January 21, 2022

NUMBER THEORY I, 24/1/2022

3h. Each question is 4 points (total 32 points).

Exercise 1. (1) Find an integral basis and the discriminant of $\mathbb{Q}[\sqrt{m}]$, where $m > 1$ is a square-free integer.

(2) Show that $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ is an integral basis of $\mathbb{Q}[\sqrt[3]{2}]$, and calculate the discriminant.

Exercise 2. (1) Find the units of the ring of integers $\mathcal{O}_K$ of the field $K = \mathbb{Q}[\sqrt{-3}]$ and the isomorphism class of the group $\mathcal{O}_K^\times$.

(2) Find the units of the ring of integers $\mathcal{O}_K$ of the field $K = \mathbb{Q}[\sqrt{5}]$ and the isomorphism class of the group $\mathcal{O}_K^\times$.

Exercise 3. (1) Show that $\mathbb{Q}[\sqrt{-3}]$ has class number equal to 1.

(2) Let $K/\mathbb{Q}$ be a number field. Show that there exists an integer h such that for any ideal $\mathfrak{a} \subseteq \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integer of K, the ideal $\mathfrak{a}^h$ is principal.

Exercise 4. (1) Let $K = \mathbb{Q}[\sqrt{2}]$. Find a prime number ℓ_1 which splits in K, a prime number ℓ_2 which does not split and does not ramify in K (*i.e.* is inert in K) and a prime number ℓ_3 which ramifies in K.

(2) Let $K/\mathbb{Q}$ be a Galois extension, and suppose that the Galois group is non-cyclic. Show that there are at most a finite number of non-split primes (namely, primes ℓ of $\mathbb{Q}$ such that $(\ell) = \ell\mathcal{O}_K$ is a prime ideal of the ring of integers $\mathcal{O}_K$ of K).