rank one over $(T_{N}^{N^-})_m$. In particular, there is a $(T_{N}^{N^-})_m$-module isomorphism
\[
(\mathcal{J}_{N^+})_m \cong (T_{N}^{N^-})_m.
\]

Proof. As in [EPW06, Prop. 3.3.1]. Note that the version of Hida’s control theorem in our context is provided by [Hid88, Thm. 9.4]. \hfill \Box

We can now conclude the proof of Theorem 3.1 just as in [EPW06, §3.3]. For the convenience of the reader, we include here the argument.
Proof of Theorem 3.1. Let $\psi_{N,r,k}$ be the product of all the arithmetic primes of $\mathbb{T}_N^N$ of weight $k$ which become trivial upon restriction to $1 + p^r\mathbb{Z}_p$. By [Hid88, Thm. 9.4], we then have

$$\mathfrak{m}_{r,k} \cap \mathbb{T}_N^N/\psi_{N,r,k} \simeq (\mathfrak{J}_{N+})_{m_{r,k}}$$

where $m_{r,k}$ is the maximal ideal of $\mathbb{T}_N^N$ induced by $m$. Since $(\mathfrak{J}_{N+})_m$ is free of rank one over $\mathbb{T}_N^N$ by Theorem 3.4, it follows that $(\mathfrak{J}_{N+,r,k})_{m_{r,k}}$ is free of rank one over $\mathbb{T}_N^N/\psi_{N,r,k}$, as was to be shown. □

Remark 3.5. In the above proofs we made crucial use of [Hid88, Thm. 9.4], which is stated in the context of totally definite quaternion algebras which are unramified at all finite places, since this is the only relevant case for the study of Hilbert modular forms over totally real number fields of even degree. However, the proofs immediately extend to the (simpler) situation of definite quaternion algebras over $\mathbb{Q}$.

3.4. Two-variable $p$-adic $L$-functions. By the “vertical compatibility” satisfied by Heegner points, the points

$$U_p^{-r} \cdot \mathfrak{Q}_{n,r} \in (\mathfrak{J}_{N+,r})_m$$

are compatible for varying $r$, thus defining an element

$$\mathfrak{Q}_n := \lim_{r \to \mathfrak{J}_{N+}} U_p^{-r} \cdot \mathfrak{Q}_{n,r} \in (\mathfrak{J}_{N+})_m.$$  

Definition 3.6. For any open subset $\sigma \Gamma_n$ of $\Gamma$, define

$$\mu(\sigma \Gamma_n) := U^{-n}_p \cdot \mathfrak{Q}_n \in (\mathfrak{J}_{N+})_m.$$  

Proposition 3.7. The rule $\mu$ is a measure on $\Gamma$.

Proof. This follows immediately from the “horizontal compatibility” of Heegner points. □

Upon the choice of an isomorphism $\sigma_N$ as in Theorem 3.4, we may regard $\mu$ as an element

$$\mathcal{L}(m, N) \in (\mathbb{T}_N^N)_m \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]].$$

Denoting by $\mathcal{L}(m, N)^*$ the image of $\mathcal{L}(m, N)$ under the involution induced by $\gamma \mapsto \gamma^{-1}$ on group-like elements, we set $L(m, N) := \mathcal{L}(m, N) \cdot \mathcal{L}(m, N)^*$, to which we will refer as the two-variable $p$-adic $L$-function attached to $(\mathbb{T}_N^N)_m$.

3.5. Two-variable $p$-adic $L$-functions on branches of the Hida family. Let $F$ be a finite field of characteristic $p$, let $\tilde{\rho} : G_Q \to \text{GL}_2(F)$ be an odd irreducible (and hence modular!) Galois representation satisfying Assumption (SU), and let $T_\Sigma$ be the universal ordinary Hecke algebra

$$T_\Sigma := (T_{N(\Sigma)})_m \simeq (T_{N(\Sigma)}^N)_n$$

associated with $\tilde{\rho}$ and a finite set of primes $\Sigma$ as described in Section 1.3.

Remark 3.8. Note that $N^-|N(\tilde{\rho})$ by hypothesis. Throughout the following, it will be assumed that $N^-$ contains all prime factors of $N(\tilde{\rho})$ which are inert in $K$ and at which $\tilde{\rho}$ is ramified, and that every prime factor of $N(\Sigma)/N^-$ splits in $K$. In particular, every prime $\ell \in \Sigma$ splits in $K$.

The construction of the preceding section produces a two-variable $p$-adic $L$-function

$$L(n, N(\Sigma)) \in (T_{N(\Sigma)}^N)_n \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]]$$

which combined with the isomorphism (8) yields an element

$$L_\Sigma(\tilde{\rho}) \in T_\Sigma \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]].$$
If $a$ is a minimal prime of $T_{\Sigma}$, we thus obtain an element
\[ L_{\Sigma}(\bar{\rho}, a) \in \mathbb{T}(a)^{\circ} \hat{\otimes} \mathbb{Z}_p \mathbb{Z}_p[\Gamma] \]
by reducing $L_{\Sigma}(\bar{\rho}) \bmod a$ (see §1.4). On the other hand, if we let $m$ be the inverse image of the maximal ideal of $\mathbb{T}(a)^{\circ}$ under the composite surjection
\[ (9) \quad T_{N(a)}^{N^{-}} \rightarrow T_{N(a)}^{new} \rightarrow T_{N(a)}^{new}/a' = T(a)^{\circ}, \]
the construction of the preceding section yields an $L$-function
\[ L(m, N(a)) \in (T_{N(a)}^{N^{-}})_{m} \hat{\otimes} \mathbb{Z}_p \mathbb{Z}_p[\Gamma] \]
giving rise, via (9), to a second element
\[ L(\bar{\rho}, a) \in \mathbb{T}(a)^{\circ} \hat{\otimes} \mathbb{Z}_p \mathbb{Z}_p[\Gamma]. \]

It is natural to compare $L_{\Sigma}(\bar{\rho}, a)$ and $L(\bar{\rho}, a)$, a task that is carried out in the next section, and provides the key for understanding the variation of analytic Iwasawa invariants.

3.6. **Comparison.** Write $\Sigma = \{\ell_1, \ldots, \ell_n\}$ and for each $\ell = \ell_i \in \Sigma$, let $e_\ell$ be the valuation of $N(\Sigma)/N(a)$ at $\ell$, and define the reciprocal Euler factor $E_\ell(a, X) \in \mathbb{T}(a)^{\circ}[X]$ by
\[ E_\ell(a, X) := \begin{cases} 1 & \text{if } e_\ell = 0 \\ 1 - (T_\ell \bmod a')\Theta^{-1}(\ell)X & \text{if } e_\ell = 1 \\ 1 - (T_\ell \bmod a')\Theta^{-1}(\ell)X + \ell X^2 & \text{if } e_\ell = 2. \end{cases} \]

Also, writing $\ell = \ell_0$, define $E_\ell(a) \in \mathbb{T}(a)^{\circ} \hat{\otimes} \mathbb{Z}_p \mathbb{Z}_p[\Gamma]$ by
\[ (10) \quad E_\ell(a) := E_\ell(a, \ell^{-1}\gamma) \cdot E_\ell(a, \ell^{-1}\gamma) \]
where $\gamma_i, \gamma_i$ are arithmetic Frobenii at $\ell, \bar{\ell}$ in $\Gamma$, respectively, and put $E_\Sigma(a) := \prod_{\ell \in \Sigma} E_\ell(a)$.

Recall that $N^{-}\bmod N(a) \bmod N(\Sigma)$ and set
\[ N(a)^{+} := N(a)/N^{-}; \quad N(\Sigma)^{+} := N(\Sigma)/N^{-} \]
both of which consist entirely of prime factors which split in $K$.

The purpose of this section to prove the following result.

**Theorem 3.9.** There is an isomorphism of $\mathbb{T}(a)^{\circ}$-modules
\[ \mathbb{T}(a)^{\circ} \hat{\otimes}_{(T_{N(a)}^{N^{-}})_{m}} \mathbb{Z}_p \mathbb{Z}_p[\Gamma] \simeq \mathbb{T}(a)^{\circ} \hat{\otimes}_{(T_{N(a)}^{N^{-}})_{m}} \mathbb{Z}_p \mathbb{Z}_p[\Gamma] \]
such that the map induced on the corresponding spaces of measures valued in these modules sends $L_{\Sigma}(\bar{\rho}, a)$ to $E_{\Sigma}(a)$ and $L(\bar{\rho}, a)$.

**Proof.** The proof follows very closely the constructions and arguments given in [EPW06, §3.8].

Let $r \geq 1$. If $M$ is any positive integer with $(M, pN^{-}) = 1$, and $d' | d$ are divisors of $M$, we have degeneracy maps
\[ B_{d,d'} : \tilde{X}_{M,r} \rightarrow \tilde{X}_{M/d,r} \]
induced by $(\Psi, g) \mapsto (\Psi, \pi_{d'}g)$, where $\pi_d \in \hat{\mathcal{B}}^\times$ has local component $\left( \begin{smallmatrix} 1 & 0 \\ 0 & e_{\text{val}(d')} \end{smallmatrix} \right)$ at every prime $\ell | d'$ and 1 outside $d'$. We thus obtain a map on homology
\[ (B_{d,d'})_* : e_{\text{ord}}H_0(\tilde{X}_{M,r}, \mathbb{Z}_p) \rightarrow e_{\text{ord}}H_0(\tilde{X}_{M/d,r}, \mathbb{Z}_p) \]
and we may define
\[ (11) \quad \epsilon_{\Gamma} : e_{\text{ord}}H_0(\tilde{X}_{N(\Sigma)^{+},r}, \mathbb{Z}_p) \rightarrow e_{\text{ord}}H_0(\tilde{X}_{N(a)^{+},r}, \mathbb{Z}_p) \]
by } ε_\ell := ε(\ell_n) \circ \cdots \circ ε(\ell_1), \text{ where for every } \ell = \ell_i \in \Sigma \text{ we put}
\epsilon(\ell) := \begin{cases} 1 & \text{if } ε_\ell = 0 \\ (B_{\ell,1})_\ast - (B_{\ell,\ell})_\ast ε_\ell^{-1} T_\ell & \text{if } ε_\ell = 1 \\ (B_{p,1})_\ast - (B_{p,\ell})_\ast ε_\ell^{-1} T_\ell + (B_{p,\ell})_\ast ε_\ell^{-1} (\ell|N_\ell)p & \text{if } ε_\ell = 2. \end{cases}

As before, let } M \text{ be a positive integer with } (M, pN^-) = 1 \text{ all of whose prime factors split in } K, \text{ and let } \ell \nmid Mp \text{ be a prime which also splits in } K. \text{ We shall adopt the following simplifying notations for the system of points } \tilde{P}_{p^n,r} \in \tilde{X}_{N+r} \text{ constructed in Section 2.2:}

\[ P := \tilde{P}_{p^n,r}^{(M)} \in \tilde{X}_{M+r}, \quad P^{(\ell)} := \tilde{P}_{p^n,r}^{(M\ell)} \in \tilde{X}_{M\ell+r}, \quad P^{(\ell^2)} := \tilde{P}_{p^n,r}^{(M\ell^2)} \in \tilde{X}_{M\ell^2+r}. \]

It is easy to check that for a suitable factorization } \ell = ll \text{ we then have the following relations:

\begin{itemize}
  \item \((B_{\ell,1})_\ast (P^{(\ell)}) = P\)
  \item \((B_{\ell,\ell})_\ast(P^{(\ell)}) = P^{\sigma_\ell}\)
  \item \((B_{p,1})_\ast(P^{(\ell)}) = L^{\sigma_\ell}\)
  \item \((B_{p,\ell})_\ast(P^{(\ell)}) = L^{\sigma_\ell}\)
  \item \((B_{p,\ell})_\ast(P^{(\ell^2)}) = P^{\sigma_\ell^2}\)
\end{itemize}

in } \tilde{X}_{M+r}, \text{ where } \sigma_\ell \in \text{Gal}(L_{p^n,r}/K) \text{ is a Frobenius element at } l. \text{ Letting } \mathcal{P} \text{ denote the image of } e^{ord} P \text{ in } \mathcal{D}_{M+r}, \text{ and defining } P^{(\ell)} \in \mathcal{D}_{M\ell+r} \text{ and } P^{(\ell^2)} \in \mathcal{D}_{M\ell^2+r} \text{ similarly, it follows that}

\begin{itemize}
  \item \((B_{\ell,1})_\ast (\mathcal{P} \otimes \zeta_r) = \mathcal{P} \otimes \zeta_r\)
  \item \((B_{\ell,\ell})_\ast(\mathcal{P} \otimes \zeta_r) = \mathcal{P}^{\sigma_\ell} \otimes \zeta_r = \Theta^{-1}(\sigma_\ell) \cdot (\mathcal{P} \otimes \zeta_r)^{\sigma_\ell}\)
  \item \((B_{p,1})_\ast(\mathcal{P} \otimes \zeta_r) = \mathcal{P}^{\sigma_\ell} \otimes \zeta_r = \Theta^{-1}(\sigma_\ell) \cdot (\mathcal{P} \otimes \zeta_r)^{\sigma_\ell}\)
  \item \((B_{p,\ell})_\ast(\mathcal{P} \otimes \zeta_r) = \mathcal{P}^{\sigma_\ell^2} \otimes \zeta_r = \Theta^{-2}(\sigma_\ell) \cdot (\mathcal{P} \otimes \zeta_r)^{\sigma_\ell^2}\)
\end{itemize}

as elements in } \mathcal{D}_{M+r}. \text{ Finally, setting } \mathcal{Q} := \text{Cor}_{H_{p^n+1+r}/K} (\mathcal{P}) \in H^0(K_n, \mathcal{D}_{M+r}^\dagger), \text{ and defining } \mathcal{Q}^{(\ell)} \in H^0(K_n, \mathcal{D}_{M\ell+r}^\dagger) \text{ and } \mathcal{Q}^{(\ell^2)} \in H^0(K_n, \mathcal{D}_{M\ell^2+r}^\dagger) \text{ similarly, we see that}

\begin{itemize}
  \item \((B_{\ell,1})_\ast(\mathcal{Q}^{(\ell)}) = \mathcal{Q}\)
  \item \((B_{\ell,\ell})_\ast(\mathcal{Q}^{(\ell)}) = \Theta^{-1}(\sigma_\ell) \cdot \mathcal{Q}^{\sigma_\ell}\)
  \item \((B_{p,1})_\ast(\mathcal{Q}^{(\ell^2)}) = \mathcal{Q}\)
  \item \((B_{p,\ell})_\ast(\mathcal{Q}^{(\ell^2)}) = \Theta^{-1}(\sigma_\ell) \cdot \mathcal{Q}^{\sigma_\ell^2}\)
  \item \((B_{p,\ell})_\ast(\mathcal{Q}^{(\ell^2)}) = \Theta^{-2}(\sigma_\ell) \cdot \mathcal{Q}^{\sigma_\ell^2}\)
\end{itemize}

in } H^0(K_n, \mathcal{D}_{M+r}^\dagger). \text{ Each of these equalities is checked by an explicit calculation. For example, for the second one:}

\begin{align*}
(B_{\ell,\ell})_\ast(\mathcal{Q}^{(\ell)}) &= (B_{\ell,\ell})_\ast \left( \text{Cor}_{H_{p^n+1+r}/K} (\mathcal{P} \otimes \zeta_r) \right) \\
&= (B_{\ell,\ell})_\ast \left( \sum_{\sigma \in \text{Gal}(H_{p^n+1+r}/K_n)} \Theta(\sigma^{-1}) \cdot (\mathcal{P} \otimes \zeta_r) \otimes \zeta_r \right) \\
&= \sum_{\sigma \in \text{Gal}(H_{p^n+1+r}/K_n)} \Theta(\sigma^{-1})(B_{\ell,\ell})_\ast ((\mathcal{P} \otimes \zeta_r) \otimes \zeta_r) \\
&= \sum_{\sigma \in \text{Gal}(H_{p^n+1+r}/K_n)} \Theta(\sigma^{-1})\Theta^{-1}(\sigma_\ell) \cdot (\mathcal{P} \otimes \zeta_r)^{\sigma_\ell^2} \\
&= \Theta^{-1}(\sigma_\ell) \cdot \mathcal{Q}^{\sigma_\ell^2}.
\end{align*}
Now let $Q_{n,r} \in \mathcal{J}_{N(\Sigma), +}^+$ be as in (6) with $N = N(\Sigma)$. Using the above formulae, we easily see that of any finite order character $\chi$ of $\Gamma$ of conductor $p^n$, the effect of $\epsilon_r$ on the element $\sum_{\sigma \in \Gamma / \Gamma_n} \chi(\sigma) Q_{n,r}^\sigma$ is given by multiplication by

$$
\prod_{\ell_i: \epsilon_{\ell_i} = 1} (1 - (\chi(\Omega))^{-1}(\sigma_{\ell_i}) \ell_i^{-1} T_{\ell_i}) \prod_{\ell_i: \epsilon_{\ell_i} = 2} (1 - (\chi(\Omega))^{-1}(\sigma_{\ell_i}) \ell_i^{-1} T_{\ell_i} + (\chi(\Omega))^{-2}(\sigma_{\ell_i}) \ell_i^{-1}(\ell_i)_{N(a)p}).
$$

Similarly, we see that $\epsilon_r$ has the effect of multiplying the element $\sum_{\sigma \in \Gamma / \Gamma_n} \chi^{-1}(\sigma) Q_{n,r}^\sigma$ by

$$
\prod_{\ell_i: \epsilon_{\ell_i} = 1} (1 - (\chi^{-1}(\Omega))^{-1}(\sigma_{\ell_i}) \ell_i^{-1} T_{\ell_i}) \prod_{\ell_i: \epsilon_{\ell_i} = 2} (1 - (\chi^{-1}(\Omega))^{-1}(\sigma_{\ell_i}) \ell_i^{-1} T_{\ell_i} + (\chi^{-1}(\Omega))^{-2}(\sigma_{\ell_i}) \ell_i^{-1}(\ell_i)_{N(a)p}).
$$

Hence, using the relations

$$
\chi(\sigma_{\ell_i}) = \chi^{-1}(\sigma_{\ell_i}); \quad \Theta(\sigma_{\ell_i}) = \Theta(\sigma_{\ell_i}) = \theta(\ell_i); \quad \theta^2(\ell_i) = (\ell_i)_{N(a)p}
$$

it follows that the effect of $\epsilon_r$ on the product of the above two elements is given by multiplication by

$$
\prod_{l_i: \epsilon_{l_i} = 1} (1 - \chi(\sigma_{\ell_i}) \theta^{-1}(\ell_i) \ell_i^{-1} T_{\ell_i}) \prod_{l_i: \epsilon_{l_i} = 2} (1 - \chi(\sigma_{\ell_i}) \theta^{-1}(\ell_i) \ell_i^{-1} T_{\ell_i} + \chi^2(\sigma_{\ell_i}) \ell_i^{-1}).
$$

Taking the limit over $r$, we thus obtain a $\mathbb{T}(a)^0$-linear map

$$
(12) \quad \mathbb{T}(a)^0 \otimes_{(\mathcal{J}_{N(\Sigma), +}^+)_n} (\mathcal{J}_{N(\Sigma), +}^+)_m \rightarrow \mathbb{T}(a)^0 \otimes_{(\mathcal{J}_{N(\Sigma), +}^+)_n} (\mathcal{J}_{N(\Sigma), +}^+)_m
$$

having as effect on the corresponding measures as stated in Theorem 3.9. Hence to conclude the proof it remains to show that (12) is an isomorphism.

By Theorem 3.4, both the source and the target of this map are free of rank one over $\mathbb{T}(a)^0$, and as in [EPW06, p.559] (using [Hid88, Thm. 9.4]), one is reduced to showing the injectivity of the dual map modulo $p$:

$$
(13) \quad H^0(\tilde{X}_{N(\Sigma), +}; F_p)^0[m] \rightarrow (\mathcal{J}_{N(\Sigma), +})^0_n (H^0(\tilde{X}_{N(\Sigma), +}; F_p)^0[m'])
$$

or equivalently (by a version of [EPW06, Lemma 3.8.1]), to showing that the composite of the first two arrows in (13) is injective.

In turn, the latter injectivity follows from Lemma 3.10, where the notations are as follows: $M^+$ is any positive integer with $(M^+, pn^-) = 1$, $\ell \neq p$ is a prime, $n_\ell = 1$ or 2 according to whether or not $\ell$ divides $M^+$, $N^+ := \ell^m M^+$, and

$$
(14) \quad \epsilon^*_r : H^0(\tilde{X}_{M^+, +}; F_p)^0[m] \rightarrow (\mathcal{T}_{N^+, +})^0_{n^+ - n^-} (H^0(\tilde{X}_{N^+, +}; F_p)^0[m'])
$$

is the map defined by

$$
\epsilon^*_r := \begin{cases} 
B^*_{\ell, 1} - B^*_{\ell, \ell - 1} T_{\ell} & \text{if } n_\ell = 1 \\
B^*_{\ell, 2} - B^*_{\ell, \ell - 1} T_{\ell} + B^*_{\ell, \ell - 1} T_{\ell} & \text{if } n_\ell = 2.
\end{cases}
$$

Lemma 3.10. The map (14) is injective.

Proof of Lemma 3.10. As in the proof of the analogous result [EPW06, Lemma 3.8.2] in the modular curve case, it suffices to show the injectivity of the map

$$
(H^0(\tilde{X}_{M^+, +}; F_p)^0[m_P])^{n_\ell + 1} \xrightarrow{\beta^*_{\ell}} H^0(\tilde{X}_{N^+, +}; F_p)^0[m_P]
$$
defined by
\[ \beta_\ell := \begin{cases} B_{\ell,1}^* \pi_1 + B_{\ell,2}^* \pi_2 & \text{if } n_\ell = 1 \\ B_{\ell,1}^* \pi_1 + B_{\ell,2}^* \pi_2 + B_{\ell,3}^* \ell^2 \pi_3 & \text{if } n_\ell = 2. \end{cases} \]

But in our quaternionic setting the proof of this injectivity follows from [SW99, Lemma 3.26] for \( n_\ell = 1 \) and [loc.cit., Lemma 3.28] for \( n_\ell = 2 \). \( \square \)

Applying inductively Lemma 3.10 to the primes in \( \Sigma \), the proof of Theorem 3.9 follows. \( \square \)

3.7. **Analytic Iwasawa invariants.** Upon the choice of an isomorphism
\[ \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]] \]
we may regard the \( p \)-adic \( L \)-functions \( L_\Sigma(\bar{\rho}, \mathfrak{a}) \) and \( L(\bar{\rho}, \mathfrak{a}) \), as well as the Euler factor \( E_\Sigma(\bar{\rho}, \mathfrak{a}) \), as elements in \( \mathbb{T}(\mathfrak{a})^\Sigma[[T]] \). In this section we apply the main result of the preceding section to study the variation of the Iwasawa invariants attached to the anticyclotomic \( p \)-adic \( L \)-functions of \( p \)-ordinary modular forms.

For any power series \( f(T) \in R[[T]] \) with coefficients in a ring \( R \), recall that the content of \( f(T) \) is defined to be the ideal \( I(\{f(T)\}) \subseteq R \) generated by the coefficients of \( f(T) \). If \( \varphi \) is a height one prime of \( \mathbb{T}_\Sigma \) belonging to the branch \( \mathbb{T}(\mathfrak{a}) \) (in the sense that \( \mathfrak{a} \) is the unique minimal prime of \( \mathbb{T}_\Sigma \) contained in \( \varphi \)), we denote by \( L(\bar{\rho}, \mathfrak{a})(\varphi) \) the element of \( \mathcal{O}_\varphi[[\Gamma]] \) obtained from \( L(\bar{\rho}, \mathfrak{a}) \) via reduction modulo \( \varphi \). In particular, we note that \( L(\bar{\rho}, \mathfrak{a})(\varphi) \) has unit content if and only if its \( \mu \)-invariant vanishes.

**Theorem 3.11.** The following are equivalent:

1. \( \mu(L(\bar{\rho}, \mathfrak{a})(\varphi)) = 0 \) for some newform \( f_\psi \) in \( H(\bar{\rho}) \).
2. \( \mu(L(\bar{\rho}, \mathfrak{a})(\varphi)) = 0 \) for every newform \( f_\psi \) in \( H(\bar{\rho}) \).
3. \( L(\bar{\rho}, \mathfrak{a}) \) has unit content for some irreducible component \( \mathbb{T}(\mathfrak{a}) \) of \( H(\bar{\rho}) \).
4. \( L(\bar{\rho}, \mathfrak{a}) \) has unit content for every irreducible component \( \mathbb{T}(\mathfrak{a}) \) of \( H(\bar{\rho}) \).

**Proof.** The argument in [EPW06, Thm 3.7.5] applies verbatim, replacing the appeal to [loc.cit., Cor. 3.6.3] by our Theorem 3.9 above. \( \square \)

When any of the conditions in Theorem 3.11 hold, we shall write
\[ \mu^{\text{an}}(\bar{\rho}) = 0. \]

For a power series \( f(T) \) with unit content and coefficients in a local ring \( R \), recall that the \( \lambda \)-invariant \( \lambda(f(T)) \) is defined to be the smallest degree in which \( f(T) \) has a unit coefficient.

**Theorem 3.12.** Assume that \( \mu^{\text{an}}(\bar{\rho}) = 0 \).

1. Let \( \mathbb{T}(\mathfrak{a}) \) be an irreducible component of \( H(\bar{\rho}) \). As \( \varphi \) varies over the arithmetic primes of \( \mathbb{T}(\mathfrak{a}) \), the \( \lambda \)-invariant \( \lambda(L(\bar{\rho}, \mathfrak{a})(\varphi)) \) takes on a constant value, denoted \( \lambda^{\text{an}}(\bar{\rho}, \mathfrak{a}) \).
2. For any two irreducible components \( \mathbb{T}(\mathfrak{a}_1), \mathbb{T}(\mathfrak{a}_2) \) of \( H(\bar{\rho}) \), we have that
\[ \lambda^{\text{an}}(\bar{\rho}, \mathfrak{a}_1) - \lambda^{\text{an}}(\bar{\rho}, \mathfrak{a}_2) = \sum_{\ell \neq p} e_\ell(\mathfrak{a}_2) - e_\ell(\mathfrak{a}_1) \]
where \( e_\ell(\mathfrak{a}) = \lambda(E_\ell(\mathfrak{a})) \).

**Proof.** The first part follows immediately from the definitions. For the second part, the argument in [EPW06, Thm. 3.7.7] applies verbatim, replacing their appeal to [loc.cit., Cor. 3.6.3] by our Theorem 3.9 above. \( \square \)

By Theorem 3.11 and Theorem 3.12, the Iwasawa invariants of \( L(\bar{\rho}, \mathfrak{a})(\varphi) \) are well-behaved as \( \varphi \) varies. However, for the applications of these results to the Iwasawa main conjecture it is of course necessary to relate \( L(\bar{\rho}, \mathfrak{a})(\varphi) \) to \( p \)-adic \( L \)-functions defined by the interpolation of special values of \( L \)-functions. This question was addressed in [CL14], as we now recall.
Theorem 3.13. If \( \wp \) is the arithmetic prime of \( \mathbb{T}(a) \) corresponding to a \( p \)-ordinary \( p \)-stabilized newform \( f_{\wp} \) of weight \( k \geq 2 \) and trivial nebentypus, then

\[
L(\bar{\rho}, a)(\wp) = L_p(f_{\wp}/K)
\]

where \( L_p(f_{\wp}/K) \) is the \( p \)-adic \( L \)-function of Chida–Hsieh [CH13]. In particular, if \( \chi : \Gamma \rightarrow \mathbb{C}_{p}^{'-} \) is the \( p \)-adic avatar of an anticyclotomic Hecke character of \( K \) of infinity type \((m, -m)\) with \(-k/2 < m < k/2\), then \( L(\bar{\rho}, a)(\wp) \) interpolates the central critical values

\[
\frac{L(f_{\wp}/K, \chi, k/2)}{\Omega_{f_{\wp}, N-}}
\]

as \( \chi \) varies, where \( \Omega_{f_{\wp}, N-} \) is the complex period (5).

Proof. This is a reformulation of the main result of [CL14]. (Note that the constant \( \lambda_p \in \mathbb{F}_p^{\times} \) in [CL14], Thm. 4.6] is not needed here, since the specialization map of [loc.cit., 3.1] is being replaced by the map \((\mathfrak{A}_{N+})_m \rightarrow (\mathfrak{A}_{N^+}, r, k)_{m, r, k} \) induced by the isomorphism (7), which preserves integrality.) \( \square \)

Corollary 3.14. Let \( f_1, f_2 \in \mathcal{H}(\bar{\rho}) \) be newforms with trivial nebentypus lying in the branches \( \mathbb{T}(a_1), \mathbb{T}(a_2) \), respectively. Then \( \mu^m(\bar{\rho}) = 0 \) and

\[
\lambda(L_p(f_1/K)) - \lambda(L_p(f_2/K)) = \sum_{\ell \neq p} e_\ell(a_2) - e_\ell(a_1)
\]

where \( e_\ell(a_j) = \lambda(E_\ell(a_j)) \).

Proof. By [CH13, Thm. 5.7] (extending Vatsal’s result [Vat03] to higher weights), if \( f \in \mathcal{H}(\bar{\rho}) \) has weight \( k \leq p + 1 \) and trivial nebentypus, then \( \mu(L_p(f/K)) = 0 \). By Theorem 3.11 and Theorem 3.13, this implies \( \mu^m(\bar{\rho}) = 0 \). The result thus follows from Theorem 3.12, using again Theorem 3.13 to replace \( \lambda^m(\bar{\rho}, a_j) \) by \( \lambda(L_p(f_j/K)) \). \( \square \)

4. Anticyclotomic Selmer groups

We continue with the notation of the previous sections. In particular, \( \bar{\rho} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F}) \) is an odd irreducible Galois representation satisfying (SU), \( \mathcal{H}(\bar{\rho}) \) is the associated Hida family, and \( \Sigma \) is a finite set of primes split in the imaginary quadratic field \( K \).

For each \( f \in \mathcal{H}(\bar{\rho}) \), let \( V_f \) denote the self-dual Tate twist of the \( p \)-adic Galois representation associated to \( f \), fix an \( \mathcal{O} \)-stable lattice \( T_f \subset V_f \), and set \( A_f := V_f/T_f \). Since \( f \) is \( p \)-ordinary, there is a unique one-dimensional \( G_{\mathbb{Q}_p} \)-invariant subspace \( F_p^+ V_f \subset V_f \) where the inertia group at \( p \) acts via \( \zeta^{k/2} \psi \), with \( \psi \) of finite order. Let \( F_p^+ A_f \) be the image of \( F_p^+ V_f \) in \( A_f \), and define the minimal Selmer group of \( f \) by

\[
\text{Sel}(K_\infty, f) := \ker \left\{ H^1(K_\infty, A_f) \longrightarrow \prod_{w \mid p} H^1(K_\infty, w, A_f) \times \prod_{w \mid p} H^1(K_\infty, w, F_p^- A_f) \right\}
\]

where \( w \) runs over the places of \( K_\infty \) and we set \( F_p^- A_f := A_f/F_p^+ A_f \).

It is well-known that \( \text{Sel}(K_\infty, f) \) is cofinitely generated over \( \Lambda \). When it is also \( \Lambda \)-cotorsion, we define the \( \mu \)-invariant \( \mu(\text{Sel}(K_\infty, f)) \) (resp. \( \lambda \)-invariant \( \lambda(\text{Sel}(K_\infty, f)) \)) to the largest power of \( \mathfrak{m} \) dividing (resp. the number of zeros of) the characteristic power series of the Pontryagin dual of \( \text{Sel}(K_\infty, f) \). The same remarks and definitions apply to \( \mathfrak{S} \text{el}(K_\infty, f) \).

A distinguishing feature of the anticyclotomic setting (in comparison with cyclotomic Iwasawa theory) is the presence of primes which split infinitely in the corresponding \( \mathbb{Z}_p \)-extension.
Indeed, being inert in $K$, all primes $\ell|N^-$ are infinitely split in $K_\infty/K$. As a result, the above Selmer group differs in general from the Greenberg Selmer group of $f$, which is defined by

$$\mathcal{Sel}(K_\infty, f) := \ker \left\{ H^1(K_\infty, A_f) \rightarrow \prod_{w|\ell} H^1(I_{\infty,w}, A_f) \times \prod_{w|p} H^1(K_{\infty,w}, F_p^{-1}A_f) \right\}$$

where $I_{\infty,w} \subset G_{K_\infty}$ denotes the inertia group at $w$.

If $S$ is a finite set of primes in $K$, we let $\text{Sel}^S(K_\infty, f)$ and $\mathcal{Sel}^S(K_\infty, f)$ be the “$S$-primitive” Selmer groups defined as above by omitting the local conditions at the primes in $S$ (except those above $\ell$, when any such prime is in $S$). Moreover, if $S$ consists of the primes dividing a rational integer $M$, we replace the superscript $S$ by $M$ in the above notation.

Immediately from the definitions, we see that there is an exact sequence

$$0 \rightarrow \text{Sel}(K_\infty, f) \rightarrow \mathcal{Sel}(K_\infty, f) \rightarrow \bigoplus_{\ell|N^-} \mathcal{H}_{\ell}^{\text{un}}$$

where

$$\mathcal{H}_{\ell}^{\text{un}} := \ker \left\{ \prod_{w|\ell} H^1(K_{\infty,w}, A_f) \rightarrow \prod_{w|\ell} H^1(I_{\infty,w}, A_f) \right\}$$

is the set of unramified cocycles. In [PW11], Pollack and Weston carried out a careful analysis of the difference between $\text{Sel}(K_\infty, f)$ and $\mathcal{Sel}(K_\infty, f)$. Even though in loc. cit. they focused on the case where $f$ is associated with an elliptic curve, many of their arguments apply more generally. In fact, the next result follows essentially from their work.

**Theorem 4.1.** Assume that $\bar{\rho}$ satisfies (SU). Then the following are equivalent:

1. $\text{Sel}(K_\infty, f_0)$ is $\Lambda$-cotorsion with $\mu$-invariant zero for some newform $f_0 \in \mathcal{H}(\bar{\rho})$.
2. $\text{Sel}(K_\infty, f)$ is $\Lambda$-cotorsion with $\mu$-invariant zero for all newforms $f \in \mathcal{H}(\bar{\rho})$.
3. $\mathcal{Sel}(K_\infty, f)$ is $\Lambda$-cotorsion with $\mu$-invariant zero for all newforms $f \in \mathcal{H}(\bar{\rho})$.

Moreover, in that case $\text{Sel}(K_\infty, f) \simeq \mathcal{Sel}(K_\infty, f)$.

**Proof.** Assume $f_0$ is a newform in $\mathcal{H}(\bar{\rho})$ for which $\text{Sel}(K_\infty, f_0)$ is $\Lambda$-cotorsion with $\mu$-invariant zero, and set $N^+ := N(\Sigma)/N^-$. By [PW11, Prop. 5.1], we then have the exact sequences

$$0 \rightarrow \text{Sel}(K_\infty, f_0) \rightarrow \text{Sel}^{N^+}(K_\infty, f_0) \rightarrow \bigoplus_{\ell|N^+} \mathcal{H}_{\ell} \rightarrow 0 \quad (16)$$

and

$$0 \rightarrow \mathcal{Sel}(K_\infty, f_0) \rightarrow \mathcal{Sel}^{N^+}(K_\infty, f_0) \rightarrow \bigoplus_{\ell|N^+} \mathcal{H}_{\ell} \rightarrow 0 \quad (17)$$

where $\mathcal{H}_{\ell}$ is the product of $H^1(K_{\infty,w}, A_{f_0})$ over the places $w|\ell$ in $K_\infty$. Since every prime $\ell|N^+$ splits in $K$ (see Remark 3.8), the $\Lambda$-cotorsionness and the vanishing of the $\mu$-invariant of $\mathcal{H}_{\ell}$ can be deduced from [GV00, Prop. 2.4]. Since $\text{Sel}(K_\infty, f_0)[\overline{\omega}]$ is finite by assumption, it thus follows from (16) that $\text{Sel}^{N^+}(K_\infty, f_0)[\overline{\omega}]$ is finite. Combined with (15) and [PW11, Cor. 5.2], the same argument using (17) shows that then $\mathcal{Sel}^{N^+}(K_\infty, f_0)[\overline{\omega}]$ is also finite.

On the other hand, following the arguments in the proof [PW11, Prop. 3.6] we see that for any $f \in \mathcal{H}(\bar{\rho})$ we have

$$\text{Sel}^{N^+}(K_\infty, \bar{\rho}) \simeq \text{Sel}^{N^+}(K_\infty, f)[\overline{\omega}]$$

$$\mathcal{Sel}^{N^+}(K_\infty, \bar{\rho}) \simeq \mathcal{Sel}^{N^+}(K_\infty, f)[\overline{\omega}]$$

As a result, the argument in the previous paragraph implies that, for any newform $f \in \mathcal{H}(\bar{\rho})$, both $\text{Sel}^{N^+}(K_\infty, f)[\overline{\omega}]$ and $\mathcal{Sel}^{N^+}(K_\infty, f)[\overline{\omega}]$ are finite, from where (using (16) and (17) with $f$ in place of $f_0$) the $\Lambda$-cotorsionness and the vanishing of both the $\mu$-invariant of $\text{Sel}(K_\infty, f)$ and of $\mathcal{Sel}(K_\infty, f)$ follows. In view of (15) and [PW11, Lemma 3.4], the result follows.  \[ \square \]
Let \( w \) be a prime of \( K_\infty \) above \( \ell \neq p \) and denote by \( G_w \subset G_{K_\infty} \) its decomposition group. Let \( T(a) \) be the irreducible component of \( T_\Sigma \) passing through \( f \), and define
\[
\delta_w(a) := \dim_p A_f^{G_w}/\mathcal{O}.
\]
(Note that this is well-defined by [EPW06, Lemma 4.3.1].) Assume \( \ell = \mathfrak{f} \) splits in \( K \) and put
\[
\delta_\ell(a) := \sum_{w|\ell} \delta_w(a)
\]
where the sum is over the (finitely many) primes \( w \) of \( K_\infty \) above \( \ell \).

In view of Theorem 4.1, we write \( \mu_{\text{alg}}(\overline{\rho}) = 0 \) whenever any of the \( \mu \)-invariants appearing in that result vanish. In that case, for any newform \( f \) in \( \mathcal{H}(\overline{\rho}) \) we may consider the \( \lambda \)-invariants \( \lambda(\text{Sel}(K_\infty,f)) = \lambda(\text{Sel}(K_\infty,f)) \).

**Theorem 4.2.** Let \( \overline{\rho} \) and \( \Sigma \) be as above, and assume that \( \mu_{\text{alg}}(\overline{\rho}) = 0 \). If \( f_1 \) and \( f_2 \) are any two newforms in the Hida family of \( \overline{\rho} \) lying in the branches \( T(a_1) \) and \( T(a_2) \), respectively, then
\[
\lambda(\text{Sel}(K_\infty,f_1)) - \lambda(\text{Sel}(K_\infty,f_2)) = \sum_{\ell \neq p} \delta_\ell(a_1) - \delta_\ell(a_2).
\]

**Proof.** Since \( N^{-1}|N(a_1)|N(\Sigma) \) and \( N(\Sigma)/N^{-} \) is only divisible by primes that are split in \( K \), the arguments of [EPW06, §4] apply verbatim (cf. [PW11, Thm. 7.1]). \( \square \)

5. Applications to the main conjecture

5.1. **Variation of Iwasawa invariants.** Recall the definition of the analytic invariant \( e_\ell(a) = \lambda(E_\ell(a)) \), where \( E_\ell(a) \) is the Euler factor from Section 3.6, and of the algebraic invariant \( \delta_\ell(a) \) introduced in (18).

**Lemma 5.1.** Let \( a_1, a_2 \) be minimal primes of \( T_\Sigma \). For any prime \( \ell \neq p \) split in \( K \), we have
\[
\delta_\ell(a_1) - \delta_\ell(a_2) = e_\ell(a_2) - e_\ell(a_1).
\]

**Proof.** Let \( a \) be a minimal prime of \( T_\Sigma \), let \( f \) be a newform in the branch \( T(a) \), and let \( \wp_f \subset a \) be the corresponding height one prime. Since \( \ell = \mathfrak{f} \) splits in \( K \), we have
\[
\oplus_{w|\ell} H^1(K_{\infty,w}, A_f) = \left( \oplus_{w|\ell} H^1(K_{\infty,w}, A_f) \right) \oplus \left( \oplus_{\wp_f \neq a} H^1(K_{\infty,w}, A_f) \right)
\]
and [GV00, Prop. 2.4] immediately implies that
\[
Ch_\lambda \left( \oplus_{w|\ell} H^1(K_{\infty,w}, A_f)^\vee \right) = E_\ell(f, \ell^{-1} \gamma) \cdot E_\ell(f, \ell^{-1} \gamma)
\]
where \( E_\ell(f, \ell^{-1} \gamma) \cdot E_\ell(f, \ell^{-1} \gamma) \) is the specialization of \( E_\ell(a) \) at \( \wp_f \). The result thus follows from [EPW06, Lemma 5.1.5]. \( \square \)

**Theorem 5.2.** Assume that \( \overline{\rho} \) satisfies (SU). If for some newform \( f_0 \in \mathcal{H}(\overline{\rho}) \) we have
\[
\mu(\text{Sel}(K_\infty,f_0)) = \mu(L_p(f_0/K)) = 0 \quad \text{and} \quad \lambda(\text{Sel}(K_\infty,f_0)) = \lambda(L_p(f_0/K))
\]
then
\[
\mu(\text{Sel}(K_\infty,f)) = \mu(L_p(f/K)) = 0 \quad \text{and} \quad \lambda(\text{Sel}(K_\infty,f)) = \lambda(L_p(f/K))
\]
for all newforms \( f \in \mathcal{H}(\overline{\rho}) \).

**Proof.** Let \( f \) be any newform in \( \mathcal{H}(\overline{\rho}) \). Since the \( \mu \)-invariants of \( f_0 \) vanish, the vanishing of \( \mu(\text{Sel}(K_\infty,f)) \) and \( \mu(L_p(f/K)) \) follows from Theorem 4.1 and Theorem 3.11, respectively.

On the other hand, combining Theorems 3.12 and 4.2, and Lemma 5.1, we see that
\[
\lambda(\text{Sel}(K_\infty,f)) - \lambda(\text{Sel}(K_\infty,f_0)) = \lambda(L_p(f/K)) - \lambda(L_p(f_0/K)),
\]
and hence the equality \( \lambda(\text{Sel}(K_\infty,f_0)) = \lambda(L_p(f_0/K)) \) implies the same equality for \( f \). \( \square \)
5.2. Applications to the main conjecture. As an immediate consequence of Weierstrass preparation theorem, our Theorem 5.2 together with one the divisibilities predicted by the anticyclotomic main conjecture implies the full anticyclotomic main conjecture.

**Theorem 5.3** (Skinner–Urban). Let \( f \in S_k(\Gamma_0(N)) \) be a newform of weight \( k \equiv 2 \pmod{p-1} \) and trivial nebentypus, and assume that \( \hat{\rho}_f \) satisfies (SU) and that \( p \) splits in \( K \). Then

\[
(L_p(f/K)) \geq \Ch_A(\Sel(K_\infty, f)^\vee).
\]

**Proof.** This follows from specializing the divisibility in [SU14, Thm. 3.26] to the anticyclotomic line. Indeed, let \( f = \sum_{n \geq 1} a_n(f)q^n \in \mathbb{I}[[q]] \) be the \( \Lambda \)-adic form with coefficients in \( \mathbb{I} := \mathbb{T}(a)^\circ \) associated with the branch of the Hida family containing \( f \), and let \( \Sigma \) be a finite set of primes as in Section 3.5, let \( \Sigma' \supseteq \Sigma \) be a finite set of primes of \( K \) containing \( \Sigma \) and all primes dividing \( pN(a)D_K \), and assume that \( \Sigma' \) contains at least one prime \( \ell \neq p \) that splits in \( K \). Under these assumptions, in [SU14, Thm. 3.26] it is shown that

\[
(L_p(f/K)) \geq \Ch_{A_f}(\Sel^\Sigma(L_\infty, A_f)^\vee)
\]

where \( L_\infty = K_\infty K_{cyc} \) is the \( \mathbb{Z}_p^2 \)-extension of \( K \), \( A_f(L_\infty) \) is the three-variable Iwasawa algebra \( \mathbb{I}[[\Gal(L_\infty/K)]] \), and \( \Sel^\Sigma(L_\infty, A_f) \) are the "\( \Sigma' \)-primitive" \( p \)-adic \( L \)-function and Selmer group defined in [SU14, §3.4.5] and [SU14, §3.1.3.10], respectively.

Recall the character \( \Theta : G_Q \to \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]^\times \) from Section 2.3, regarded as a character on \( \Gal(L_\infty/K) \), and let

\[
T_{\Theta^{-1}} : A_f(L_\infty) \to A_f(L_\infty)
\]

be the \( \mathbb{I} \)-linear isomorphism induced by \( T_{\Theta^{-1}}(g) = \Theta^{-1}(g)g \) for \( g \in \Gal(L_\infty/K) \). Choose a topological generator \( \gamma \in \Gal(K_{cyc}/K) \), and expand

\[
T_{\Theta^{-1}}(\Sel^\Sigma(f/K)) = \Sel^\Sigma_0(f/K) + \Sel^\Sigma_1(f/K)(\gamma - 1) + \cdots
\]

with \( \Sel^\Sigma_0(f/K) \in A_f(K_\infty) = \mathbb{I}[[\Gamma]] \). In particular, note that \( \Sel^\Sigma_0(f/K) \) is the restriction of the twisted three-variable \( p \)-adic \( L \)-function \( T_{\Theta^{-1}}(\Sel^\Sigma(f/K)) \) to the "self-dual" plane.

Because of our assumptions on \( f \), the \( \Lambda \)-adic form \( f \) has trivial tame character, and hence denoting by \( \Frob_\ell \) an arithmetic Frobenius at any prime \( \ell \nmid N(a)p \), the Galois representation

\[
\rho(\bar{a}) : G_Q \to \GL(T_\ell) \simeq \GL_2(\mathbb{T}(a)^\circ)
\]

considered in §1.4 (which is easily seen to agree with the twisted representation considered in [SU14, p.37]) is such that

\[
\det(X - \Frob_\ell(T_\ell)) = X^2 - a_\ell(f)X + \Theta^2(\ell)\ell.
\]

The twist \( T_\ell^\dagger := T_\ell \otimes \Theta^{-1} \) is therefore self-dual. Thus combining [Rub00, Lemma 6.1.2] with a straightforward variant of [SU14, Prop. 3.9] having \( \Gal(K_\infty/K) \) in place of \( \Gal(K_{cyc}/K) \), we see that the divisibility (19) implies that

\[
(L_p^\Sigma_0(f/K)) \geq \Ch_{A_f}(\Sel^\Sigma(K_\infty, A_f^\dagger)^\vee).
\]

(Here, as above, \( A_f \) denotes the Pontryagin dual \( T_\ell \otimes \Hom_{cts}(\mathbb{I}, \mathbb{Q}_p/\mathbb{Z}_p) \), and \( A_f^\dagger \) is the corresponding twist.) We next claim that, setting \( \Sigma'' := \Sigma' \setminus \Sigma \), we have

\[
(L_p^\Sigma_0(f/K)) = (L_\Sigma(\bar{\rho}, a) \cdot \prod_{\nu \in \Sigma'' \setminus \nu \mid p} E_\nu(a))
\]

where \( L_\Sigma(\bar{\rho}, a) \) is the two-variable \( p \)-adic \( L \)-function constructed in §3.4, and if \( \nu \) lies over the rational prime \( \ell \), \( E_\nu(a) \) is the Euler factor given by

\[
E_\nu(a) = \det(1 - \Frob_\nu X | (V^\dagger_\ell)_{\nu})_{X = \ell^{-1}\Frob_\nu}.
\]
where $V_t := T_t \otimes \text{Frac}(l)$, and $\text{Frob}_v$ is an arithmetic Frobenius at $v$. (Note that for $\ell = \bar{\ell}$ split in $K$, $E_l(a) \cdot E_l(a)$ is simply the Euler factor (10).) Indeed, combined with Theorem 3.9 and Theorem 3.13, the equality (21) specialized to any arithmetic prime $\varphi \subset \mathbb{T}(a)$ of weight 2 is shown in [SU14, (12.3)], from where the claim follows easily from the density of these primes. (See also [PW11, Thm. 6.8] for the comparison between the different periods involved in the two constructions, which differ by a $p$-adic unit under our assumptions.)

Finally, (20) and (21) combined with Theorem 3.9 and [GV00, Props. 2.3,8] imply that

$$\mu(\text{Sel}(K_\infty, f)) = 0 \quad \text{and} \quad \lambda(\text{Sel}(K_\infty, f)) = \lambda(L_p(f/K)).$$

from where the result follows by specializing at $\varphi f$ using Theorem 3.13 and Theorem 4.1. □

**Corollary 5.4.** Suppose that $\bar{\rho}$ satisfies (SU) and that $p$ splits in $K$. If the anticyclotomic main conjecture holds for some newform $f_0$ in $\mathcal{H}(\bar{\rho})$ of weight $k_0 \equiv 2 \pmod{p-1}$ and trivial nebentypus, then it holds for all newforms $f$ in $\mathcal{H}(\bar{\rho})$ of weight $k \equiv 2 \pmod{p-1}$ and trivial nebentypus.

**Proof.** After Theorem 5.3, to check the anticyclotomic main conjecture for any newform $f$ as in the statement, it suffices to check that

$$(22) \quad \mu(\text{Sel}(K_\infty, f)) = 0 \quad \text{and} \quad \lambda(\text{Sel}(K_\infty, f)) = \lambda(L_p(f/K)).$$

If the anticyclotomic main conjecture holds for some newform $f_0$ as in the statement, then the first and third equalities in (22) clearly hold for $f_0$, while the vanishing of $\mu(L_p(f_0/K))$ follows from Corollary 3.14; by Theorem 5.2, the equalities (22) then also hold $f$, and hence the anticyclotomic main conjecture for $f$ follows. □

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