THE SAITO-KUROKAWA LIFTING AND STARK-HEEGNER POINTS

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Abstract. Let $E_{/\mathbb{Q}}$ be an elliptic curve of conductor Np with $p \nmid N$ and let f be its associated newform of weight 2. Denote by f_∞ the p-adic Hida family passing though f, and by F_∞ its Λ-adic Saito-Kurokawa lift. The p-adic family F_∞ of Siegel modular forms admits a formal Fourier expansion, from which we can define a family of normalized Fourier coefficients $\{\widetilde{A}_T(k)\}_T$ indexed by positive definite symmetric half-integral matrices T of size 2×2. We relate explicitly certain global points on E (coming from the theory of Stark-Heegner points) with the values of these Fourier coefficients and of their p-adic derivatives, evaluated at weight $k = 2$.

1. Introduction

Let f be an elliptic newform of weight 2 and level $\Gamma_0(M)$. Eichler and Shimura showed how to associate to f an abelian variety A_f of arithmetic conductor M such that the complex L-functions attached to f and A_f agree, cf. [18, Theorem 7.14]. Although the theory of Siegel modular forms provides a satisfactory generalization of the notion of classical elliptic modular forms in higher dimension, no such construction is known for Siegel modular forms of genus > 1. Generalizing the Shimura-Taniyama conjecture, Yoshida [22] conjectured the existence of a genus two holomorphic Siegel modular cusp eigenform of parallel weight 2 associated to any irreducible abelian surface A defined over $\mathbb{Q}$. Note that this Siegel modular form, in contrast with the elliptic case, cannot be obtained in general by cutting a piece of the étale cohomology of the Siegel modular variety. This makes the connection to geometric constructions much less immediate.

In this paper, we study instead the reducible case, thereby lifting with Saito-Kurokawa known connections between elliptic modular forms and certain global points on rational elliptic curves to Siegel modular forms in the Maaß Spezialschar. Our main result (partially summarized in (2) and (3) below) provides a relation between Fourier coefficients of the Saito-Kurokawa lift of f and global points on the elliptic curve associated with f as above. This result offers a geometric interpretation of Fourier coefficients of Saito-Kurokawa lifts. Such a result was, after inquiry, unsuspected by specialists of explicit formulae for classical automorphic forms. It is obtained by combining p-adic techniques developed by Darmon-Tornaría in [6] for the Λ-adic Shintani lifting with various results on the Saito-Kurokawa lifting recently obtained by Ibukiyama in [10] and an explicit description of Fourier coefficients of modular forms of half-integral weight by Kohnen [12]. The global points

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providing such a description come from the theory of Stark-Heegner points introduced by Darmon in [5], explaining the title of the paper.

To describe our work and main results in a more precise form, we fix an elliptic newform f of weight 2, level $\Gamma_0(Np)$ and rational Fourier coefficients, where $N > 1$ is an integer and $p \nmid N$ is a fixed prime number. We assume in the paper that $p \equiv 1 \mod 4$. This plays a rôle in our combinatorial arguments and cannot, at the present, be removed.

Denote by

$$f_\infty(k) = \sum_{n \geq 1} a_n(k) q^n \tag{1}$$

the Hida family passing through f, where k is a p-adic variable in a neighborhood U of 2 in

$$\mathcal{X} := \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

and $a_1(k) = 1$ for all $k \in U$. Here $\mathbb{Z} \hookrightarrow \mathcal{X}$ via $k \mapsto (x \mapsto x^k)$. For simplicity, we will also assume that U is contained in the residue class of 2 modulo $p-1$. Thus $f_\infty(2)$ coincides with the Fourier expansion of f and, more generally, $f_\infty(k)$ is a normalized ordinary eigenform of level $\Gamma_0(Np)$ for all positive integers $k \in U$. Using the explicit Λ-adic Saito-Kurokawa lifting due to Li [15] and based on work of Stevens [20], in Section 2 we recall the construction of a p-adic family of Siegel modular forms

$$F_\infty(k) := \sum_{T > 0} \alpha_T(k) q^T$$

(where $k \in U$ and the sum runs over all positive definite, half-integral symmetric matrices T of size 2×2) interpolating a convenient stabilization of the Saito-Kurokawa lifting of the classical forms appearing in the Hida family $f_\infty(k)$, cf. Theorem 2.4. In particular, $F_\infty(2)$ is just the Fourier expansion of the Saito-Kurokawa lift of f (well-defined up to non-zero complex factors). In Section 3, we introduce a normalization of the coefficients $\alpha_T(k)$, which we denote $\widetilde{A}_T(k)$; these are p-adic analytic functions on a neighborhood U of 2 in $\mathcal{X}$. The analysis of the modified Fourier coefficients $\widetilde{A}_T(k)$ is carried out in Section 3, which contains the technical heart of the proof. We combine various results for the explicit Saito-Kurokawa lifting due to Ibukiyama [10] and explicit relations for Fourier coefficients on modular forms of half-integral weight due to Kohnen [12]. In particular, the paper [10] generalizes to arbitrary level various results on the Saito-Kurokawa lifting due to, among others, Eichler-Zagier [7], Kohnen [13], Manickam-Ramakrishnan-Vasudevan [16] and Manickam-Ramakrishnan [17]. In Section 4, we combine our explicit analysis of the modified Fourier coefficients $\widetilde{A}_T(k)$ with the work of Darmon-Tornaría [6] on half-integral weight modular forms to relate the values $\widetilde{A}_T(2)$ and $\widetilde{A}'_T(2)$ to global points on the elliptic curve E associated with f via the Eichler-Shimura construction. To be more precise, the family of p-adic Fourier coefficients $\{\widetilde{A}_T(k)\}$ can be divided into two subfamilies called of type I and II, respectively, corresponding to those T for which $\widetilde{A}_T(2)$ do not need to vanish and respectively to those T such that $A_T(2)$ is forced to vanish, see §4.1 and eq. (9) for precise definitions. Thus, for T of type II we have $A_T(2) = 0$ and it natural to look at the value of the first derivative $A'_T(2)$ (with respect to k) at $k = 2$. Our main result, Theorem 4.4, thus shows the existence of a point $Q_T \in E(K_T) \otimes_{\mathbb{Z}} \mathbb{Q}$ (where K_T is an imaginary quadratic field, depending on T) such that

$$\frac{\partial}{\partial k}\widetilde{A}_T(k)_{|k=2} = \log_E(Q_T), \tag{2}$$

where, if q is Tate's period of the p-adic Tate uniformization Φ_{Tate} of E and $\log_q$ is the branch of the p-adic logarithm satisfying $\log_q(q) = 0$, we define $\log_E$ to be the $\mathbb{Q}$-linear extension of $\log_E(P) := \log_q(\Phi_{\mathrm{Tate}}^{-1}(P))$ (see (15) for details). Further, at least in the special case when T is primitive (meaning that if $T = \left(\begin{smallmatrix} u & v/2 \\ v/2 & w \end{smallmatrix}\right)$, then $\gcd(u, v, w) = 1$) we also have the relation

$$Q_T \neq 0 \Longleftrightarrow L'(\mathrm{SK}(f), \chi_{K_T}, 1) \neq 0, \tag{3}$$

where $L(\mathrm{SK}(f), \chi_{K_T}, s)$ is the Adrianov L-function attached to the Saito-Kurokawa lift $\mathrm{SK}(f)$ of f, twisted by the quadratic character χ_{K_T} of K_T.

The above equations (2) and (3) may be viewed as an analogue of [6, Theorems 1.5, 5.1] where similar results are established for the Shintani lifting in lieu of the Saito-Kurokawa lifting.

We also remark that a more general version (3) above is proven in the text, relaxing the primitivity assumption on T. For more general T of type II, the relation (3) between the global point Q_T and the L-function $L(\mathrm{SK}(f), \chi_{K_T}, s)$ depends on the value of a certain explicit integer attached to T (denoted n_T in the text: see Proposition 3.1 for its definition), which might be zero. In this case the condition $L'(\mathrm{SK}(f), \chi_{K_T}, 1) \neq 0$ does not imply the non-vanishing of the point Q_T. This is not surprising and corresponds to the fact, expressed by (13) in the text, that the order of vanishing of $\widetilde{A}_T(k)$ at 2 might be bigger than the order of vanishing of the analogous coefficient appearing in [6] and related to the Λ-adic Shintani lifting of f_∞.

Our second main result, Theorem 4.5, deals instead with T' of type I. Having fixed a T of type II, we show that, at least when T' is primitive of type I, the product $\widetilde{A}_{T'}(2)\log_E(Q_T)$ is equal to a quantity (denoted by $J(f, \mathfrak{d}_T, \mathfrak{d}_{T'})$ in the text) obtained as a p-adic variation of classical Shintani geodesic integrals attached to f and certain quadratic forms depending on T and T', see (16) for details and definitions. This result also holds without the assumption, made in the introduction, that $N > 1$.

We mention that Kawamura recently investigated in [11], with a different perspective, applications of p-adic methods à la Hida to the Ikeda lifting, introducing a p-stabilized family of Siegel modular forms different from that used in our paper (see Remark 2.5). Combination of our work and [11] might be of interest. Also, we finally point out that Brumer-Kramer refined Yoshida's conjecture in the paramodular case, conjecturing a bijection between isogeny classes of abelian surfaces $A/\mathbb{Q}$ of conductor M with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ and weight 2 paramodular newforms of level M with rational eigenvalues which are not Gritsenko lifts, with an equality of L-series. Brumer-Kramer's conjecture has been verified numerically for small prime levels by Poor-Yuen. As a complement, it would thus be interesting to see if our arguments can be adapted to the paramodular case.

2. The Λ-adic Saito-Kurokawa lifting

In this section, we construct an explicit Λ-adic Saito-Kurokawa lifting, using a key result of Stevens [20] and following Li [15, Thm. 4.9, p.56]. See also the anterior work [9] by Guerzhoy.

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a newform of weight 2 and level $\Gamma_0(Np)$, where $N \geq 1$ is a fixed odd squarefree integer and $p \nmid N$ is an odd prime number and such that $a_n \in \mathbb{Z}$ for all $n \geq 1$. Consider the Hida family $f_\infty(k)$ passing through f introduced in (1); recall that k belongs to a neighborhood U of 2 in the weight space $\mathcal{X}$, with U contained in the residue class of 2 mod $p-1$, and $a_1(k) = 1$ for all $k \in U$. For any integer $k \geq 2$ in U, let $f_k := f_\infty(k)$ denote the k-specialization of f_∞. We also let $\Lambda := \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]$ denote the Iwasawa algebra of $1 + p\mathbb{Z}_p$ with coefficients in $\mathbb{Z}_p$.

We first recall a well-known result of Stevens [20] on the Λ-adic Shintani lifting. Let $\theta(f_k)$ denote the Shintani lifting of f_k, whose definition is recalled, for example, in [20, §2.2]. In particular, note that $\theta(f_k)$ is well-defined only up to a complex non-zero factor. A result of Shimura [19] (cf. [20, Proposition 2.3.1]), asserts the existence, for any positive even integer k, of a complex number $\Omega^-_{f_k}$ such that

$$\frac{\theta(f_k)}{\Omega^-_{f_k}} \in \mathcal{O}_{f_k}$$

where $\mathcal{O}_{f_k}$ denotes the ring of integers of the finite extension K_{f_k} generated over $\mathbb{Q}$ be the Fourier coefficients of f_k.

Remark 2.1. $\Omega^-_{f_k}$ is defined in [8, Theorem 4.8] in such a way that $\Omega^-_{f_k}\Phi_{f_k}$ (where Φ_{f_k} is the standard modular symbol associated to f_k as in [8, Definition 4.7]) spans the one-dimensional K_{f_k}-vector space consisting of $\mathrm{Sym}^{k-2}(K^2_{f_k})$-valued $\Gamma_0(Np)$-invariant modular symbols where the involution $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts as -1 and the Hecke algebra acts through the character associated to f_k, cf. [8, Section 4].

We recall now the following result due to Stevens (see [20, Theorem 3.3]).

Theorem 2.2 (Stevens)**.** *There is a formal power expansion*

$$\Theta(k) = \sum_{n=1}^{\infty} \beta_n(k) q^n$$

(where β_n are p-adic analytic functions defined on U) and, for any positive even integer k, a p-adic number $\Omega_k \in \bar{\mathbb{Q}}_p$, such that:

1. $\Omega_2 \neq 0$;
2. *For any positive even integer k,*

$$\Theta(k) = \frac{\Omega_k}{\Omega^-_{f_k}}\theta(f_k).$$

We shall use the above result to construct an explicit Λ-adic Saito-Kurokawa lifting. This is well-known to experts and follows in particular from Li's thesis [15]. We recall the constructions which will be used later. For an integer $k > 2$ in U, let $f_k^\sharp$ denote the unique newform of weight k and level $\Gamma_0(N)$ such that

$$f_k(z) = f_k^\sharp(z) - a_p(k)^{-1} p^{k-1} f_k^\sharp(pz).$$

Let $F_k^\sharp = \mathrm{SK}(f_k^\sharp)$ denote the Saito-Kurokawa lifting of $f_k^\sharp$, for an integer $k > 2$ in U, the existence of which is proved in this generality in [10]. (Note that $F_k^\sharp$ is well-defined only up to a non-zero complex number.) The Siegel modular form $F_k^\sharp$ has then weight $k/2+1$ and level $\Gamma_0^2(N)$ where, for any integer $M \geq 1$, we define

$$\Gamma_0^2(M) = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in \mathrm{Sp}_2(\mathbb{Z}) | C \equiv 0 \mod M \right\}.$$

The Fourier expansion of $F_k^\sharp$ can be written as follows:

$$F_k^\sharp(Z) = \sum_{T>0} A_T(k) q^T$$

where the sum is over all positive definite, half-integral symmetric matrices T of size 2×2 and $q^T := \exp\big(2\pi i \mathrm{tr}(TZ)\big)$. If we write

$$g_k^\sharp := \theta(f_k^\sharp) = \sum_{D \geq 1;\ (-1)^{k/2} D \equiv 0,1 \bmod 4} c_D(k) q^D$$

for the Shintani lifting of $f_k^\sharp$ (which is well-defined only up to a non-zero complex number), then the image of the Saito-Kurokawa lifting satisfies the following relation, thanks to [10, §3.4]:

$$A_T(k) = \sum_{d>0;\ d|c(T);\ (N,d)=1} c_{D_T/d^2}(k) d^{k/2} \tag{4}$$

where $D_T = \det(2T)$ and, for $T = \left(\begin{smallmatrix} u & v/2 \\ v/2 & w \end{smallmatrix} \right)$, $c(T) := \gcd(u,v,w)$. Of course, the only d's contributing to the above sum are those for which $(-1)^{k/2} D_T/d^2 \equiv 0, 1 \bmod 4$. For later use, we also let

$$g = g_2 := \theta(f) = \sum_{D \geq 1;\ -D \equiv 0,1 \bmod 4} c_D(2) q^D$$

denote the Shintani lifting of $f = f_2$, as usually well-defined only up to scalars. We also denote by $F = F_2$ the Saito-Kurokawa lifting of f, which is a Siegel modular form of weight 2 and level $\Gamma_0^2(Np)$. The Fourier expansion of F is

$$F(Z) = \sum_{T>0} A_T(2) q^T$$

with the same conventions as above. By [10, §3.4], the Fourier coefficients of F satisfy the following relation

$$A_T(k) = \sum_{d>0;\ d|c(T);\ (d,Np)=1} c_{D_T/d^2}(k) d^{k/2}. \tag{5}$$

Remark 2.3. Equations (4) and (5) can be interpreted as an analogue for arbitrary level of the Maaß relations describing the image (called the Maaß Spezialschar) of the Saito-Kurokawa lifting in level 1. See [7, Chap. 6], [13, Proposition 3] and [1] for precise references. For general levels, we refer to the discussion in [10] on the possibility of describing the image of the Saito-Kurokawa lifting in terms of such relations.

To correct the asymmetry in (4) and (5), for integers $k \geq 4$ in U, define the Siegel modular form:

$$F_k(Z) = F_k^\sharp(Z) - p^{k/2} F_k^\sharp(pZ).$$

By [10, Prop. 3.7], we know that F_k is a modular form of weight $k/2+1$ and level $\Gamma_0^2(Np)$. Also, define for any positive integer $k \in U$:

$$A_T^{(p)}(k) := \sum_{d>0;\ d|c(T);\ (d,Np)=1} c_{D_T/d^2}(k) d^{k/2}$$

obtained from $A_T(k)$ by excluding the coefficients such that $p \mid c(T)$. The definition of the coefficients $A_T^{(p)}(k)$ is motivated by the following well-known interpolation property of the Saito-Kurokawa lifting. Although we will not use this result in the paper, we add it for completeness. Define for $k \in U$

$$\alpha_T(k) = \sum_{d>0;\ d|c(T);\ (Np,d)=1} \beta_{D_T/d^2}(k)$$

Theorem 2.4 (Li). *The formal power series expansion:*

$$F_\infty(k) := \sum_{T>0} \alpha_T(k) q^T$$

defined for $k \in U$ satisfies

$$F(k) = \frac{\Omega_k}{\Omega_{f_k}^-} F_k$$

for any positive integer $k \in U$, where the p-adic numbers Ω_k are defined in Theorem 2.2.

Proof. An easy computation shows that, for any integer $k \geq 2$ in U,

$$p^{k/2} F_k^\sharp(pZ) = \sum_{T>0;\ p|c(T);} \left(\sum_{p|d \text{ and } d|c(T)} c_{D_T/d^2}(k) d^{k/2} \right) q^T,$$

from which we have:

$$F_k = \sum_{T>0} A_T^{(p)}(k) q^T. \tag{6}$$

The result now follows from Theorem 2.2 and Equation (6). $\square$

Remark 2.5. The family of modular forms obtained in Theorem 2.4 above interpolates the modular forms F_k, for even integers $k \geq 2$ in a suitable neighborhood of 2 in $\mathcal{X}$. However, F_k is not the Saito-Kurokawa lifting of f_k if the integer k is ≥ 4, and it is not, in general, an ordinary form (in the sense that F_k does not need to be an eigenform for p-Hecke operators such that the eigenvalues are p-adic units). One may ask for (semi-)ordinarity conditions by considering another family of Siegel modular forms interpolating p-adic stabilizations (in a suitable sense) of the Saito-Kurokawa lifting of the classical forms f_k in the Hida family. This is the point of view taken in Kawamura's paper [11]. See [11, Sec. 4] for the notion of p-stabilization and [11, Theorem 4.4] for the construction of the Λ-adic family of Siegel modular forms interpolating suitable p-stabilizations of the Ikeda lifting of the classical forms f_k in the Hida family. However, for the purpose of this paper, only the family appearing in Theorem 2.4 above will play a role.

3. Normalized Fourier coefficients

In this section we modify the Fourier coefficients $A_T^{(p)}(k)$ introduced in the previous section in order to relate them with the work of Darmon-Tornaría [6]. We first recall a corollary of Theorem 2.2, cf. [6, Proposition 1.3]. Choose D_0 such that $c_{D_0}(2) \neq 0$.

Proposition 3.1. *There is a p-adic neighborhood $\mathcal{U} \subseteq U$ of 2 such that:*

1. $c_{D_0}(k) \neq 0$ *for $k \in \mathbb{Z} \cap \mathcal{U}$ with $k \geq 2$.*
2. *The function $k \mapsto \widetilde{c}_D(k)$, defined for integers $k \geq 4$ in U, by:*

$$\widetilde{c}_D(k) := \frac{\left(1 - \left(\frac{D}{p}\right) a_p(k)^{-1} p^{k/2-1}\right) c_D(k)}{\left(1 - \left(\frac{D_0}{p}\right) a_p(k)^{-1} p^{k/2-1}\right) c_{D_0}(k)} = \frac{c_{Dp^2}(k)}{c_{D_0p^2}(k)}$$

extends to a p-adic analytic function on $\mathcal{U}$ satisfying:

$$\widetilde{c}_D(2) = \frac{c_{Dp^2}(2)}{c_{D_0p^2}(2)} = \frac{c_D(2)}{c_{D_0}(2)}.$$

In order to relate our coefficients $A_T^{(p)}(k)$ with Prop. 3.1, we proceed as follows. Fix a fundamental discriminant $\mathfrak{d}$ with $(-1)^{k/2}\mathfrak{d} > 0$ and an integer $n \neq 0$. Define, for any even integer $k \geq 4$ with $(-1)^{k/2}\mathfrak{d} > 0$

$$\rho_{\mathfrak{d},n}(k) := \sum_{d|n;\ (d,N)=1} \mu(d) \left(\frac{\mathfrak{d}}{d}\right) d^{k/2-1} a_{n/d}(k),$$

where $\mu(d)$ is the Möbius function. Thanks to [12, (11)], for all even integers $k > 2$ we have:

$$c_{|\mathfrak{d}|n^2}(k) = c_{|\mathfrak{d}|}(k)\rho_{\mathfrak{d},n}(k). \tag{7}$$

Also, recall that if $c_{|\mathfrak{d}|}(k) \neq 0$ then $(-1)^{k/2}\mathfrak{d} \equiv 0, 1 \bmod 4$.

We now assume that $p \equiv 1 \bmod 4$. If a positive integer k belongs to $\mathcal{U}$, then $k \equiv 2$ modulo $p-1$ and hence $k/2$ is always odd. So, if we fix $\mathfrak{d} < 0$ a fundamental discriminant, we see that (7) holds for all integers $k \in \mathcal{U}$ with $k > 2$.

Since $p \equiv 1 \bmod 4$, we see that $c_D(k) = 0$ for all D such that $-D \equiv 2, 3 \bmod 4$ and for all positive integers $k \in \mathcal{U}$. It then follows that $\widetilde{c}_D(k)$ is identically zero on $\mathcal{U}$ for all D such that $-D \equiv 2, 3 \bmod 4$. Define

$$S_T := \{d \in \mathbb{Z} \text{ such that } d \mid c(T),\ (d, Np) = 1,\ -D_T/d^2 \equiv 0, 1 \bmod 4\}.$$

We can thus write

$$A_T^{(p)}(k) = \sum_{d \in S_T} d^{k/2} c_{D_T/d^2}(k). \tag{8}$$

We now use (7) to simplify the above sum (8). For this, we need the following preliminary discussion. For any integer D we can write $D = \mathfrak{d}\mathfrak{f}^2$ with $\mathfrak{d}$ the fundamental discriminant of the quadratic field $\mathbb{Q}(\sqrt{\mathfrak{d}})$ and $\mathfrak{f} > 0$ an half-integer. In this case, we say that $\mathfrak{d}$ is the fundamental discriminant associated with D. Also, note that $\mathfrak{f} \in \mathbb{Z}$ if $D \equiv 0, 1 \mod 4$. More precisely, if we denote by δ the maximal integer such that $\delta^2 \mid D$, then $\mathfrak{d} = D/\delta^2$ and $\mathfrak{f} = \delta$ except if $D \equiv 0 \bmod 4$ and $D/\delta^2 \equiv 2, 3 \bmod 4$, when $\mathfrak{d} = D/(\delta/2)^2$ and $\mathfrak{f} = \delta/2$. If $D \equiv 2, 3 \bmod 4$, then $\mathfrak{d} = 4D/\delta^2$ and $\mathfrak{f} = \delta/2 \notin \mathbb{Z}$. More generally, let $d \mid \delta$, so that $d^2 \mid D$. Write $D = \mathfrak{d}_D\mathfrak{f}_D^2$

and $D/d^2 = \mathfrak{d}_{D/d^2}(\mathfrak{f}_{D/d^2})^2$, with $\mathfrak{d}_D$ and $\mathfrak{d}_{D/d^2}$ the fundamental discriminants associated with D and D/d, respectively. First note that the maximal integer whose square divides D/d^2 is δ/d. We then have $\mathfrak{d}_D = \mathfrak{d}_{D/d^2}$ and $\mathfrak{f}_D = d\mathfrak{f}_{D/d^2}$ except if $D \equiv 0 \bmod 4$, $D/\delta^2 \equiv 1 \bmod 4$ and $D/d^2 \equiv 2, 3 \bmod 4$, when $4\mathfrak{d}_D = \mathfrak{d}_{D/d^2}$ and $\mathfrak{f}_D = 2d\mathfrak{f}_{D/d^2}$.

As anticipated, we now use (7) to simplify (8). For any matrix $T > 0$ write

$$-D_T = \mathfrak{d}_T\mathfrak{f}_T^2, \tag{9}$$

with $\mathfrak{d}_T < 0$. We have $-D_T \equiv 0, 1 \bmod 4$, so, in particular, the above discussion shows that $\mathfrak{f}_T \in \mathbb{Z}$. We thus have

$$c_{D_T}(k) = c_{|\mathfrak{d}_T|\mathfrak{f}_T^2}(k) = c_{|\mathfrak{d}_T|}(k)\rho_{\mathfrak{d}_T,\mathfrak{f}_T}(k)$$

for all integers $k \in \mathcal{U}$ with $k > 2$. More generally, fix $d \in S_T$. For $T = \left(\begin{smallmatrix} u & v/2 \\ v/2 & w \end{smallmatrix}\right)$, define $T/d := \left(\begin{smallmatrix} u/d & v/2d \\ v/2d & w/d \end{smallmatrix}\right)$. Then $D_{T/d} = D_T/d^2$. Under the assumption that $-D_T/d^2 \equiv 0, 1 \bmod 4$, the above discussion shows that $\mathfrak{f}_{T/d} \in \mathbb{Z}$ and that $\mathfrak{d}_T = \mathfrak{d}_{T/d}$ and $\mathfrak{f}_T = d\mathfrak{f}_{T/d}$. We can thus write:

$$c_{D_T/d^2}(k) = c_{D_{T/d}}(k) = c_{|\mathfrak{d}_{T/d}|\mathfrak{f}_{T/d}^2}(k) = c_{|\mathfrak{d}_T|}(k)\rho_{\mathfrak{d}_T,\mathfrak{f}_T/d}(k) \tag{10}$$

for all integers $k \in \mathcal{U}$ with $k > 2$. Inserting (10) into (8) we get

$$A_T^{(p)}(k) = c_{|\mathfrak{d}_T|}(k) \sum_{d \in S_T} d^{k/2}\rho_{\mathfrak{d}_T,\mathfrak{f}_T/d}(k). \tag{11}$$

for all integers $k \in \mathcal{U}$ with $k > 2$.

We are now ready to define our modified Fourier coefficients. Fix a matrix T_0 such that $c(T_0) = 1$ and $c_{D_{T_0}}(2) \neq 0$, which exists by a combination of the explicit formula of [14] relating $c_m(2)$ with special values of L-series and non-vanishing results for L-series by [4], [21] (see for example [11, Lem. 2.4] for details). Define, for integers in $\mathcal{U}$ with $k > 2$,

$$\widetilde{A}_T(k) := \frac{\left(1 - \left(\frac{\mathfrak{d}_T}{p}\right) a_p(k)^{-1}p^{k/2-1}\right) A_T^{(p)}(k)}{\left(1 - \left(\frac{D_{T_0}}{p}\right) a_p(k)^{-1}p^{k/2-1}\right) A_{T_0}^{(p)}(k)}. \tag{12}$$

To simplify notations, define, for all integers $k \in \mathcal{U}$ with $k > 2$:

$$n_T(k) := \sum_{d \in S_T} d^{k/2}\rho_{\mathfrak{d}_T,\mathfrak{f}_T/d}(k).$$

Putting (11) into (12) we get, for all integers $k \in \mathcal{U}$ with $k \geq 4$:

$$\widetilde{A}_T(k) = \frac{\left(1 - \left(\frac{\mathfrak{d}_T}{p}\right) a_p(k)^{-1}p^{k/2-1}\right) c_{|\mathfrak{d}_T|}(k)}{\left(1 - \left(\frac{D_{T_0}}{p}\right) a_p(k)^{-1}p^{k/2-1}\right) c_{D_{T_0}}(k)} n_T(k) = \widetilde{c}_{|\mathfrak{d}_T|}(k)n_T(k).$$

Proposition 3.2. *Let $p \equiv 1 \bmod 4$. The function $k \mapsto \widetilde{A}_T(k)$, defined for integers $k > 2$ in $\mathcal{U}$, extends to a p-adic analytic function on $\mathcal{U}$. Moreover, for $k = 2$ we get the equality:*

$$\widetilde{A}_T(2) = n_T \cdot \widetilde{c}_{|\mathfrak{d}_T|}(2) = n_T \cdot \frac{c_{|\mathfrak{d}_T|}(2)}{c_{D_{T_0}}(2)} \tag{13}$$

where $n_T := \widetilde{n}_T(2)$.

Proof. Clearly, $\rho_{\mathfrak{d},n}(k)$, defined for integers $k>2$ in $\mathcal{U}$, can be extended analytically to all of $\mathcal{U}$ because the same is true for the Fourier coefficients $a_n(k)$. We denote by $\widetilde{\rho}_{\mathfrak{d},n}(k)$ the resulting function. Likewise, we set $\widetilde{n}_T(k)$ the obvious analytic extension of $n_T(k)$ obtained from that of $\rho_{\mathfrak{d}_T,\mathfrak{f}_T/d}(k)$ for all $d\in S_T$. We can thus define for all $k\in\mathcal{U}$

$$\widetilde{A}_T(k):=\widetilde{c}_{|\mathfrak{d}_T|}(k)\widetilde{n}_T(k) \tag{14}$$

which is, thanks to Proposition 3.1, the sought for extension of $\widetilde{A}_T(k)$. The value at 2 is then a consequence of the second part of Proposition 3.1. □

Remark 3.3. The integer n_T above can be explicitly written as

$$n_T=\sum_{d\in S_T}\sum_{e|\mathfrak{f}_T/d;\ (e,N)=1} d\cdot\mu(e)\cdot\left(\frac{\mathfrak{d}_T}{e}\right)\cdot a_{\mathfrak{f}_T/(de)}(2)$$

and, at the moment, it has no clear geometric meaning to us. We simply note that nothing prevents a priori n_T to be zero.

4. Global points on elliptic curves

In this section, we compute the p-adic derivative of the explicit formula for the Fourier coefficients of the Saito-Kurokawa family exhibited in Section 3, and relate it to global points on elliptic curves via the fundamental results of [2] and [3].

4.1. **Global points associated to the Shintani lifting.**

We start by recalling the work of Darmon and Tornaría [6], for which we need to introduce some extra notations. Recall that $g=g_2$ denotes the Shintani lifting of $f=f_2$, as usually well-defined only up to scalars, with Fourier coefficients $c_D(2)$. For any prime number $\ell\mid Np$, denote by $w_\ell\in\{\pm1\}$ the eigenvalue of the Atkin-Lehner involution W_ℓ acting on f. Accordingly with [6], we introduce the following terminology:

(I) We call *discriminants of type* (I) those fundamental discriminants $\mathfrak{d}<0$ such that
$$\left(\frac{\mathfrak{d}}{\ell}\right)=w_\ell$$
for all primes $\ell\mid Np$.

(II) We call *discriminants of type* (II) those $\mathfrak{d}$ such that
$$\left(\frac{\mathfrak{d}}{\ell}\right)=w_\ell$$
for all primes $\ell\mid N$ and
$$\left(\frac{\mathfrak{d}}{p}\right)=-w_p.$$

If $\mathfrak{d}$ is of type II, then $c_{\mathfrak{d}}(2)=0$ and then $\widetilde{c}_{\mathfrak{d}}(2)=0$ too. In this latter case, it becomes interesting to look at the value at 2 of the derivative of $c_{\mathfrak{d}}(k)$ with respect to k.

Let $E/\mathbb{Q}$ denote the elliptic curve of conductor Np associated with f via the Eichler-Shimura construction. Let $\mathfrak{d}<0$ be a fundamental discriminant and, to simplify notations, define the imaginary quadratic field

$$K_{\mathfrak{d}}:=\mathbb{Q}(\sqrt{\mathfrak{d}}).$$

Denote by $E(K_{\mathfrak{d}})^-$ the submodule of the Mordell-Weil group $E(K_{\mathfrak{d}})$ on which the non-trivial involution of $\mathrm{Gal}(K_{\mathfrak{d}}/\mathbb{Q})$ acts as -1.

Since p does not divide N, the curve E has multiplicative reduction at p and we let

$$\Phi_{\mathrm{Tate}} : \mathbb{C}_p^\times/q^{\mathbb{Z}} \longrightarrow E(\mathbb{C}_p)$$

denote Tate's p-adic uniformization, where q is Tate's p-adic period. Let $\log_q$ denote the branch of the p-adic logarithm such that $\log_q(q) = 0$ and, for any $P \in E(\mathbb{C}_p)$, define

$$\log_E(P) := \log_q\left(\Phi_{\mathrm{Tate}}^{-1}(P)\right). \tag{15}$$

We extend $\log_E$ by $\mathbb{Q}$-linearity to $E(\mathbb{C}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$.

For a fundamental discriminant $\mathfrak{d} < 0$, let $L(f, \chi_{\mathfrak{d}}, s)$ denote the complex L-function of E twisted by the quadratic character

$$\chi_{\mathfrak{d}}(n) := \left(\frac{\mathfrak{d}}{n}\right)$$

of $K_{\mathfrak{d}}$. So, if the real part of s is sufficiently large, we have

$$L(f, \chi_{\mathfrak{d}}, s) = \sum_{n=1}^{\infty} \chi_{\mathfrak{d}}(n) a_n n^{-s}.$$

We now state a result of Darmon-Tornaría ([6, Theorem 1.5]).

Theorem 4.1 (Darmon-Tornaría)**.** *Let $N > 1$ and suppose that $\mathfrak{d} < 0$ is a fundamental discriminant if type II. Then there exists an element $P_{\mathfrak{d}} \in E(K_{\mathfrak{d}})^- \otimes_{\mathbb{Z}} \mathbb{Q}$ such that*

(1) $\frac{\partial}{\partial k}\tilde{c}_{|\mathfrak{d}|}(k)_{|k=2} = \log_E(P_{\mathfrak{d}})$.
(2) $P_{\mathfrak{d}} \neq 0$ *if any and only if* $L'(E, \chi_{\mathfrak{d}}, 1) \neq 0$.

Remark 4.2. The point $P_{\mathfrak{d}}$ in Theorem 4.1 comes, as already mentioned, from the theory of Stark-Heegner points. These points were introduced in [5] as local points on elliptic curves, and are conjectured to be global points, defined over ring class fields of real quadratic fields. A special case of this conjecture, which is needed in Theorem 4.1 above, is proved by Bertolini and Darmon in [3], using results from [2], by establishing a connection between classical Heegner points and Stark-Heegner points. For more details, we refer the reader to the discussion in [6, Section 3].

For the next result, let $\mathfrak{d}' < 0$ be a fundamental discriminant of type I, prime to $\mathfrak{d}$, and define $\Delta(\mathfrak{d}, \mathfrak{d}') := \mathfrak{d}\mathfrak{d}'$. In the following lines, we simply write Δ when the role of $\mathfrak{d}$ and $\mathfrak{d}'$ are clear from the context. In particular, since $\mathfrak{d}$ is of type II, Δ is not a perfect square and is prime to Np. Let χ_Δ denote the genus character associated to the pair of quadratic Dirichler characters $\chi_{\mathfrak{d}}$ and $\chi_{\mathfrak{d}'}$ of $\mathbb{Q}(\sqrt{\mathfrak{d}})$ and $\mathbb{Q}(\sqrt{\mathfrak{d}'})$. Then Δ is the discriminant of the real quadratic field $\mathbb{Q}(\sqrt{\Delta})$ and the field $\mathbb{Q}(\sqrt{\mathfrak{d}}, \sqrt{\mathfrak{d}'})$ cut out by χ_Δ is a quadratic extension of $\mathbb{Q}(\sqrt{\Delta})$ (unless χ_Δ is trivial, in which case coincides with $\mathbb{Q}(\sqrt{\Delta})$). Since all primes $\ell \mid N$ are split in K, we may choose an integer δ such that $\delta^2 \equiv \Delta \bmod 4N$. Recall from [6, §2] that a primitive binary quadratic form $Q(x, y) = Ax^2 + Bxy + Cy^2$ of discriminant Δ is said to be a *Heegner form* relative to the level N if $N \mid A$ and $B \equiv \delta \bmod N$, and let $\mathcal{F}_\Delta$ denote the set of such forms. Let $a + b\sqrt{\Delta}$ denote a fundamental unit of norm one in $\mathbb{Z}[(\Delta+\sqrt{\Delta})/2]$, normalized such that $a>0$ and $b>0$, and, for $Q(x,y)=Ax^2+Bxy+Cy^2\in\mathcal{F}_\Delta$, define the matrix

$$\gamma_Q=\begin{pmatrix} a+bB & 2Cb \\ -2Ab & a-bB \end{pmatrix}\in\Gamma_0(N).$$

The group $\Gamma_0(N)$ acts on $\mathcal{F}_\Delta$ from the right by the formula

$$(Q|\gamma)(x,y)=Q(ax+by,cx+dy)$$

for $\gamma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. Let $d=Q(m,n)$ be any integer represented by Q (namely, such that there are integers (m,n) such that $Q(m,n)=d$) satisfying the condition $\gcd(d,\mathfrak{d})=1$. The genus character χ_Δ defines a function, denoted by the same symbol,

$$\chi_\Delta:\mathcal{F}_\Delta/\Gamma_0(N)\longrightarrow\{\pm1\},\qquad \chi_\Delta(Q)=\begin{cases} 0 & \text{if } \gcd(A,B,C,\mathfrak{d})>1 \\ \left(\frac{\mathfrak{d}}{d}\right) & \text{otherwise.}\end{cases}$$

For any integer $k\in\mathcal{U}$ with $k\geq 4$, we consider the integrals

$$I_{\mathbb{C}}(f_k^\sharp,P,r,s):=\int_r^s f_k^\sharp(z)P(z)dz$$

where P is a polynomial of degree at most $k-2$ with coefficients in $\mathbb{C}$, r,s are in $\mathbb{P}^1(\mathbb{Q})$ and the above integral is along any path in the upper half plane connecting r and s. A result of Shimura shows that one can find a complex period $\Omega^-_{f_k^\sharp}$, analogous to $\Omega^-_{f_k}$ considered in Remark 2.1, such that

$$I(f_k^\sharp,P,r,s):=I_{\mathbb{C}}(f_k^\sharp,P,r,s)/\Omega^-_{f_k^\sharp}$$

belongs to the field generated over $\mathbb{Q}$ by the Fourier coefficients of $f_k^\sharp$, for all P, r and s. Define

$$J(f_k^\sharp,P,r,s):=(1-a_p(k)^{-2}p^{k-2})I(f_k^\sharp,P,r,s).$$

We fix now an embedding $\bar{\mathbb{Q}}_p\hookrightarrow\mathbb{C}$, so that polynomials with coefficients in $\bar{\mathbb{Q}}_p$ will be automatically viewed as polynomials with coefficients in $\mathbb{C}$. By [3, Proposition 3.4], one knows that, for any choice of $\tau\in\mathbb{Q}_{p^2}\backslash\mathbb{Q}_p$, where $\mathbb{Q}_{p^2}$ is the quadratic unramified extension of $\mathbb{Q}_p$, the function $k\mapsto J(f_k^\sharp,P,r,s)$, a priori only defined for integers $k>2$ in $\mathcal{U}$, extends to a p-adic analytic function on $\mathcal{U}$ that vanishes at $k=2$. We shall denote $k\mapsto J(k,P,r,s)$ this function.

Fix a square root $\sqrt{\Delta}$ of Δ in $\mathbb{Q}_{p^2}$ and, for $Q\in\mathcal{F}_\Delta$ as above, define

$$\tau_Q:=\frac{-B+\sqrt{\Delta}}{2A}.$$

We may then define

$$J(f,Q):=\frac{d}{dk}J(k,(z-\tau_Q)^{k-2},r,\gamma_Q(r))_{|k=2}.$$

Twisting $J(f,Q)$ by χ_Δ, we may also define:

$$J(f,\mathfrak{d},\mathfrak{d}')=\sum_{Q\in\mathcal{F}_\Delta/\Gamma_0(N)}\chi_\Delta(Q)J(f,Q). \tag{16}$$

The second result we quote from the work of Darmon-Tornaría is [6, Theorem 5.1], which can be restated as follows:

Theorem 4.3 (Darmon-Tornaría). *Fix $\mathfrak{d} < 0$ is a fundamental discriminant if type II.*

(1) *For any fundamental discriminant $\mathfrak{d}'$ of type I, we have the relation:*
$$\tilde{c}_{|\mathfrak{d}'|}(2) \log_E(P_{\mathfrak{d}}) = J(f, \mathfrak{d}, \mathfrak{d}').$$
(2) *If $N > 1$ then the function $\mathfrak{d}' \mapsto J(f, \mathfrak{d}, \mathfrak{d}')$ is non-zero if and only if $L'(E, \chi_{\mathfrak{d}}, 1) \neq 0$.*

4.2. **Global points associated to the Saito-Kurokawa lifting.** We combine the explicit Λ-adic Saito-Kurokawa lifting with Darmon-Tornaría's result. We first introduce the L-function attached to $\mathrm{SK}(f)$. Denote by $L(\mathrm{SK}(f), s)$ the Adrianov L-function associated with $\mathrm{SK}(f)$, whose definition can be found, e. g., in [16, p. 179], where it is denoted by $Z_F^*(s)$ for $F = \mathrm{SK}(f)$. We have the following relation between $L(\mathrm{SK}(f), s)$ and the standard L-function of f (see [16, Theorem 8])
$$L(\mathrm{SK}(f), s) = \zeta(s)\zeta(s-1)L(f, s)$$
from which it is apparent that the central critical point is $s = 1$. Accordingly, we may consider the $\chi_{\mathfrak{d}_T}$-twisted L-functions
$$L(\mathrm{SK}(f), \chi_{\mathfrak{d}_T}, s) = L(\chi_{\mathfrak{d}_T}, s)L(\chi_{\mathfrak{d}_T}, s-1)L(f, \chi_{\mathfrak{d}_T}, s).$$
Since $\mathfrak{d}_T$ is of type II, $L(f, \chi_{\mathfrak{d}}, 1) = 0$ and we have
$$L'(\mathrm{SK}(f), \chi_{\mathfrak{d}}, 1) = L(\chi_{\mathfrak{d}}, 0)L(\chi_{\mathfrak{d}}, 1)L'(f, \chi_{\mathfrak{d}}, 1)$$
In particular, since the factor $L(\chi_{\mathfrak{d}}, 0)L(\chi_{\mathfrak{d}}, 1)$ is non-zero, we see that
$$L'(\mathrm{SK}(f), \chi_{\mathfrak{d}}, 1) \neq 0 \iff L'(f, \chi_{\mathfrak{d}}, 1) \neq 0. \tag{17}$$

Before stating the main result of this paper, let us recall the integer n_T introduced in Proposition 3.2 and define the point
$$Q_T := n_T \cdot P_{\mathfrak{d}_T}.$$
Thanks to Theorem 4.1, we already know that
$$Q_T \neq 0 \Longrightarrow L'(\mathrm{SK}(f), \chi_{\mathfrak{d}_T}, 1) \neq 0.$$
However, the opposite implication is not true in general because the integer n_T might be zero.

Theorem 4.4. *Let $N > 1$ and $p \equiv 1 \bmod 4$. Suppose that $-D_T = \mathfrak{d}_T \cdot \mathfrak{f}_T^2$, where $\mathfrak{d}_T < 0$ is a fundamental discriminant of type II. Then:*

(1) *We have the following relation:*
$$\frac{\partial}{\partial k}\widetilde{A}_T(k)_{|k=2} = \log_E Q_T.$$
(2) *Suppose $n_T \neq 0$. Then Q_T is not zero in $E(K_{\mathfrak{d}_T})^- \otimes_{\mathbb{Z}} \mathbb{Q}$ if and only if $L'(\mathrm{SK}(f), \chi_{\mathfrak{d}_T}, 1) \neq 0$.*

Proof. Using (14) to compute formally the p-adic derivative of the Fourier coefficients $\widetilde{A}_T(k)$, we find:
$$\frac{\partial}{\partial k}\widetilde{A}_T(k)_{|k=2} = \frac{\partial}{\partial k}\widetilde{c}_{|\mathfrak{d}_T|}(k)_{|k=2} \cdot \widetilde{n}_T(2) + \widetilde{c}_{|\mathfrak{d}_T|}(2) \cdot \frac{\partial}{\partial k}\widetilde{n}_T(k)_{|k=2}.$$
Since $\mathfrak{d}_T$ is of type II, we have $\widetilde{c}_{|\mathfrak{d}_T|}(2) = 0$, and thus Theorem 4.1 implies the result, in light of the definition of Q_T. For the second part, simply note that if $P_{\mathfrak{d}_T} \neq 0$ then any non-zero multiple of it is non-zero and then use the relation (17). $\square$

Theorem 4.5. *Let $p \equiv 1$ mod 4. Fix T such that $-D_T = \mathfrak{d}_T \cdot \mathfrak{f}_T^2$ with $\mathfrak{d}_T < 0$ a fundamental discriminant of type II. For T' such that $-D_{T'} = \mathfrak{d}_{T'} \cdot \mathfrak{f}_{T'}^2$ with $\mathfrak{d}_{T'}$ a fundamental discriminant of type I, we have:*

(1) $\widetilde{A}_{T'}(2)\log_E(P_{\mathfrak{d}_T}) = n_{T'}J(f, \mathfrak{d}_T, \mathfrak{d}_{T'})$.
(2) *If $N > 1$ then the function $\mathfrak{d}_{T'} \mapsto J(f, \mathfrak{d}_T, \mathfrak{d}_{T'})$ is non-zero if and only if $L'(\mathrm{SK}(f), \chi_{\mathfrak{d}_T}, 1) \neq 0$.*

Proof. Combining (13) and Theorem 4.3, we have

$$\widetilde{A}_{T'}(2)\log_E(P_{\mathfrak{d}_T}) = \widetilde{c}_{|\mathfrak{d}_{T'}|}(2) n_{T'} \log_E(P_{\mathfrak{d}_T}) = n_{T'}J(f, \mathfrak{d}_T, \mathfrak{d}'_T).$$

This shows the first part, while the second is just a restatement of the second part of Theorem 4.3 combined with (17). □

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