THE Λ-ADIC SHINTANI-SHIMURA-WALDSPURGER CORRESPONDENCE

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Abstract. We generalize the Λ-adic Shintani lifting for GL₂(ℚ) to indefinite quaternion algebras over ℚ.

1. Introduction

Langlands’s principle of functoriality predicts the existence of a staggering wealth of transfers (or lifts) between automorphic forms for different reductive groups. In recent years, attempts at the formulation of ℓ-adic variants of Langlands’s functoriality have been articulated in various special cases. We prove the existence of the Shintani-Shimura-Waldspurger lift for ℓ-adic families.

More precisely, Stevens, building on the work of Hida and Greenberg-Stevens, showed in [18] the existence of a Λ-adic variant of the classical Shintani lifting of [17] for GL₂(ℚ). This Λ-adic lifting can be seen as a formal power series with coefficients in a finite extension of the Iwasawa algebra Λ := ℤ[ [X] ] equipped with specialization maps interpolating classical Shintani lifting of classical modular forms appearing in a given Hida family.

Shimura in [16], resp. Waldspurger in [19] generalized the classical Shimura-Shintani correspondence to quaternion algebras over ℚ, resp. over any number field. In this paper, motivated by ulterior applications to Shimura curves over ℚ, we generalize Stevens’s result to any non-split rational indefinite quaternion algebra B, building on works of Shimura [16] and combining this with a result of Longo-Vigni [8]. Our main result, for which the reader is referred to Theorem 3.8 below, states the existence of a formal power series and specialization maps interpolating Shimura-Shintani lifts of classical forms in a given ℓ-adic family of automorphic forms on the quaternion algebra B. The Λ-adic variant of Waldspurger’s result appears computationally challenging (see remark in [13, Intro.]), but it seems within reach for real quadratic fields (cf. [11]).

As an example of our main result, we consider the case of families with trivial character. More precisely, embed the set ℤ≥2 of integers greater or equal to 2 in Hom(ℤp×, ℤp×) by sending k ∈ ℤ≥2 to the character x → xk−2. Let f∞ be an Hida family of tame level N, a positive integer. For any k ∈ ℤ≥2 we thus have an element fk ∈ Sk(Γ₀(Np)), where p ∤ N is a prime. Fix a factorization N = MD with D > 1 a square-free product of an even number of primes and (M, D) = 1. Applying the Jacquet-Langlands correspondence we get for any k ∈ ℤ≥2 a modular form Jlk on Γ, which is the group of norm-one elements in an Eichler order R of level Mp contained in indefinite the rational quaternion algebra B of discriminant D. One

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can show that these modular forms can be \( p \)-adically interpolated, up to scaling. More precisely, let \( \mathcal{O} \) be the ring of integers of a finite extension \( F \) of \( \mathbb{Q}_p \) and let \( \mathcal{D} \) denote the \( \mathcal{O} \)-module of \( \mathcal{O} \)-valued measures on \( \mathbb{Z}_p^2 \) which are supported on the set of primitive elements in \( \mathbb{Y} \). Let \( \Gamma_0 \) be the group of norm-one elements in an Eichler order \( R_0 \subseteq B \) containing \( R \). There is a canonical action of \( \Gamma_0 \) on \( \mathcal{D} \) (see [8, §2.4] for its description). Denote by \( F_k \) the extension of \( F \) generated by the Fourier coefficients of \( f_k \). Then there is an element \( \Phi \in H^1(\Gamma_0, \mathcal{D}) \) and maps
\[
\rho_k : H^1(\Gamma_0, \mathcal{D}) \longrightarrow H^1(\Gamma, F_k)
\]
such that \( \rho(k)(\Phi) = \phi_k \), the cohomology class associated to \( f_k \) (for this we need a suitable normalization of the cohomology class associated to \( f_k^{JL} \), which we do not touch for simplicity in this introduction). We view \( \Phi \) as a quaternionic family of modular forms. To each \( \phi_k \) we may apply the Shintani-Shimura lifting ([16]) and obtain a modular form \( h_k \) of weight \( k + 1/2 \), level \( 4Np \) and trivial character. We show that this collection of forms can be \( p \)-adically interpolated. For clarity’s sake, we present the liftings and their \( \Lambda \)-adic variants in a diagram, in which the horizontal maps are specialization maps of the \( p \)-adic family to weight \( k \); JL stands for the Jacquet-Langlands correspondence; SSW stands for the Shintani-Shimura-Waldspurger lift; and the dotted arrows are constructed in this paper.

\[
\begin{array}{ccc}
\Phi & \rho_k & \rightarrow \\
\downarrow & & \downarrow \\
\Lambda \text{-adic SSW} & \text{SSW} & \Theta \\
& & \downarrow \\
& & h_k
\end{array}
\]

More precisely, as a particular case of our main result, Theorem 3.8, we get the following

**Theorem 1.1.** Fix \( k_0 \in \mathbb{Z}^{>2} \). Then there exists a \( p \)-adic neighborhood \( \mathcal{U}_0 \) of \( k_0 \) in \( \text{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \), \( p \)-adic periods \( \Omega_k \) for \( k \in \mathcal{U}_0 \) and a formal expansion
\[
\Theta = \sum_{\xi \geq 1} a_\xi q^\xi
\]
with coefficients \( a_\xi \) in the ring of \( \mathbb{C}_p \)-valued functions on \( \mathcal{U}_0 \), such that for all \( k \in \mathcal{U}_0 \) we have
\[
\Theta(k) = \Omega_k \cdot h_k
\]
Further, \( \Omega_{k_0} \neq 0 \).

**2. SHINTANI INTEGRALS AND FOURIER COEFFICIENTS OF HALF-INTEGRAL WEIGHT MODULAR FORMS**

We express the Fourier coefficients of half-integral weight modular forms in terms of period integrals, thus allowing a cohomological interpretation which is key to the production of the \( \Lambda \)-adic version of the Shintani-Shimura-Waldspurger correspondence. For the quaternionic Shintani-Shimura-Waldspurger correspondence of interest to us (see [13], [19]), the period integrals expressing the values of the Fourier coefficients have been computed generally by Prasanna in [14].
2.1. The Shimura-Shintani lifting. Let $4M$ be a positive integer, $2k$ an even non-negative integer and $\chi$ a Dirichlet character modulo $4M$ such that $\chi(-1) = 1$. Recall that the space of half-integral weight modular forms $S_{k+1/2}(4M, \chi)$ consists of holomorphic cuspidal functions $h$ on the upper-half place $\mathcal{H}$ such that

$$h(\gamma(z)) = j^{1/2}(\gamma, z)^{2k+1}\chi(d)h(z),$$

for all $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(4M)$, where $j^{1/2}(\gamma, z)$ is the standard square root of the usual automorphy factor $j(\gamma, z)$ (cf. [13, 2.3]).

To any quaternionic integral weight modular form we may associate a half-integral weight modular form following Shimura’s work [16], as we will describe below.

Fix an odd square free integer $N$ and a factorization $N = M \cdot D$ into coprime integers such that $D > 1$ is a product of an even number of distinct primes. Fix a Dirichlet character $\psi$ modulo $M$ and a positive even integer $2k$. Suppose that

$$\psi(-1) = (-1)^k.$$ Define the Dirichlet character $\chi$ modulo $4N$ by

$$\chi(x) := \psi(x) \left( \frac{-1}{x} \right)^k.$$ Let $B$ be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant the $D$. Fix a maximal order $\mathcal{O}_B$ of $B$. For every prime $\ell | M$, choose an isomorphism $i_\ell : B \otimes \mathbb{Q}_\ell \simeq \mathbb{M}_2(\mathbb{Q}_\ell)$ such that $i_\ell(\mathcal{O}_B \otimes \mathbb{Z}_\ell) = \mathbb{M}_2(\mathbb{Z}_\ell)$. Let $R \subseteq \mathcal{O}_B$ be the Eichler order of $B$ of level $M$ defined by requiring that $i_\ell(R \otimes \mathbb{Z}_\ell)$ is the suborder of $\mathbb{M}_2(\mathbb{Z}_\ell)$ of upper triangular matrices modulo $\ell$ for all $\ell | M$. Let $\Gamma$ denote the subgroup of the group $R^*_\ell$ of norm $1$ elements in $R_\ell$ consisting of those $\gamma$ such that $i_\ell(\gamma) \equiv \left( \begin{smallmatrix} b \mid \ell \\ \alpha \ast \beta \end{smallmatrix} \right)$ mod $\ell$ for all $\ell | M$. We denote by $S_{2k}(\Gamma)$ the $\mathbb{C}$-vector space of weight $2k$ modular forms on $\Gamma$, and by $S_{2k}(\Gamma, \psi^2)$ the subspace of $S_{2k}(\Gamma)$ consisting of forms having character $\psi^2$ under the action of $R^*_\ell$. Fix a Hecke eigenform

$$f \in S_{2k}(\Gamma, \psi^2)$$
as in [16, Section 3].

Let $V$ denote the $\mathbb{Q}$-subspace of $B$ consisting of elements with trace equal to zero. For any $v \in V$, which we view as a trace zero matrix in $\mathbb{M}_2(\mathbb{R})$ (after fixing an isomorphism $i_\infty : B \otimes \mathbb{R} \simeq \mathbb{M}_2(\mathbb{R})$), set

$$G_v := \{ \gamma \in \text{SL}_2(\mathbb{R}) | \gamma^{-1}v\gamma = v \}$$and put $\Gamma_v := \Gamma \cap G_v$. One can show that there exists an isomorphism $\omega : \mathbb{R}^\times \xrightarrow{\sim} G_v$ defined by $\omega(s) = \beta^{-1}\left( \begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix} \right) \beta$, for some $\beta \in \text{SL}_2(\mathbb{R})$. Let $t_v$ be the order of $\Gamma_v \cap \{ \pm 1 \}$ and let $\gamma_v$ be an element of $\Gamma_v$ which generates $\Gamma_v \{ \pm 1 \} / \{ \pm 1 \}$. Changing $\gamma_v$ to $\gamma_v^{-1}$ if necessary, we may assume $\gamma_v = \omega(t)$ with $t > 0$. Define $V^*$ to be the $\mathbb{Q}$-subspace of $V$ consisting of elements with strictly negative norm. For any $\alpha = \left( \begin{smallmatrix} a & b \\ c & -a \end{smallmatrix} \right) \in V^*$ and $z \in \mathcal{H}$, define the quadratic form

$$Q_\alpha(z) := cz^2 - 2az - b.$$Fix $\tau \in \mathcal{H}$ and set

$$P(f, \alpha, \Gamma) := -\left( 2(-\text{nr}(\alpha))^{1/2}/t_v \right) \int_{\tau}^{\gamma_v(\tau)} Q_\alpha(z)^{k-1}f(z)dz.$$
where \(nr : B \to \mathbb{Q}\) is the norm map. By [16, Lemma 2.1], the integral is independent on the choice \(\tau\), which justifies the notation.

**Remark 2.1.** The definition of \(P(f, \alpha, \Gamma)\) given in [16, (2.5)] looks different: the above expression can be derived as in [16, page 629] by means of [16, (2.20) and (2.22)].

Let \(R(\Gamma)\) denote the set of equivalence classes of \(V^*\) under the action of \(\Gamma\) by conjugation. By [16, (2.6)], \(P(f, \alpha, \Gamma)\) only depends on the conjugacy class of \(\alpha\), and thus, for \(\mathcal{C} \in R(\Gamma)\), we may define \(P(f, \mathcal{C}, \Gamma) := P(f, \alpha, \Gamma)\) for any choice of \(\alpha \in \mathcal{C}\). Also, \(q(\mathcal{C}) := -nr(\alpha)\) for any \(\alpha \in \mathcal{C}\).

Define \(O'_B\) to be the maximal order in \(B\) such that \(O'_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \simeq O_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell\) for all \(\ell \nmid M\) and \(O'_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell\) is equal to the local order of \(B \otimes \mathbb{Q}_\ell\) consisting of elements \(\gamma\) such that \(i_\ell(\gamma) = \left(\begin{array}{cc} a/b & b/M \\ c/M & d \end{array}\right)\) with \(a, b, c, d \in \mathbb{Z}_\ell\), for all \(\ell \mid M\). Given \(\alpha \in O'_B\), we can find an integer \(b_\alpha\) such that

\[
i_\ell(\alpha) \equiv \left(\begin{array}{cc} * & b_\alpha/M \\ * & * \end{array}\right) \mod i_\ell(\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell), \quad \forall \ell \mid M.
\]

Define a locally constant function \(\eta_\psi\) on \(V\) by \(\eta_\psi(\alpha) = \psi(b_\alpha)\) if \(\alpha \in O'_B \cap V\) and \(\eta(\alpha) = 0\) otherwise, with \(\psi(a) = 0\) if \((a/M) \neq 1\) (for the definition of locally constant functions on \(V\) in this context, we refer to [16, p. 611]).

For any \(\mathcal{C} \in R(\Gamma)\), fix \(\alpha_\mathcal{C} \in \mathcal{C}\). For any integer \(\xi \geq 1\), define

\[
a_\xi(\tilde{h}) := (2\mu(\Gamma\setminus \mathcal{F}))/2 \cdot \sum_{\mathcal{C} \in R(\Gamma), q(\mathcal{C}) = \xi} \eta_\psi(\alpha_\mathcal{C}) \xi^{-1/2} P(f, \mathcal{C}, \Gamma).
\]

Then, by [16, Theorem 3.1],

\[
\hat{h} := \sum_{\xi \geq 1} a_\xi(\tilde{h}) q^\xi \in S_{k+1/2}(4N, \chi)
\]

is called the Shimura-Shintani lifting of \(f\).

### 2.2. Cohomological interpretation.

We introduce necessary notations to define the action of the Hecke action on cohomology groups; for details, see [8, §2.1]. If \(G\) is a subgroup of \(B^x\) and \(S\) a subsemigroup of \(B^x\) such that \((G, S)\) is an Hecke pair, we let \(H(G, S)\) denote the Hecke algebra corresponding to \((G, S)\), whose elements are written as \(T(s) = GsG = \bigsqcup_i Gs_i\) for \(s, s_i \in S\) (finite disjoint union). For any \(s \in S\), let \(s^* := \text{norm}(s)s^{-1}\) and denote by \(S^*\) the set of elements of the form \(s^*\) for \(s \in S\). For any \(\mathbb{Z}[S^*]\)-module \(M\) we let \(T(s)\) act on \(H^1(G, M)\) at the level of cochains \(c \in Z^1(G, M)\) by the formula \((c|T(s)))(\gamma) = \sum_i s_i^* c(t_i(\gamma))\), where \(t_i(\gamma)\) are defined by the equations \(Gs_i^* \gamma = Gs_j\) and \(s_i \gamma = t_i(\gamma)s_j\). In the following, we will consider the case of \(G = \Gamma\) and

\[
S = \{s \in B^x | i_\ell(s) \text{ is congruent to } \left(\begin{array}{cc} 1 & * \\ 0 & * \end{array}\right) \mod \ell \text{ for all } \ell \mid M\}.
\]

For any field \(L\) and any integer \(n \geq 0\), let \(V_n(L)\) denote the \(L\)-dual of the \(L\)-vector space \(P_n(L)\) of homogeneous polynomials in 2 variables of degree \(n\). We let \(M_{2}(L)\) act from the right on \(P(x, y)\) as \(P|\gamma(x, y) := P(\gamma(x, y))\), where \(\gamma(x, y) := (ax + by, cx + dy)\) if \(\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\). This also equips \(V_n(L)\) with a left action by \(\gamma \cdot \varphi(P) := \varphi(P|\gamma)\). To simplify notations, we will write \(P(z)\) for \(P(z, 1)\).

Let \(F\) denote the finite extension of \(\mathbb{Q}\) generated by the eigenvalues of the Hecke action on \(f\). For any field \(K\) containing \(F\), set

\[
\mathbb{W}_f(K) := H^1(\Gamma, V_{k-2}(K))^f
\]
where the superscript $f$ denotes the subspace on which the Hecke algebra acts via the character associated with $f$. Also, for any sign $\pm$, let $\mathbb{W}_f^\pm(K)$ denote the $\pm$-eigenspace for the action of the archimedean involution $\iota$. Remember that $\iota$ is defined by choosing an element $\omega_\infty$ of norm $-1$ in $R^\times$ such that such that $i_\ell(\omega_\infty) \equiv (1, 0, 0, 1) \text{ mod } M$ for all primes $\ell | M$ and then setting $\iota := T(\omega_\infty)$ (see [8, §2.1]). Then $\mathbb{W}_f^\pm(K)$ is one dimensional (see, e.g., [8, Proposition 2.2]); fix a generator $\phi_f^\pm$ of $\mathbb{W}_f^\pm(F)$.

To explicitly describe $\phi_f^\pm$, let us introduce some more notations. If $\iota_\infty(\omega_\infty) = \left( \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ then define $f|\omega_\infty(z) := (Cz + D)^{-k/2}f(\omega_\infty(z))$. Then $f|\omega_\infty \in S_{2k}(\Gamma)$ as well. If the eigenvalues of the Hecke action on $f$ are real, then we may assume, after multiplying $f$ by a scalar, that $f|\omega_\infty = f$ (see [16, p. 627] or [9, Lemma 4.15]). In general, let $I(f)$ denote the class in $H^1(\Gamma, V_{k-2}(\mathbb{C}))$ represented by the cocycle

$$\gamma \mapsto \left[ P \mapsto I_\gamma(f)(P) := \int_\gamma f(z)P(z)dz \right]$$

for any $\gamma \in \mathcal{H}$ (the corresponding class is independent on the choice of $\gamma$). With these notations,

$$P(f, \alpha, \Gamma) = -(2(-nr(\alpha))^{1/2}/t_\alpha) \cdot I_{\gamma_\alpha}(f)(Q_{\alpha}\omega_\infty(z)^{k-1}).$$

Denote by $I^\pm(f) := (1/2) \cdot I(f) \pm (1/2) \cdot I(f)|\omega_\infty$ the projection of $I(f)$ to the eigenspaces for the action of $\omega_\infty$. Then we have $I(f) = I^+(f) + I^-(f)$, and $I_f^\pm = \Omega_f^\pm \cdot \phi_f^\pm$, for some $\Omega_f^\pm \in \mathbb{C}^\times$.

Given $\alpha \in V^*$ of norm $-\xi$, put $\alpha' := \omega_\infty^{-1}\alpha\omega_\infty$. By [16, 4.19], we have

$$\eta(\alpha)\xi^{-1/2}P(f, \alpha, \Gamma) + \eta(\alpha')\xi^{-1/2}P(f, \alpha', \Gamma) = -\eta(\alpha) \cdot t_\alpha^{-1} \cdot I_{\gamma_\alpha}^+(Q_{\alpha}\omega_\infty(z)^{k-1})$$

We then have

$$a_\xi(h) = \sum_{c \in R_2(\Gamma), q(c)=\xi} -\eta_\nu(\alpha_\xi) \cdot \frac{2\mu(\Gamma \setminus \mathcal{H}) \cdot t_{\alpha_\xi} \cdot I_{\gamma_\alpha}(Q_{\alpha_\xi}(z)^{k-1})}{\Omega_f^\pm}.$$

We close this paragraph by choosing a suitable multiple of $h$ which will be the object of the next section. Given $Q_\alpha(z) = cz^2 - 2az - b$ as above, with $\alpha \in V^*$, define $\widehat{Q}_\alpha(z) := M \cdot Q_\alpha(z)$. Then, clearly, $I^\pm(f)(\widehat{Q}_\alpha(z)^{k-1})$ is equal to $M^{k-1}I^\pm(f)(\widehat{Q}_\alpha(z)^{k-1})$. We thus normalize the Fourier coefficients by setting

$$a_\xi(h) := \frac{-\eta_\nu(\alpha_\xi) \cdot M^{k-1} \cdot 2\mu(\Gamma \setminus \mathcal{H})}{\Omega_f^\pm} \cdot \sum_{c \in R(\Gamma), q(c)=\xi} \frac{\eta_\nu(\alpha_\xi)}{t_{\alpha_\xi}} \cdot \phi_f^\pm(\widehat{Q}_\alpha(z)^{k-1}).$$

So

$$\hat{h} := \sum_{\xi \geq 1} a_\xi(h)q^\xi$$

belongs to $S_{k+1/2}(4N, \chi)$ and is a non-zero multiple of $\hat{h}$. 

3. The $\Lambda$-adic Shintani-Shimura-Waldspurger correspondence

At the heart of Stevens’s proof lies the control theorem of Greenberg-Stevens, which has been worked out in the quaternionic setting by Longo–Vigni [8].

Recall that $N \geq 1$ is a square free integer and fix a decomposition $N = M \cdot D$ where $D$ is a square free product of an even number of primes and $M$ is coprime to $D$. Let $p \mid N$ be a prime number and fix an embedding $\bar{Q} \hookrightarrow \bar{Q}_p$.

3.1. The Hida Hecke algebra. Fix an ordinary $p$-stabilized newform

$$f_0 \in S_{k_0}(\Gamma_1(Mp) \cap \Gamma_0(D), \epsilon_0)$$

of level $\Gamma_1(Mp) \cap \Gamma_0(D)$, Dirichlet character $\epsilon_0$ and weight $k_0$, and write $\mathcal{O}$ for the ring of integers of the field generated over $\bar{Q}_p$ by the Fourier coefficients of $f_0$.

Let $\Lambda$ (respectively, $\mathcal{O}[\mathbb{Z}_p^\times]$) denote the Iwasawa algebra of $W := 1 + p\mathbb{Z}_p$ (respectively, $\mathbb{Z}_p^\times$) with coefficients in $\mathcal{O}$. We denote group-like elements in $\Lambda$ and $\mathcal{O}[\mathbb{Z}_p^\times]$ as $[t]$. Let $\mathcal{h}_{\infty}$ denote the $p$-ordinary Hida Hecke algebra with coefficients in $\mathcal{O}$ of tame level $\Gamma_1(N)$. Denote by $\mathcal{L} := \text{Frac}(\Lambda)$ the fraction field of $\Lambda$. Let $\mathcal{R}$ denote the integral closure of $\Lambda$ in the primitive component $\mathcal{K}$ of $\mathcal{h}_{\infty} \otimes_\Lambda \mathcal{L}$ corresponding to $f_0$. It is well known that the $\Lambda$-algebra $\mathcal{R}$ is finitely generated as $\Lambda$-module.

Denote by $\mathcal{X}$ the $\mathcal{O}$-module $\text{Hom}_{\mathcal{O}_{\text{alg}}}(\mathcal{R}, \bar{Q}_p)$ of continuous homomorphisms of $\mathcal{O}$-algebras. Let $\mathcal{X}_{\text{arith}}$ the set of arithmetic homomorphisms in $\mathcal{X}$, defined in [8, §2.2] by requiring that the composition

$$W \hookrightarrow \Lambda \xrightarrow{\kappa} \bar{Q}_p$$

has the form $\gamma \mapsto \psi_\kappa(\gamma)\gamma^{n_\kappa}$ with $n_\kappa = k_\kappa - 2$ for an integer $k_\kappa \geq 2$ (called the weight of $\kappa$) and a finite order character $\psi_\kappa : W \rightarrow \bar{Q}$ (called the wild character of $\kappa$). Denote by $r_\kappa$ the smallest among the positive integers $t$ such that $1 + p^t\mathbb{Z}_p \subseteq \ker(\psi_\kappa)$.

For any $\kappa \in \mathcal{X}_{\text{arith}}$, let $P_\kappa$ denote the kernel of $\kappa$ and $\mathcal{R}_{P_\kappa}$ the localization of $\mathcal{R}$ at $\kappa$. The field $F_\kappa := \mathcal{R}_{P_\kappa}/P_\kappa\mathcal{R}_{P_\kappa}$ is a finite extension of $\text{Frac}(\mathcal{O})$. Further, by duality, $\kappa$ corresponds to a normalized eigenform

$$f_\kappa \in S_{k_\kappa}(\Gamma_0(Np^{r_\kappa}), \epsilon_\kappa),$$

where $\epsilon_\kappa := \psi_\kappa \cdot \epsilon_0 \cdot \omega^{-n_\kappa}$; here $\omega$ is the Teichmüller character, $n_\kappa := k_\kappa - 2$ and $\epsilon_\kappa$ is viewed as a Dirichlet character of $(\mathbb{Z}/Np^{r_\kappa})^\times$ via the decomposition

$$(\mathbb{Z}/Np^{r_\kappa})^\times \simeq (1 + p\mathbb{Z}_p)/(1 + p^{r_\kappa}\mathbb{Z}_p) \times (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$$

where $\psi_\kappa$ acts on the first factor, $\omega^{-n_\kappa}$ acts on the second factor and $\epsilon_0$ acts on the third factor. We call $(\epsilon_\kappa, k_\kappa)$ the signature of $\kappa$. We let $\kappa_0$ denote the arithmetic character associated with $f_0$, so that $f_0 = f_{\kappa_0}$. The eigenvalues of $f_\kappa$ under the action of the Hecke operators $T_n$ ($n \geq 1$ an integer) belong to $F_\kappa$. Actually, one can show that $f_\kappa$ is a $p$-stabilized newform on $\Gamma_1(Mp^{r_\kappa}) \cap \Gamma_0(D)$.

Let $\Lambda_N$ denote the Iwasawa algebra of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ with coefficients in $\mathcal{O}$. To simplify notations, define $\Delta := (\mathbb{Z}/Np\mathbb{Z})^\times$. We have a canonical isomorphism of rings $\Lambda_N \simeq \Lambda[\Delta]$, which makes $\Lambda_N$ a $\Lambda$-algebra, finitely generated as $\Lambda$-module. Define the tensor product of $\Lambda$-algebras

$$\mathcal{R}_N := \mathcal{R} \otimes_\Lambda \Lambda_N,$$

which is again a $\Lambda$-algebra (resp. $\Lambda_N$-algebra) finitely generated as a $\Lambda$-module, (resp. as a $\Lambda_N$-module). One easily checks that there is a canonical isomorphism
of $\Lambda$-algebras

$$\mathcal{R}_N \simeq \mathcal{R}[\Delta]$$

(where $\Lambda$ acts on $\mathcal{R}$); this is also an isomorphism of $\Lambda_N$-algebras, when we let $\Lambda_N \simeq \Lambda[\Delta]$ act on $\mathcal{R}[\Delta]$ in the obvious way.

We can extend any $\kappa \in \mathcal{X}^{\text{arith}}$ to a continuous $\mathcal{O}$-algebra morphism $\kappa_N : \mathcal{R}_N \to \bar{\mathbb{Q}}_p$ setting

$$\kappa_N \left( \sum_{i=1}^{n} r_i \cdot \delta_i \right) := \sum_{i=1}^{n} \kappa(r_i) \cdot (\epsilon \cdot \omega^{-n}) (\delta_i)$$

for $r_i \in \mathcal{R}$ and $\delta_i \in \Delta$. If we denote by $\mathcal{X}_N$ the $\mathcal{O}$-module of continuous $\mathcal{O}$-algebra homomorphisms from $\mathcal{R}_N$ to $\bar{\mathbb{Q}}_p$, the above correspondence sets up an injective map $\mathcal{X}^{\text{arith}} \hookrightarrow \mathcal{X}_N$. Let $\mathcal{X}_N^{\text{arith}}$ denote the image of $\mathcal{X}^{\text{arith}}$ under this map. For $\kappa_N \in \mathcal{X}_N^{\text{arith}}$, we define the signature of $\kappa_N$ to be that of the corresponding $\kappa$.

### 3.2. The control theorem in the quaternionic setting

Recall that $\mathcal{B}/\mathbb{Q}$ is a quaternion algebra of discriminant $D$. Fix an auxiliary real quadratic field $F$ such that all primes dividing $D$ are inert in $F$ and all primes dividing $Mp$ are split in $F$, and an isomorphism $i_F : \mathcal{B} \otimes \mathbb{Q} \simeq \mathcal{M}_2(F)$. Let $\mathcal{O}_B$ denote the maximal order of $B$ obtained by taking the intersection of $B$ with $\mathcal{M}_2(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of $F$. More precisely, define

$$\mathcal{O}_B := \iota^{-1}(i_F^{-1}(i_F(B \otimes 1) \cap \mathcal{M}_2(\mathcal{O}_F)))$$

where $\iota : B \hookrightarrow B \otimes \mathbb{Q}$ is the inclusion defined by $b \mapsto b \otimes 1$. This is a maximal order in $B$ because $i_F(B \otimes 1) \cap \mathcal{M}_2(\mathcal{O}_F)$ is a maximal order in $i_F(B \otimes 1)$. In particular, $i_F$ and our fixed embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ induce an isomorphism $i_p : B \otimes \mathbb{Q}_p \simeq \mathcal{M}_2(\mathbb{Q}_p)$ such that $i_p(\mathcal{O}_B \otimes \mathbb{Z}_p) = \mathcal{M}_2(\mathbb{Z}_p)$. For any prime $\ell | M$, also choose an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$ which, composed with $i_F$, yields isomorphisms $i_p : B \otimes \mathbb{Q}_\ell \simeq \mathcal{M}_2(\mathbb{Q}_\ell)$ such that $i_p(\mathcal{O}_B \otimes \mathbb{Z}_\ell) = \mathcal{M}_2(\mathbb{Z}_\ell)$. Define an Eichler order $R \subseteq \mathcal{O}_B$ of level $M$ by requiring that for all primes $\ell | M$ the image of $R \otimes \mathbb{Z}_\ell$ via $i_p \otimes \mathbb{Q}_\ell$ consists of upper triangular matrices modulo $\ell$. For any $r \geq 0$, let $\Gamma_r$ denote the subgroup of the group $R^*_\ell$ of norm-one elements in $R$ consisting of those $\gamma$ such that $i_p(\gamma) = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ with $c \equiv 0 \mod Mp^r$ and $a \equiv d \equiv 1 \mod Mp^r$, for all primes $\ell | Mp$. To conclude this list of notations and definitions, fix an embedding $F \hookrightarrow \mathbb{R}$ and let $\iota_{\infty} : B \otimes \mathbb{Q} \simeq \mathcal{M}_2(\mathbb{R})$ be the induced isomorphism.

Let $\mathbb{Y} := \mathbb{Z}_p^2$ and denote by $\mathcal{X}$ the set of primitive vectors in $\mathbb{Y}$. Let $\mathbb{D}$ denote the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Y}$ which are supported on $\mathcal{X}$. Note that $\mathcal{M}_2(\mathbb{Z}_p)$ acts on $\mathbb{Y}$ by left multiplication; this induces an action of $\mathcal{M}_2(\mathbb{Z}_p)$ on the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Y}$, which induces an action on $\mathbb{D}$. The group $R^x$ acts on $\mathbb{D}$ via $i_p$. In particular, we may define the group:

$$\mathbb{W} := H^1(\Gamma_0, \mathbb{D}).$$

Then $\mathbb{D}$ has a canonical structure of $\mathcal{O}[\mathbb{Z}_p^x]$-module, as well as $\mathfrak{h}^\infty_{\mathfrak{ad}}$ action, as described in [8, §2.4]. In particular, let us recall that, for any $[t] \in \mathcal{O}[\mathbb{Z}_p^x]$, we have

$$\int_{\mathcal{X}} \varphi(x,y)d([t] \cdot \nu) = \int_{\mathcal{X}} \varphi(tx,ty)d\nu,$$

for any locally constant function $\varphi$ on $\mathcal{X}$.

For any $\kappa \in \mathcal{X}^{\text{arith}}$ and any sign $\pm$, set

$$\mathbb{W}^\pm_{\kappa} := \mathbb{W}^\pm_{\mathfrak{h}^\infty_{\mathfrak{ad}}}(F_\kappa) = H^1(\Gamma_{\kappa \pm}, V_{\kappa \pm}(F_\kappa))^{f_{\kappa \pm}}$$
where $f^\text{IL}_\kappa$ is any Jacquet-Langlands lift of $f_\kappa$ to $\Gamma_{\kappa}$; recall that the superscript $f_\kappa$ denotes the subspace on which the Hecke algebra acts via the character associated with $f_\kappa$, and the superscript $\epsilon = \pm$ denotes the $\epsilon$-eigenspace for the action of the archimedean involution $\iota$. Also, recall that $W^\pm_k$ is one dimensional and fix a generator $\phi^\pm_k$ of it.

We may define specialization maps

$$\rho_\kappa : \mathbb{D} \rightarrow V_{n_k}(F_\kappa)$$

by the formula

$$(5) \quad \rho_\kappa(\nu)(P) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p^\epsilon \kappa(y)} P(x, y) d\nu$$

which induces (see [8, §2.5]) a map:

$$\rho_\kappa : W^\text{ord} \rightarrow W^\text{ord}_\kappa.$$  

Here $W^\text{ord}$ and $W^\text{ord}_\kappa$ denote the ordinary submodules of $W$ and $W_\kappa$, respectively, defined as in [3, Definition 2.2] (see also [8, §3.5]). We also let $W_R := W \otimes A \mathcal{R}$, and extend the above map $\rho_\kappa$ to a map

$$\rho_\kappa : W^\text{ord}_R \rightarrow W^\text{ord}_\kappa$$

by setting $\rho_\kappa(x \otimes r) := \rho_\kappa(x) \cdot \kappa(r)$.

**Theorem 3.1.** There exists a $p$-adic neighborhood $U_0$ of $\kappa_0$ in $\mathcal{X}$, elements $\Phi^\pm \in W^\text{ord}_R$ and choices of $p$-adic periods $\Omega^\pm_k \in F_\kappa$ for $\kappa \in U_0 \cap \mathcal{X}^{\text{arith}}$ such that, for all $\kappa \in U_0 \cap \mathcal{X}^{\text{arith}}$, we have

$$\rho_\kappa(\Phi^\pm) = \Omega^\pm_k \cdot \phi^\pm_k$$

and $\Omega^\pm_{\kappa_0} \neq 0$.

**Proof.** This is an easy consequence of [8, Theorem 2.18] and follows along the lines of the proof of [18, Theorem 5.5], cf. [9, Proposition 3.2]. $\square$

We now normalize our choices as follows. With $U_\mathcal{H}$ as above, define

$$U_0^{\text{arith}} := U_0 \cap \mathcal{X}^{\text{arith}}.$$

Fix $\kappa \in U_0^{\text{arith}}$ and an embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$. Let $f^\text{IL}_\kappa$ denote a modular form on $\Gamma_{\kappa}$ corresponding to $f_\kappa$ by the Jacquet-Langlands correspondence, which is well defined up to elements in $\mathbb{C}^\times$. View $\phi^\pm_k$ as an element in $H^1(\Gamma_{\kappa}, V_n(C))^\pm$. Choose a representative $\Phi^\pm_\gamma$ of $\Phi^\pm_k$, by which we mean that if $\Phi^\pm_k = \sum_i \Phi^\pm_i \otimes r_i$, then we choose a representative $\Phi^\pm_\gamma$ for all $i$. Also, we will write $\rho_\kappa(\Phi)(P)$ as

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\epsilon \kappa(y)} \kappa(r_i) \cdot \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\epsilon \kappa(y)} \epsilon_k(y) P(x, y) d\Phi^\pm_\gamma.$$

With these notations, we see that the two cohomology classes

$$\gamma \mapsto \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\epsilon \kappa(y)} \epsilon_k(y) P(x, y) d\Phi^\pm_\gamma(x, y)$$

and

$$\gamma \mapsto \Omega^\pm_k \cdot \int_{\gamma(\tau)} P(z, 1) f^\text{IL}_\kappa^\pm(z) dz$$

are cohomologous in $H^1(\Gamma_{\kappa}, V_{n_k}(C))$, for any choice of $\tau \in \mathcal{H}$.  


3.3. Metaplectic Hida Hecke algebras. Let $\sigma : \Lambda_N \to \Lambda_N$ be the ring homomorphism associated to the group homomorphism $t \mapsto t^2$ on $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, and denote by the same symbol its restriction to $\Lambda$ and $\mathcal{O}[\mathbb{Z}_p^\times]$. We let $\Lambda_\sigma$, $\mathcal{O}[\mathbb{Z}_p^\times]_\sigma$ and $\Lambda_{N,\sigma}$ denote, respectively, $\Lambda$, $\mathcal{O}[\mathbb{Z}_p^\times]$ and $\Lambda_N$ viewed as algebras over themselves via $\sigma$. The ordinary metaplectic $p$-adic Hida Hecke algebra we will consider is the $\Lambda$-algebra

$$\tilde{\mathcal{R}} := \mathcal{R} \otimes_\Lambda \Lambda_\sigma.$$ 

Define as above

$$\tilde{\mathcal{X}} := \text{Hom}_{\text{cont}}(\tilde{\mathcal{R}}, \hat{\mathbb{Q}}_p)$$

and let the set $\tilde{\mathcal{X}}_{\text{arith}}$ of arithmetic points in $\tilde{\mathcal{X}}$ to consist of those $\tilde{\kappa}$ such that the composition

$$W \overset{\lambda \mapsto 1 \otimes \lambda}{\longrightarrow} \Lambda \overset{\lambda \mapsto 1 \otimes \lambda}{\longrightarrow} \tilde{\mathcal{R}} \overset{\tilde{\kappa}}{\longrightarrow} \hat{\mathbb{Q}}_p$$

has the form $\gamma \mapsto \psi_\kappa(\gamma)^{n_\kappa}$ with $n_\kappa := k_\kappa - 2$ for an integer $k_\kappa \geq 2$ (called the weight of $\tilde{\kappa}$) and a finite order character $\psi_\kappa : W \to \hat{\mathbb{Q}}$ (called the wild character of $\tilde{\kappa}$). Let $r_\kappa$ the smallest among the positive integers $t$ such that $1 + p^t \mathbb{Z}_p \subseteq \ker(\psi_\kappa)$.

We have a map $p : \tilde{\mathcal{X}} \to \mathcal{X}$ induced by pull-back from the canonical map $\mathcal{R} \to \tilde{\mathcal{R}}$. The map $p$ restricts to arithmetic points.

As above, define the $\Lambda$-algebra (or $\Lambda_N$-algebra)

$$\tilde{\mathcal{R}}_N := \mathcal{R} \otimes_\Lambda \Lambda_{N,\sigma}$$

via $\lambda \mapsto 1 \otimes \lambda$.

We easily see that

$$\tilde{\mathcal{R}}_N \simeq \tilde{\mathcal{R}}[\Delta]$$

as $\Lambda_N$-algebras, where we enhance $\tilde{\mathcal{R}}[\Delta]$ with the following structure of $\Lambda_N \simeq \Lambda[\Delta]$-algebra: for $\sum_i \lambda_i \cdot \delta_i \in \Lambda[\Delta]$ ($\lambda_i \in \Lambda$ and $\delta_i \in \Delta$) and $\sum r_j \cdot \delta_j' \in \tilde{\mathcal{R}}[\Delta]$ ($r_j = \sum_h r_{j,h} \otimes \lambda_{j,h} \in \tilde{\mathcal{R}}$ with $r_{j,h} \in \mathcal{R}$ and $\lambda_{j,h} \in \Lambda_{\sigma}$, and $\delta_j' \in \Delta$), we set

$$\left( \sum_i \lambda_i \cdot \delta_i \right) \cdot \left( \sum_j r_j \cdot \delta_j' \right) := \sum_{i,j,h} \left( r_{j,h} \otimes (\lambda_i \cdot \delta_i) \cdot (\delta_j' \delta_i) \right).$$

We now fix a Dirichlet character $\psi$ modulo $N$, which is trivial on $(\mathbb{Z}/D\mathbb{Z})^\times$ under the decomposition $(\mathbb{Z}/N\mathbb{Z})^\times \simeq (\mathbb{Z}/M\mathbb{Z})^\times \times (\mathbb{Z}/D\mathbb{Z})^\times$. As above, we can extend $\tilde{\kappa} \in \tilde{\mathcal{X}}_{\text{arith}}$ to a continuous $\mathcal{O}$-algebra morphism $\tilde{\kappa}_N : \tilde{\mathcal{R}}_N \to \hat{\mathbb{Q}}_p$ by setting

$$\tilde{\kappa}_N \left( \sum_{i=1}^n x_i \cdot \delta_i \right) := \sum_{i=1}^n \tilde{\kappa}(x_i) \cdot (\psi \cdot \omega^{-n_\kappa})(\delta_i)$$

for $x_i \in \tilde{\mathcal{R}}$ and $\delta_i \in \Delta$. If we denote by $\tilde{\mathcal{X}}_N$ the $\mathcal{O}$-module of continuous $\mathcal{O}$-linear homomorphisms from $\tilde{\mathcal{R}}_N$ to $\hat{\mathbb{Q}}_p$, the above correspondence sets up an injective map $\tilde{\mathcal{X}}_{\text{arith}} \hookrightarrow \tilde{\mathcal{X}}_N$ and we let $\tilde{\mathcal{X}}_{\text{arith}}$ denote the image of $\tilde{\mathcal{X}}_{\text{arith}}$. Define

$$\epsilon_\kappa := \psi_\kappa \cdot \psi \cdot \omega^{-n_\kappa},$$

viewed as a Dirichlet character of $(\mathbb{Z}/Np^\times)^\times$ via the decomposition

$$(\mathbb{Z}/Np^\times)^\times \simeq (1 + p\mathbb{Z}_p)/(1 + p^2\mathbb{Z}_p) \times (\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$$

and call the pair $(\epsilon_\kappa, k_\kappa)$ the signature of $\tilde{\kappa}_N$, where $\tilde{\kappa}$ is the arithmetic point corresponding to $\tilde{\kappa}_N$. 


We also have a map \( p_N : \mathcal{X}_N \rightarrow \mathcal{X}_N \) induced by pull-back from the map \( R_N \rightarrow \widehat{R}_N \) taking \( r \mapsto r \otimes 1 \). The map \( p_N \) also restricts to arithmetic points. The maps \( p \) and \( p_N \) make the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{X}_{\text{arith}} & \xrightarrow{\iota} & \mathcal{X}_N \\
\downarrow{p} & & \downarrow{p_N} \\
\mathcal{X}_{\text{arith}} & \xrightarrow{\iota} & \mathcal{X}_N
\end{array}
\]

where the projections take a signature \((\epsilon, k)\) to \((\epsilon^2, 2k)\).

### 3.4. The \( \Lambda \)-adic correspondence

In this part, we combine the explicit integral formula of Shimura and the fact that the toric integrals can be \( p \)-adically interpolated to show the existence of a \( \Lambda \)-adic Shintani-Shimura-Waldspurger correspondence with the expected interpolation property. This follows very closely [18, §6].

Let \( \tilde{k}_N \in X_\mathcal{N} \) with the signature of \( \tilde{k}_N \) is \((\epsilon_\kappa, k_\kappa)\) and decompose \( \epsilon_\kappa = \psi \cdot \psi_\kappa \cdot \omega^{-n_k} \). Let \( L_r \) denote the order of \( M_2(F) \) consisting of matrices \( \begin{pmatrix} a & b/Mp \ 
\frac{1}{a} & d \end{pmatrix} \) with \( a, b, c, d \in \mathcal{O}_F \). Define

\[
\mathcal{O}_{B, r} := \epsilon^{-1}(i_{F}^{-1}(i_{F}(B \otimes 1) \cap L_r))
\]

Then \( \mathcal{O}_{B, r} \) is the maximal order introduced in §2.1 (and denoted \( \mathcal{O}'_B \) there) defined in terms of the maximal order \( \mathcal{O}_B \) and the integer \( Mp^r \). Also, \( S := \mathcal{O}_B \cap \mathcal{O}_{B, r} \) is an Eichler order of \( B \) of level \( Mp \) containing the fixed Eichler order \( R \) of level \( M \).

With \( \alpha \in V^* \cap \mathcal{O}_{B, 1} \), we have

\[
i_{F}(\alpha) = \begin{pmatrix} a & b/(Mp) \\
\frac{1}{a} & -a \end{pmatrix}
\]

in \( M_2(F) \) with \( a, b, c \in \mathcal{O}_F \) and we can consider the quadratic forms

\[
Q_{\alpha}(x, y) := cx^2 - 2axy - (b/(Mp))y^2,
\]

and

\[
\tilde{Q}_{\alpha}(x, y) := Mp \cdot Q_{\alpha}(x, y) = Mpcx^2 - 2Mpxay - by^2.
\]

Then \( \tilde{Q}_{\alpha}(x, y) \) has coefficients in \( \mathcal{O}_F \) and, composing with \( F \hookrightarrow \mathbb{R} \) and letting \( x = z, y = 1 \), we recover \( Q_{\alpha}(z) \) and \( \tilde{Q}_{\alpha}(z) \) of §2.1 (defined by means of the isomorphism \( i_\infty \)). Since each prime \( \ell \mid Mp \) is split in \( F \), the elements \( a, b, c \) can be viewed as elements in \( \mathbb{Z}_\ell \) via our fixed embedding \( \mathbb{Q} \hookrightarrow \mathbb{Q}_\ell \), for any prime \( \ell \mid Mp \) (we will continue writing \( a, b, c \) for these elements, with a slight abuse of notations). So, letting \( b_\alpha \in \mathbb{Z} \) such that \( i_\ell(\alpha) \equiv (z \cdot b_\alpha/(Mp)) \) modulo \( \mathbb{S} \otimes \mathbb{Z}_\ell \), for all \( \ell \mid Mp \), we have \( b \equiv b_\alpha \) modulo \( Mp\mathbb{Z}_\ell \) as elements in \( \mathbb{Z}_\ell \), for all \( \ell \mid Mp \), and thus we get

\[
\eta_{\epsilon_\kappa}(\alpha) = \epsilon_\kappa(b_\alpha) = \epsilon_\kappa(b)
\]

for \( b \) as in (8).

For any \( \nu \in \mathbb{D} \), we may define an \( \mathcal{O} \)-valued measure \( j_{\alpha}(\nu) \) on \( \mathbb{Z}_p^* \) by the formula:

\[
\int_{\mathbb{Z}_p^*} f(t)j_{\alpha}(\nu)(t) := \int_{\mathbb{Z}_p^* \times \mathbb{Z}_p^*} f(\tilde{Q}_{\alpha}(x, y))\nu(x, y).
\]

For any continuous function \( f : \mathbb{Z}_p^* \rightarrow \mathbb{C}_p \). Recall that the group of \( \mathcal{O} \)-valued measures on \( \mathbb{Z}_p^* \) is isomorphic to the Iwasawa algebra \( \mathcal{O}[\mathbb{Z}_p^*] \), and thus we may
view $j_\alpha(\nu)$ as an element in $\mathcal{O}[\mathbb{Z}_p^\times]$ (see, for example, [1, §3.2]). In particular, for any group-like element $[\lambda] \in \mathcal{O}[\mathbb{Z}_p^\times]$ we have:

$$\int_{\mathbb{Z}_p^\times} f(t)d([\lambda] \cdot j_\alpha(\nu))(t) = \int_{\mathbb{Z}_p^\times} \left( \int_{\mathbb{Z}_p^\times} f(ts)d([\lambda](s)) \right) dj_\alpha(\nu)(t) = \int_{\mathbb{Z}_p^\times} f(\lambda t) dj_\alpha(\nu)(t).$$

On the other hand,

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} f(Q_\alpha(x,y))d(\lambda \cdot \nu) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} f(Q_\alpha(\lambda x, \lambda y))d\nu = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} f(\lambda^2 Q_\alpha(x,y))d\nu$$

and we conclude that $j_\alpha(\lambda \cdot \nu) = [\lambda^2] \cdot \nu$. In other words, $j_\alpha$ is a $\mathcal{O}[\mathbb{Z}_p^\times]$-linear map

$$j_\alpha : \mathbb{D} \rightarrow \mathcal{O}[\mathbb{Z}_p^\times]_{\sigma}.$$

Before going ahead, let us introduce some notations. Let $\chi$ be a Dirichlet character modulo $Mp^r$, for a positive integer $r$, which we decompose accordingly with the isomorphism $\left( \mathbb{Z}/Np^r\mathbb{Z} \right)^\times \simeq \left( \mathbb{Z}/N\mathbb{Z} \right)^\times \times \left( \mathbb{Z}/p^r\mathbb{Z} \right)^\times$ into the product $\chi = \chi_N \cdot \chi_p$ with $\chi_N : \left( \mathbb{Z}/N\mathbb{Z} \right)^\times \rightarrow \mathbb{C}^\times$ and $\chi_p : \left( \mathbb{Z}/p^r\mathbb{Z} \right)^\times \rightarrow \mathbb{C}^\times$. Thus, we will write $\chi(x) = \chi_N(x_N) \cdot \chi_p(x_p)$, where $x_N$ and $x_p$ are the projections of $x \in \left( \mathbb{Z}/Np^r\mathbb{Z} \right)^\times$ to $\left( \mathbb{Z}/N\mathbb{Z} \right)^\times$ and $\left( \mathbb{Z}/p^r\mathbb{Z} \right)^\times$, respectively. To simplify notations, we will suppress the $N$ and $p$ from the notations for $x_N$ and $x_p$, thus simply writing $x$ for any of the two. We assume that, under the isomorphism $\left( \mathbb{Z}/N\mathbb{Z} \right)^\times \simeq \left( \mathbb{Z}/M\mathbb{Z} \right)^\times \times \left( \mathbb{Z}/D\mathbb{Z} \right)^\times$, $\chi_N$ is trivial on the second factor and decompose $\chi_N = \chi_M \cdot \chi_D$ as above with $\chi_M$ and $\chi_D$ characters on $\left( \mathbb{Z}/M\mathbb{Z} \right)^\times$ and $\left( \mathbb{Z}/D\mathbb{Z} \right)^\times$, respectively.

Using the above notations, we may define a $\mathcal{O}[\mathbb{Z}_p^\times]$-linear map $J_\alpha : \mathbb{D} \rightarrow \mathcal{O}[\mathbb{Z}_p^\times]$ by

$$J_\alpha(\nu) = \epsilon_{\bar{\kappa},M}(b) \cdot \epsilon_{\bar{\kappa},p}(-1) \cdot j_\alpha(\nu)$$

with $b$ as in (8). Set $\mathbb{D}_N := \mathbb{D} \otimes_{\mathcal{O}[\mathbb{Z}_p^\times]} \Lambda_N$, where the map $\mathcal{O}[\mathbb{Z}_p^\times] \rightarrow \Lambda_N$ is defined induced from the map $\mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p^\times \times \left( \mathbb{Z}/N\mathbb{Z} \right)^\times$ on group-like elements given by $x \mapsto x \otimes 1$. Then $J_\alpha$ can be extended to a $\Lambda_N$-linear map $J_\alpha : \mathbb{D}_N \rightarrow \Lambda_N.\sigma$. Setting $\mathbb{D}_{\mathcal{R}_N} := \mathcal{R}_N \otimes_{\Lambda_N} \mathbb{D}_N$ and extending by $\mathcal{R}_N$-linearity over $\Lambda_N$ we finally obtain a $\mathcal{R}_N$-linear map, again denoted by the same symbol,

$$J_\alpha : \mathbb{D}_{\mathcal{R}_N} \rightarrow \mathcal{R}_N.$$

For $\nu \in \mathbb{D}_N$ and $r \in \mathcal{R}_N$ we thus have

$$J_\alpha(r \otimes \nu) = \epsilon_{\bar{\kappa},M}(b) \cdot \epsilon_{\bar{\kappa},p}(-1) \cdot r \otimes J_\alpha(\nu).$$

For the next result, for any arithmetic point $\kappa_N \in \mathcal{X}_N^{\operatorname{arith}}$ coming from $\kappa \in \mathcal{X}_N^{\operatorname{arith}}$, extend $\rho_\kappa$ in (5) by $\mathcal{R}_N$-linearity over $\mathcal{O}[\mathbb{Z}_p^\times]$, to get a map

$$\rho_{\kappa_N} : \mathbb{D}_{\mathcal{R}_N} \rightarrow V_{\alpha \kappa_N}$$

defined by $\rho_{\kappa_N}(r \otimes \nu) := \rho_\kappa(\nu) \cdot \kappa_N(r)$, for $\nu \in \mathbb{D}$ and $r \in \mathcal{R}_N$. To simplify notations, set

$$\rho_{\kappa_N}(x)(\hat{\Omega}_{\alpha \kappa_N}^{n_x}/2) := \langle x, \alpha \rangle_{\kappa_N}.$$  

The following is essentially [18, Lemma (6.1)].

**Lemma 3.2.** Let $\bar{\kappa}_N \in \mathcal{X}_N^{\operatorname{arith}}$ with signature $(\epsilon_{\bar{\kappa}}, k_{\bar{\kappa}})$ and define $\kappa_N := p_N(\bar{\kappa}_N)$. Then for any $x \in \mathbb{D}_{\mathcal{R}_N}$ we have:

$$\bar{\kappa}_N(J_\alpha(x)) = \eta_{\bar{\kappa}}(\alpha) \cdot \langle x, \alpha \rangle_{\kappa_N}.$$
Proof. For $\nu \in \mathbb{D}_N$ and $r \in \mathcal{R}_N$ we have
\begin{align*}
\tilde{\kappa}_N(J_\alpha(r \otimes \nu)) &= \tilde{\kappa}_N(\epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot r \otimes j_\alpha(\nu)) \\
&= \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot \tilde{\kappa}_N(1 \otimes r \otimes j_\alpha(\nu)) \\
&= \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot \kappa_N(r) \cdot \int_{\mathbb{Z}_p^r} \tilde{\kappa}_N(t) dj_\alpha(\nu)
\end{align*}
and thus, noticing that $\tilde{\kappa}$ restricted to $\mathbb{Z}_p^\times$ is $\tilde{\kappa}_N(t) = \epsilon_{\tilde{\kappa},p}(t)t^{ns}$, we have
\begin{align*}
\tilde{\kappa}_N(J_\alpha(r \otimes \nu)) &= \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(-1) \cdot \kappa_N(r) \int_{\mathbb{Z}_p^r} \epsilon_{\tilde{\kappa},p}(\tilde{Q}_\alpha(x, y)) \tilde{Q}_\alpha(x, y)^{ns/2} d\nu.
\end{align*}
Recalling (9), and viewing $a, b, c$ as elements in $\mathbb{Z}_p$, we have, for $(x, y) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$, $\epsilon_{\tilde{\kappa},p}(\tilde{Q}_\alpha(x, y)) = \epsilon_{\tilde{\kappa},p}(-by^2) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\tilde{\kappa},p}(y^2) = \epsilon_{\tilde{\kappa},p}(-b)\epsilon_{\tilde{\kappa},p}(y)$. Thus, since $\epsilon_{\tilde{\kappa}}(-1)^2 = 1$, we get:
\begin{align*}
\tilde{\kappa}_N(J_\alpha(r \otimes \nu)) &= \kappa_N(r) \cdot \epsilon_{\tilde{\kappa},M}(b) \cdot \epsilon_{\tilde{\kappa},p}(b) \cdot \rho_\kappa(\nu)(\tilde{Q}_\alpha^{ns/2}) = \eta_{\kappa}(\alpha) \cdot \langle \nu, \alpha \rangle \kappa_N
\end{align*}
where for the last equality use (10) and (11). □

Define
\[ W_{\mathcal{R}_N} := \mathbb{W} \otimes_{\mathcal{O}[\mathbb{Z}_p^\times]} \mathcal{R}_N, \]
the structure of $\mathcal{O}[\mathbb{Z}_p^\times]$-module of $\mathcal{R}_N$ being that induced by the composition of the two maps $\mathcal{O}[\mathbb{Z}_p^\times] \to \Lambda_N \to \mathcal{R}_N$ described above. There is a canonical map
\[ \theta : W_{\mathcal{R}_N} \longrightarrow H^1(\Gamma_0, \mathbb{D}_{\mathcal{R}_N}) \]
described as follows: if $\nu_\gamma$ is a representative of an element $\nu$ in $W$ and $r \in \mathcal{R}_N$, then $\theta(\nu \otimes r)$ is represented by the cocycle $\nu_\gamma \otimes r$.

For $\nu \in W_{\mathcal{R}_N}$ represented by $\nu_\gamma$ and $\xi \geq 1$ an integer, define
\[ \theta_{\xi}(\nu) := \sum_{c \in \mathbb{R}(\Gamma_0^+) \cdot q(c) = \xi} J_{\mathfrak{a}_c}(\nu_{\mathfrak{a}_c}) / \mathfrak{a}_{\mathfrak{a}_c}. \]

**Definition 3.3.** For $\nu \in W_{\mathcal{R}_N}$, the formal Fourier expansion
\[ \Theta(\nu) := \sum_{\xi \geq 1} \theta_{\xi}(\nu)q^\xi \]
in $\mathcal{R}_N[q]$ is called the $\Lambda$-adic Shintani-Shimura-Waldspurger lift of $\nu$. For any $\tilde{\kappa} \in \mathbb{X}_{\text{arith}}$, the formal power series expansion
\[ \Theta(\nu)(\tilde{\kappa}_N) := \sum_{\xi \geq 1} \tilde{\kappa}_N(\theta_{\xi}(\nu))q^\xi \]
is called the $\tilde{\kappa}$-specialization of $\Theta(\nu)$.

There is a natural map
\[ W_{\mathcal{R}} \longrightarrow W_{\mathcal{R}_N} \]
taking $\nu \otimes r$ to itself (use that $\mathcal{R}$ has a canonical map to $\mathcal{R}_N \simeq \mathcal{R}[\Delta]$, as described above). So, for any choice of sign, $\Phi^\pm \in W_{\mathcal{R}}$ will be viewed as an element in $W_{\mathcal{R}_N}$.

From now on we will use the following notations. Fix $\tilde{\kappa}_0 \in \mathbb{X}_{\text{arith}}$ and put $\kappa_0 := p(\tilde{\kappa}_0) \in \mathbb{X}_{\text{arith}}$. Recall the neighborhood $U_0$ of $\kappa_0$ in Theorem 3.1. Define $U_0 := p^{-1}(U_0)$ and $U_0^\text{arith} := U_0 \cap \mathbb{X}_{\text{arith}}$. 

For each $\kappa \in \overline{U}_0^{\operatorname{arith}}$ put $\kappa = p(\tilde{\kappa}) \in \overline{U}_0^{\operatorname{arith}}$. Recall that if $(\epsilon, k, \kappa)$ is the signature of $\kappa$, then $(\epsilon, k, \kappa) := (\epsilon^2, 2k\kappa)$ is that of $\kappa_0$. For any $\kappa := p(\tilde{\kappa})$ as above, we may consider the modular form

$$f^{\operatorname{IRL}}_{\kappa} \in S_{\kappa}(\Gamma_{\kappa}, \epsilon)$$

and its Shimura-Shintani lift

$$h_{\kappa} = \sum_{\xi} a_{\xi}(h_{\kappa})q^{\xi} \in S_{\kappa+1/2}(4Np^{n}, \chi_{\kappa}),$$

where $\chi_{\kappa}(x) := \epsilon_{\kappa}(x) \left( -\frac{1}{x} \right)^{k_{\kappa}},$

normalized as in (2) and (3). For our fixed $\kappa_0$, recall the elements $\Phi := \Phi^+$ chosen as in Theorem 3.1 and define $\phi_{\kappa} := \phi_{\kappa}^+$ and $\Omega_{\kappa} := \Omega_{\kappa}^+$ for $\kappa \in \overline{U}_0^{\operatorname{arith}}$.

**Proposition 3.4.** For all $\kappa \in \overline{U}_0^{\operatorname{arith}}$ such that $r_{\kappa} = 1$ we have

$$\tilde{\kappa}_N(\theta_{\xi}(\Phi)) = \Omega_{\kappa} \cdot a_{\xi}(h_{\kappa}) \quad \text{and} \quad \Theta(\Phi)(\tilde{\kappa}_N) = \Omega_{\kappa} \cdot h_{\kappa}.$$

**Proof.** By Lemma 3.2 we have

$$\tilde{\kappa}_N(\theta_{\xi}(\Phi)) = \sum_{C \in R(\Gamma, q(C)) = \xi} \frac{\eta_{\kappa}(\alpha_C)}{t_{\alpha_C}} \rho_{\kappa,\xi}(\Phi)(\tilde{Q}^{n_{\alpha_C}/2}_{\alpha_C}).$$

Using Theorem 3.1, we get

$$\tilde{\kappa}_N(\theta_{\xi}(\Phi)) = \sum_{C \in R(\Gamma, q(C)) = \xi} \frac{\eta_{\kappa}(\alpha_C) \cdot \Omega_{\kappa}}{t_{\alpha_C}} \phi_{\kappa}(\tilde{Q}^{n_{\alpha_C} - 1}_{\alpha_C}).$$

Now (2) shows the statement on $\tilde{\kappa}_N(\theta_{\xi}(\Phi))$, while that on $\Theta(\Phi)(\tilde{\kappa}_N)$ is a formal consequence of the previous one. $\square$

**Corollary 3.5.** Let $a_p$ denote the image of the Hecke operator $T_p$ in $\mathcal{R}$. Then

$$\Theta(\Phi)|T_p^2 = a_p \cdot \Theta(\Phi).$$

**Proof.** For any $\kappa \in \overline{X}^{\operatorname{arith}}$, let $a_p(\kappa) := n(T_p)$, which is a $p$-adic unit by the ordinarity assumption. For all $\kappa \in \overline{U}_0^{\operatorname{arith}}$ with $r_{\kappa} = 1$, we have $\kappa = \kappa_0$. Consequently

$$\kappa_N(\theta_{\xi^2}(\Phi)) = a_p(\kappa) \cdot \tilde{\kappa}_N(\theta_{\xi}(\Phi))$$

for all $\kappa$ such that $r_{\kappa} = 1$. Since this subset is dense in $\overline{X}_N$, we conclude that $\theta_{\xi^2}(\Phi) = a_p \cdot \theta_{\xi}(\Phi)$ and so $\Theta(\Phi)|T_p^2 = a_p \cdot \Theta(\Phi)$. $\square$

For any integer $n \geq 1$ and any quadratic form $Q$ with coefficients in $F$, write $[Q]^n$ for the class of $Q$ modulo the action of $i_F(\Gamma_n)$. Define $F_{n,\xi}$ to be the subset of the $F$-vector space of quadratic forms with coefficients in $F$ consisting of quadratic forms $Q_\alpha$ such that $\alpha \in V^* \cap \mathcal{O}_{B,n}$ and $-\operatorname{nr}(\alpha) = \xi$. Writing $\delta_{Q_\alpha}$ for the discriminant of $Q_\alpha$, the above set can be equivalently described as

$$F_{n,\xi} := \{ \tilde{Q}_\alpha | \alpha \in V^* \cap \mathcal{O}_{B,n}, \delta_{Q_\alpha} = Np^n \xi \}.$$

Define $F_{n,\xi}/\Gamma_n$ to be the set $\{ [\tilde{Q}_\alpha]_n | \tilde{Q}_\alpha \in F_{n,\xi} \}$ of equivalence classes of $F_{n,\xi}$ under the action of $i_F(\Gamma_n)$. A simple computation shows that $Q_{g^{-1}a} = Q_\alpha|g$ for all $\alpha \in V^*$ and all $g \in \Gamma_n$, and thus we find

$$F_{n,\xi}/\Gamma_n = \{ [\tilde{Q}_\alpha]_n | C \in R(\Gamma_n), \delta_{Q_\alpha} = Np^n \xi \}.$$
We also note that, in the notations of §2.1, if \( f \) has weight character \( \psi \), defined modulo \( Np^n \), and level \( \Gamma_n \), the Fourier coefficients \( a_\xi(h) \) of the Shintani-Shimura lift \( h \) of \( f \) are given by

\[
a_\xi(h) = \sum_{[Q] \in \mathcal{F}_{\xi, \Gamma_n}} \frac{\psi(Q)}{t_Q} \phi_+^\dagger(Q(z)^{k-1})
\]

and, if \( Q = \tilde{Q}_\alpha \), we put \( \psi(Q) := \eta_\psi(b_\alpha) \) and \( t_Q := t_\alpha \). Also, if we let

\[
\mathcal{F}_{n, \xi} / \Gamma_n := \prod_{\xi} \mathcal{F}_{n, \xi} / \Gamma_n
\]

we can write

\[
h = \sum_{[Q] \in \mathcal{F}_{n, \xi} / \Gamma_n} \frac{\psi(Q)}{t_Q} \phi_+^\dagger(Q(z)^{k-1}) q^{\delta_Q/(Np^n)}.
\]

Fix now an integer \( m \geq 1 \) and let \( n \in \{1, m\} \). For any \( t \in (\mathbb{Z}/p^n\mathbb{Z})^\times \) and any integer \( \xi \geq 1 \), define \( \mathcal{F}_{n, \xi, t} \) to be the subset of \( \mathcal{F}_{n, \xi} \) consisting of forms such that \( Np^n b_\alpha \equiv t \mod Np^m \). Also, define \( \mathcal{F}_{n, \xi, t} / \Gamma_n \) to be the set of equivalence classes of \( \mathcal{F}_{n, \xi, t} \) under the action of \( i_{F}(\Gamma_n) \). If \( \alpha \in V^* \cap O_{B, m} \) and \( i_{F}(\alpha) = (\frac{a}{c} \ b \ c) \), then

\[
\tilde{Q}_\alpha(x, y) = Np^n cx^2 - 2Np^n axy - Np^b y^2
\]

from which we see that there is an inclusion \( \mathcal{F}_{m, \xi, t} \subseteq \mathcal{F}_{1, \xi, p^m-1, t} \). If \( Q_\alpha \) and \( \tilde{Q}_{\alpha'} \) belong to \( \mathcal{F}_{m, \xi, t} \), and \( \alpha' = gag^{-1} \) for some \( g \in \Gamma_m \), then, since \( \Gamma_m \subseteq \Gamma_1 \), we see that \( Q_\alpha \) and \( Q_{\alpha'} \) represent the same class in \( \mathcal{F}_{1, \xi, p^m-1, t} / \Gamma_1 \). This shows that \([\tilde{Q}_\alpha]_m \mapsto [\tilde{Q}_\alpha]_1 \) gives a well-defined map

\[
\pi_{m, \xi, t} : \mathcal{F}_{m, \xi, t} / \Gamma_m \longrightarrow \mathcal{F}_{1, \xi, p^m-1, t} / \Gamma_1.
\]

**Lemma 3.6.** The map \( \pi_{m, \xi, t} \) is bijective.

**Proof.** We first show the injectivity. For this, suppose \( Q_\alpha \) and \( \tilde{Q}_{\alpha'} \) are in \( \mathcal{F}_{m, \xi, t} \) and \([\tilde{Q}_\alpha]_1 = [\tilde{Q}_{\alpha'}]_1 \). So there exists \( g = (\frac{a}{c} \ b \ c) \in i_F(\Gamma_1) \) such that such that \( \tilde{Q}_\alpha = \tilde{Q}_{\alpha'} g \).

If \( \tilde{Q}_\alpha = cx^2 - 2axy - by^2 \), and easy computation shows that \( \tilde{Q}_{\alpha'} = c'x^2 - 2a'xy - b'y^2 \) with

\[
c' = ca^2 - 2a_\alpha \beta \gamma - b_\gamma^2
\]

\[
a' = -ca_\beta \beta + a_\alpha \gamma + a_\alpha \delta + b_\gamma \delta
\]

\[
b' = -c_\beta \beta^2 + 2a_\alpha \delta + b_\delta^2.
\]

The first condition shows that \( \gamma \equiv 0 \mod Np^m \). Further, we have \( b \equiv b' \equiv t \mod Np^m \), so \( \delta^2 \equiv 1 \mod Np^m \). Since \( \delta \equiv 1 \mod Np \), we see that \( \delta \equiv 1 \mod Np^m \) too.

We now first show the surjectivity. For this, fix \([\tilde{Q}_{\alpha_c}]_1 \) in the target of \( \pi \), and choose a representative \( \tilde{Q}_{\alpha_c} = cx^2 - 2axy - by^2 \) (recall \( Mp^m \xi \mid \delta \tilde{Q}_{\alpha_c}, Np \mid c, Np \mid a, \) and \( b \in O_F^\times \), the last condition due to \( \eta_\psi(\alpha_c) \neq 0 \)). By the Strong Approximation Theorem, we can find \( \tilde{g} \in \Gamma_1 \) such that \( i_F(\tilde{g}) \equiv (\frac{1}{ab} \ 0 \ 0) \mod Np^m \) for all \( \ell \mid Np \). Take \( g := i_F(\tilde{g}) \), and put \( \alpha := g^{-1} \alpha_c g \).

An easy computation, using the expressions for \( a', b', c' \) in terms of \( a, b, c \) and \( g = \)
Proof. For the proof of Proposition 3.4, combining Lemma 3.2 and Theorem 3.1 we get Proposition 3.7.

where the last equality follows because \( g \in \Gamma_1 \).

\[ \pi([\tilde{Q}_\alpha]_m) = [\tilde{Q}_\alpha]_1 = [\tilde{Q}_{g^{-1} \alpha \tilde{g}}]_1 = [\tilde{Q}_{\alpha}]_1 \]

where the last equality follows because \( g \in \Gamma_1 \).

\[
\text{Proposition 3.7. For all } \kappa \in \mathcal{U}_0^\text{arith} \text{ we have }
\Theta(\Phi)(\tilde{\kappa}_N)|_{T_p^{r\kappa-1}} = \Omega_\kappa \cdot h_\kappa.
\]

Proof. For \( r_\kappa = 1 \), this is Proposition 3.4 above, so we may assume \( r_\kappa \geq 2 \). As in the proof of Proposition 3.4, combining Lemma 3.2 and Theorem 3.1 we get

\[
\Theta(\Phi)(\tilde{\kappa}_N) = \sum_{\varepsilon \geq 1} \left( \sum_{C \in R(\Gamma_1), q(C) = \varepsilon} \frac{\eta_{\varepsilon}(\alpha_C) \cdot \Omega_\kappa}{t_\varepsilon} \phi_\kappa(\tilde{Q}_{k\varepsilon-1}) \right) q^{\ell_\varepsilon}.
\]

which, by (12) and (13) above we may rewrite as

\[
\Theta(\Phi)(\tilde{\kappa}_N) = \sum_{[Q] \in \mathcal{F}_1/\Gamma_1} \frac{\eta_{\varepsilon}(Q) \cdot \Omega_\kappa}{t_Q} \phi_\kappa(Q^{k\varepsilon-1}) q^{\ell_\varepsilon/(Np)}.
\]

By definition of the action of \( T_p \) on power series, we have

\[
\Theta(\Phi)(\tilde{\kappa}_N)|_{T_p^{r\kappa-1}} = \sum_{[Q] \in \mathcal{F}_1/\Gamma_1, p^{r\kappa} | \delta_Q} \frac{\eta_{\varepsilon}(Q) \cdot \Omega_\kappa}{t_Q} \phi_\kappa(Q^{k\varepsilon-1}) q^{\ell_\varepsilon/(Np^{r\kappa})}.
\]

Setting \( \mathcal{F}_{n,t} := \prod_{\varepsilon \geq 1} \mathcal{F}_{n,t,\varepsilon}/\Gamma_n \) for \( n \in \{1, r_\kappa\} \), Lemma 3.6 shows that

\[
\mathcal{F}_{1,t} := \{ [Q] \in \mathcal{F}_{1,t}/\Gamma_1, t \text{ such that } p^{r\kappa} | \delta_Q \} \text{ is equal to } \mathcal{F}_{r_\kappa,t}.
\]

Therefore, splitting the above sum over \( t \in (\mathbb{Z}/Np^{r\kappa})^\times \), we get

\[
\Theta(\Phi)(\tilde{\kappa}_N)|_{T_p^{r\kappa-1}} = \sum_{t \in (\mathbb{Z}/p^{r\kappa-1})^\times} \sum_{[Q] \in \mathcal{F}_{1,t}} \frac{\eta_{\varepsilon}(Q) \cdot \Omega_\kappa}{t_Q} \phi_\kappa(Q^{k\varepsilon-1}) q^{\ell_\varepsilon/(Np^{r\kappa})}
\]

\[
= \sum_{t \in (\mathbb{Z}/p^{r\kappa-1})^\times} \sum_{[Q] \in \mathcal{F}_{m,t}/\Gamma_m} \frac{\eta_{\varepsilon}(Q) \cdot \Omega_\kappa}{t_Q} \phi_\kappa(Q^{k\varepsilon-1}) q^{\ell_\varepsilon/(Np^{r\kappa})}
\]

\[
= \sum_{[Q] \in \mathcal{F}_{m,t}/\Gamma_m} \frac{\eta_{\varepsilon}(Q) \cdot \Omega_\kappa}{t_Q} \phi_\kappa(Q^{k\varepsilon-1}) q^{\ell_\varepsilon/(Np^{r\kappa})}.
\]

Comparing this expression with (13) gives the result. \( \square \)

We are now ready to state the analogue of [18, Thm. 3.3], which is our main result. For the reader’s convenience, we briefly recall the notations appearing below. We denote by \( \mathcal{X} \) the ordinary Hida Hecke algebra, and by \( \mathcal{X}^\text{arith} \) its arithmetic points. For \( \kappa_0 \in \mathcal{X}^\text{arith} \), we denote by \( \mathcal{U}_0 \) the \( p \)-adic neighborhood of \( \kappa_0 \) appearing in the statement of Theorem 3.1 and put \( \mathcal{U}_0^\text{arith} := \mathcal{U}_0 \cap \mathcal{X}^\text{arith} \). We also denote by \( \Phi = \Phi^+ \in \text{W}_{\text{ord}} \) the cohomology class appearing in Theorem 3.1. The metaplectic Hida Hecke algebra \( \mathcal{X} \) is defined in §3.3 and is equipped with a canonical map \( p : \mathcal{X}^\text{arith} \to \mathcal{X}^\text{arith} \) on arithmetic points. Let \( \mathcal{U}_0^\text{arith} := \mathcal{U}_0 \cap \mathcal{X}^\text{arith} \). For each \( \kappa \in \mathcal{U}_0^\text{arith} \) put \( \kappa = p(\kappa) \in \mathcal{U}_0^\text{arith} \). Recall that if \( (\varepsilon_\kappa, k_\kappa) \) is the signature of \( \kappa \), then \( (\varepsilon_\kappa, k_\kappa) := (\varepsilon_\kappa^2, 2k_\kappa) \) is that of \( \kappa_0 \). For any \( \kappa := p(\kappa) \) as above, we may consider the modular form
and its Shimira-Shintani lift

\[ h_\kappa = \sum_\xi a_\xi(h_\kappa)q^\xi \in S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi_\kappa), \quad \text{where } \chi_\kappa(x) := \epsilon_\kappa(x) \left( \frac{-1}{x} \right)^{k_\kappa}, \]

normalized as in (2) and (3). Finally, for \( \tilde{\kappa} \in \tilde{X}_{\text{arith}} \), we denote by \( \tilde{\kappa}_N \) its extension to the metaplectic Hecke algebra \( \hat{\mathcal{R}}_N \) defined in §3.3.

**Theorem 3.8.** Let \( \kappa_0 \in X_{\text{arith}} \). Then there exists a choice of \( p \)-adic periods \( \Omega_\kappa \) for \( \kappa \in U_0 \) such that the \( \Lambda \)-adic Shintani-Shimura-Waldspurger lift of \( \Phi \)

\[ \Theta(\Phi) := \sum_{\xi \geq 1} \theta_\xi(\Phi)q^\xi \]

in \( \mathcal{R}_N[q] \) has the following properties:

1. \( \Omega_{\kappa_0} \neq 0. \)
2. For any \( \tilde{\kappa} \in \tilde{X}_{\text{arith}} \), the \( \tilde{\kappa} \)-specialization of \( \Theta(\Phi) \)

\[ \Theta(\nu)(\tilde{\kappa}_N) := \sum_{\xi \geq 1} \tilde{\kappa}(\theta_\xi(\Phi))q^\xi \text{ belongs to } S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi_\kappa'), \]

where \( \chi_\kappa'(x) := \chi_\kappa(x) \cdot (\frac{x}{p})^{k_\kappa-1} \), and satisfies

\[ \Theta(\Phi)(\tilde{\kappa}_N) = \Omega_\kappa \cdot h_\kappa|T_p^{1-r_\kappa}. \]

**Proof.** The elements \( \Omega_\kappa \) are those \( \Omega_\kappa^+ \) appearing in Theorem 3.1, which we used in Propositions 3.4 and 3.7 above, so (1) is clear. Applying \( T_p^{r_\kappa-1} \) to the formula of Proposition 3.7, using Corollary 3.5 and applying \( a_p(\kappa) \) on both sides gives

\[ \Theta(\Phi)(\tilde{\kappa}_N) = a_p(\kappa)^{1-r_\kappa} \Omega_\kappa \cdot h_\kappa|T_p^{r_\kappa-1}. \]

By [15, Prop. 1.9], each application of \( T_p \) has the effect of multiplying the character by \( (\frac{p}{x}) \), hence

\[ h_\kappa' := h_\kappa|T_p^{r_\kappa-1} \in S_{k_\kappa+1/2}(4Np^{r_\kappa}, \chi_\kappa') \]

with \( \chi_\kappa' \) as in the statement. This gives the first part of (2), while the last formula follows immediately from Proposition 3.7.

**Remark 3.9.** Theorem 1.1 is a direct consequence of Theorem 3.8. Briefly, embed the set \( \mathbb{Z}^{\geq 2} \) of integers greater or equal to 2 in \( \tilde{X}_{\text{arith}} \) by sending \( k \in \mathbb{Z}^{\geq 2} \) to the unique arithmetic point in \( X_{\text{arith}} \) of signature \((k,1)\). First, one can define an embedding

\[ \mathbb{Z}^{\geq 2} \rightarrow X(\Lambda) := \text{Hom}_{\text{cont}}(\Lambda, \mathbb{Q}_p) \]

via \( x \mapsto x^{k-2} \). Call \( k \in X(\Lambda) \) the element thus obtained from \( k \in \mathbb{Z}^{\geq 2} \). Second, a well-known result by Hida (see [6, Corollary 1.4]) shows that \( \mathcal{R}/\Lambda \) is unramified at \( k \). We can thus extend uniquely \( k \) to \( \hat{\mathcal{R}} \). Applying Theorem 3.8 to the image of \( \mathbb{Z}^{\geq 2} \) in \( X_{\text{arith}} \), we get Theorem 1.1.

**Remark 3.10.** For \( \kappa \in \tilde{U}_0^{\text{arith}} \) of signature \((\epsilon_\kappa, k_\kappa)\) with \( r_\kappa = 1 \) as in the above theorem, \( h_\kappa \) is trivial if \((-1)^{k_\kappa} \neq \epsilon_\kappa(-1)\). However, since \( \phi_{k_\kappa} \neq 0 \), it follows that \( h_{\kappa_0} \) is not trivial as long as the necessary condition \((-1)^{k_0} = \epsilon_0(-1)\) is verified.

**Remark 3.11.** This result can be used to produce a quaternionic \( \Lambda \)-adic version of the Saito-Kurokawa lifting, following closely the arguments in [7, Cor. 1].
References


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