THE Λ-ADIC SHINTANI-SHIMURA-WALDSPURGER CORRESPONDENCE

MATTEO LONGO, MARC-HUBERT NICOLE

Abstract. We generalize the Λ-adic Shintani lifting for $\mathrm{GL}_2(\mathbb{Q})$ to indefinite quaternion algebras over $\mathbb{Q}$.

1. Introduction

Langlands's principle of functoriality predicts the existence of a staggering wealth of transfers (or lifts) between automorphic forms for different reductive groups. In recent years, attempts at the formulation of p-adic variants of Langlands's functoriality have been articulated in various special cases. We prove the existence of the Shintani-Shimura-Waldspurger lift for p-adic families.

More precisely, Stevens, building on the work of Hida and Greenberg-Stevens, showed in [18] the existence of a Λ-adic variant of the classical Shintani lifting of [17] for $\mathrm{GL}_2(\mathbb{Q})$. This Λ-adic lifting can be seen as a formal power series with coefficients in a finite extension of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[\![X]\!]$ equipped with specialization maps interpolating classical Shintani lifting of classical modular forms appearing in a given Hida family.

Shimura in [16], resp. Waldspurger in [19] generalized the classical Shimura-Shintani correspondence to quaternion algebras over $\mathbb{Q}$, resp. over any number field. In this paper, motivated by ulterior applications to Shimura curves over $\mathbb{Q}$, we generalize Stevens's result to any non-split rational indefinite quaternion algebra B, building on works of Shimura [16] and combining this with a result of Longo-Vigni [8]. Our main result, for which the reader is referred to Theorem 3.8 below, states the existence of a formal power series and specialization maps interpolating Shimura-Shintani lifts of classical forms in a given p-adic family of automorphic forms on the quaternion algebra B. The Λ-adic variant of Waldspurger's result appears computationally challenging (see remark in [13, Intro.]), but it seems within reach for real quadratic fields (cf. [11]).

As an example of our main result, we consider the case of families with trivial character. More precisely, embed the set $\mathbb{Z}^{\geq 2}$ of integers greater or equal to 2 in $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ by sending $k \in \mathbb{Z}^{\geq 2}$ to the character $x \mapsto x^{k-2}$. Let f_∞ be an Hida family of tame level N, a positive integer. For any $k \in \mathbb{Z}^{\geq 2}$ we thus have an element $f_k \in S_k(\Gamma_0(Np))$, where $p \nmid N$ is a prime. Fix a factorization $N = MD$ with $D > 1$ a square-free product of an even number of primes and $(M, D) = 1$. Applying the Jacquet-Langlands correspondence we get for any $k \in \mathbb{Z}^{\geq 2}$ a modular form f_k^{JL} on Γ, which is the group of norm-one elements in an Eichler order R of level Mp contained in indefinite the rational quaternion algebra B of discriminant D. One

Date: October 30, 2012.
2000 *Mathematics Subject Classification.* Primary 11F37, 11F30, 11F85.

can show that these modular forms can be p-adically interpolated, up to scaling. More precisely, let $\mathcal{O}$ be the ring of integers of a finite extension F of $\mathbb{Q}_p$ and let $\mathbb{D}$ denote the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Z}_p^2$ which are supported on the set of primitive elements in $\mathbb{Y}$. Let Γ_0 be the group of norm-one elements in an Eichler order $R_0 \subseteq B$ containing R. There is a canonical action of Γ_0 on $\mathbb{D}$ (see [8, §2.4] for its description). Denote by F_k the extension of F generated by the Fourier coefficients of f_k. Then there is an element $\Phi \in H^1(\Gamma_0, \mathbb{D})$ and maps

$$\rho_k : H^1(\Gamma_0, \mathbb{D}) \longrightarrow H^1(\Gamma, F_k)$$

such that $\rho(k)(\Phi) = \phi_k$, the cohomology class associated to f_k^{JL} (for this we need a suitable normalization of the cohomology class associated to f_k^{JL}, which we do not touch for simplicity in this introduction). We view Φ as a quaternionic family of modular forms. To each ϕ_k we may apply the Shintani-Shimura lifting ([16]) and obtain a modular form h_k of weigh $k + 1/2$, level $4Np$ and trivial character. We show that this collection of forms can be p-adically interpolated. For clarity's sake, we present the liftings and their Λ-adic variants in a diagram, in which the horizontal maps are specialization maps of the p-adic family to weight k; JL stands for the Jacquet-Langlands correspondence; SSW stands for the Shintani-Shimura-Waldspurger lift; and the dotted arrows are constructed in this paper.

$$\begin{tikzcd}
f_\infty \arrow[r, mapsto] \arrow[d, mapsto, "\Lambda\text{-adic JL}"'] & f_k \arrow[d, mapsto, "\text{JL}"] \\
\Phi \arrow[r, mapsto, "\rho_k"] \arrow[d, mapsto, dotted, "\Lambda\text{-adic SSW}"'] & \phi_k \arrow[d, mapsto, "\text{SSW}"] \\
\Theta \arrow[r, mapsto, dotted] & h_k
\end{tikzcd}$$

More precisely, as a particular case of our main result, Theorem 3.8, we get the following

Theorem 1.1. *Fix $k_0 \in \mathbb{Z}^{\geq 2}$. Then there exists a p-adic neighborhood $\mathcal{U}_0$ of k_0 in $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$, p-adic periods Ω_k for $k \in \mathcal{U}_0$ and a formal expansion*

$$\Theta = \sum_{\xi \geq 1} a_\xi q^\xi$$

with coefficients a_ξ in the ring of $\mathbb{C}_p$-valued functions on $\mathcal{U}_0$, such that for all $k \in \mathcal{U}_0$ we have

$$\Theta(k) = \Omega_k \cdot h_k$$

Further, $\Omega_{k_0} \neq 0$.

2. Shintani integrals and Fourier coefficients of half-integral weight modular forms

We express the Fourier coefficients of half-integral weight modular forms in terms of period integrals, thus allowing a cohomological interpretation which is key to the production of the Λ-adic version of the Shintani-Shimura-Waldspurger correspondence. For the quaternionic Shintani-Shimura-Waldspurger correspondence of interest to us (see [13], [19]), the period integrals expressing the values of the Fourier coefficients have been computed generally by Prasanna in [14].

2.1. **The Shimura-Shintani lifting.** Let $4M$ be a positive integer, $2k$ an even non-negative integer and χ a Dirichlet character modulo $4M$ such that $\chi(-1)=1$. Recall that the space of half-integral weight modular forms $S_{k+1/2}(4M,\chi)$ consists of holomorphic cuspidal functions h on the upper-half place $\mathfrak{H}$ such that

$$h(\gamma(z)) = j^{1/2}(\gamma,z)^{2k+1}\chi(d)h(z),$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4M)$, where $j^{1/2}(\gamma,z)$ is the standard square root of the usual automorphy factor $j(\gamma,z)$ (cf. [13, 2.3]).

To any quaternionic integral weight modular form we may associate a half-integral weight modular form following Shimura's work [16], as we will describe below.

Fix an odd square free integer N and a factorization $N = M\cdot D$ into coprime integers such that $D>1$ is a product of an even number of distinct primes. Fix a Dirichlet character ψ modulo M and a positive even integer $2k$. Suppose that

$$\psi(-1) = (-1)^k.$$

Define the Dirichlet character χ modulo $4N$ by

$$\chi(x) := \psi(x)\left(\frac{-1}{x}\right)^k.$$

Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant the D. Fix a maximal order $\mathcal{O}_B$ of B. For every prime $\ell|M$, choose an isomorphism $i_\ell : B\otimes_\mathbb{Q}\mathbb{Q}_\ell \simeq \mathbb{M}_2(\mathbb{Q}_\ell)$ such that $i_\ell(\mathcal{O}_B\otimes_\mathbb{Z}\mathbb{Z}_\ell) = \mathbb{M}_2(\mathbb{Z}_\ell)$. Let $R\subseteq\mathcal{O}_B$ be the Eichler order of B of level M defined by requiring that $i_\ell(R\otimes_\mathbb{Z}\mathbb{Z}_\ell)$ is the suborder of $\mathbb{M}_2(\mathbb{Z}_\ell)$ of upper triangular matrices modulo ℓ for all $\ell|M$. Let Γ denote the subgroup of the group $R_1^\times$ of norm 1 elements in $R^\times$ consisting of those γ such that $i_\ell(\gamma)\equiv\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ mod ℓ for all $\ell\mid M$. We denote by $S_{2k}(\Gamma)$ the $\mathbb{C}$-vector space of weight $2k$ modular forms on Γ, and by $S_{2k}(\Gamma,\psi^2)$ the subspace of $S_{2k}(\Gamma)$ consisting of forms having character ψ^2 under the action of $R_1^\times$. Fix a Hecke eigenform

$$f\in S_{2k}(\Gamma,\psi^2)$$

as in [16, Section 3].

Let V denote the $\mathbb{Q}$-subspace of B consisting of elements with trace equal to zero. For any $v\in V$, which we view as a trace zero matrix in $\mathbb{M}_2(\mathbb{R})$ (after fixing an isomorphism $i_\infty : B\otimes\mathbb{R}\simeq\mathbb{M}_2(\mathbb{R})$), set

$$G_v := \{\gamma\in\mathrm{SL}_2(\mathbb{R})|\,\gamma^{-1}v\gamma = v\}$$

and put $\Gamma_v := G_v\cap\Gamma$. One can show that there exists an isomorphism $\omega : \mathbb{R}^\times \xrightarrow{\sim} G_v$ defined by $\omega(s) = \beta^{-1}\left(\begin{smallmatrix} s & 0 \\ 0 & s^{-1} \end{smallmatrix}\right)\beta$, for some $\beta\in\mathrm{SL}_2(\mathbb{R})$. Let $\mathfrak{t}_v$ be the order of $\Gamma_v\cap\{\pm1\}$ and let γ_v be an element of Γ_v which generates $\Gamma_v\{\pm1\}/\{\pm1\}$. Changing γ_v to γ_v^{-1} if necessary, we may assume $\gamma_v = \omega(t)$ with $t>0$. Define V^* to be the $\mathbb{Q}$-subspace of V consisting of elements with strictly negative norm. For any $\alpha = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\in V^*$ and $z\in\mathcal{H}$, define the quadratic form

$$Q_\alpha(z) := cz^2 - 2az - b.$$

Fix $\tau\in\mathcal{H}$ and set

$$P(f,\alpha,\Gamma) := -\big(2(-\mathrm{nr}(\alpha))^{1/2}/\mathfrak{t}_\alpha\big)\int_\tau^{\gamma_\alpha(\tau)} Q_\alpha(z)^{k-1}f(z)dz$$

where $\mathrm{nr}: B \to \mathbb{Q}$ is the norm map. By [16, Lemma 2.1], the integral is independent on the choice τ, which justifies the notation.

Remark 2.1. The definition of $P(f,\alpha,\Gamma)$ given in [16, (2.5)] looks different: the above expression can be derived as in [16, page 629] by means of [16, (2.20) and (2.22)].

Let $R(\Gamma)$ denote the set of equivalence classes of V^* under the action of Γ by conjugation. By [16, (2.6)], $P(f,\alpha,\Gamma)$ only depends on the conjugacy class of α, and thus, for $\mathcal{C} \in R(\Gamma)$, we may define $P(f,\mathcal{C},\Gamma) := P(f,\alpha,\Gamma)$ for any choice of $\alpha \in \mathcal{C}$. Also, $q(\mathcal{C}) := -\mathrm{nr}(\alpha)$ for any $\alpha \in \mathcal{C}$.

Define $\mathcal{O}'_B$ to be the maximal order in B such that $\mathcal{O}'_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \simeq \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ for all $\ell \nmid M$ and $\mathcal{O}'_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is equal to the local order of $B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ consisting of elements γ such that $i_\ell(\gamma) = \left(\begin{smallmatrix} a & b/M \\ cM & d \end{smallmatrix}\right)$ with $a,b,c,d \in \mathbb{Z}_\ell$, for all $\ell \mid M$. Given $\alpha \in \mathcal{O}'_B$, we can find an integer b_α such that

$$i_\ell(\alpha) \equiv \begin{pmatrix} * & b_\alpha/M \\ * & * \end{pmatrix} \mod i_\ell(R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell), \quad \forall \ell \mid M. \tag{1}$$

Define a locally constant function η_ψ on V by $\eta_\psi(\alpha) = \psi(b_\alpha)$ if $\alpha \in \mathcal{O}'_B \cap V$ and $\eta(\alpha) = 0$ otherwise, with $\psi(a) = 0$ if $(a,M) \neq 1$ (for the definition of locally constant functions on V in this context, we refer to [16, p. 611]).

For any $\mathcal{C} \in R(\Gamma)$, fix $\alpha_{\mathcal{C}} \in \mathcal{C}$. For any integer $\xi \geq 1$, define

$$a_\xi(\tilde{h}) := \big(2\mu(\Gamma\backslash\mathfrak{H})\big)^{-1} \cdot \sum_{\mathcal{C} \in R(\Gamma), q(\mathcal{C}) = \xi} \eta_\psi(\alpha_{\mathcal{C}}) \xi^{-1/2} P(f,\mathcal{C},\Gamma).$$

Then, by [16, Theorem 3.1],

$$\tilde{h} := \sum_{\xi \geq 1} a_\xi(\tilde{h}) q^\xi \in S_{k+1/2}(4N,\chi)$$

is called the Shimura-Shintani lifting of f.

2.2. **Cohomological interpretation.** We introduce necessary notations to define the action of the Hecke action on cohomology groups; for details, see [8, §2.1]. If G is a subgroup of $B^\times$ and S a subsemigroup of $B^\times$ such that (G,S) is an Hecke pair, we let $\mathcal{H}(G,S)$ denote the Hecke algebra corresponding to (G,S), whose elements are written as $T(s) = GsG = \coprod_i Gs_i$ for $s, s_i \in S$ (finite disjoint union). For any $s \in S$, let $s^* := \mathrm{norm}(s)s^{-1}$ and denote by S^* the set of elements of the form s^* for $s \in S$. For any $\mathbb{Z}[S^*]$-module M we let $T(s)$ act on $H^1(G,M)$ at the level of cochains $c \in Z^1(G,M)$ by the formula $(c|T(s))(\gamma) = \sum_i s_i^* c(t_i(\gamma))$, where $t_i(\gamma)$ are defined by the equations $Gs_i\gamma = Gs_j$ and $s_i\gamma = t_i(\gamma)s_j$. In the following, we will consider the case of $G = \Gamma$ and

$$S = \{s \in B^\times | i_\ell(s) \text{ is congruent to } \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \bmod \ell \text{ for all } \ell \mid M\}.$$

For any field L and any integer $n \geq 0$, let $V_n(L)$ denote the L-dual of the L-vector space $\mathcal{P}_n(L)$ of homogeneous polynomials in 2 variables of degree n. We let $\mathbb{M}_2(L)$ act from the right on $P(x,y)$ as $P|\gamma(x,y) := P(\gamma(x,y))$, where $\gamma(x,y) := (ax+yb, cx+dy)$ if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. This also equips $V_n(L)$ with a left action by $\gamma \cdot \varphi(P) := \varphi(P|\gamma)$. To simplify notations, we will write $P(z)$ for $P(z,1)$.

Let F denote the finite extension of $\mathbb{Q}$ generated by the eigenvalues of the Hecke action on f. For any field K containing F, set

$$\mathbb{W}_f(K) := H^1\big(\Gamma, V_{k-2}(K)\big)^f$$

where the superscript f denotes the subspace on which the Hecke algebra acts via the character associated with f. Also, for any sign $\pm$, let $\mathbb{W}_f^{\pm}(K)$ denote the $\pm$-eigenspace for the action of the archimedean involution ι. Remember that ι is defined by choosing an element ω_∞ of norm -1 in $R^\times$ such that such that $i_\ell(\omega_\infty) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \bmod M$ for all primes $\ell \mid M$ and then setting $\iota := T(w_\infty)$ (see [8, §2.1]). Then $\mathbb{W}_f^{\pm}(K)$ is one dimensional (see, e.g., [8, Proposition 2.2]); fix a generator $\phi_f^{\pm}$ of $\mathbb{W}_f^{\pm}(F)$.

To explicitly describe $\phi_f^{\pm}$, let us introduce some more notations. If $i_\infty(\omega_\infty) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ then define

$$f|\omega_\infty(z) := (Cz+D)^{-k/2}\overline{f(\omega_\infty(\bar{z}))}.$$

Then $f|\omega_\infty \in S_{2k}(\Gamma)$ as well. If the eigenvalues of the Hecke action on f are real, then we may assume, after multiplying f by a scalar, that $f|\omega_\infty = f$ (see [16, p. 627] or [9, Lemma 4.15]). In general, let $I(f)$ denote the class in $H^1(\Gamma, V_{k-2}(\mathbb{C}))$ represented by the cocycle

$$\gamma \longmapsto \left[P \mapsto I_\gamma(f)(P) := \int_\tau^{\gamma(\tau)} f(z)P(z)dz \right]$$

for any $\tau \in \mathcal{H}$ (the corresponding class is independent on the choice of τ). With these notations,

$$P(f,\alpha,\Gamma) = -\big(2(-\mathrm{nr}(\alpha))^{1/2}/\mathfrak{t}_\alpha\big) \cdot I_{\gamma_{\alpha_\mathcal{C}}}(f)\big(Q_{\alpha_\mathcal{C}}(z)^{k-1}\big).$$

Denote by $I^{\pm}(f) := (1/2)\cdot I(f) \pm (1/2)\cdot I(f)|\omega_\infty$ the projection of $I(f)$ to the eigenspaces for the action of ω_∞. Then we have $I(f) = I^+(f) + I^-(f)$. and $I_f^{\pm} = \Omega_f^{\pm} \cdot \phi_f^{\pm}$, for some $\Omega_f^{\pm} \in \mathbb{C}^\times$.

Given $\alpha \in V^*$ of norm $-\xi$, put $\alpha' := \omega_\infty^{-1}\alpha\omega_\infty$. By [16, 4.19], we have

$$\eta(\alpha)\xi^{-1/2}P(f,\alpha,\Gamma) + \eta(\alpha')\xi^{-1/2}P(f,\alpha',\Gamma) = -\eta(\alpha)\cdot {\mathfrak{t}_\alpha}^{-1} \cdot I^+_{\gamma_\alpha}\big(Q_{\alpha_\mathcal{C}}(z)^{k-1}\big)$$

We then have

$$a_\xi(\tilde{h}) = \sum_{\mathcal{C}\in R_2(\Gamma), q(\mathcal{C})=\xi} \frac{-\eta_\psi(\alpha_\mathcal{C})}{2\mu(\Gamma\backslash\mathcal{H})\cdot \mathfrak{t}_{\alpha_\mathcal{C}}} \cdot I^+_{\gamma_{\alpha_\mathcal{C}}}\big(Q_{\alpha_\mathcal{C}}(z)^{k-1}\big).$$

We close this paragraph by choosing a suitable multiple of h which will be the object of the next section. Given $Q_\alpha(z) = cz^2 - 2az - b$ as above, with $\alpha \in V^*$, define $\tilde{Q}_\alpha(z) := M \cdot Q_\alpha(z)$. Then, clearly, $I^{\pm}(f)(\tilde{Q}_{\alpha_\mathcal{C}}(z)^{k-1})$ is equal to $M^{k-1}I^{\pm}(f)(Q_{\alpha_\mathcal{C}}(z)^{k-1})$. We thus normalize the Fourier coefficients by setting

$$(2) \quad a_\xi(h) := -\frac{a_\xi(\tilde{h})\cdot M^{k-1}\cdot 2\mu(\Gamma\backslash\mathcal{H})}{\Omega_f^+} = \sum_{\mathcal{C}\in R(\Gamma), q(\mathcal{C})=\xi} \frac{\eta_\psi(\alpha_\mathcal{C})}{\mathfrak{t}_{\alpha_\mathcal{C}}} \cdot \phi_f^+\big(\tilde{Q}_{\alpha_\mathcal{C}}(z)^{k-1}\big).$$

So

$$(3) \qquad h := \sum_{\xi \geq 1} a_\xi(h) q^\xi$$

belongs to $S_{k+1/2}(4N,\chi)$ and is a non-zero multiple of $\tilde{h}$.

3. The Λ-adic Shintani-Shimura-Waldspurger correspondence

At the heart of Stevens's proof lies the control theorem of Greenberg-Stevens, which has been worked out in the quaternionic setting by Longo–Vigni [8].

Recall that $N \geq 1$ is a square free integer and fix a decomposition $N = M \cdot D$ where D is a square free product of an even number of primes and M is coprime to D. Let $p \nmid N$ be a prime number and fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

3.1. **The Hida Hecke algebra.** Fix an ordinary p-stabilized newform

$$f_0 \in S_{k_0}\big(\Gamma_1(Mp) \cap \Gamma_0(D), \epsilon_0\big) \tag{4}$$

of level $\Gamma_1(Mp) \cap \Gamma_0(D)$, Dirichlet character ϵ_0 and weight k_0, and write $\mathcal{O}$ for the ring of integers of the field generated over $\mathbb{Q}_p$ by the Fourier coefficients of f_0.

Let Λ (respectively, $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$) denote the Iwasawa algebra of $W := 1+p\mathbb{Z}_p$ (respectively, $\mathbb{Z}_p^\times$) with coefficients in $\mathcal{O}$. We denote group-like elements in Λ and $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ as $[t]$. Let $\mathfrak{h}_\infty^{\mathrm{ord}}$ denote the p-ordinary Hida Hecke algebra with coefficients in $\mathcal{O}$ of tame level $\Gamma_1(N)$. Denote by $\mathcal{L} := \mathrm{Frac}(\Lambda)$ the fraction field of Λ. Let $\mathcal{R}$ denote the integral closure of Λ in the primitive component $\mathcal{K}$ of $\mathfrak{h}_\infty^{\mathrm{ord}} \otimes_\Lambda \mathcal{L}$ corresponding to f_0. It is well known that the Λ-algebra $\mathcal{R}$ is finitely generated as Λ-module.

Denote by $\mathcal{X}$ the $\mathcal{O}$-module $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}\text{-alg}}(\mathcal{R}, \bar{\mathbb{Q}}_p)$ of continuous homomorphisms of $\mathcal{O}$-algebras. Let $\mathcal{X}^{\mathrm{arith}}$ the set of arithmetic homomorphisms in $\mathcal{X}$, defined in [8, §2.2] by requiring that the composition

$$W \hookrightarrow \Lambda \xrightarrow{\kappa} \bar{\mathbb{Q}}_p$$

has the form $\gamma \mapsto \psi_\kappa(\gamma)\gamma^{n_\kappa}$ with $n_\kappa = k_\kappa - 2$ for an integer $k_\kappa \geq 2$ (called the weight of κ) and a finite order character $\psi_\kappa : W \to \bar{\mathbb{Q}}$ (called the wild character of κ). Denote by r_κ the smallest among the positive integers t such that $1+p^t\mathbb{Z}_p \subseteq \ker(\psi_\kappa)$. For any $\kappa \in \mathcal{X}^{\mathrm{arith}}$, let P_κ denote the kernel of κ and $\mathcal{R}_{P_\kappa}$ the localization of $\mathcal{R}$ at κ. The field $F_\kappa := \mathcal{R}_{P_\kappa}/P_\kappa\mathcal{R}_{P_\kappa}$ is a finite extension of $\mathrm{Frac}(\mathcal{O})$. Further, by duality, κ corresponds to a normalized eigenform

$$f_\kappa \in S_{k_\kappa}\big(\Gamma_0(Np^{r_\kappa}), \epsilon_\kappa\big),$$

where $\epsilon_\kappa := \psi_\kappa \cdot \epsilon_0 \cdot \omega^{-n_\kappa}$; here ω is the Teichmüller character, $n_\kappa := k_\kappa - 2$ and ϵ_κ is viewed as a Dirichlet character of $(\mathbb{Z}/Np^{r_\kappa})^\times$ via the decomposition

$$(\mathbb{Z}/Np^{r_\kappa})^\times \simeq (1+p\mathbb{Z}_p)/(1+p^{r_\kappa}\mathbb{Z}_p) \times (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$$

where ψ_κ acts on the first factor, ω^{-n_κ} acts on the second factor and ϵ_0 acts on the third factor. We call $(\epsilon_\kappa, k_\kappa)$ the signature of κ. We let κ_0 denote the arithmetic character associated with f_0, so that $f_0 = f_{\kappa_0}$. The eigenvalues of f_κ under the action of the Hecke operators T_n ($n \geq 1$ an integer) belong to F_κ. Actually, one can show that f_κ is a p-stabilized newform on $\Gamma_1(Mp^{r_\kappa}) \cap \Gamma_0(D)$.

Let Λ_N denote the Iwasawa algebra of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ with coefficients in $\mathcal{O}$. To simplify notations, define $\Delta := (\mathbb{Z}/Np\mathbb{Z})^\times$. We have a canonical isomorphism of rings $\Lambda_N \simeq \Lambda[\Delta]$, which makes Λ_N a Λ-algebra, finitely generated as Λ-module. Define the tensor product of Λ-algebras

$$\mathcal{R}_N := \mathcal{R} \otimes_\Lambda \Lambda_N,$$

which is again a Λ-algebra (resp. Λ_N-algebra) finitely generated as a Λ-module, (resp. as a Λ_N-module). One easily checks that there is a canonical isomorphism of Λ-algebras

$$\mathcal{R}_N \simeq \mathcal{R}[\Delta]$$

(where Λ acts on $\mathcal{R}$); this is also an isomorphism of Λ_N-algebras, when we let $\Lambda_N \simeq \Lambda[\Delta]$ act on $\mathcal{R}[\Delta]$ in the obvious way.

We can extend any $\kappa \in \mathcal{X}^{\mathrm{arith}}$ to a continuous $\mathcal{O}$-algebra morphism $\kappa_N : \mathcal{R}_N \to \bar{\mathbb{Q}}_p$ setting

$$\kappa_N\left(\sum_{i=1}^n r_i \cdot \delta_i\right) := \sum_{i=1}^n \kappa(r_i) \cdot (\epsilon \cdot \omega^{-n_\kappa})(\delta_i)$$

for $r_i \in \mathcal{R}$ and $\delta_i \in \Delta$. If we denote by $\mathcal{X}_N$ the $\mathcal{O}$-module of continuous $\mathcal{O}$-algebra homomorphisms from $\mathcal{R}_N$ to $\bar{\mathbb{Q}}_p$, the above correspondence sets up an injective map $\mathcal{X}^{\mathrm{arith}} \hookrightarrow \mathcal{X}_N$. Let $\mathcal{X}_N^{\mathrm{arith}}$ denote the image of $\mathcal{X}^{\mathrm{arith}}$ under this map. For $\kappa_N \in \mathcal{X}_N^{\mathrm{arith}}$, we define the signature of κ_N to be that of the corresponding κ.

3.2. **The control theorem in the quaternionic setting.** Recall that $B/\mathbb{Q}$ is a quaternion algebra of discriminant D. Fix an auxiliary real quadratic field F such that all primes dividing D are inert in F and all primes dividing Mp are split in F, and an isomorphism $i_F : B \otimes_{\mathbb{Q}} F \simeq \mathbb{M}_2(F)$. Let $\mathcal{O}_B$ denote the maximal order of B obtained by taking the intersection of B with $\mathbb{M}_2(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of F. More precisely, define

$$\mathcal{O}_B := \iota^{-1}\big(i_F^{-1}\big(i_F(B \otimes 1) \cap \mathbb{M}_2(\mathcal{O}_F)\big)\big)$$

where $\iota : B \hookrightarrow B \otimes_{\mathbb{Q}} F$ is the inclusion defined by $b \mapsto b \otimes 1$. This is a maximal order in B because $i_F(B \otimes 1) \cap \mathbb{M}_2(\mathcal{O}_F)$ is a maximal order in $i_F(B \otimes 1)$. In particular, i_F and our fixed embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ induce an isomorphism $i_p : B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \mathbb{M}_2(\mathbb{Q}_p)$ such that $i_p(\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \mathbb{M}_2(\mathbb{Z}_p)$. For any prime $\ell \mid M$, also choose an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$ which, composed with i_F, yields isomorphisms $i_\ell : B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \mathbb{M}_2(\mathbb{Q}_\ell)$ such that $i_p(\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) = \mathbb{M}_2(\mathbb{Z}_\ell)$. Define an Eichler order $R \subseteq \mathcal{O}_B$ of level M by requiring that for all primes $\ell \mid M$ the image of $R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ via i_ℓ consists of upper triangular matrices modulo ℓ. For any $r \geq 0$, let Γ_r denote the subgroup of the group $R_1^\times$ of norm-one elements in R consisting of those γ such that $i_\ell(\gamma) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \equiv 0 \mod Mp^r$ and $a \equiv d \equiv 1 \mod Mp^r$, for all primes $\ell \mid Mp$. To conclude this list of notations and definitions, fix an embedding $F \hookrightarrow \mathbb{R}$ and let $i_\infty : B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{M}_2(\mathbb{R})$ be the induced isomorphism.

Let $\mathbb{Y} := \mathbb{Z}_p^2$ and denote by $\mathbb{X}$ the set of primitive vectors in $\mathbb{Y}$. Let $\mathbb{D}$ denote the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Y}$ which are supported on $\mathbb{X}$. Note that $\mathbb{M}_2(\mathbb{Z}_p)$ acts on $\mathbb{Y}$ by left multiplication; this induces an action of $\mathbb{M}_2(\mathbb{Z}_p)$ on the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Y}$, which induces an action on $\mathbb{D}$. The group $R^\times$ acts on $\mathbb{D}$ via i_p. In particular, we may define the group:

$$\mathbb{W} := H^1(\Gamma_0, \mathbb{D}).$$

Then $\mathbb{D}$ has a canonical structure of $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-module, as well as $\mathfrak{h}_\infty^{\mathrm{ord}}$-action, as described in [8, §2.4]. In particular, let us recall that, for any $[t] \in \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$, we have

$$\int_{\mathbb{X}} \varphi(x,y)d\big([t]\cdot\nu\big) = \int_{\mathbb{X}} \varphi(tx,ty)d\nu,$$

for any locally constant function φ on $\mathbb{X}$.

For any $\kappa \in \mathcal{X}^{\mathrm{arith}}$ and any sign $\pm$, set

$$\mathbb{W}_\kappa^\pm := \mathbb{W}_{f_\kappa^{\mathrm{JL}}}^\pm(F_\kappa) = H^1\big(\Gamma_{r_\kappa}, V_{n_\kappa}(F_\kappa)\big)^{f_\kappa,\epsilon}$$

where f_κ^{JL} is any Jacquet-Langlands lift of f_κ to Γ_{r_κ}; recall that the superscript f_κ denotes the subspace on which the Hecke algebra acts via the character associated with f_κ, and the superscript $\epsilon = \pm$ denotes the ϵ-eigenspace for the action of the archimedean involution ι. Also, recall that $\mathbb{W}_k^\pm$ is one dimensional and fix a generator $\phi_k^\pm$ of it.

We may define specialization maps

$$\rho_\kappa : \mathbb{D} \longrightarrow V_{n_\kappa}(F_\kappa)$$

by the formula

$$\rho_\kappa(\nu)(P) := \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \epsilon_\kappa(y)P(x,y)d\nu \tag{5}$$

which induces (see [8, §2.5]) a map:

$$\rho_\kappa : \mathbb{W}^{\mathrm{ord}} \longrightarrow \mathbb{W}_\kappa^{\mathrm{ord}}.$$

Here $\mathbb{W}^{\mathrm{ord}}$ and $\mathbb{W}_\kappa^{\mathrm{ord}}$ denote the ordinary submodules of $\mathbb{W}$ and $\mathbb{W}_\kappa$, respectively, defined as in [3, Definition 2.2] (see also [8, §3.5]). We also let $\mathbb{W}_{\mathcal{R}} := \mathbb{W}\otimes_\Lambda \mathcal{R}$, and extend the above map ρ_κ to a map

$$\rho_\kappa : \mathbb{W}_{\mathcal{R}}^{\mathrm{ord}} \longrightarrow \mathbb{W}_\kappa^{\mathrm{ord}}$$

by setting $\rho_\kappa(x\otimes r) := \rho_\kappa(x)\cdot\kappa(r)$.

Theorem 3.1. *There exists a p-adic neighborhood $\mathcal{U}_0$ of κ_0 in $\mathcal{X}$, elements $\Phi^\pm \in \mathbb{W}_{\mathcal{R}}^{\mathrm{ord}}$ and choices of p-adic periods $\Omega_\kappa^\pm \in F_\kappa$ for $\kappa \in \mathcal{U}_0\cap\mathcal{X}^{\mathrm{arith}}$ such that, for all $\kappa\in\mathcal{U}_0\cap\mathcal{X}^{\mathrm{arith}}$, we have*

$$\rho_\kappa(\Phi^\pm) = \Omega_\kappa^\pm\cdot\phi_\kappa^\pm$$

and $\Omega_{\kappa_0}^\pm \neq 0$.

Proof. This is an easy consequence of [8, Theorem 2.18] and follows along the lines of the proof of [18, Theorem 5.5], cf. [9, Proposition 3.2]. □

We now normalize our choices as follows. With $\mathcal{U}_f$ as above, define

$$\mathcal{U}_0^{\mathrm{arith}} := \mathcal{U}_0\cap\mathcal{X}^{\mathrm{arith}}.$$

Fix $\kappa\in\mathcal{U}_0^{\mathrm{arith}}$ and an embedding $\bar{\mathbb{Q}}_p\hookrightarrow\mathbb{C}$. Let f_κ^{JL} denote a modular form on Γ_{r_κ} corresponding to f_κ by the Jacquet-Langlands correspondence, which is well defined up to elements in $\mathbb{C}^\times$. View $\phi_\kappa^\pm$ as an element in $H^1(\Gamma_{r_\kappa}, V_n(\mathbb{C}))^\pm$. Choose a representative $\Phi_\gamma^\pm$ of $\Phi^\pm$, by which we mean that if $\Phi^\pm = \sum_i \Phi_i^\pm\otimes r_i$, then we choose a representative $\Phi_{i,\gamma}^\pm$ for all i. Also, we will write $\rho_\kappa(\Phi)(P)$ as

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}\epsilon_\kappa(y)P(x,y)d\Phi_\gamma^\pm := \sum_i \kappa(r_i)\cdot\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}\epsilon_\kappa(y)P(x,y)d\Phi_{i,\gamma}^\pm.$$

With these notations, we see that the two cohomology classes

$$\gamma\longmapsto\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}\epsilon_\kappa(y)P(x,y)d\Phi_\gamma^\pm(x,y)$$

and

$$\gamma\longmapsto\Omega_\kappa^\pm\cdot\int_\tau^{\gamma(\tau)}P(z,1)f_\kappa^{\mathrm{JL},\pm}(z)dz$$

are cohomologous in $H^1(\Gamma_{r_\kappa}, V_{n_\kappa}(\mathbb{C}))$, for any choice of $\tau\in\mathcal{H}$.

3.3. **Metaplectic Hida Hecke algebras.** Let $\sigma : \Lambda_N \to \Lambda_N$ be the ring homomorphism associated to the group homomorphism $t \mapsto t^2$ on $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, and denote by the same symbol its restriction to Λ and $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$. We let Λ_σ, $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]_\sigma$ and $\Lambda_{N,\sigma}$ denote, respectively, Λ, $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ and Λ_N viewed as algebras over themselves via σ. The ordinary metaplectic p-adic Hida Hecke algebra we will consider is the Λ-algebra

$$\widetilde{\mathcal{R}} := \mathcal{R} \otimes_\Lambda \Lambda_\sigma.$$

Define as above

$$\widetilde{\mathcal{X}} := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}\text{-alg}}(\widetilde{\mathcal{R}}, \bar{\mathbb{Q}}_p)$$

and let the set $\widetilde{\mathcal{X}}^{\mathrm{arith}}$ of arithmetic points in $\widetilde{\mathcal{X}}$ to consist of those $\tilde{\kappa}$ such that the composition

$$W \hookrightarrow \Lambda \xhookrightarrow{\lambda \mapsto 1\otimes\lambda} \widetilde{\mathcal{R}} \xrightarrow{\tilde{\kappa}} \bar{\mathbb{Q}}_p$$

has the form $\gamma \mapsto \psi_{\tilde{\kappa}}(\gamma)\gamma^{n_{\tilde{\kappa}}}$ with $n_{\tilde{\kappa}} := k_{\tilde{\kappa}} - 2$ for an integer $k_{\tilde{\kappa}} \geq 2$ (called the weight of $\tilde{\kappa}$) and a finite order character $\psi_{\tilde{\kappa}} : W \to \bar{\mathbb{Q}}$ (called the wild character of $\tilde{\kappa}$). Let $r_{\tilde{\kappa}}$ the smallest among the positive integers t such that $1 + p^t\mathbb{Z}_p \subseteq \ker(\psi_{\tilde{\kappa}})$.

We have a map $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ induced by pull-back from the canonical map $\mathcal{R} \to \widetilde{\mathcal{R}}$. The map p restricts to arithmetic points.

As above, define the Λ-algebra (or Λ_N-algebra)

$$\widetilde{\mathcal{R}}_N := \mathcal{R} \otimes_\Lambda \Lambda_{N,\sigma} \tag{6}$$

via $\lambda \mapsto 1 \otimes \lambda$.

We easily see that

$$\widetilde{\mathcal{R}}_N \simeq \widetilde{\mathcal{R}}[\Delta]$$

as Λ_N-algebras, where we enhance $\widetilde{\mathcal{R}}[\Delta]$ with the following structure of $\Lambda_N \simeq \Lambda[\Delta]$-algebra: for $\sum_i \lambda_i \cdot \delta_i \in \Lambda[\Delta]$ ($\lambda_i \in \Lambda$ and $\delta_i \in \Delta$) and $\sum r_j \cdot \delta'_j \in \widetilde{\mathcal{R}}[\Delta]$ ($r_j = \sum_h r_{j,h} \otimes \lambda_{j,h} \in \widetilde{\mathcal{R}}$ with $r_{j,h} \in \mathcal{R}$ and $\lambda_{j,h} \in \Lambda_\sigma$, and $\delta'_j \in \Delta$), we set

$$\Big(\sum_i \lambda_i \cdot \delta_i\Big) \cdot \Big(\sum_j r_j \cdot \delta'_j\Big) := \sum_{i,j,h} \big(r_{j,h} \otimes (\lambda_i \lambda_{j,h})\big) \cdot (\delta_i \delta'_j).$$

We now fix a Dirichlet character ψ modulo N, which is trivial on $(\mathbb{Z}/D\mathbb{Z})^\times$ under the decomposition $(\mathbb{Z}/N\mathbb{Z})^\times \simeq (\mathbb{Z}/M\mathbb{Z})^\times \times (\mathbb{Z}/D\mathbb{Z})^\times$. As above, we can extend $\tilde{\kappa} \in \widetilde{\mathcal{X}}^{\mathrm{arith}}$ to a continuous $\mathcal{O}$-algebra morphism $\tilde{\kappa}_N : \widetilde{\mathcal{R}}_N \to \bar{\mathbb{Q}}_p$ by setting

$$\tilde{\kappa}_N\left(\sum_{i=1}^n x_i \cdot \delta_i\right) := \sum_{i=1}^n \tilde{\kappa}(x_i) \cdot (\psi \cdot \omega^{-n_{\tilde{\kappa}}})(\delta_i)$$

for $x_i \in \widetilde{\mathcal{R}}$ and $\delta_i \in \Delta$. If we denote by $\widetilde{\mathcal{X}}_N$ the $\mathcal{O}$-module of continuous $\mathcal{O}$-linear homomorphisms from $\widetilde{\mathcal{R}}_N$ to $\bar{\mathbb{Q}}_p$, the above correspondence sets up an injective map $\widetilde{\mathcal{X}}^{\mathrm{arith}} \hookrightarrow \widetilde{\mathcal{X}}_N$ and we let $\widetilde{\mathcal{X}}_N^{\mathrm{arith}}$ denote the image of $\widetilde{\mathcal{X}}^{\mathrm{arith}}$. Define

$$\epsilon_{\tilde{\kappa}} := \psi_{\tilde{\kappa}} \cdot \psi \cdot \omega^{-n_{\tilde{\kappa}}}, \tag{7}$$

viewed as a Dirichlet character of $(\mathbb{Z}/Np^{r_{\tilde{\kappa}}})^\times$ via the decomposition

$$(\mathbb{Z}/Np^{r_{\tilde{\kappa}}})^\times \simeq (1 + p\mathbb{Z}_p)/(1 + p^{r_{\tilde{\kappa}}}\mathbb{Z}_p) \times (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$$

and call the pair $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$ the signature of $\tilde{\kappa}_N$, where $\tilde{\kappa}$ is the arithmetic point corresponding to $\tilde{\kappa}_N$.

We also have a map $p_N : \widetilde{\mathcal{X}}_N \to \mathcal{X}_N$ induced by pull-back from the map $\mathcal{R}_N \to \widetilde{\mathcal{R}}_N$ taking $r \mapsto r \otimes 1$. The map p_N also restricts to arithmetic points. The maps p and p_N make the following diagram commute:

$$\begin{tikzcd} \widetilde{\mathcal{X}}^{\mathrm{arith}} \arrow[r, hook] \arrow[d, "p"] & \widetilde{\mathcal{X}}_N^{\mathrm{arith}} \arrow[d, "p_N"] \\ \mathcal{X}^{\mathrm{arith}} \arrow[r, hook] & \mathcal{X}_N^{\mathrm{arith}} \end{tikzcd}$$

where the projections take a signature (ϵ, k) to $(\epsilon^2, 2k)$.

3.4. **The Λ-adic correspondence.** In this part, we combine the explicit integral formula of Shimura and the fact that the toric integrals can be p-adically interpolated to show the existence of a Λ-adic Shintani-Shimura-Waldspurger correspondence with the expected interpolation property. This follows very closely [18, §6].

Let $\tilde{\kappa}_N \in \widetilde{\mathcal{X}}_N^{\mathrm{arith}}$ with the signature of $\tilde{\kappa}_N$ is $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$ and decompose $\epsilon_{\tilde{\kappa}} = \psi \cdot \psi_{\tilde{\kappa}} \cdot \omega^{-n_{\tilde{\kappa}}}$. Let L_r denote the order of $\mathbb{M}_2(F)$ consisting of matrices $\left(\begin{smallmatrix} a & b/Mp^r \\ Mp^r c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathcal{O}_F$. Define

$$\mathcal{O}_{B,r} := \iota^{-1}\big(i_F^{-1}\big(i_F(B \otimes 1) \cap L_r\big)\big)$$

Then $\mathcal{O}_{B,r}$ is the maximal order introduced in §2.1 (and denoted $\mathcal{O}_B'$ there) defined in terms of the maximal order $\mathcal{O}_B$ and the integer Mp^r. Also, $S := \mathcal{O}_B \cap \mathcal{O}_{B,r}$ is an Eichler order of B of level Mp containing the fixed Eichler order R of level M. With $\alpha \in V^* \cap \mathcal{O}_{B,1}$, we have

$$i_F(\alpha) = \begin{pmatrix} a & b/(Mp) \\ c & -a \end{pmatrix} \tag{8}$$

in $\mathbb{M}_2(F)$ with $a, b, c \in \mathcal{O}_F$ and we can consider the quadratic forms

$$Q_\alpha(x, y) := cx^2 - 2axy - \big(b/(Mp)\big)y^2,$$

and

$$\tilde{Q}_\alpha(x, y) := Mp \cdot Q_\alpha(x, y) = Mpcx^2 - 2Mpaxy - by^2. \tag{9}$$

Then $\tilde{Q}_\alpha(x, y)$ has coefficients in $\mathcal{O}_F$ and, composing with $F \hookrightarrow \mathbb{R}$ and letting $x = z$, $y = 1$, we recover $Q_\alpha(z)$ and $\tilde{Q}_\alpha(z)$ of §2.1 (defined by means of the isomorphism i_∞). Since each prime $\ell \mid Mp$ is split in F, the elements a, b, c can be viewed as elements in $\mathbb{Z}_\ell$ via our fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$, for any prime $\ell \mid Mp$ (we will continue writing a, b, c for these elements, with a slight abuse of notations). So, letting $b_\alpha \in \mathbb{Z}$ such that $i_\ell(\alpha) \equiv \left(\begin{smallmatrix} * & b_\alpha/(Mp) \\ * & * \end{smallmatrix}\right)$ modulo $i_\ell(S \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$, for all $\ell \mid Mp$, we have $b \equiv b_\alpha$ modulo $Mp\mathbb{Z}_\ell$ as elements in $\mathbb{Z}_\ell$, for all $\ell \mid Mp$, and thus we get

$$\eta_{\epsilon_{\tilde{\kappa}}}(\alpha) = \epsilon_{\tilde{\kappa}}(b_\alpha) = \epsilon_{\tilde{\kappa}}(b) \tag{10}$$

for b as in (8).

For any $\nu \in \mathbb{D}$, we may define an $\mathcal{O}$-valued measure $j_\alpha(\nu)$ on $\mathbb{Z}_p^\times$ by the formula:

$$\int_{\mathbb{Z}_p^\times} f(t)dj_\alpha(\nu)(t) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} f\big(\tilde{Q}_\alpha(x, y)\big)d\nu(x, y).$$

for any continuous function $f : \mathbb{Z}_p^\times \to \mathbb{C}_p$. Recall that the group of $\mathcal{O}$-valued measures on $\mathbb{Z}_p^\times$ is isomorphic to the Iwasawa algebra $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$, and thus we may view $j_\alpha(\nu)$ as an element in $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ (see, for example, [1, §3.2]). In particular, for any group-like element $[\lambda] \in \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ we have:

$$\int_{\mathbb{Z}_p^\times} f(t)d\big([\lambda]\cdot j_\alpha(\nu)\big)(t) = \int_{\mathbb{Z}_p^\times}\left(\int_{\mathbb{Z}_p^\times} f(ts)d[\lambda](s)\right)dj_\alpha(\nu)(t) = \int_{\mathbb{Z}_p^\times} f(\lambda t)dj_\alpha(\nu)(t).$$

On the other hand,

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} f\big(\tilde{Q}_\alpha(x,y)\big)d(\lambda\cdot\nu) = \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} f\big(\tilde{Q}_\alpha(\lambda x,\lambda y)\big)d\nu = \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} f\big(\lambda^2\tilde{Q}_\alpha(x,y)\big)d\nu$$

and we conclude that $j_\alpha(\lambda\cdot\nu) = [\lambda^2]\cdot\nu$. In other words, j_α is a $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-linear map

$$j_\alpha : \mathbb{D} \longrightarrow \mathcal{O}[\![\mathbb{Z}_p^\times]\!]_\sigma.$$

Before going ahead, let us introduce some notations. Let χ be a Dirichlet character modulo Mp^r, for a positive integer r, which we decompose accordingly with the isomorphism $(\mathbb{Z}/Np^r\mathbb{Z})^\times \simeq (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/p^r\mathbb{Z})^\times$ into the product $\chi = \chi_N\cdot\chi_p$ with $\chi_N : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ and $\chi_p : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \mathbb{C}^\times$. Thus, we will write $\chi(x) = \chi_N(x_N)\cdot\chi_p(x_p)$, where x_N and x_p are the projections of $x \in (\mathbb{Z}/Np^r\mathbb{Z})^\times$ to $(\mathbb{Z}/N\mathbb{Z})^\times$ and $(\mathbb{Z}/p^r\mathbb{Z})^\times$, respectively. To simplify notations, we will suppress the N and p from the notations for x_N and x_p, thus simply writing x for any of the two. We assume that, under the isomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \simeq (\mathbb{Z}/M\mathbb{Z})^\times \times (\mathbb{Z}/D\mathbb{Z})^\times$, χ_N is trivial on the second factor and decompose $\chi_N = \chi_M\cdot\chi_D$ as above with χ_M and χ_D characters on $(\mathbb{Z}/M\mathbb{Z})^\times$ and $(\mathbb{Z}/D\mathbb{Z})^\times$, respectively.

Using the above notations, we may define a $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-linear map $J_\alpha : \mathbb{D} \to \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ by

$$J_\alpha(\nu) = \epsilon_{\tilde{\kappa},M}(b)\cdot\epsilon_{\tilde{\kappa},p}(-1)\cdot j_\alpha(\nu)$$

with b as in (8). Set $\mathbb{D}_N := \mathbb{D}\otimes_{\mathcal{O}[\![\mathbb{Z}_p^\times]\!]}\Lambda_N$, where the map $\mathcal{O}[\![\mathbb{Z}_p^\times]\!] \to \Lambda_N$ is defined induced from the map $\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ on group-like elements given by $x \mapsto x\otimes 1$. Then J_α can be extended to a Λ_N-linear map $J_\alpha : \mathbb{D}_N \to \Lambda_{N,\sigma}$. Setting $\mathbb{D}_{\mathcal{R}_N} := \mathcal{R}_N \otimes_{\Lambda_N} \mathbb{D}_N$ and extending by $\mathcal{R}_N$-linearity over Λ_N we finally obtain a $\mathcal{R}_N$-linear map, again denoted by the same symbol,

$$J_\alpha : \mathbb{D}_{\mathcal{R}_N} \longrightarrow \widetilde{\mathcal{R}}_N.$$

For $\nu \in \mathbb{D}_N$ and $r \in \mathcal{R}_N$ we thus have

$$J_\alpha(r\otimes\nu) = \epsilon_{\tilde{\kappa},M}(b)\cdot\epsilon_{\tilde{\kappa},p}(-1)\cdot r\otimes j_\alpha(\nu).$$

For the next result, for any arithmetic point $\kappa_N \in \mathcal{X}_N^{\mathrm{arith}}$ coming from $\kappa \in \mathcal{X}^{\mathrm{arith}}$, extend ρ_κ in (5) by $\mathcal{R}_N$-linearity over $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$, to get a map

$$\rho_{\kappa_N} : \mathbb{D}_{\mathcal{R}_N} \longrightarrow V_{n_\kappa}$$

defined by $\rho_{\kappa_N}(r\otimes\nu) := \rho_\kappa(\nu)\cdot\kappa_N(r)$, for $\nu \in \mathbb{D}$ and $r \in \mathcal{R}_N$. To simplify notations, set

$$\rho_{\kappa_N}(x)(\tilde{Q}_\alpha^{n_{\tilde{\kappa}}/2}) := \langle x,\alpha\rangle_{\kappa_N}. \tag{11}$$

The following is essentially [18, Lemma (6.1)].

Lemma 3.2. *Let $\tilde{\kappa}_N \in \widetilde{\mathcal{X}}_N^{\mathrm{arith}}$ with signature $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$ and define $\kappa_N := p_N(\tilde{\kappa}_N)$. Then for any $x \in \mathbb{D}_{\mathcal{R}_N}$ we have:*

$$\tilde{\kappa}_N\big(J_\alpha(x)\big) = \eta_{\epsilon_{\tilde{\kappa}}}(\alpha)\cdot\langle x,\alpha\rangle_{\kappa_N}.$$

Proof. For $\nu \in \mathbb{D}_N$ and $r \in \mathcal{R}_N$ we have

$$\begin{aligned}\tilde\kappa_N\big(J_\alpha(r\otimes\nu)\big) &= \tilde\kappa_N\big(\epsilon_{\tilde\kappa,M}(b)\cdot\epsilon_{\tilde\kappa,p}(-1)\cdot r\otimes j_\alpha(\nu)\big)\\ &= \epsilon_{\tilde\kappa,M}(b)\cdot\epsilon_{\tilde\kappa,p}(-1)\cdot\tilde\kappa_N(r\otimes 1)\cdot\tilde\kappa_N\big(1\otimes j_\alpha(\nu)\big)\\ &= \epsilon_{\tilde\kappa,M}(b)\cdot\epsilon_{\tilde\kappa,p}(-1)\cdot\kappa_N(r)\cdot\int_{\mathbb{Z}_p^\times}\tilde\kappa_N(t)dj_\alpha(\nu)\end{aligned}$$

and thus, noticing that $\tilde\kappa_N$ restricted to $\mathbb{Z}_p^\times$ is $\tilde\kappa_N(t)=\epsilon_{\tilde\kappa,p}(t)t^{n_{\tilde\kappa}}$, we have

$$\tilde\kappa_N\big(J_\alpha(r\otimes\nu)\big)=\epsilon_{\tilde\kappa,M}(b)\cdot\epsilon_{\tilde\kappa,p}(-1)\cdot\kappa_N(r)\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times}\epsilon_{\tilde\kappa,p}(\tilde Q_\alpha(x,y))\tilde Q_\alpha(x,y)^{n_{\tilde\kappa}/2}d\nu.$$

Recalling (9), and viewing a,b,c as elements in $\mathbb{Z}_p$, we have, for $(x,y)\in\mathbb{Z}_p\times\mathbb{Z}_p^\times$,

$$\epsilon_{\tilde\kappa,p}\big(\tilde Q_\alpha(x,y)\big)=\epsilon_{\tilde\kappa,p}(-by^2)=\epsilon_{\tilde\kappa,p}(-b)\epsilon_{\tilde\kappa,p}(y^2)=\epsilon_{\tilde\kappa,p}(-b)\epsilon_{\tilde\kappa,p}^2(y)=\epsilon_{\tilde\kappa,p}(-b)\epsilon_{\kappa,p}(y).$$

Thus, since $\epsilon_{\tilde\kappa}(-1)^2=1$, we get:

$$\tilde\kappa_N\big(J_\alpha(r\otimes\nu)\big)=\kappa_N(r)\cdot\epsilon_{\tilde\kappa,M}(b)\cdot\epsilon_{\tilde\kappa,p}(b)\cdot\rho_\kappa(\nu)(\tilde Q_\alpha^{n_{\tilde\kappa}/2})=\eta_{\epsilon_\kappa}(\alpha)\cdot\langle x,\alpha\rangle_{\kappa_N}$$

where for the last equality use (10) and (11). □

Define

$$\mathbb{W}_{\mathcal{R}_N}:=\mathbb{W}\otimes_{\mathcal{O}[\![\mathbb{Z}_p^\times]\!]}\mathcal{R}_N,$$

the structure of $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-module of $\mathcal{R}_N$ being that induced by the composition of the two maps $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]\to\Lambda_N\to\mathcal{R}_N$ described above. There is a canonical map

$$\vartheta:\mathbb{W}_{\mathcal{R}_N}\longrightarrow H^1(\Gamma_0,\mathbb{D}_{\mathcal{R}_N})$$

described as follows: if ν_γ is a representative of an element ν in $\mathbb{W}$ and $r\in\mathcal{R}_N$, then $\vartheta(\nu\otimes r)$ is represented by the cocycle $\nu_\gamma\otimes r$.

For $\nu\in\mathbb{W}_{\mathcal{R}_N}$ represented by ν_γ and $\xi\geq 1$ an integer, define

$$\theta_\xi(\nu):=\sum_{\mathcal{C}\in R(\Gamma_1),q(\mathcal{C})=\xi}\frac{J_{\alpha_\mathcal{C}}(\nu_{\gamma_{\alpha_\mathcal{C}}})}{\mathfrak{t}_{\alpha_\mathcal{C}}}.$$

Definition 3.3. For $\nu\in\mathbb{W}_{\mathcal{R}_N}$, the formal Fourier expansion

$$\Theta(\nu):=\sum_{\xi\geq1}\theta_\xi(\nu)q^\xi$$

in $\mathcal{R}_N[\![q]\!]$ is called the Λ-adic Shintani-Shimura-Waldspurger lift of ν. For any $\tilde\kappa\in\widetilde{\mathcal{X}}^{\mathrm{arith}}$, the formal power series expansion

$$\Theta(\nu)(\tilde\kappa_N):=\sum_{\xi\geq1}\tilde\kappa_N\big(\theta_\xi(\nu)\big)q^\xi$$

is called the $\tilde\kappa$-specialization of $\Theta(\nu)$.

There is a natural map

$$\mathbb{W}_\mathcal{R}\longrightarrow\mathbb{W}_{\mathcal{R}_N}$$

taking $\nu\otimes r$ to itself (use that $\mathcal{R}$ has a canonical map to $\mathcal{R}_N\simeq\mathcal{R}[\Delta]$, as described above). So, for any choice of sign, $\Phi^\pm\in\mathbb{W}_\mathcal{R}$ will be viewed as an element in $\mathbb{W}_{\mathcal{R}_N}$.

¿From now on we will use the following notations. Fix $\tilde\kappa_0\in\widetilde{\mathcal{X}}^{\mathrm{arith}}$ and put $\kappa_0:=p(\tilde\kappa_0)\in\mathcal{X}^{\mathrm{arith}}$. Recall the neighborhood $\mathcal{U}_0$ of κ_0 in Theorem 3.1. Define $\widetilde{\mathcal{U}}_0:=p^{-1}(\mathcal{U}_0)$ and

$$\widetilde{\mathcal{U}}_0^{\mathrm{arith}}:=\widetilde{\mathcal{U}}_0\cap\widetilde{\mathcal{X}}^{\mathrm{arith}}.$$

For each $\tilde\kappa \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ put $\kappa = p(\tilde\kappa) \in \mathcal{U}_0^{\mathrm{arith}}$. Recall that if $(\epsilon_{\tilde\kappa}, k_{\tilde\kappa})$ is the signature of $\tilde\kappa$, then $(\epsilon_\kappa, k_\kappa) := (\epsilon_{\tilde\kappa}^2, 2k_{\tilde\kappa})$ is that of κ_0. For any $\kappa := p(\tilde\kappa)$ as above, we may consider the modular form

$$f_\kappa^{\mathrm{JL}} \in S_{k_\kappa}(\Gamma_{r_\kappa}, \epsilon_\kappa)$$

and its Shimira-Shintani lift

$$h_\kappa = \sum_\xi a_\xi(h_\kappa) q^\xi \in S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi_\kappa), \quad \text{where } \chi_\kappa(x) := \epsilon_{\tilde\kappa}(x)\left(\frac{-1}{x}\right)^{k_\kappa},$$

normalized as in (2) and (3). For our fixed κ_0, recall the elements $\Phi := \Phi^+$ chosen as in Theorem 3.1 and define $\phi_\kappa := \phi_\kappa^+$ and $\Omega_\kappa := \Omega_\kappa^+$ for $\kappa \in \mathcal{U}_0^{\mathrm{arith}}$.

Proposition 3.4. *For all $\tilde\kappa \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ such that $r_\kappa = 1$ we have*

$$\tilde\kappa_N\big(\theta_\xi(\Phi)\big) = \Omega_\kappa \cdot a_\xi(h_\kappa) \quad \text{and} \quad \Theta(\Phi)(\tilde\kappa_N) = \Omega_\kappa \cdot h_\kappa.$$

Proof. By Lemma 3.2 we have

$$\tilde\kappa_N\big(\theta_\xi(\Phi)\big) = \sum_{\mathcal{C}\in R(\Gamma_1), q(\mathcal{C})=\xi} \frac{\eta_{\epsilon_{\tilde\kappa}}(\alpha_{\mathcal{C}})}{\mathfrak{t}_{\alpha_{\mathcal{C}}}} \rho_{\kappa_N}(\Phi)(\tilde Q_{\alpha_{\mathcal{C}}}^{n_{\tilde\kappa}/2}).$$

Using Theorem 3.1, we get

$$\tilde\kappa_N\big(\theta_\xi(\Phi)\big) = \sum_{\mathcal{C}\in R(\Gamma_1), q(\mathcal{C})=\xi} \frac{\eta_{\epsilon_{\tilde\kappa}}(\alpha_{\mathcal{C}})\cdot\Omega_\kappa}{\mathfrak{t}_{\alpha_{\mathcal{C}}}} \phi_\kappa(\tilde Q_{\alpha_{\mathcal{C}}}^{k_\kappa-1}).$$

Now (2) shows the statement on $\tilde\kappa_N(\theta_\xi(\Phi))$, while that on $\Theta(\Phi)(\tilde\kappa_N)$ is a formal consequence of the previous one. $\square$

Corollary 3.5. *Let a_p denote the image of the Hecke operator T_p in $\mathcal{R}$. Then $\Theta(\Phi)|T_p^2 = a_p \cdot \Theta(\Phi)$.*

Proof. For any $\kappa \in \mathcal{X}^{\mathrm{arith}}$, let $a_p(\kappa) := \kappa(T_p)$, which is a p-adic unit by the ordinarity assumption. For all $\tilde\kappa \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ with $r_\kappa = 1$, we have We have

$$\Theta(\Phi)(\tilde\kappa_N)|U_p^2 = \Omega_\kappa \cdot h_\kappa|U_p^2 = a_p(\kappa)\cdot\Omega_\kappa\cdot h_\kappa = a_p(\kappa)\cdot\Theta(\Phi)(\tilde\kappa_N).$$

Consequently,

$$\tilde\kappa_N\big(\theta_{\xi p^2}(\Phi)\big) = a_p(\kappa)\cdot\tilde\kappa_N\big(\theta_\xi(\Phi)\big)$$

for all $\tilde\kappa$ such that $r_\kappa = 1$. Since this subset is dense in $\widetilde{\mathcal{X}}_N$, we conclude that $\theta_{\xi p^2}(\Phi) = a_p \cdot \theta_\xi(\Phi)$ and so $\Theta(\Phi)|T_p^2 = a_p \cdot \Theta(\Phi)$.

$\square$

For any integer $n \geq 1$ and any quadratic form Q with coefficients in F, write $[Q]_n$ for the class of Q modulo the action of $i_F(\Gamma_n)$. Define $\mathcal{F}_{n,\xi}$ to be the subset of the F-vector space of quadratic forms with coefficients in F consisting of quadratic forms $\tilde Q_\alpha$ such that $\alpha \in V^* \cap \mathcal{O}_{B,n}$ and $-\mathrm{nr}(\alpha) = \xi$. Writing $\delta_{\tilde Q_\alpha}$ for the discriminant of Q_α, the above set can be equivalently described as

$$\mathcal{F}_{n,\xi} := \{\tilde Q_\alpha |\, \alpha \in V^* \cap \mathcal{O}_{B,n},\, \delta_{\tilde Q_\alpha} = Np^n\xi\}.$$

Define $\mathcal{F}_{n,\xi}/\Gamma_n$ to be the set $\{[\tilde Q_\alpha]_n |\, \tilde Q_\alpha \in \mathcal{F}_{n,\xi}\}$ of equivalence classes of $\mathcal{F}_{n,\xi}$ under the action of $i_F(\Gamma_n)$. A simple computation shows that $Q_{g^{-1}\alpha g} = Q_\alpha|g$ for all $\alpha \in V^*$ and all $g \in \Gamma_n$, and thus we find

$$\mathcal{F}_{n,\xi}/\Gamma_n = \{[\tilde Q_{\mathcal{C}_\alpha}]_n |\, \mathcal{C} \in R(\Gamma_n),\, \delta_{\tilde Q_\alpha} = Np^n\xi\}.$$

We also note that, in the notations of §2.1, if f has weight character ψ, defined modulo Np^n, and level Γ_n, the Fourier coefficients $a_\xi(h)$ of the Shintani-Shimura lift h of f are given by

$$a_\xi(h) = \sum_{[Q]\in\mathcal{F}_{n,\xi}/\Gamma_n} \frac{\psi(Q)}{\mathfrak{t}_Q}\phi_f^+\big(Q(z)^{k-1}\big) \tag{12}$$

and, if $Q = \tilde{Q}_\alpha$, we put $\psi(Q) := \eta_\psi(b_\alpha)$ and $\mathfrak{t}_Q := \mathfrak{t}_\alpha$. Also, if we let

$$\mathcal{F}_n/\Gamma_n := \coprod_\xi \mathcal{F}_{n,\xi}/\Gamma_n$$

we can write

$$h = \sum_{[Q]\in\mathcal{F}_n/\Gamma_n} \frac{\psi(Q)}{\mathfrak{t}_Q}\phi_f^+\big(Q(z)^{k-1}\big)q^{\delta_Q/(Np^n)}. \tag{13}$$

Fix now an integer $m \geq 1$ and let $n \in \{1, m\}$. For any $t \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ and any integer $\xi \geq 1$, define $\mathcal{F}_{n,\xi,t}$ to be the subset of $\mathcal{F}_{n,\xi}$ consisting of forms such that $Np^n b_\alpha \equiv t \mod Np^m$. Also, define $\mathcal{F}_{n,\xi,t}/\Gamma_n$ to be the set of equivalence classes of $\mathcal{F}_{n,\xi,t}$ under the action of $i_F(\Gamma_n)$. If $\alpha \in V^* \cap \mathcal{O}_{B,m}$ and $i_F(\alpha) = \left(\begin{smallmatrix} a & b \\ c & -a\end{smallmatrix}\right)$, then

$$\tilde{Q}_\alpha(x,y) = Np^n cx^2 - 2Np^n axy - Np^n by^2 \tag{14}$$

from which we see that there is an inclusion $\mathcal{F}_{m,\xi,t} \subseteq \mathcal{F}_{1,\xi p^{m-1},t}$. If $\tilde{Q}_\alpha$ and $\tilde{Q}_{\alpha'}$ belong to $\mathcal{F}_{m,\xi,t}$, and $\alpha' = g\alpha g^{-1}$ for some $g \in \Gamma_m$, then, since $\Gamma_m \subseteq \Gamma_1$, we see that $\tilde{Q}_\alpha$ and $\tilde{Q}_{\alpha'}$ represent the same class in $\mathcal{F}_{1,\xi p^{m-1},t}/\Gamma_1$. This shows that $[\tilde{Q}_\alpha]_m \mapsto [\tilde{Q}_\alpha]_1$ gives a well-defined map

$$\pi_{m,\xi,t} : \mathcal{F}_{m,\xi,t}/\Gamma_m \longrightarrow \mathcal{F}_{1,\xi p^{m-1},t}/\Gamma_1.$$

Lemma 3.6. *The map $\pi_{m,\xi,t}$ is bijective.*

Proof. We first show the injectivity. For this, suppose $\tilde{Q}_\alpha$ and $\tilde{Q}_{\alpha'}$ are in $\mathcal{F}_{m,\xi,t}$ and $[\tilde{Q}_\alpha]_1 = [\tilde{Q}_{\alpha'}]_1$. So there exists $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ in $i_F(\Gamma_1)$ such that such that $\tilde{Q}_\alpha = \tilde{Q}_{\alpha'}|g$. If $\tilde{Q}_\alpha = cx^2 - 2axy - by^2$, and easy computation shows that $\tilde{Q}_{\alpha'} = c'x^2 - 2a'xy - b'y^2$ with

$$c' = c\alpha^2 - 2a\alpha\gamma - b\gamma^2$$
$$a' = -c\alpha\beta + a\beta\gamma + a\alpha\delta + b\gamma\delta$$
$$b' = -c\beta^2 + 2a\beta\delta + b\delta^2.$$

The first condition shows that $\gamma \equiv 0 \mod Np^m$. Further, we have $b \equiv b' \equiv t \mod Np^m$, so $\delta^2 \equiv 1 \mod Np^m$. Since $\delta \equiv 1 \mod Np$, we see that $\delta \equiv 1 \mod Np^m$ too.

We now first show the surjectivity. For this, fix $[\tilde{Q}_{\alpha_\mathcal{C}}]_1$ in the target of π, and choose a representative

$$\tilde{Q}_{\alpha_\mathcal{C}} = cx^2 - 2axy - by^2$$

(recall $Mp^m\xi \mid \delta_{\tilde{Q}_{\alpha_\mathcal{C}}}$, $Np \mid c$, $Np \mid a$, and $b \in \mathcal{O}_F^\times$, the last condition due to $\eta_\psi(\alpha_\mathcal{C}) \neq 0$). By the Strong Approximation Theorem, we can find $\tilde{g} \in \Gamma_1$ such that $i_\ell(\tilde{g}) \equiv \left(\begin{smallmatrix} 1 & 0 \\ ab^{-1} & 1\end{smallmatrix}\right) \mod Np^m$ for all $\ell \mid Np$. Take $g := i_F(\tilde{g})$, and put $\alpha := g^{-1}\alpha_\mathcal{C} g$. An easy computation, using the expressions for a', b', c' in terms of a, b, c and $g =$

$\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ as above, shows that $\alpha \in V^* \cap \mathcal{O}_{B,m}$, $\eta_\psi(\alpha) = t$ and $\delta_{\tilde{Q}_\alpha} = Np^m\xi$, and it follows that $\tilde{Q}_\alpha \in \mathcal{F}_{m,\xi,t}$. Now

$$\pi\big([\tilde{Q}_\alpha]_m\big) = [\tilde{Q}_\alpha]_1 = [\tilde{Q}_{g^{-1}\alpha_\mathcal{C} g}]_1 = [\tilde{Q}_{\alpha_\mathcal{C}}]_1$$

where the last equality follows because $g \in \Gamma_1$. □

Proposition 3.7. *For all $\tilde{\kappa} \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ we have*

$$\Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_\kappa - 1} = \Omega_\kappa \cdot h_\kappa.$$

Proof. For $r_\kappa = 1$, this is Proposition 3.4 above, so we may assume $r_\kappa \geq 2$. As in the proof of Proposition 3.4, combining Lemma 3.2 and Theorem 3.1 we get

$$\Theta(\Phi)(\tilde{\kappa}_N) = \sum_{\xi \geq 1}\left(\sum_{\mathcal{C} \in R(\Gamma_1), q(\mathcal{C}) = \xi} \frac{\eta_{\epsilon_{\tilde{\kappa}}}(\alpha_\mathcal{C}) \cdot \Omega_\kappa}{\mathsf{t}_{\alpha_\mathcal{C}}} \phi_\kappa(\tilde{Q}_{\alpha_\mathcal{C}}^{k_\kappa - 1})\right) q^\xi$$

which, by (12) and (13) above we may rewrite as

$$\Theta(\Phi)(\tilde{\kappa}_N) = \sum_{[Q] \in \mathcal{F}_1/\Gamma_1} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np)}$$

By definition of the action of T_p on power series, we have

$$\Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_\kappa - 1} = \sum_{[Q] \in \mathcal{F}_1/\Gamma_1,\, p^{r_\kappa} | \delta_Q} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np^{r_\kappa})}.$$

Setting $\mathcal{F}_{n,t}/\Gamma_n := \coprod_{\xi \geq 1} \mathcal{F}_{n,t,\xi}/\Gamma_n$ for $n \in \{1, r_\kappa\}$, Lemma 3.6 shows that

$$\mathcal{F}_{1,t}^* := \{[Q] \in \mathcal{F}_{1,t}/\Gamma_{1,t} \text{ such that } p^{r_\kappa} \mid \delta_Q\} \text{ is equal to } \mathcal{F}_{r_\kappa,t}.$$

Therefore, splitting the above sum over $t \in (\mathbb{Z}/Np^{r_\kappa}\mathbb{Z})^\times$, we get

$$\begin{aligned}
\Theta(\Phi)(\tilde{\kappa}_N)|T_p^{r_\kappa - 1} &= \sum_{t \in (\mathbb{Z}/p^{r_\kappa - 1}\mathbb{Z})^\times} \sum_{[Q] \in \mathcal{F}_{1,t}^*} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np^{r_\kappa})} \\
&= \sum_{t \in (\mathbb{Z}/p^{r_\kappa - 1}\mathbb{Z})^\times} \sum_{[Q] \in \mathcal{F}_{m,t}/\Gamma_m} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np^{r_\kappa})} \\
&= \sum_{[Q] \in \mathcal{F}_m/\Gamma_m} \frac{\epsilon_{\tilde{\kappa}}(Q) \cdot \Omega_\kappa}{\mathsf{t}_Q} \phi_\kappa(Q^{k_\kappa - 1}) q^{\delta_Q/(Np^{r_\kappa})}.
\end{aligned}$$

Comparing this expression with (13) gives the result. □

We are now ready to state the analogue of [18, Thm. 3.3], which is our main result. For the reader's convenience, we briefly recall the notations appearing below. We denote by $\mathcal{X}$ the ordinary Hida Hecke algebra, and by $\mathcal{X}^{\mathrm{arith}}$ its arithmetic points. For $\kappa_0 \in \mathcal{X}^{\mathrm{arith}}$, we denote by $\mathcal{U}_0$ the p-adic neighborhood of κ_0 appearing in the statement of Theorem 3.1 and put $\mathcal{U}_0^{\mathrm{arith}} := \mathcal{U}_0 \cap \mathcal{X}^{\mathrm{arith}}$. We also denote by $\Phi = \Phi^+ \in \mathbb{W}_\mathcal{R}^{\mathrm{ord}}$ the cohomology class appearing in Theorem 3.1. The metaplectic Hida Hecke algebra $\widetilde{\mathcal{X}}$ is defined in §3.3 and is equipped with a canonical map $p : \widetilde{\mathcal{X}}^{\mathrm{arith}} \to \mathcal{X}^{\mathrm{arith}}$ on arithmetic points. Let $\widetilde{\mathcal{U}}_0^{\mathrm{arith}} := \widetilde{\mathcal{U}}_0 \cap \widetilde{\mathcal{X}}^{\mathrm{arith}}$. For each $\tilde{\kappa} \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ put $\kappa = p(\tilde{\kappa}) \in \mathcal{U}_0^{\mathrm{arith}}$. Recall that if $(\epsilon_{\tilde{\kappa}}, k_{\tilde{\kappa}})$ is the signature of $\tilde{\kappa}$, then $(\epsilon_\kappa, k_\kappa) := (\epsilon_{\tilde{\kappa}}^2, 2k_{\tilde{\kappa}})$ is that of κ_0. For any $\kappa := p(\tilde{\kappa})$ as above, we may consider the modular form

$$f_\kappa^{\mathrm{JL}} \in S_{k_\kappa}(\Gamma_{r_\kappa}, \epsilon_\kappa)$$

and its Shimira-Shintani lift

$$h_\kappa = \sum_\xi a_\xi(h_\kappa)q^\xi \in S_{k_\kappa+1/2}(4Np^{r_\kappa},\chi_\kappa), \quad \text{where } \chi_\kappa(x) := \epsilon_{\tilde\kappa}(x)\left(\frac{-1}{x}\right)^{k_\kappa},$$

normalized as in (2) and (3). Finally, for $\tilde\kappa \in \widetilde{\mathcal{X}}^{\mathrm{arith}}$, we denote by $\tilde\kappa_N$ its extension to the metaplectic Hecke algebra $\widetilde{\mathcal{R}}_N$ defined in §3.3.

Theorem 3.8. *Let $\kappa_0 \in \mathcal{X}^{\mathrm{arith}}$. Then there exists a choice of p-adic periods Ω_κ for $\kappa \in \mathcal{U}_0$ such that the Λ-adic Shintani-Shimura-Waldspurger lift of Φ*

$$\Theta(\Phi) := \sum_{\xi\geq 1}\theta_\xi(\Phi)q^\xi$$

in $\mathcal{R}_N[\![q]\!]$ has the following properties:

(1) $\Omega_{\kappa_0} \neq 0$.
(2) *For any $\tilde\kappa \in \widetilde{\mathcal{X}}^{\mathrm{arith}}$, the $\tilde\kappa$-specialization of $\Theta(\Phi)$*

$$\Theta(\nu)(\tilde\kappa_N) := \sum_{\xi\geq 1}\tilde\kappa\big(\theta_\xi(\Phi)\big)q^\xi \text{ belongs to } S_{k_\kappa+1/2}(4Np^{r_\kappa},\chi'_\kappa),$$

where $\chi'_\kappa(x) := \chi_\kappa(x)\cdot\left(\frac{p}{x}\right)^{k_\kappa-1}$, and satisfies

$$\Theta(\Phi)(\tilde\kappa_N) = \Omega_\kappa \cdot h_\kappa|T_p^{1-r_\kappa}.$$

Proof. The elements Ω_κ are those Ω_κ^+ appearing in Theorem 3.1, which we used in Propositions 3.4 and 3.7 above, so (1) is clear. Applying $T_p^{r_\kappa-1}$ to the formula of Proposition 3.7, using Corollary 3.5 and applying $a_p(\kappa)$ on both sides gives

$$\Theta(\Phi)(\tilde\kappa_N) = a_p(\kappa)^{1-r_\kappa}\Omega_\kappa\cdot h_\kappa|T_p^{r_\kappa-1}.$$

By [15, Prop. 1.9], each application of T_p has the effect of multiplying the character by $\left(\frac{p}{\cdot}\right)$, hence

$$h'_\kappa := h_\kappa|T_p^{r_\kappa-1} \in S_{k_\kappa+1/2}(4Np^{r_\kappa},\chi'_\kappa)$$

with χ'_κ as in the statement. This gives the first part of (2), while the last formula follows immediately from Proposition 3.7. □

Remark 3.9. Theorem 1.1 is a direct consequence of Theorem 3.8. Briefly, embed the set $\mathbb{Z}^{\geq 2}$ of integers greater or equal to 2 in $\mathcal{X}^{\mathrm{arith}}$ by sending $k \in \mathbb{Z}^{\geq 2}$ to the unique arithmetic point in $\mathcal{X}^{\mathrm{arith}}$ of signature $(k,1)$. First, one can define an embedding

$$\mathbb{Z}^{\geq 2} \hookrightarrow \mathcal{X}(\Lambda) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}\text{-alg}}(\Lambda,\bar{\mathbb{Q}}_p)$$

via $x \mapsto x^{k-2}$. Call $k \in \mathcal{X}(\Lambda)$ the element thus obtained from $k \in \mathbb{Z}^{\geq 2}$. Second, a well-known result by Hida (see [6, Corollary 1.4]) shows that $\mathcal{R}/\Lambda$ is unramified at k. We can thus extend uniquely k to $\mathcal{R}$. Applying Theorem 3.8 to the image of $\mathbb{Z}^{\geq 2}$ in $\mathcal{X}^{\mathrm{arith}}$, we get Theorem 1.1.

Remark 3.10. For $\tilde\kappa \in \widetilde{\mathcal{U}}_0^{\mathrm{arith}}$ of signature $(\epsilon_{\tilde\kappa}, k_{\tilde\kappa})$ with $r_{\tilde\kappa} = 1$ as in the above theorem, h_κ is trivial if $(-1)^{k_{\tilde\kappa}} \neq \epsilon_{\tilde\kappa}(-1)$. However, since $\phi_{\kappa_0} \neq 0$, it follows that h_{κ_0} is not trivial as long as the necessary condition $(-1)^{k_0} = \epsilon_0(-1)$ is verified.

Remark 3.11. This result can be used to produce a quaternionic Λ-adic version of the Saito-Kurokawa lifting, following closely the arguments in [7, Cor. 1].

References

[1] J. Coates, R. Sujatha, *Cyclotomic fields and zeta values.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006.

[2] H. Darmon, G. Tornaria, Stark-Heegner points and the Shimura correspondence. *Compositio Math.*, **144** (2008) 1155-1175.

[3] R. Greenberg, G. Stevens, p-adic L-functions and p-adic periods of modular forms. *Invent. Math.* **111** (1993), no. 2, 407–447.

[4] Koblitz, N., *Introduction to elliptic curves and modular forms.* Graduate Texts in Mathematics, **97**. Springer-Verlag, New York, 1984. viii+248 pp.

[5] W. Kohnen, Fourier coefficients of modular forms of half-integral weight. *Math. Ann.* **271** (1985), no. 2, 237–268.

[6] H. Hida, Galois representations into $\mathrm{GL}_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.* **85** (1986), no. 3, 545–613.

[7] M. Longo, M.-H. Nicole, The Saito-Kurokawa lifting and Darmon points, to appear in Math. Ann. DOI 10.1007/s00208-012-0846-5

[8] M. Longo, S. Vigni, A note on control theorems for quaternionic Hida families of modular forms, Int. J. Number Theory, 2012 (to appear).

[9] M. Longo, S. Vigni, The rationality of quaternionic Darmon points over genus fields of real quadratic fields, preprint 2011.

[10] J. Nekovář, A. Plater, On the parity of ranks of Selmer groups. *Asian J. Math.* **4** (2000), no. 2, 437–497.

[11] A. Popa, Central values of Rankin L-series over real quadratic fields. *Compos. Math.* **142** (2006), no. 4, 811–866.

[12] K. Prasanna, Integrality of a ratio of Petersson norms and level-lowering congruences. *Ann. of Math.* (2) **163** (2006), no. 3, 901–967.

[13] K. Prasanna, Arithmetic properties of the Shimura-Shintani-Waldspurger correspondence. With an appendix by Brian Conrad. *Invent. Math.* **176** (2009), no. 3, 521–600.

[14] K. Prasanna, On the Fourier coefficients of modular forms of half-integral weight. *Forum Math.* **22** (2010), no. 1, 153–177.

[15] G. Shimura, On modular forms of half integral weight. *Ann. of Math.* (2) **97** (1973), 440–481.

[16] G. Shimura, The periods of certain automorphic forms of arithmetic type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** (1981), no. 3, 605-632 (1982).

[17] T. Shintani, On construction of holomorphic cusp forms of half integral weight. *Nagoya Math. J.* **58** (1975), 83–126.

[18] G. Stevens, Λ-adic modular forms of half-integral weight and a Λ-adic Shintani lifting. Arithmetic geometry (Tempe, AZ, 1993), 129–151, Contemp. Math., **174**, Amer. Math. Soc., Providence, RI, 1994.

[19] J.-L. Waldspurger, Correspondances de Shimura et quaternions. *Forum Math.* **3** (1991), no. 3, 219–307.

Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Trieste 63, 35121 Padova, Italy

E-mail address: `mlongo@math.unipd.it`

Institut de mathématiques de Luminy, Université d'Aix-Marseille, campus de Luminy, case 907, 13288 Marseille cedex 9, France

E-mail address: `nicole@iml.univ-mrs.fr`