A NOTE ON CONTROL THEOREMS FOR QUATERNIONIC HIDA FAMILIES OF MODULAR FORMS

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ABSTRACT. We extend a result of Greenberg and Stevens ([11]) on the interpolation of modular symbols in Hida families to the context of non-split rational quaternion algebras. Both the definite case and the indefinite case are considered.

1. INTRODUCTION

Fix an integer $N \geq 1$ and a prime number $p \geq 5$ not dividing N. Let $\mathbb{X}$ denote the set of primitive vectors in $\mathbb{Y} := \mathbb{Z}_p^2$, i.e., the subset of $\mathbb{Y}$ consisting of those elements which are not divisible by p. Write $\tilde{\mathbb{D}}$ for the group of $\mathbb{Z}_p$-valued measures on $\mathbb{Y}$ and $\mathbb{D}$ for the direct summand of $\tilde{\mathbb{D}}$ consisting of those measures which are supported on $\mathbb{X}$. Then it is possible to introduce an $\mathrm{M}_2(\mathbb{Z}_p)$-action as well as a $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$-module structure on $\mathbb{D}$ and $\tilde{\mathbb{D}}$. Define the $\mathbb{Z}_p$-module of $\mathbb{D}$-valued *modular symbols* on $\Gamma_1(N)$ as

$$\mathbb{W} := \mathrm{Symb}_{\Gamma_1(N)}(\mathbb{D}) := \mathrm{Hom}_{\Gamma_1(N)}(\mathcal{D}_0, \mathbb{D}) \simeq H^1_{\mathrm{cpt}}(\Gamma_1(N)\backslash\mathcal{H}, \mathbb{D})$$

where $\mathcal{D}_0$ is the subgroup of degree 0 divisors on $\mathrm{Div}\big(\mathbb{P}^1(\mathbb{Q})\big)$ and $\mathcal{H}$ is the complex upper half plane (for the above isomorphism, see [11, Theorem 4.2]). Then $\mathbb{W}$ is endowed with a structure of $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$-algebra as well as with a structure of Hecke module over the $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$-Hecke algebra $\mathcal{H}$ defined in [11, (1.6)].

Let $\Lambda := \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!]$ denote the Iwasawa algebra of $1 + p\mathbb{Z}_p$ and let $\mathcal{L} := \mathrm{Frac}(\Lambda)$ be its fraction field. Let $\mathcal{R}$ denote the *universal ordinary p-adic Hecke algebra* of tame level N defined in [11, Definition 2.4]; then we have a natural map of Λ-algebras $h : \mathcal{H} \to \mathcal{R}$. Let $\mathcal{X}^{\mathrm{arith}}$ denote the subset of $\mathcal{X} := \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}_p}(\tilde{\mathcal{R}}, \bar{\mathbb{Q}}_p)$ made up of the arithmetic points defined in [11, Definition 2.4], where $\tilde{\mathcal{R}}$ is the normalization of $\mathcal{R}$ in $\mathcal{K} := \mathcal{R} \otimes_\Lambda \mathcal{L}$. For any arithmetic point $\kappa \in \mathcal{X}^{\mathrm{arith}}$ we can consider the localization $\mathcal{R}_{(\kappa)}$ of $\mathcal{R}$ at κ. Define $\mathbb{W}_{\mathcal{R}_{(\kappa)}} := \mathbb{W} \otimes_\Lambda \mathcal{R}_{(\kappa)}$ and denote $h_{(\kappa)} : \mathcal{H} \to \mathcal{R}_{(\kappa)}$ the composition of h with the localization map. Let $\mathbb{W}_{(\kappa)}$ denote the $h_{(\kappa)}$-eigenmodule in $\mathbb{W}_{\mathcal{R}_{(\kappa)}}$. Finally, noticing that the matrix $\iota := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ induces an involution on $\mathbb{W}$, we get a decomposition $\mathbb{W}_{\mathcal{R}_{(\kappa)}} = \mathbb{W}^+_{\mathcal{R}_{(\kappa)}} \oplus \mathbb{W}^-_{\mathcal{R}_{(\kappa)}}$, where ι acts on $\mathbb{W}^\epsilon_{\mathcal{R}_{(\kappa)}}$ as multiplication by ϵ for $\epsilon = \pm 1$.

With any $\kappa \in \mathcal{X}^{\mathrm{arith}}$ we can associate an ordinary p-stabilized newform $f_\kappa \in S_k(\Gamma_1(Np), \bar{\mathbb{Q}}_p)$ of tame conductor N (see [11, Definition 2.5 and Theorem 2.6]). Let F_κ be the (finite) extension of $\mathbb{Q}_p$ generated by the Fourier coefficients of f_κ. Then we may consider, for any choice of sign $\pm$, the modular symbol

$$\Phi^\pm_{f_\kappa} \in \mathrm{Symb}_{\Gamma_1(Np)}\big(L_{k-2}(F_\kappa)\big) := \mathrm{Hom}_{\Gamma_1(Np)}\big(\mathcal{D}_0, L_{k-2}(F_\kappa)\big).$$

Here, for any field F and any integer $n \geq 0$, the symbol $L_n(F)$ denotes the F-vector space of homogeneous polynomials of degree n endowed with the right action of $\mathrm{GL}_2(F)$ given by $(f|g)(X, Y) := f((X, Y)g^*)$, where $g^* := \det(g)g^{-1}$ for $g \in \mathrm{GL}_2(F)$. Recall that $\Phi^\pm_{f_\kappa}$ generates

the 1-dimensional F_κ-subspace $\mathbb{W}_\kappa^\pm$ of $\mathrm{Symb}_{\Gamma_1(Np)}(L_{k-2}(F_\kappa))$ on which complex conjugation acts as ± 1 and the Hecke algebra acts via the character associated with f_κ.

The $\mathcal{R}_{(\kappa)}$-modules $\mathbb{W}_{(\kappa)}^\pm$ and $\mathbb{W}_\kappa^\pm$ are connected by a *specialization map*

$$\phi_{\kappa,*} : \mathbb{W}_{(\kappa)}^\pm \longrightarrow \mathbb{W}_\kappa^\pm$$

(see [11, Definition 5.6]) deduced from the map $\phi_\kappa : \mathbb{D} \to L_{k-2}(\bar{\mathbb{Q}}_p)$ defined by the integration formula

$$\mu \longmapsto \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} \epsilon(x)(xY - yX)^{k-2} d\mu(x,y),$$

where κ has character ϵ and weight k. The result we are interested in is [11, Theorem 5.13], which can be stated as follows.

Theorem 1.1 (Greenberg–Stevens). *For any $\kappa \in \mathcal{X}^{\mathrm{arith}}$ and any choice of sign $\epsilon \in \{\pm 1\}$ the map $\phi_{\kappa,*}$ induces an isomorphism*

$$\phi_{\kappa,*} : \mathbb{W}_{(\kappa)}^\epsilon / P_\kappa \mathbb{W}_{(\kappa)}^\epsilon \xrightarrow{\simeq} \mathbb{W}_\kappa^\epsilon$$

where $P_\kappa \subset \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ is the kernel of κ.

It is worth remarking that a generalization of this result to Hilbert modular forms over totally real fields was proved in [4, Theorem 3.7].

The aim of the present paper is to extend Theorem 1.1 to the context of quaternion algebras over $\mathbb{Q}$ (the reader can find a dictionary between classical Hida families and their quaternionic counterparts in [16, Sections 5 and 6]). Although Hida in [14] does not distinguish between the case of definite quaternion algebras and the case of indefinite quaternion algebras, we prefer to keep these two settings separate. The reason for doing so is that the natural substitutes for $\mathbb{W}$ look quite different in the two cases, and some arguments in the definite case are simpler than the corresponding ones in the indefinite case. The price for this choice is that similar arguments are repeated twice, while the advantage is that the exposition becomes clearer and one can read each of the two parts independently.

The main result that we obtain can be described as follows. Let B be a quaternion algebra over $\mathbb{Q}$ of discriminant the square-free integer $D > 1$. Fix an Eichler order R of B of level M prime to D and let p be a prime not dividing MD. Fix also an ordinary p-stabilized eigenform f of level $\Gamma_1(MDp)$ and weight k, and write F_f for the field generated over $\mathbb{Q}_p$ by its Fourier coefficients, whose ring of integers will be denoted $\mathcal{O}_f$. For simplicity, we assume that the p-adic representation attached to f is residually absolutely irreducible and p-distinguished. For every prime $\ell | M$ choose an isomorphism $i_\ell : B \otimes_\mathbb{Q} \mathbb{Q}_\ell \simeq \mathrm{M}_2(\mathbb{Q}_\ell)$ such that $i_\ell(R \otimes_\mathbb{Z} \mathbb{Z}_\ell)$ is the subgroup of upper triangular matrices modulo $\ell^{\mathrm{ord}_\ell(M)}$. Moreover, choose $i_p : B \otimes_\mathbb{Q} \mathbb{Q}_p \simeq \mathrm{M}_2(\mathbb{Q}_p)$ such that $i_p(R \otimes_\mathbb{Z} \mathbb{Z}_p) = \mathrm{M}_2(\mathbb{Z}_p)$. Define

$$\mathbb{W} := \begin{cases} H^1(\Gamma_0, \mathbb{D}) & \text{if } B \text{ is indefinite,} \\ S_2(U_0, \mathbb{D}) & \text{if } B \text{ is definite,} \end{cases}$$

where the notations are as follows:
- $\mathbb{D}$ is the $\mathcal{O}_f$-module of $\mathcal{O}_f$-valued measures on $\mathbb{Y}$ which are supported on $\mathbb{X}$;
- Γ_0 is a finite index subgroup of the group $R_1^\times$ of norm 1 elements in $R^\times$, containing the subgroup of $R_1^\times$ consisting of the elements γ such that $i_\ell(\gamma) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{\ell^{\mathrm{ord}_\ell(M)}}$ for all primes $\ell | M$;
- U_0 is a finite index subgroup of $\hat{R}^\times$ containing the subgroup of $\hat{R}^\times$ consisting of the elements $u = (u_\ell)_\ell$ such that $i_\ell(u_\ell) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{\ell^{\mathrm{ord}_\ell(M)}}$ for all primes $\ell | M$;
- $S_2(U_0, \mathbb{D})$ is the $\mathcal{O}_f$-module of $\mathbb{D}$-valued modular forms of weight 2 and level U_0 on $\hat{B}^\times$ (see §3.1).

To state our main result, we introduce the following notations, which slightly differ from those used before. Let $\mathcal{R}$ denote the integral closure of Λ in the primitive component $\mathcal{K}$ of $\mathfrak{h}_\infty^{D,\mathrm{ord}} \otimes_\Lambda \mathcal{L}$ corresponding to f, where now $\mathfrak{h}_\infty^{\mathrm{ord}}$ is the p-ordinary Hecke algebra of level Γ_0 (indefinite case) or U_0 (definite case) with coefficients in $\mathcal{O}_f$ associated with B (see §2.1 and §3.2 for the relevant definitions). Let $\mathcal{A}(\mathcal{R})$ denote the set of arithmetic homomorphisms in $\mathrm{Hom}(\mathcal{R}, \bar{\mathbb{Q}}_p)$ (this notion is introduced in §2.2). A point $\kappa \in \mathcal{A}(\mathcal{R})$ corresponds to a normalized eigenform f_κ; write F_κ for the field generated over $\mathbb{Q}_p$ by the Fourier coefficients of f_κ. For any $\kappa \in \mathcal{A}(\mathcal{R})$ define

$$\mathbb{W}_\kappa := \begin{cases} H^1\big(\Gamma_r, V_{k_\kappa-2}(F_\kappa)\big)^{f_\kappa} & \text{if } B \text{ is indefinite,} \\ S_2(U_r, F_\kappa)^{f_\kappa} & \text{if } B \text{ is definite,} \end{cases}$$

where

- the superscript f_κ denotes the subspace on which the Hecke algebra acts via the character associated with f_κ;
- Γ_r is the subgroup of Γ_0 consisting of the elements γ such that $i_p(\gamma) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{p^r}$;
- $U_r \subset U_0$ is the compact open subgroup of $\hat{B}^\times$ whose p-component is isomorphic via i_p to the group of matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$ congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$ modulo p^r;
- $S_2(U_r, F_\kappa)$ is the F_κ-vector space of modular forms of weight 2 and level U_r.

For any field F we may define *specialization maps*

$$\rho_{k-2,\epsilon} : \mathbb{D} \longrightarrow V_{k-2}(F)$$

by the formulas

$$\rho_{k-2,\epsilon}(\nu)(P) := \begin{cases} \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon(y) P(x,y) d\nu & \text{if } B \text{ is indefinite,} \\ \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} \epsilon(x) P(x,y) d\nu & \text{if } B \text{ is definite.} \end{cases}$$

For any $\kappa \in \mathcal{A}(\mathcal{R})$ of weight k_κ and character ϵ_κ (see §2.2 for definitions) we may consider the map $\rho_{k_\kappa-2,\epsilon_\kappa}$ which induces maps:

$$\rho_\kappa : \mathbb{W}^{\mathrm{ord}} \longrightarrow \mathbb{W}_\kappa^{\mathrm{ord}}.$$

Here $\mathbb{W}^{\mathrm{ord}}$ and $\mathbb{W}_\kappa^{\mathrm{ord}}$ denote the *ordinary submodules* of $\mathbb{W}$ and $\mathbb{W}_\kappa$, respectively, defined as in [11, Definition 2.2] (see also §2.4 and §3.5). Finally, in this case too there is a universal Hecke algebra $\mathfrak{h}_{\mathrm{univ}}^D$ equipped with a canonical morphism $h : \mathfrak{h}_{\mathrm{univ}}^D \to \mathfrak{h}_\infty^{\mathrm{ord}}$. For any $\kappa \in \mathcal{A}(\mathcal{R})$ let P_κ denote its kernel and $\mathcal{R}_{P_\kappa}$ the localization of $\mathcal{R}$ at P_κ (note the slight notational change with respect to [11]). Let $\mathbb{W}_{h_\kappa}^{\mathrm{ord}}$ be the h_κ-submodule in $\mathbb{W}^{\mathrm{ord}}$, where $h_\kappa : \mathfrak{h}_{\mathrm{univ}}^D \to \mathcal{R}_{P_\kappa}$ is the composition of h with the localization map $\mathcal{R} \to \mathcal{R}_{P_\kappa}$.

Theorem 1.2. *For any $\kappa \in \mathcal{A}(\mathcal{R})$ the specialization map defines an isomorphism*

$$\mathbb{W}_{h_\kappa}^{\mathrm{ord}} / P_\kappa \mathbb{W}_{h_\kappa}^{\mathrm{ord}} \xrightarrow{\simeq} \mathbb{W}_\kappa^{\mathrm{ord}}.$$

Related results in the context of Coleman families are available: see the article [8] by Chenevier (definite case) and the paper [3] by Ash and Stevens. However, in this note we avoid using locally analytic distributions because we work in the simpler setting of ordinary deformations, where we can offer a more explicit and detailed version of this result. Actually, this is our motivation in writing this paper.

The above result is a combination of Theorem 2.18 (indefinite case) and Theorem 3.7 (definite case) and was crucially applied in [17] to obtain rationality results for quaternionic Darmon points on elliptic curves. In the indefinite case a more precise version of Theorem 1.2 can be stated, taking into account the action of the archimedean involution as in Theorem 1.1. We also observe that in the definite case the above result generalizes [5, Theorem 2.5] and, actually, provides a full proof of it (in fact, a proof was only briefly sketched in [5]).

We caution the reader that some of the notations adopted in the main body of the paper may slightly differ from those used in this introduction. For example, as noticed above, in the sequel we use the symbol $\mathcal{R}$ to denote a single component of the universal ordinary Hecke algebra appearing in Theorem 1.1. Furthermore, localizations of $\mathcal{R}$ at arithmetic primes κ are denoted $\mathcal{R}_{P_\kappa}$ instead of $\mathcal{R}_{(\kappa)}$ (the latter being the symbol used in [11]). However, every piece of notation will be carefully defined, and we are confident that no confusion will arise.

Convention. Throughout the paper we fix field embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$.

2. THE INDEFINITE CASE

In this section B is an *indefinite* quaternion algebra over $\mathbb{Q}$, whose discriminant $D \geq 1$ is then a square-free product of an *even* number of primes (if $D = 1$ then $B \simeq \mathrm{M}_2(\mathbb{Q})$).

2.1. Hecke algebras.

Let $\mathcal{G}$ be a group. For any subgroup $G \subset \mathcal{G}$ and any subsemigroup $S \subset \mathcal{G}$ such that (G, S) is a Hecke pair in the sense of [1, §1.1] we denote $\mathcal{H}(G, S)$ the Hecke algebra (over $\mathbb{Z}$) of the pair (G, S), whose elements are combinations with integer coefficients of double cosets $T(s) := GsG$ for $s \in S$. If $\mathcal{G} = B^\times$ let $g \mapsto g^* := \mathrm{norm}(g)g^{-1}$ denote the main involution of $B^\times$, where norm $: B^\times \to \mathbb{Q}$ is the norm map. Similarly, for any $S \subset B^\times$ as above let S^* denote the image of S under $g \mapsto g^*$. If M is a left $\mathbb{Z}[S^*]$-module then the group $H^1(G, M)$ has a natural right action of $R(G, S)$ defined as follows. For $s \in S$ write $GsG = \coprod Gs_i$, then define functions $t_i : G \to G$ by the equations $Gs_i\gamma = Gs_j$ (for some j) and $g_i\gamma = t_i(\gamma)g_j$. The action on $H^1(G, M)$ is given at the level of cochains $c \in Z^1(G, M)$ by the formula

$$\big(c|T(s)\big)(\gamma) = \sum_i s_i^* c\big(t_i(\gamma)\big).$$

Fix a maximal order R^{max} in B. For every prime number $\ell \nmid D$ fix also an isomorphism of $\mathbb{Q}_\ell$-algebras

$$i_\ell : B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq \mathrm{M}_2(\mathbb{Q}_\ell)$$

in such a way that $i_\ell(R^{\mathrm{max}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) = \mathrm{M}_2(\mathbb{Z}_\ell)$. For $x \in B$, we will occasionally write $i_\ell(x)$ in place of $i_\ell(x \otimes 1)$. Fix also an integer $M \geq 1$ prime to D and a prime p such that $p \nmid MD$. For any integer $r \geq 0$ write $R_0^D(Mp^r)$ for the Eichler order of level Mp^r contained in R^{max} and defined by the condition that $i_\ell(R_0^D(Mp^r) \otimes \mathbb{Z}_\ell) = R_\ell^{\mathrm{loc}}(\mathrm{ord}_\ell(Mp^r))$ for all primes $\ell | Mp^r$ where, for every integer $n \geq 0$ and every prime ℓ, we denote $R_\ell^{\mathrm{loc}}(n)$ the order of $\mathrm{M}_2(\mathbb{Z}_\ell)$ consisting of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \equiv 0 \pmod{\ell^n}$. Moreover, let $\Gamma_0^D(Mp^r)$ be the group of norm 1 elements of $R_0^D(Mp^r)$ and let Γ_r be the subgroup of $\Gamma_0^D(Mp^r)$ consisting of those γ such that $i_\ell(\gamma) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a \equiv 1 \pmod{Mp^r}$.

For a prime $\ell \nmid D$ let $\Sigma_\ell^{\mathrm{loc}}$ denote the semigroup of elements in $R^{\mathrm{max}} \otimes \mathbb{Z}_\ell$ with non-zero norm, and for a prime $\ell | Mp$ and an integer $n \geq 0$ let $\Sigma_\ell^{\mathrm{loc}}(\ell^n) \subset \Sigma_\ell^{\mathrm{loc}}$ be the inverse image under i_ℓ of the semigroup of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_\ell) \cap \mathrm{M}_2(\mathbb{Z}_\ell)$ with $a \equiv 1 \pmod{\ell^n}$ and $c \equiv 0 \pmod{\ell^n}$ (so $\Sigma_\ell^{\mathrm{loc}}(0) = \Sigma_\ell^{\mathrm{loc}}$). Then for every integer $r \geq 0$ define the semigroups

$$\Sigma_r := B^\times \cap \left(\prod_{\ell | Mp} \Sigma_\ell^{\mathrm{loc}}(\mathrm{ord}_\ell(Mp^r)) \times \prod_{\ell \nmid Mp} \Sigma_\ell^{\mathrm{loc}} \right)$$

and

$$\Delta_r := B^\times \cap \left(\Sigma_p^{\mathrm{loc}}(r) \times \prod_{\ell \neq p} \Sigma_\ell^{\mathrm{loc}} \right).$$

Finally, set $\Sigma_r^+ := \Sigma_r \cap B^+$ and $\Delta_r^+ := \Delta_r \cap B^+$ where B^+ is the subgroup of elements in $B^\times$ of positive norm.

For every integer $n \geq 1$ there is a Hecke operator $T_n = \sum_i T(\alpha_i)$ in $\mathcal{H}(\Gamma_r, \Sigma_r^+)$ and $\mathcal{H}(\Gamma_r, \Delta_r^+)$, where the sum is taken over all double cosets of the form $\Gamma_r \alpha_i \Gamma_r$ with $\alpha_i \in \Sigma_r^+$

and $\mathrm{norm}(\alpha_i) = n$. If $r \geq 1$ we denote the Hecke operator T_p by U_p. We also have operators $T_{n,n} \in \mathcal{H}(\Gamma_r, \Sigma_r^+)$ and $\mathcal{H}(\Gamma_r, \Delta_r^+)$ for integers $n \geq 1$ prime to MDp^r, defined as follows. For every $n \in \mathbb{Z}$ with $(n, MDp^r) = 1$ choose $\gamma_n \in \Gamma_0^D(Mp^r)$ such that $i_\ell(\gamma_n) \equiv \left(\begin{smallmatrix} * & * \\ 0 & n \end{smallmatrix}\right)$ (mod Mp^r) for all primes $\ell | Mp^r$ (use the Approximation Theorem: see, e.g., [19, Theorem 5.2.10]), then set $\delta_n := n\gamma_n \in \Sigma_r^+ \subset \Delta_r^+$ and define $T_{n,n} := T_{\delta_n}$ in $\mathcal{H}(\Gamma_r, \Sigma_r^+)$ and $\mathcal{H}(\Gamma_r, \Delta_r^+)$, which is independent of the choice of γ_n. We also denote ι the Hecke operator T_β in $\mathcal{H}(\Gamma_r, \Sigma_r)$ or $\mathcal{H}(\Gamma_r, \Delta_r)$ where β is any element of $R_0^D(Mp^r)$ of norm -1 such that $i_\ell(\beta) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (mod Mp^r) for all primes $\ell | Mp^r$ (use again the Approximation Theorem). It can easily be checked that ι commutes with the elements T_n and $T_{n,n}$ in $\mathcal{H}(\Gamma_r, \Sigma_r)$ and $\mathcal{H}(\Gamma_r, \Delta_r)$. Finally, recall that the $\mathcal{H}(\Gamma_r, \Sigma_r^+)$ are commutative rings generated (over $\mathbb{Z}$) by the Hecke operators T_n and $T_{n,n}$ defined above.

Let $\mathcal{O}$ be the ring of integers of a finite extension F of $\mathbb{Q}_p$. For any integer $r \geq 1$ denote $\mathfrak{h}_r^D$ the image of $\mathcal{H}(\Gamma_r, \Sigma_r^+) \otimes_{\mathbb{Z}} \mathcal{O}$ acting on the $\mathbb{C}$-vector space $S_2^{D\text{-new}}(\Gamma_1(MDp^r))$ of weight 2 cusp forms on $\Gamma_1(MDp^r)$ which are new at all primes dividing D.

Remark 2.1. Of course, the algebra $\mathfrak{h}_r^D$ depends on $\mathcal{O}$; however, since the field F will always be clear in our applications, for simplicity we drop this dependence from the notation.

Let $\mathfrak{h}_r^{D,\mathrm{ord}}$ denote the product of the localizations of $\mathfrak{h}_r^D$ where U_p is invertible and write e_r for the corresponding projector. Set $\mathfrak{h}_\infty^D := \varprojlim_r \mathfrak{h}_r^D$ and let $e_\infty := \varprojlim_r e_r$ be the ordinary projector in $\mathfrak{h}_\infty^D$. Then define $\mathfrak{h}_\infty^{D,\mathrm{ord}} := e_\infty \mathfrak{h}_\infty^D$, so that

$$\mathfrak{h}_\infty^{D,\mathrm{ord}} = \varprojlim_r \mathfrak{h}_r^{D,\mathrm{ord}}.$$

Consider the Iwasawa algebra $\tilde{\Lambda} := \mathcal{O}[\![\mathbb{Z}_p^\times]\!]$ with coefficients in $\mathcal{O}$ and denote $\gamma \mapsto [\gamma]$ the natural inclusion $\mathbb{Z}_p^\times \hookrightarrow \tilde{\Lambda}$ of group-like elements. Then $\mathfrak{h}_\infty^{D,\mathrm{ord}}$ is a finitely generated $\tilde{\Lambda}$-algebra.

Following [11], we are interested in defining a commutative $\tilde{\Lambda}$-algebra $\mathfrak{h}_{\mathrm{univ}}^D$ equipped with a canonical morphism of $\tilde{\Lambda}$-algebras $h : \mathfrak{h}_{\mathrm{univ}}^D \to \mathfrak{h}_\infty^D$. For this, we first consider the Hecke algebras $\mathcal{H}(\Gamma_r, \Delta_r)$ and $\mathcal{H}(\Gamma_r, \Delta_r^+)$ for integers $r \geq 0$. Let $\mathbb{Z}'$ denote the subset of $\mathbb{Z}$ consisting of integers which are prime to p. For every $a \in \mathbb{Z}'$ choose $\gamma_a' \in \Gamma_0^D(Mp^r)$ such that $i_p(\gamma_a') \equiv \left(\begin{smallmatrix} * & * \\ * & a \end{smallmatrix}\right)$ (mod p^r). Then $\delta_a' := a\gamma_a' \in \Delta_r^+$ and we can define the Hecke operator $[a] := T_{\delta_a'}$ in $\mathcal{H}(\Gamma_r, \Delta_r^+)$ and $\mathcal{H}(\Gamma_r, \Delta_r)$, which does not depend on the choice of γ_a'. The maps $\mathbb{Z}' \to \mathcal{H}(\Gamma_r, \Delta_r)$ and $\mathbb{Z}' \to \mathcal{H}(\Gamma_r, \Delta_r^+)$ defined by $a \mapsto [a]$ are multiplicative, hence extend to ring homomorphisms $\mathbb{Z}[\mathbb{Z}'] \to \mathcal{H}(\Gamma_r, \Delta_r)$ and $\mathbb{Z}[\mathbb{Z}'] \to \mathcal{H}(\Gamma_r, \Delta_r^+)$. Since $\mathbb{Z}[\mathbb{Z}']$ also embeds naturally in $\tilde{\Lambda}$, we can form the $\tilde{\Lambda}$-algebras

$$\mathcal{H}(p^r) := \mathcal{H}(\Gamma_r, \Delta_r) \otimes_{\mathbb{Z}[\mathbb{Z}']} \tilde{\Lambda}, \qquad \mathcal{H}^+(p^r) := \mathcal{H}(\Gamma_r, \Delta_r^+) \otimes_{\mathbb{Z}[\mathbb{Z}']} \tilde{\Lambda}.$$

If M is a $\mathbb{Z}_p[\Delta_r]$-module (respectively, a $\mathbb{Z}_p[\Delta_r^+]$-module) such that the action of $\mathbb{Z}'$ extends to a continuous action of $\mathbb{Z}_p^\times$ then for $i = 0, 1$ the action of the Hecke algebra $\mathcal{H}(\Gamma_r, \Delta_r)$ (respectively, $\mathcal{H}(\Gamma_r, \Delta_r^+)$) on $H^i(\Gamma_r, M)$ extends uniquely to a continuous action of $\mathcal{H}(p^r)$ (respectively, $\mathcal{H}^+(p^r)$).

Now the Hecke pairs (Γ_r, Δ_r) and (Γ_r, Δ_r^+) are weakly compatible (according to [1, Definition 2.1]) to (Γ_0, Δ_0) and (Γ_0, Δ_0^+), respectively. Hence, as explained in [1, §2], there are canonical surjective $\tilde{\Lambda}$-algebra homomorphisms

$$\rho_r : \mathcal{H}(1) \longrightarrow \mathcal{H}(p^r), \qquad \rho_r^+ : \mathcal{H}^+(1) \longrightarrow \mathcal{H}^+(p^r)$$

for all integers $r \geq 1$. We let $\mathcal{H}(1)$ (respectively, $\mathcal{H}^+(1)$) act on $H^1(\Gamma_r, M)$ by composing the action of $\mathcal{H}(p^r)$ (respectively, $\mathcal{H}^+(p^r)$) with ρ_r (respectively, ρ_r^+).

Define the universal Hecke algebra $\mathfrak{h}_{\mathrm{univ}}^D$ as

$$\mathfrak{h}_{\mathrm{univ}}^D := \tilde{\Lambda}\big[\, T_n \text{ for every } n \geq 1 \text{ and } T_{n,n} \text{ for every } n \geq 1 \text{ with } (n, MD) = 1 \,\big] \subset \mathcal{H}^+(1).$$

One can check that $\iota \in \mathcal{H}(\Gamma_0, \Delta_0)$ commutes with all the elements in $\mathfrak{h}_{\mathrm{univ}}^D$ and thus we can also consider the commutative Hecke algebra $\mathfrak{h}_{\mathrm{univ}}^D[\iota] \subset \mathcal{H}(1)$.

The $\tilde{\Lambda}$-algebra $\mathfrak{h}_{\mathrm{univ}}^D$ acts compatibly on $S_2^D(\Gamma_1(Mp^r))$, in the sense that the diagram of $\tilde{\Lambda}$-algebras

$$
\begin{array}{ccc}
\mathfrak{h}_{\mathrm{univ}}^D & \xrightarrow{\rho_r^+} & \mathcal{H}^+(p^r) \\
& \searrow{\scriptstyle \rho_{r-1}^+} & \downarrow \\
& & \mathcal{H}^+(p^{r-1})
\end{array}
$$

commutes for all $r \geq 1$ (here the vertical arrow is the map which arises from the weakly compatibility of the Hecke pairs (Γ_r, Δ_r^+) and $(\Gamma_{r-1}, \Delta_{r-1}^+)$). The image of $\mathfrak{h}_{\mathrm{univ}}^D$ in the endomorphism algebra of $S_2^D(\Gamma_1(Mp^r))$ is canonically isomorphic to $\mathfrak{h}_r^D$ (note that if $n \in (\mathbb{Z}/Mp^r\mathbb{Z})^\times$ then $n \in \mathbb{Z}'$ and $T_{n,n}$ is the image of $[n] \in \mathfrak{h}_{\mathrm{univ}}^D$), hence, by the universal property of the inverse limit, there exists a canonical morphism of $\tilde{\Lambda}$-algebras

$$
h : \mathfrak{h}_{\mathrm{univ}}^D \longrightarrow \mathfrak{h}_\infty^D.
$$

2.2. **Hida families.** Fix a non-zero normalized cusp form $f \in S_k\big(\Gamma_0(MDp^r), \epsilon\big)$ with q-expansion

$$
f(q) = \sum_{n=1}^\infty a_n q^n,
$$

and suppose that f is an eigenform for the action of the Hecke operators T_n and $T_{n,n}$. Write F_f for the field $\mathbb{Q}_p(a_n \mid n \geq 1)$ generated over $\mathbb{Q}_p$ by the Fourier coefficients of f, let $\mathcal{O}_f$ denote the ring of integers of F_f and let $\wp$ denote the maximal ideal of $\mathcal{O}_f$. In addition, assume that f is an ordinary p-stabilized newform whose $\wp$-adic representation is residually irreducible and p-distinguished (see, e.g., [10, §2]).

With notation as in §2.1, take $\mathcal{O} = \mathcal{O}_f$, so that $\tilde{\Lambda} := \mathcal{O}_f[\![\mathbb{Z}_p^\times]\!]$. Identify the group of $(p-1)$-st roots of unity in $\bar{\mathbb{Q}}_p$ with $T := (\mathbb{Z}/p\mathbb{Z})^\times$ via the Teichmüller character ω and set $W := 1+p\mathbb{Z}_p$, whence $\mathbb{Z}_p^\times \simeq T \times W$. This decomposition induces a decomposition $\tilde{\Lambda} = \mathcal{O}_f[T] \oplus \Lambda$ where Λ is (non-canonically) isomorphic to the algebra of power series in one variable with coefficients in $\mathcal{O}_f$. A $\tilde{\Lambda}$-module M inherits a canonical Λ-module structure via the inclusion $\Lambda \hookrightarrow \tilde{\Lambda}$. Finally, let $\mathcal{L}$ denote the fraction field of Λ.

There is a decomposition

$$
\mathfrak{h}_\infty^{1,\mathrm{ord}} \otimes_\Lambda \mathcal{L} \simeq \left(\bigoplus_{i \in I} \mathcal{K}_i \right) \oplus \mathcal{N}
$$

where the $\mathcal{K}_i$ are finite field extensions of $\mathcal{L}$ (called the *primitive components* of $\mathfrak{h}_\infty^{1,\mathrm{ord}} \otimes \mathcal{L}$), I is a finite set and $\mathcal{N}$ is non-reduced. Denote $\mathcal{K}$ the primitive component through which the morphism associated with f factors and let $\mathcal{R}$ be the integral closure of Λ in $\mathcal{K}$. We call the induced map
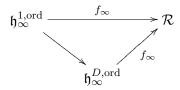
$$
f_\infty : \mathfrak{h}_\infty^{1,\mathrm{ord}} \longrightarrow \mathcal{R}
$$

the *primitive morphism* associated with f.

Now recall that, thanks to the Jacquet–Langlands correspondence, $S_k(\Gamma_r)$ is isomorphic to the subspace $S_k^{D\text{-new}}(\Gamma_1(MDp^r))$ of $S_k(\Gamma_1(MDp^r))$ consisting of those forms which are new at all the primes dividing D. Hence for all $r \geq 1$ there is a canonical projection $\mathfrak{h}_r^1 \to \mathfrak{h}_r^D$ which restricts to the ordinary parts for, and thus we get a canonical map $\mathfrak{h}_\infty^{1,\mathrm{ord}} \to \mathfrak{h}_\infty^{D,\mathrm{ord}}$. Now, as above, there is a splitting

$$
\mathfrak{h}_\infty^{D,\mathrm{ord}} \otimes_\Lambda \mathcal{L} \simeq \left(\bigoplus_{j \in J} \mathcal{F}_j \right) \oplus \mathcal{M}
$$

where $\mathcal{F}_j$ are finite field extensions of $\mathcal{L}$ and $\mathcal{M}$ is non-reduced. Since the morphism associated with f factors through $\mathfrak{h}_\infty^{D,\mathrm{ord}}$, it must factor through some $\mathcal{F} \in \{\mathcal{F}_j\}_{j \in J}$ which is canonically isomorphic to $\mathcal{K}$. Summing up, we get a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{h}_\infty^{1,\mathrm{ord}} & \xrightarrow{\quad f_\infty \quad} & \mathcal{R} \\
& \searrow \qquad \nearrow{\scriptstyle f_\infty} & \\
& \mathfrak{h}_\infty^{D,\mathrm{ord}} &
\end{array}
$$

where we write f_∞ also for the factoring map and the unlabeled arrow is the canonical projection considered before.

For any topological $\mathcal{O}_f$-algebra R let

$$\mathcal{X}(R) := \mathrm{Hom}_{\mathcal{O}_f\text{-alg}}^{\mathrm{cont}}(R, \bar{\mathbb{Q}}_p)$$

denote the $\mathcal{O}_f$-module of continuous homomorphisms $R \to \bar{\mathbb{Q}}_p$ of $\mathcal{O}_f$-algebras. We call *arithmetic homomorphisms* those $\kappa \in \mathcal{X}(\mathcal{R})$ whose restriction to the canonical image of $W = 1 + p\mathbb{Z}_p$ in Λ has the form $x \mapsto \epsilon(x)x^k$ for an integer $k \geq 2$ and a finite order character ϵ of W. Write $\mathcal{A}(\mathcal{R})$ for the subset of $\mathcal{X}(R)$ consisting of the arithmetic homomorphisms. The kernel $P_\kappa \in \mathrm{Spec}(\mathcal{R})$ of a $\kappa \in \mathcal{A}(\mathcal{R})$ is called an *arithmetic prime*, and the residue field $F_\kappa := \mathcal{R}_{P_\kappa}/P_\kappa\mathcal{R}_{P_\kappa}$ is a finite extension of F_f. The composition $W \to \mathcal{R}^\times \to F_\kappa^\times$ has the form $\gamma \mapsto \psi_\kappa(\gamma)\gamma^{k_\kappa}$ for a finite order character $\psi_\kappa : W \to F_\kappa^\times$ and an integer $k_\kappa \geq 2$. We call ψ_κ the *wild character* of κ and k_κ the *weight* of κ.

Let $\kappa \in \mathcal{A}(\mathcal{R})$. If κ has weight $k = k_\kappa$ and character ϵ_κ then the composition

$$f_\kappa := \kappa \circ f_\infty : \mathfrak{h}_\infty^{D,\mathrm{ord}} \longrightarrow \bar{\mathbb{Q}}_p$$

corresponds by duality to a modular form (denoted by the same symbol)

$$f_\kappa \in S_k\big(\Gamma_0(Np^{m_\kappa}), \epsilon_\kappa, F_\kappa\big)$$

of weight k, conductor divisible by N, level $\Gamma_0(Np^{m_\kappa})$ where m_κ is the maximum between 1 and the order at p of the conductor of ψ_κ and character

$$\epsilon_\kappa := \epsilon\psi_\kappa\omega^{-(k-2)} : \mathbb{Z}_p^\times \longrightarrow \bar{\mathbb{Q}}_p^\times.$$

It is known that $f_\kappa \in S_k^{D\text{-new}}(\Gamma_1(MDp^r))$ for all $\kappa \in \mathcal{A}(\mathcal{R})$.

2.3. **Modular forms on quaternion algebras.** For any commutative ring R and any integer $n \geq 0$ let

$$P_n(R) := \mathrm{Sym}^n(R)$$

denote the R-module of degree n homogeneous polynomials in two variables with coefficients in R. It is equipped with a right action of the group $\mathrm{GL}_2(R)$ by the rule

$$(P|\gamma)(x,y) := P(ax+by, cx+dy) \qquad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

The R-linear dual $V_n(R)$ of $P_n(R)$ is then endowed with a left action of $\mathrm{GL}_2(R)$ by the formula

$$(\gamma\phi)(P) := \phi(P|\gamma).$$

Finally, if F is a splitting field for B we may fix an isomorphism $i_F : B \otimes_\mathbb{Q} F \simeq \mathrm{M}_2(F)$, and then $P_n(F)$ (respectively, $V_n(F)$) is equipped with a right (respectively, left) action of $B^\times$ via i_F. In the applications, F will be either $\mathbb{Q}_p$ or $\mathbb{R}$, so that we can (and do) choose i_F to be i_p or i_∞, respectively.

For every integer $r \geq 0$ let X_r denote the compact Shimura curve $\Gamma_r\backslash\mathcal{H}$ and write $\mathfrak{h}_{r,k}^D$ for the image of $\mathcal{H}(\Gamma_r, \Sigma_r^+)$ in the endomorphism algebra of $H^1\big(X_r, V_{k-2}(\mathbb{C})\big)$. Let g be a cusp form of level Γ_r and weight k which is a Hecke eigenform and denote $\lambda : \mathfrak{h}_{r,k}^D \to \mathbb{C}$ the

corresponding ring homomorphism. Let F_g be a subfield of $\mathbb{C}$ containing the image of λ, let F/F_g be a field extension splitting B and fix an isomorphism i_F as above. Define

$$H^1(X_r, V_{k-2}(F))^{g,\pm} := \left\{ \xi \in H^1(X_r, V_{k-2}(F)) \mid \xi|T = \lambda(T)\xi \text{ for all } T \in \mathfrak{h}_{r,k} \text{ and } \xi|\iota = \pm\xi \right\}.$$

Thanks to a result of Matsushima and Shimura ([18]), we know that

$$\dim_L \left(H^1(X_r, V_{k-2}(F))^{g,\pm} \right) = 1.$$

Recall that there is a canonical isomorphism

$$H^1(X_r, V_{k-2}(\mathbb{C})) \simeq H^1(\Gamma_r, V_{k-2}(\mathbb{C}))$$

which is equivariant for the action of the involution ι. If $\tau \in \mathcal{H}$ then the complex vector space $H^1(\Gamma_r, V_{k-2}(\mathbb{C}))^{g,\pm}$ is spanned by the projection on the $\pm$-eigenspace for ι of the cohomology class represented by the cocycle $\gamma \mapsto \omega(g)_\gamma$ given by

$$(1) \qquad\qquad \omega(g)_\gamma(P(x,y)) := \int_\tau^{\gamma(\tau)} g(z)P(z,1)dz$$

(the class does not depend on the choice of the base point $\tau \in \mathcal{H}$). For details, see [20, §8.2].

Now let $g \in S_k^{D\text{-new}}(\Gamma_1(MDp^r))$. The Jacquet–Langlands correspondence associates with g a modular form g^{JL} of weight k on Γ_r, which is well defined only up to a non-zero scalar factor. As above, let F be a splitting field for B containing the eigenvalues of the Hecke operators acting on g (and so also on g^{JL}). Then for any sign $\pm$ we may choose a multiple $g_\pm^{\mathrm{JL}}$ of g^{JL} in such a way the projection to the $\pm$-eigenspace of the cohomology class represented by the cocycle $\gamma \mapsto \omega(g_\pm^{\mathrm{JL}})_\gamma$ as in (1) generates $H^1(\Gamma_r, V_{k-2}(F))^{g,\pm}$.

Let F be a subfield of $\mathbb{C}$ containing F_f via the fixed embedding $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. By a slight abuse of notation, we use the symbol $\mathfrak{h}_{r,k}^D$ also to denote the image of $\mathcal{H}(\Gamma_r, \Sigma_r^+) \otimes_{\mathbb{Z}} \mathcal{O}_f$ in the endomorphism algebra of $H^1(X_r, V_{k-2}(F))$.

For lack of a convenient reference, we prove a generalization of [14, Theorem 7.2].

Proposition 2.2. *For every choice of sign $\pm$ the $\mathfrak{h}_{r,k}^D \otimes_{\mathcal{O}_f} F$-module $H^1(\Gamma_r, V_{k-2}(F))^\pm$ is free of rank 1.*

Proof. The main result of [18] shows that the map

$$g \longmapsto \left(\gamma \mapsto \Re(\omega(g)_\gamma) \right),$$

where $\Re$ denotes the real part of a complex number, induces an $\mathbb{R}$-linear isomorphism between $S_k(\Gamma_r)$ and $H^1(\Gamma_r, V_{k-2}(\mathbb{R}))$ (see also [20, Theorem 8.4]). One can rephrase this theorem by saying that there is an isomorphism

$$(2) \qquad\qquad H^1(\Gamma_r, V_{k-2}(\mathbb{C})) \simeq S_k(\Gamma_r) \oplus \bar{S}_k(\Gamma_r)$$

where $\bar{S}_k(\Gamma_r)$ is the complex conjugate of the image of $S_k(\Gamma_r)$ in $H^1(\Gamma_r, V_{k-2}(\mathbb{C}))$ under the map ω introduced in (1) (cf. [14, Section 2] and [14, Theorem 6.2]). This isomorphism is compatible with the Hecke action. There is a hermitian positive definite bilinear pairing

$$(\,,\,): S_k(\Gamma_r) \times S_k(\Gamma_r) \longrightarrow \mathbb{C}$$

defined by

$$(f,g) := \int_{\Gamma \backslash \mathcal{H}} f(z)\overline{g(z)}y^{k-2}dxdy, \qquad z = x + iy$$

(see, e.g., [20, §8.2]) which, using (2), induces isomorphisms

$$H^1(\Gamma_r, V_{k-2}(\mathbb{C}))^\pm \simeq \mathrm{Hom}_{\mathbb{C}}(H^1(\Gamma_r, V_{k-2}(\mathbb{C}))^\mp, \mathbb{C}).$$

Recall that, thanks to [12, Proposition 3.1], there is a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}^1_{r,k} \otimes_{\mathcal{O}_f} \mathbb{C}, \mathbb{C}) \simeq S_k(\Gamma_1(MDp^r)).$$

A morphism in the left hand side factors through $\mathfrak{h}^D_{r,k}$ if and only if the corresponding modular form is new at all the primes dividing D, and thus we obtain a non-canonical isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}^D_{r,k} \otimes_{\mathcal{O}_f} \mathbb{C}, \mathbb{C}) \simeq S_k(\Gamma_r)$$

(here we fix an isomorphism $S^{D\text{-new}}_k(\Gamma_1(MDp^r)) \simeq S_k(\Gamma_r)$). Therefore we get an isomorphism

$$\mathfrak{h}^D_{r,k} \otimes_{\mathcal{O}_f} \mathbb{C} \simeq H^1\big(\Gamma_r, V_{k-2}(\mathbb{C})\big)^{\pm}$$

for each choice of sign $\pm$. Now the universal coefficient theorem shows that

$$H^1\big(\Gamma_r, V_{k-2}(\mathbb{C})\big)^{\pm} \simeq H^1\big(\Gamma_r, V_{k-2}(F)\big)^{\pm} \otimes_F \mathbb{C}.$$

On the other hand, $\mathfrak{h}^D_{r,k} \otimes_{\mathcal{O}_f} \mathbb{C} \simeq (\mathfrak{h}^D_{r,k} \otimes_{\mathcal{O}_f} F) \otimes_F \mathbb{C}$, and the result follows because $\mathbb{C}$ is fully faithful over F. $\qquad\square$

2.4. **Measure-valued cohomology groups.** Fix a finite extension F of $\mathbb{Q}_p$ and let $\mathcal{O}$ denote its ring of integers. Denote $\tilde{\mathbb{D}}(\mathcal{O})$ the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Y} := \mathbb{Z}_p^2$ and by $\mathbb{D}(\mathcal{O})$ the $\mathcal{O}$-submodule of $\tilde{\mathbb{D}}(\mathcal{O})$ consisting of measures which are supported on the subset $\mathbb{X}$ of primitive vectors of $\mathbb{Y}$ (i.e., those vectors which are not divisible by p). Define

$$\mathbb{W}(\mathcal{O}) := H^1(\Gamma_0, \mathbb{D}(\mathcal{O})).$$

If $\mathcal{O} = \mathcal{O}_f$, we simply write $\tilde{\mathbb{D}}$, $\mathbb{D}$ and $\mathbb{W}$ for the corresponding objects. In what follows, we identify Γ_r and Δ_r with their image in $\mathrm{M}_2(\mathbb{Z}_p)$ via i_p. Then, since the action of $\mathbb{Z}'$ extends to a continuous action of $\mathbb{Z}_p^{\times}$, it follows from the discussion in §2.1 that $\mathbb{W}(\mathcal{O})$ is a $\tilde{\Lambda}$-module endowed with a canonical right action of $\mathfrak{h}^D_{\mathrm{univ}}[\iota]$. For any continuous $\tilde{\Lambda}$-algebra R we also adopt the notation $\mathbb{W}(\mathcal{O})_R := \mathbb{W}(\mathcal{O}) \otimes_{\tilde{\Lambda}} R$.

As an application of Shapiro's lemma, we get an isomorphism of $\mathbb{Z}_p$-modules

(3) $$\mathbb{W}(\mathcal{O}) \simeq \varprojlim_r H^1(\Gamma_r, \mathcal{O})$$

(for details, see [15, Proposition 7.6]). Applying [1, Lemma 2.2 (b)], we also see that (3) is an isomorphism of $\mathfrak{h}^D_{\mathrm{univ}}[\iota]$-modules.

As in [2, Section 5], if A is a compact $\mathbb{Z}_p$-module and $T : A \to A$ is a continuous homomorphism then the *ordinary submodule of A with respect to T* is

$$A^{\mathrm{ord}} := \bigcap_{n=1}^{\infty} T^n(A).$$

It follows that A^{ord} is the largest submodule of A on which T acts invertibly. If A is a profinite abelian group and T is equal to a limit of operators on the finite quotients of A then there is a canonical decomposition

$$A = A^{\mathrm{ord}} \oplus A^{\mathrm{nil}}$$

where the subgroup A^{nil} on which T acts topologically nilpotently is the set of $a \in A$ such that $\lim_{n \to \infty} T^n(a) = 0$ (see [11, Proposition 2.3]).

Remark 2.3. In the sequel, T will always be the Hecke operator at p.

Since each $H^1(\Gamma_r, \mathcal{O})$ is a profinite group, so is $\mathbb{W}(\mathcal{O})$ thanks to (3). Therefore, by specializing the above discussion to $A = \mathbb{W}$ and $T = T_p$, we can define

$$\mathbb{W}(\mathcal{O})^{\mathrm{ord}} := \bigcap_{n=1}^{\infty} \mathbb{W}(\mathcal{O})|T_p^n$$

and obtain a decomposition $\mathbb{W}(\mathcal{O}) = \mathbb{W}(\mathcal{O})^{\mathrm{ord}} \oplus \mathbb{W}(\mathcal{O})^{\mathrm{nil}}$.

The action of a Hecke operator $T = \Gamma_0 \alpha \Gamma_0 = \coprod_i \Gamma_0 \alpha_i$ on a class $\boldsymbol{\Phi} \in \mathbb{W}(\mathcal{O})$ can be described as follows. Fix a representative Φ of $\boldsymbol{\Phi}$. Then for any $\gamma \in \Gamma_0$ and any continuous function φ on $\mathbb{Y}$ one has

$$(4) \qquad \begin{aligned} (\Phi|T)_\gamma(\varphi) &= \int_{\mathbb{X}} \varphi(x,y) d(\Phi|T)_\gamma = \sum_i \int_{\mathbb{X}} \varphi(x,y) \alpha_i^* d\Phi_{t_i(\gamma)} = \sum_i \int_{\mathbb{X}} \varphi(\alpha_i^*(x,y)) d\Phi_{t_i(\gamma)} \\ &= \sum_i \int_{\alpha_i^* \mathbb{X}} \varphi(x,y) d\Phi_{t_i(\gamma)} = \sum_i \int_{\alpha_i^* \mathbb{X} \cap \mathbb{X}} \varphi(x,y) d\Phi_{t_i(\gamma)}, \end{aligned}$$

where the last equality follows from the fact that Φ is supported on $\mathbb{X}$ and, as usual, the $t_i(\gamma)$ are defined by the equations $\Gamma \alpha_i \gamma = \Gamma \alpha_j$ (for some j) and $\alpha_i \gamma = t_i(\gamma) \alpha_j$.

We are interested in making the action of $T_p \in \mathfrak{h}_{\mathrm{univ}}^D$ explicit. By [19, Theorem 5.3.5], the operator T_p gives rise to a coset decomposition

$$(5) \qquad T_p = \coprod_{a \in \{0, \dots, p-1, \infty\}} \Gamma_0 g_a$$

where $i_p(g_\infty) = u_\infty \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $i_p(g_i) = u_i \begin{pmatrix} 1 & a_i \\ 0 & p \end{pmatrix}$ for some $u_\infty, u_i \in \mathrm{GL}_2(\mathbb{Z}_p)$ and some integers a_i forming a complete system of representatives of $\mathbb{Z}/p\mathbb{Z}$.

2.5. **Specialization maps.** Before going on, let us observe a simple fact. Let $\epsilon : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ be a character factoring through $(\mathbb{Z}/p^r\mathbb{Z})^\times$ for an integer $r \geq 1$, let L be finite field extension of $\mathbb{Q}_p$ containing F and the values of ϵ and let $\mathcal{O}_L$ be its ring of integers. Moreover, fix an element $\boldsymbol{\nu} \in H^1(\Gamma_0, \mathbb{D}(\mathcal{O}))$ and choose a representative ν of $\boldsymbol{\nu}$.

Lemma 2.4. *Fix an even integer $n \geq 0$ and let $\mathbb{U} \subset \mathbb{X}$ be such that $\gamma \mathbb{U} = \mathbb{U}$ for all $\gamma \in \Gamma_r$. Then the function*

$$\gamma \longmapsto \left(P \mapsto \int_{\mathbb{U}} \epsilon(y) P(x,y) d\nu_\gamma \right)$$

defined on Γ_r with values in $V_n(\mathcal{O}_L)$ is a 1-cocycle whose class in $H^1(\Gamma_r, V_n(\mathcal{O}_L))$ does not depend on the choice of ν.

Proof. We know that

$$\int_{\mathbb{U}} \epsilon(y) P(x,y) d\nu_{\sigma\tau} = \int_{\mathbb{U}} \epsilon(y) P(x,y) d\nu_\sigma + \int_{\mathbb{U}} \epsilon(y) P(x,y) d\sigma\nu_\tau$$

for all $\sigma, \tau \in \Gamma_r$. Since $\sigma^{-1}\mathbb{U} = \mathbb{U}$, the above equation also yields

$$\int_{\mathbb{U}} \epsilon(y) P(x,y) d\nu_{\sigma\tau} = \int_{\mathbb{U}} \epsilon(cx + dy) P(\sigma(x,y)) d\nu_\sigma + \int_{\mathbb{U}} \epsilon(y) P(x,y) d\nu_\tau$$

where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\epsilon(cx + dy) = \epsilon(y)$, this shows that the considered function is a 1-cocycle.

To complete the proof, let ν and ν' be representatives of $\boldsymbol{\nu}$; thus there exists $m \in \mathbb{D}(\mathcal{O})$ such that $\nu_\gamma = \nu'_\gamma + \gamma(m) - m$ for all $\gamma \in \Gamma_0$. Then

$$(6) \qquad \begin{aligned} \int_{\mathbb{U}} \epsilon(y) P(x,y) d\nu_\gamma &= \int_{\mathbb{U}} \epsilon(y) P(x,y) d\nu'_\gamma \\ &+ \int_{\gamma^{-1}\mathbb{U}} \epsilon(cx + dy) P(\gamma(x,y)) dm - \int_{\mathbb{U}} \epsilon(y) P(x,y) dm \end{aligned}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_r$. Let $v \in V_n(\mathcal{O}_L)$ be defined by

$$v(P) := \int_{\mathbb{U}} \epsilon(y) P(x,y) dm.$$

Then, by definition, $(\gamma \cdot v)(P(x,y)) = v\big(P(\gamma(x,y))\big)$. Since $\gamma\mathbb{U} = \mathbb{U}$ for all $\gamma \in \Gamma_r$ and $\epsilon(cx + dy) = \epsilon(y)$, the result follows from (6). $\qquad\square$

As in Lemma 2.4, let ϵ be a character of $\mathbb{Z}_p^\times$ factoring through $(\mathbb{Z}/p^r\mathbb{Z})^\times$ for some integer $r \geq 1$. Extend ϵ multiplicatively to $\mathbb{Z}_p$ by setting $\epsilon(p) = 0$. For an integer $n \geq 0$ and a field extension L of $\mathbb{Q}_p$ containing the values of ϵ define the *specialization map*

$$\rho_{n,\epsilon} = \rho_{n,\epsilon,L} : \mathbb{D}(\mathcal{O}) \longrightarrow V_n(\mathcal{O}_L)$$

by setting

$$\rho_{n,\epsilon}(\nu)(P) := \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon(y)P(x,y)d\nu.$$

Since $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ is stable under the action of $\Gamma_0^D(Mp)$, Lemma 2.4 ensures that if $\boldsymbol{\Phi} \in \mathbb{W}(\mathcal{O})$ and Φ is a 1-cocycle representing $\boldsymbol{\Phi}$ then the class in $H^1(\Gamma_r, V_n(\mathcal{O}_L))$ of the cocycle $\gamma \mapsto \rho_{n,\epsilon}(\Phi_\gamma)$ is independent of the choice of Φ. This class will be denoted by $\rho_{n,\epsilon}(\boldsymbol{\Phi})$.

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p) \cap \mathrm{M}_2(\mathbb{Z}_p)$ such that $a \in \mathbb{Z}_p^\times$ and $c \equiv 0 \pmod{p}$. Then for any $\nu \in \mathbb{D}(\mathcal{O})$ one has

$$\begin{aligned}
\rho_{n,\epsilon}(\gamma^* \cdot \nu)(P) &= \int_{\mathbb{Y}} \chi_{\mathbb{Z}_p \times \mathbb{Z}_p^\times}(x,y)\epsilon(y)P(x,y)d(\gamma^* \cdot \nu)(x,y) \\
&= \int_{\mathbb{Y}} \chi_{\mathbb{Z}_p \times \mathbb{Z}_p^\times}(dx - by, -cx + ay)\epsilon(-cx + ay)P(dx - by, -cx + ay)d\nu(x,y) \\
&= \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon(a)\epsilon(y)P(dx - by, -cx + ay)d\nu(x,y) \\
&= \epsilon(a)\rho_{n,\epsilon}(\nu)(P|\gamma^*) = \epsilon(a)\big(\gamma^* \cdot \rho_{n,\epsilon}(\nu)\big)(P),
\end{aligned}$$

whence

$$\rho_{n,\epsilon}(\gamma^* \cdot \nu) = \epsilon(a)\big(\gamma^* \cdot \rho_{n,\epsilon}(\nu)\big).$$

Note that we have used the condition $p|c$ twice: first to obtain $\epsilon(-cx + ay) = \epsilon(ay)$ and then to get $\gamma^*(\mathbb{Z}_p \times \mathbb{Z}_p^\times) = \mathbb{Z}_p \times \mathbb{Z}_p^\times$.

Recall that if $p \nmid n$ then the Hecke operators T_n and $T_{n,n}$ and the involution ι can be written as $\coprod_i \Gamma_0\alpha_i$ and $\coprod_i \Gamma_r\alpha_i$ for the same α_i. Comparing with (4), this shows that $\rho_{n,\epsilon}$ is compatible with the action of the Hecke operators T_n and $T_{n,n}$ for $p \nmid n$ and with the action of ι. For the operators T_p and U_p, we observe that

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^* \mathbb{X} \cap (\mathbb{Z}_p \times \mathbb{Z}_p^\times) = \emptyset.$$

Comparing again with (4), we conclude that

$$(7) \qquad\qquad \rho_{n,\epsilon}(\boldsymbol{\Phi}|T_p) = \rho_{n,\epsilon}(\boldsymbol{\Phi})|U_p.$$

Therefore, by passing to cohomology and restricting from Γ_0 to Γ_r, we obtain an $\mathfrak{h}_{\mathrm{univ}}^D[\iota]$-equivariant map

$$(8) \qquad\qquad \rho_{n,\epsilon} : \mathbb{W}(\mathcal{O}) \longrightarrow H^1\big(\Gamma_r, V_n(\mathcal{O}_L)\big),$$

denoted by the same symbol. Taking ordinary submodules in (8) yields an $\mathfrak{h}_{\mathrm{univ}}^D[\iota]$-equivariant map

$$\rho_{n,\epsilon}^{\mathrm{ord}} : \mathbb{W}(\mathcal{O})^{\mathrm{ord}} \longrightarrow H^1\big(\Gamma_r, V_n(\mathcal{O}_L)\big)^{\mathrm{ord}}.$$

Keeping in mind that $H^1(\Gamma_r, V_n(\mathcal{O}_L))$ is a compact $\mathbb{Z}_p$-module, we define

$$H^1\big(\Gamma_r, V_n(L)\big)^{\mathrm{ord}} := H^1\big(\Gamma_r, V_n(\mathcal{O}_L)\big)^{\mathrm{ord}} \otimes_{\mathcal{O}_L} L,$$

where the ordinary submodule on the right hand side is defined with respect to the operator U_p. It follows that U_p acts invertibly on $H^1(\Gamma_r, V_n(L))^{\mathrm{ord}}$.

2.6. **The Control Theorem.**

2.6.1. *The kernel of the specialization map.* We begin by studying the specialization map introduced before and computing its kernel. The main result of this §2.6.1 is Proposition 2.16. Recall that we fix the following data: an even integer $n \geq 0$, a character $\epsilon : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ factoring through $(\mathbb{Z}/p^r\mathbb{Z})^\times$ for some integer $r \geq 1$, and a finite extension F of $\mathbb{Q}_p$ – whose ring of integers we denote $\mathcal{O}$ – containing both F_f and the values of ϵ.

For any integer $m \geq 1$ and any character $\chi : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ define the function $\psi_{m,\chi} : \mathbb{X} \to \mathbb{Z}_p$ by

$$\psi_{m,\chi}((x,y)) := \begin{cases} \chi(y) & \text{if } (x,y) \in \mathbb{U}(m), \\ 0 & \text{otherwise.} \end{cases}$$

If $\chi : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ is a character then a continuous function $\varphi : \mathbb{X} \to \mathbb{Z}_p$ is *homogeneous of degree* χ if $\varphi(t(x,y)) = \chi(t)\varphi(x,y)$ for all $t \in \mathbb{Z}_p^\times$. Clearly, $\psi_{m,\chi}$ is homogeneous of degree χ for all integers $m \geq 1$.

Lemma 2.5. *Let $\chi : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ be the homomorphism defined by $\chi(t) = \epsilon(t)t^n$ for an integer $n \geq 0$ and a character ϵ factoring through $(\mathbb{Z}/p^r\mathbb{Z})^\times$ for some integer $r \geq 1$. Let $\boldsymbol{\Phi} \in \mathbb{W}(\mathcal{O})$ and let $\kappa \in \mathcal{A}(\tilde{\Lambda})$ be of weight $k := n+2$ and character ϵ, so that the restriction of κ to $\mathbb{Z}_p^\times$ coincides with χ. Then the following conditions are equivalent:*

(1) $\boldsymbol{\Phi} \in P_\kappa \mathbb{W}(\mathcal{O})$;
(2) $\boldsymbol{\Phi}$ *can be represented by a cocycle* $\Phi \in Z^1(\Gamma_0, P_\kappa \mathbb{D}(\mathcal{O}))$;
(3) $\boldsymbol{\Phi}$ *can be represented by a cocycle* $\Phi \in Z^1(\Gamma_0, \mathbb{D}(\mathcal{O}))$ *such that*

$$\int_{\mathbb{X}} \varphi(x,y) d\Phi_\gamma(x,y) = 0$$

for all homogeneous functions $\varphi : \mathbb{X} \to \mathbb{Z}_p$ of degree χ and all $\gamma \in \Gamma_0$;
(4) $\boldsymbol{\Phi}$ *can be represented by a cocycle* $\Phi \in Z^1(\Gamma_0, \mathbb{D}(\mathcal{O}))$ *such that*

$$\int_{\mathbb{X}} \psi_{m,\chi}(x,y) d\Phi_\gamma(x,y) = 0$$

for all integers $m \geq 1$ and all $\gamma \in \Gamma_0$.

Proof. The ideal P_κ is principal, generated by $[\gamma] - \kappa(\gamma)$ where γ is a topological generator of $1 + p\mathbb{Z}_p$. By [2, Lemma 1.2], it follows that $P_\kappa \mathbb{W}(\mathcal{O}) = H^1(\Gamma_0, P_\kappa \mathbb{D}(\mathcal{O}))$, and this shows the equivalence of (1) and (2). The equivalence of (2) and (3) follows directly from [1, Lemma (6.3)], and clearly (3) implies (4). To complete the proof, it remains to show that (4) implies (3), so suppose that (4) is true. For all $\gamma_1, \gamma_2 \in \Gamma_0$ we have $\Phi_{\gamma_1\gamma_2} = \gamma_1 \cdot \Phi_{\gamma_2} + \Phi_{\gamma_1}$, so, thanks to (4), there is an equality

$$\int_{\mathbb{X}} \psi_{m,\chi}(x,y) d(\gamma_1 \cdot \Phi_{\gamma_2}) = \int_{\mathbb{X}} \psi_{m,\chi}(x,y) d\Phi_{\gamma_1\gamma_2} - \int_{\mathbb{X}} \psi_{m,\chi}(x,y) d\Phi_{\gamma_1} = 0.$$

Therefore

$$(9) \qquad \int_{\mathbb{X}} \psi_{m,\chi}(\gamma_1(x,y)) d\Phi_{\gamma_2} = 0$$

for all $\gamma_1, \gamma_2 \in \Gamma_0$. An easy argument shows that every function φ which is homogeneous of degree χ is the uniform limit of a sequence of linear combinations of functions of the form $(x,y) \mapsto \psi_{m,\chi}(\gamma(x,y))$ for $m \geq 1$ and $\gamma \in \Gamma_0$. Thus (3) follows from this fact and (9). $\square$

Now we need to slightly change notations in order to use arguments borrowed from [2]. To do this, fix also a projective resolution $F_\bullet = \{F_k\}_k$ of $\mathbb{Z}$ by left $\mathcal{O}[\Gamma_0]$-modules. In the following it will be important to observe that, since $\mathcal{O}[\Gamma_0]$ is a free $\mathcal{O}[\Gamma_r]$-module (of finite rank), $F_\bullet$ is also a projective resolution of $\mathbb{Z}$ in the category of left $\mathcal{O}[\Gamma_r]$-modules. Moreover,

we notice that, under the assumptions of the above lemma, condition (4) in Lemma 2.5 is equivalent to the following

(4') $\boldsymbol{\Phi}$ can be represented by a cocycle $\Phi \in \mathrm{Hom}_{\Gamma_0}\big(F_1, \mathbb{D}(\mathcal{O})\big)$ such that

$$\int_{\mathbb{X}} \psi_{m,\chi}(x,y) d\Phi(f)(x,y) = 0$$

for all integers $m \geq 1$ and all $f \in F_1$.

Now we are going to manipulate Lemma 2.5 by using the fixed resolution $F_\bullet$. For any integer $m \geq 1$ define the open sets

(10) $$\mathbb{U}(m) := \big\{(x,y) \in \mathbb{X} \mid x \equiv 0 \pmod{p^m}\big\}.$$

and

$$\mathbb{V}(m) := \big\{(x,y) \in \mathbb{X} \mid y \equiv 0 \pmod{p^m}\big\}.$$

Let $V_n(\mathcal{O})$ be the $\mathcal{O}$-linear dual of $P_n(\mathcal{O})$. Define a map $\sigma_{n,\epsilon}^{(m)} : \mathbb{D}(\mathcal{O}) \to V_n(\mathcal{O})$ as

$$\sigma_{n,\epsilon}^{(m)}(\nu) := \left(P \mapsto \int_{\mathbb{V}(m)} \epsilon(x) P(x,y) d\nu\right).$$

Let $V_{n,\epsilon}^{(m)}(\mathcal{O})$ denote the image of $\sigma_{n,\epsilon}^{(m)}$. One immediately verifies that $\gamma\mathbb{V}(m) = \mathbb{V}(m)$ for all $\gamma \in \Gamma_m$. The same argument as in Lemma 2.4 (simply replace the condition $\epsilon(cx + dy) = \epsilon(y)$ with the condition $\epsilon(ax + by) = \epsilon(x)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_r$, which is true for our choice of $\mathbb{V}(m)$) yields a well-defined map

$$\sigma_{n,\epsilon}^{(m)} : \mathbb{W}(\mathcal{O}) := H^1\big(\Gamma_0, \mathbb{D}(\mathcal{O})\big) \longrightarrow H^1\big(\Gamma_m, V_n(\mathcal{O})\big).$$

This is obtained, as above, by fixing a representative Φ of $\boldsymbol{\Phi} \in \mathbb{W}(\mathcal{O})$ and defining $\sigma_{n,\epsilon}^{(m)}(\boldsymbol{\Phi})$ to be the class represented by the cocycle $\sigma_{n,\epsilon}^{(m)}(\Phi)$.

The next results (Lemma 2.6–Lemma 2.13) explain how to translate in our setting the results of [2, Section 7].

Lemma 2.6. *Let $\boldsymbol{\Phi} \in \mathbb{W}(\mathcal{O})$ and suppose that $\sigma_{n,\epsilon}^{(m)}(\boldsymbol{\Phi}) = 0$. Then, up to replacing $\mathcal{O}$ with a finite unramified extension, $\boldsymbol{\Phi}$ can be represented by a cocycle Φ such that $\sigma_{n,\epsilon}^{(m)}(\Phi) = 0$ as a cochain in $\mathrm{Hom}_{\Gamma_m}(F_1, V_n(\mathcal{O}))$.*

Proof. Let I denote the induced module

$$I := \mathrm{Ind}_{\Gamma_m}^{\Gamma_0}\big(V_{n,\epsilon}^{(m)}(\mathcal{O})\big) = \big\{\phi : \Gamma_0 \to V_{n,\epsilon}^{(m)}(\mathcal{O}) \mid \phi(\gamma x) = \gamma\phi(x) \text{ for all } x \in \Gamma \text{ and } \gamma \in \Gamma_m\big\}.$$

Here we view Γ_m as a subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. We make I into a left Γ_0-module by the formula $y\phi(x) := \phi(yx)$ for all $x, y \in \Gamma_0$. We have a map

(11) $$\psi : \mathbb{D}(\mathcal{O}) \longrightarrow I, \qquad \psi(\nu)(x) := \sigma_{n,\epsilon}^{(m)}(x\nu).$$

Recall that, thanks to [2, Lemma 7.1], if ψ is surjective and $a \in B^1\big(\Gamma_0, V_{n,\epsilon}^{(m)}(\mathcal{O})\big)$ – where we adopt the usual notation $B^1(G, M)$ for the group of 1-coboundaries of the discrete left G-module M – then we can choose a coboundary $b \in B^1\big(\Gamma_0, \mathbb{D}(\mathcal{O})\big)$ such that $\sigma_{n,\epsilon}^{(m)}(b) = a$ as cochains. (Note that we are adopting a slightly different formalism with respect to [2], where *right* modules are used. However, [2, Lemma 7.1] is still true for *left* modules if one converts a right action $m \mapsto m|x$ into a left one $m \mapsto x \cdot m$ by the formula $x \cdot m := m|x^{-1}$.) Then to prove the result it is enough to show that ψ is surjective because, if this is true, then $\sigma_{n,\epsilon}^{(m)}(\Phi) = a$ is a coboundary and hence, applying the above result, we may choose $b \in B^1\big(\Gamma, \mathbb{D}(\mathcal{O})\big)$ such that $\sigma_{n,\epsilon}^{(m)}(b) = a$, whence $\sigma_{n,\epsilon}^{(m)}(\Phi - b) = 0$ as a cochain.

We are thus reduced to showing that ψ in (11) is surjective. To do this, fix an $\mathcal{O}$-basis $\mathcal{B}$ of $V_{n,\epsilon}^{(m)}(\mathcal{O})$. Since the image of ψ is a Γ_0-submodule of $V_{n,\epsilon}^{(m)}(\mathcal{O})$, it is enough to show that,

for any $b \in \mathcal{B}$, the function sending 1 to b and $\Gamma_0 - \Gamma_m$ to 0 belongs to the image of ψ. Now for any $\nu \in \mathbb{D}(\mathcal{O})$ and any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0$ we have

$$\psi(\mu)(\gamma)(P) = \sigma_{n,\epsilon}^{(m)}(\gamma\nu)(P) = \int_{\mathbb{V}(m)} \epsilon(x)P(x,y)d\gamma\nu = \int_{\mathbb{X}} \chi_{\mathbb{V}(m)}(\gamma(x,y))\epsilon(ax+by)P(\gamma(x,y))d\nu$$

where $\chi_{\mathbb{V}(m)}$ is the characteristic function of $\mathbb{V}(m)$. If ν is the Dirac measure supported at a point $(x_0, y_0) \in \mathbb{V}(m)$ then

$$\psi(\nu)(\gamma)(P) = \chi_{\mathbb{V}(m)}((ax_0 + by_0, cx_0 + dy_0))\epsilon(ax_0 + by_0)P(ax_0 + by_0, cx_0 + dy_0).$$

Comparing with the definition of $\mathbb{V}(m)$, one immediately checks that $\psi(\nu)(\gamma)(P) \neq 0$ only if $c \equiv 0 \pmod{p^m}$, so only if $\gamma \in \Gamma_m$. Furthermore, for $\gamma = 1$ we have

$$\psi(\nu)(1)(P) = \chi_{\mathbb{V}(m)}((x_0, y_0))\epsilon(x_0)P(x_0, y_0) = \epsilon(x_0)P(x_0, y_0).$$

Fix now $b = \sigma_{n,\epsilon}^{(m)}(\nu')$ and write $b(x^i y^j) = a_{i,j}$. One immediately verifies that $a_{i,j} \in p^{mj}\mathcal{O}$. Since $\epsilon(x_0) \in \mathcal{O}^\times$, we are reduced to showing that for any set $\{a'_{i,j}\} \subset \mathcal{O}$ with i, j non-negative integers such that $i + j = n$ there are $(x_0, y_0), \ldots, (x_t, y_t) \in \mathcal{O}^\times \times \mathcal{O}$ such that $a'_{i,j} = \sum_{k=0}^{t} \alpha_k x_k^i y_k^j$ for suitable $\alpha_k \in \mathcal{O}$. To do this we show that, for example, we can find $(x_0, y_0), \ldots, (x_t, y_t)$ as above with $t = \binom{n}{2}$ such that the determinant of the matrix $\left(x_k^{n-i} y_k^i\right)_{i,k=0,\ldots,t}$ belongs to $\mathcal{O}^\times$. Replacing $\mathcal{O}$ with the ring of integers of a sufficiently large unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$ we can find $(\bar{x}_0, \bar{y}_0), \ldots, (\bar{x}_t, \bar{y}_t) \in \mathbb{F}^\times \times \mathbb{F}$ with $t = \binom{n}{2}$ such that the determinant of the matrix $\left(\bar{x}_k^{n-i} \bar{y}_k^i\right)_{i,k=0,\ldots,t}$ is in $\mathbb{F}^\times$, and lifting these pairs to $\mathcal{O}^\times \times \mathcal{O}$ concludes the proof. $\qquad\square$

Remark 2.7. The condition that $\mathcal{O}$ be large enough is used only in the last step of the proof. In fact, we need the residue field of $\mathcal{O}$ to be sufficiently large. We believe that the result is still true without replacing $\mathcal{O}$, but at present we cannot find a simple proof of this fact.

We need a general description of Hecke operators in terms of cochains; the following discussion is taken from [9, p. 116]. Let M be a $\mathcal{O}$-module endowed with a left action of $\mathrm{GL}_2(\mathbb{Q}_p)$. Fix an integer $m \geq 0$, let $\alpha \in \mathrm{GL}_2(\mathbb{Q}_p)$ be such that the groups $\Gamma_m \cap \alpha\Gamma_m\alpha^{-1}$ and $\alpha^{-1}\Gamma_m\alpha$ are commensurable with Γ_m, then set $\Gamma_m^{(\alpha)} := \Gamma_m \cap \alpha\Gamma_m\alpha^{-1}$ and $\Gamma_m^{(\alpha^{-1})} := \Gamma_m \cap \alpha^{-1}\Gamma_m\alpha$. The Hecke operator $T(\alpha)$ on $H^1(\Gamma_m, M)$ is defined as the composition

$$H^1(\Gamma_m, M) \xrightarrow{\mathrm{res}} H^1\big(\Gamma_m^{(\alpha)}, M\big) \xrightarrow{\mathrm{conj}_\alpha} H^1\big(\Gamma_m^{(\alpha^{-1})}, M\big) \xrightarrow{\mathrm{cores}} H^1(\Gamma_m, M),$$

where

- res and cores are the usual restriction and corestriction maps;
- conj_α is the map taking a cocycle $\gamma \mapsto c_\gamma$ to the cocycle $\gamma \mapsto \alpha^* c_{\alpha\gamma\alpha^{-1}}$.

An easy formal computation (which can be found, e.g., in [21, Proposition 3.1] for congruence subgroups) shows that this action agrees with the one already defined in §2.1. This allows us to describe Hecke actions in terms of our fixed projective resolution, obtaining [2, Formulae 4.3]: if $z \in \mathrm{Hom}_{\Gamma_m}(F_k, M)$ and α is as above then $z|T(\alpha)$ is represented by the cochain

$$f \longmapsto \sum_i \alpha_i^* z\big(\tau(\gamma_i f)\big),$$

where

- the α_i are elements in $\mathrm{GL}_2(\mathbb{Q}_p)$ giving rise to the coset decomposition

$$\Gamma_m \alpha \Gamma_m = \coprod_i \Gamma_m \alpha_i;$$

- the γ_i are coset representatives for $(\alpha^{-1}\Gamma_m\alpha \cap \Gamma_m)\backslash\Gamma_m$;

- τ is a homotopy equivalence between the two resolutions $F_\bullet$ and $F'_\bullet$ of $\alpha^{-1}\Gamma_m\alpha$, where $F'_\bullet$ has the same underlying groups as $F_\bullet$ but the group action is defined by $(\alpha^{-1}\gamma\alpha)f'_k := \gamma f_k$.

We begin by recalling the following application of the Approximation Theorem.

Lemma 2.8. *Suppose that $m \geq 1$. There are $\pi_m \in R_0^D(M)$ of norm p^m and $w \in \Gamma_0$ such tha*

(1) $\pi_m w$ *normalizes* Γ_m;
(2) $(\pi_m w)^2 \in p^m \Gamma_m$.

Proof. To simplify notations, put $R := R_0^D(Mp^m)$ and $R_\ell := R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ for all primes ℓ. By the Approximation Theorem ([19, Theorem 5.2.10]), one can find elements $\pi_m \in R$ of norm p^m and $w \in \Gamma_0$ such that $i_\ell(\pi_m) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^m \end{smallmatrix}\right)$ modulo $Mp^m R_\ell$ and $i_\ell(w) \equiv \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ modulo $Mp^m R_\ell$ for all primes $\ell|Mp$. Then one has $i_\ell(\pi_m w) \equiv \left(\begin{smallmatrix} 0 & 1 \\ -p^m & 0 \end{smallmatrix}\right)$ modulo $Mp^m R_\ell$ for all $\ell|Mp$. Since $i_\ell(\pi_m w)$ belongs to the normalizer of R_ℓ for all primes ℓ, this shows (1). Furthermore, $(\pi_m w)^2 \equiv p^m$ modulo $Mp^m R_\ell$ for all primes $\ell|Mp$. The element $p^{-m}(\pi_m w)^2$ is congruent to 1 modulo $Mp^m R_\ell$ for all primes $\ell|Mp^m$, has determinant 1 and belongs to R_ℓ for all ℓ. Therefore $p^{-m}(\pi_m w)^2 \in \Gamma_m$, and (2) is proved. $\square$

Thanks to part (2) of Lemma 2.8, write

$$(\pi_m w)^2 = p^m \gamma_0$$

with $\gamma_0 \in \Gamma_m$. One can define the operator X_m acting on $\mathrm{Hom}_{\Gamma_m}(F_\bullet, M)$ by the formula

$$(z|X_m)(f) := \sum_{i=0}^{p^m-1} (w^{-1}\gamma_{m,i})^* z(w^{-1}\gamma_{m,i}f)$$

where the $\gamma_{m,i}$ are coset representatives for $(\pi^{-m}\Gamma_m\pi^m \cap \Gamma_m)\backslash\Gamma_m$. If we let

$$\Gamma_m \pi_m \Gamma_m = \coprod_{i=0}^{p^m-1} \Gamma_m g_{m,i}$$

then $\pi_m \gamma_{m,i} = g_{m,i}$ (cf., e.g., [20, Proposition 3.1]).

Now define the operator $W_m := \Gamma_m \pi_m w \Gamma_m$. Then

$$(12) \qquad W_m T_{p^m} = \Gamma_m \pi_m w \pi_m \Gamma_m = \Gamma_m p^m \gamma_0 w^{-1} \Gamma_m = p^m \Gamma_m w^{-1} \Gamma_m = p^m X_m$$

where the last equality follows from part (1) of Lemma 2.8.

Lemma 2.9. *In $\Phi \in \mathrm{Hom}_{\Gamma_0}(F_1, \mathbb{D}(\mathcal{O}))$ then $p^m \rho_{n,\epsilon}(\Phi) = \sigma_{n,\epsilon}^{(m)}(\Phi)|X_m$, where the equality holds at the cochain level. Hence, in cohomology, one has $p^m \rho_{n,\epsilon}(\mathbf{\Phi}) = \sigma_{n,\epsilon}^{(m)}(\mathbf{\Phi})W_m T_{p^m}$, for all $\mathbf{\Phi} \in \mathbb{W}(\mathcal{O})$.*

Proof. The last statement is immediate from (12) and the first one, which we are going to prove. A direct computation shows that

$$\left(\sigma_{n,\epsilon}^{(m)}(\Phi)|X_m\right)\left(P(x,y)\right) = \left(\sum_{i=0}^{p^m-1}(w^{-1}\gamma_{m,i})^*\sigma_{n,\epsilon}^{(m)}\left(\Phi(w^{-1}\gamma_{m,i}f)\right)\right)\left(P(x,y)\right)$$

$$= \sum_{i=0}^{p^m-1}\int_{\mathbb{V}(m)}\epsilon(x)P\left((w^{-1}\gamma_{m,i})^*(x,y)\right)d\Phi(w^{-1}\gamma_{m,i}f)$$

$$= \sum_{i=0}^{p^m-1}\int_{\mathbb{V}(m)}\epsilon(x)P\left((w^{-1}\gamma_{m,i})^*(x,y)\right)w^{-1}\gamma_{m,i}d\Phi(f),$$

where the last equality follows from the Γ_0-equivariance of Φ (note that $w^{-1}\gamma_{m,i} \in \Gamma_0$). Now $(w^{-1}\gamma_{m,i})(w^{-1}\gamma_{m,i})^* = 1$. We thus obtain

$$\big(\sigma_{n,\epsilon}^{(m)}(\Phi)|X_m\big)\big(P(x,y)\big) = \sum_{i=0}^{p^m-1} \int_{\gamma_{m,i}^{-1} w \mathbb{V}(m)} \epsilon\big(w^{-1}\gamma_{m,i}(x)\big)P(x,y)d\Phi(f).$$

But the family $\big\{\gamma_{m,i}^{-1} w \mathbb{V}(m)\big\}_i$ is a partition of $\mathbb{Z}_p \times \mathbb{Z}_p^\times$, hence

$$\big(\sigma_{n,\epsilon}^{(m)}(\Phi)|X_m\big)\big(P(x,y)\big) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon(y)P(x,y)d\Phi(f),$$

as was to be shown. $\qquad\square$

Lemma 2.10. *If $\Phi \in \mathrm{Hom}_{\Gamma_0}(F_1, \mathbb{D}(\mathcal{O}))$ then*

$$\sigma_{n,\epsilon}^{(m)}\big(\Phi|T(w^{-1}\pi_m w)\big) = w^{-1}\pi_m^* \rho_{n,\epsilon}(\Phi'),$$

where $\Phi'(f) = w\Phi(\tau(f))$ and $\tau : F_\bullet \to F_\bullet$ is a homotopy equivalence satisfying

$$\tau(\gamma f) = (w^{-1}\pi_m w)\gamma(w^{-1}\pi_m w)^{-1}\tau(f)$$

for $\gamma \in (w^{-1}\pi_m w)^{-1}\Gamma_0(w^{-1}\pi_m w) \cap \Gamma_0 = \Gamma_m$.

Proof. First we check that $w^{-1}\pi_m^*\rho_{n,\epsilon}^{(m)}(\Phi')$ is Γ_m-equivariant. Thanks to the Γ_0-equivariance of Φ and the definition of τ, we have

$$\Phi'(\gamma f) = w\Phi(\tau(\gamma f)) = ww^{-1}\pi_m w\gamma w^{-1}(\pi_m)^{-1}w\Phi(\tau(f))$$
$$= (\pi_m w)\gamma(\pi_m w)^{-1}w\Phi(\tau(f)) = (\pi_m w)\gamma(\pi_m w)^{-1}\Phi'(f).$$

Since $\rho_{n,\epsilon}$ is Γ_m-equivariant, $\pi_m\pi_m^* = p^m$, and $\pi_m w$ normalizes Γ_m by (1) in Lemma 2.8, it follows that

$$w^{-1}\pi_m^*\rho_{n,\epsilon}(\Phi')(\gamma f) = w^{-1}\pi_m^*\rho_{n,\epsilon}\big((\pi_m w)\gamma(\pi_m w)^{-1}\Phi'(f)\big) = \gamma\rho_{n,\epsilon}\big(w^{-1}\pi^*\Phi'(f)\big).$$

Recall that $\Gamma_0\pi_m\Gamma_0 = \coprod_i \Gamma_0 g_{m,i}$ with $g_{m,i} = \pi_m\gamma_{m,i}$. Thus $\Gamma_0 w^{-1}\pi_m w\Gamma_0$ is the disjoint union of the $\Gamma_0 w^{-1}g_{i,m}w$ and $w^{-1}g_{m,i}w = w^{-1}\pi_m ww^{-1}\gamma_{m,i}$. By definition,

$$\big(\Phi|T(w^{-1}\pi_m w)\big)(f) = \sum_i (w^{-1}g_{m,i}w)^*\Phi\big(\tau(w^{-1}\gamma_{m,i}wf)\big),$$

hence

$$\sigma_{n,\epsilon}^{(m)}\big(\Phi|T(w^{-1}\pi_m w)\big)(f) = \sum_i \int_{\mathbb{V}_m} \epsilon(x)P(x,y)d\Big(\big(w^{-1}g_{m,i}w\big)^*\Phi\big(\tau(w^{-1}\gamma_{m,i}wf)\big)\Big)$$

$$= \sum_i \int_{\mathbb{X} \cap (w^{-1}g_{m,i}^*w)^{-1}\mathbb{V}_m} \epsilon\big(w^{-1}g_{m,i}^*w(x)\big)P\big(w^{-1}g_{m,i}^*w(x,y)\big)d\Big(\Phi\big(\tau(w^{-1}\gamma_{m,i}wf)\big)\Big).$$

Now $w\mathbb{Y}_m = \mathbb{U}_m$ and $g_{m,i}^*\mathbb{X} \cap \mathbb{X} \neq \emptyset$ if and only if $g_{m,i} = \pi_m$, in which case $\pi_m\mathbb{X} \subset \mathbb{U}_m$ and the corresponding $\gamma_{m,i}$ is equal to 1 (a similar argument will also be used in the proof of Lemma 2.14). Hence $\mathbb{X} \cap (w^{-1}g_{m,i}^*w)^{-1}\mathbb{V}_m = w^{-1}\mathbb{X}$. Finally, notice that $\epsilon\big(w^{-1}\pi_m^*w(x)\big) = \epsilon(x)$ for (x,y) in the domain of integration. Therefore the above sum is equal to

$$\int_{w^{-1}\mathbb{X}} \epsilon(x)P\big(w^{-1}\pi_m^*w(x,y)\big)d\big(\Phi(\tau(f))\big) = \int_{\mathbb{X}} \epsilon(y)P\big(w^{-1}\pi_m^*(x,y)\big)d\big(w\Phi(\tau(f))\big)$$

(note that $\epsilon(y) \neq 0$ if and only if $y \in \mathbb{Z}_p^\times$, so the integral is actually computed over $\mathbb{Z}_p \times \mathbb{Z}_p^\times$), and the proof is complete. $\qquad\square$

Lemma 2.11. *If $\rho_{n,\epsilon}(\mathbf{\Phi}) = 0$ then $\sigma_{n,\epsilon}^{(m)}(\mathbf{\Phi}|T_p^m) = 0$.*

Proof. The operator $\Gamma_0 w^{-1}\pi_m w\Gamma_0$ is nothing other than $\Gamma_0\pi_m\Gamma_0$ because $w \in \Gamma_0$. Let us represent $\mathbf{\Phi}$ by a cocycle $\Phi \in \operatorname{Hom}_{\Gamma_0}(F_1, \mathbb{D}(\mathcal{O}))$. By assumption, we can find a cochain $b \in \operatorname{Hom}_{\Gamma_m}(F_0, \mathbb{D}(\mathcal{O}))$ such that

$$b(df)(P(x,y)) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon(y)P(x,y)d\Phi(f)(x,y)$$

for all $f \in F_1$. It follows from Lemma 2.10 and the Γ_0-equivariance of Φ that $\sigma_{n,\epsilon}^{(m)}(\Phi|T(\pi^m))$ is represented by the functional sending a polynomial P to

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon(y)P\big(w^{-1}\pi^*(x,y)\big)dw\Phi(\tau(f)) = b(dw\tau(f))\big(P(w^{-1}\pi^*(x,y))\big).$$

This shows that $\sigma_{n,\epsilon}^{(m)}(\Phi|T(\pi_m))$ is represented by the coboundary db' where b' sends a polynomial P to

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} \epsilon(y)P\big(w^{-1}\pi^*(x,y)\big)b(w\tau(f)).$$

To complete the proof, we need to check that b' is Γ_m-equivariant. This follows as in the first paragraph of the proof of Lemma 2.10 by formally replacing Φ with b and Φ' with b'. $\square$

Lemma 2.12. *If $\mathbf{\Phi}$ is ordinary and $\rho_{n,\epsilon}(\mathbf{\Phi}) = 0$ then $\sigma_{n,\epsilon}^{(m)}(\mathbf{\Phi}) = 0$.*

Proof. Choose $\mathbf{\Psi}$ such that $\mathbf{\Phi} = \mathbf{\Psi}|T_p^m$. Since $\rho_{n,\epsilon}(\mathbf{\Phi}) = 0$, we also have that $\mathbf{\Psi}$ is ordinary and $\rho_{n,\epsilon}(\mathbf{\Psi}) = 0$ (this argument will be used again in the proof of Lemma 2.14). Lemma 2.11 then implies that $\sigma_{n,\epsilon}^{(m)}(\mathbf{\Phi}) = \sigma_{n,\epsilon}^{(m)}(\mathbf{\Psi}|T_p^m) = 0$. Finally, it can be checked that b' is Γ_m-equivariant, which completes the proof. $\square$

Lemma 2.13. *Let $\mathbf{\Phi} \in \mathbb{W}(\mathcal{O})^{\mathrm{ord}}$ and suppose that $\rho_{n,\epsilon}(\mathbf{\Phi}) = 0$. If the residue field of $\mathcal{O}$ is sufficiently large then $\mathbf{\Phi}$ can be represented by a cocycle Φ such that $\rho_{n,\epsilon}(\Phi) = 0$ as a cochain in $\operatorname{Hom}_{\Gamma_r}(F_1, V_n(\mathcal{O}))$.*

Proof. Since $\rho_{n,\epsilon}(\mathbf{\Phi}) = 0$, it follows from Lemma 2.12 that $\sigma_{n,\epsilon}^{(m)}(\mathbf{\Phi}) = 0$. By Lemma 2.6 one can choose a representative Φ of $\mathbf{\Phi}$ such that $\sigma_{n,\epsilon}^{(m)}(\Phi) = 0$ as a cochain. Now Lemma 2.9 shows that

$$\rho_{n,\epsilon}(\Phi) = \rho_{n,\epsilon}(\Phi)|wT_p^{-m} = 0$$

in $\operatorname{Hom}_{\Gamma_r}(F_1, V_n(\mathcal{O}))$. $\square$

We are now going to combine Lemma 2.5 with Lemma 2.13 to study the kernel of the specialization map.

Lemma 2.14. *Fix $\kappa \in \mathcal{A}(\tilde{\Lambda})$ of weight k and a character ϵ factoring through $(\mathbb{Z}/p^r\mathbb{Z})^\times$ for some integer $r \geq 1$. Set $n := k - 2$. Suppose that the residue field of $\mathcal{O}$ is sufficiently large, so that Lemma 2.13 can be applied. Then the map $\rho_{n,\epsilon}^{\mathrm{ord}}$ induces an injective, $\mathfrak{h}_{\mathrm{univ}}^D[\iota]$-equivariant homomorphism of $\tilde{\Lambda}/P_\kappa\tilde{\Lambda}$-modules*

$$\rho_{n,\epsilon}^{\mathrm{ord}} : \mathbb{W}(\mathcal{O})^{\mathrm{ord}}/P_\kappa\mathbb{W}(\mathcal{O})^{\mathrm{ord}} \hookrightarrow H^1\big(\Gamma_r, V_n(\mathcal{O})\big)^{\mathrm{ord}}.$$

Proof. Since the integrand $\epsilon(y)P(x,y)$ appearing in $\rho_{n,\epsilon}^{\mathrm{ord}}$ is homogeneous of degree κ, the inclusion $\ker(\rho_{n,\epsilon}^{\mathrm{ord}}) \supset P_\kappa\mathbb{W}(\mathcal{O})^{\mathrm{ord}}$ follows from the implication $(1) \Rightarrow (3)$ in Lemma 2.5.

Now we show the opposite inclusion. Let $\mathbf{\Phi} \in \ker(\rho_{n,\epsilon}^{\mathrm{ord}})$ and represent it by a cocycle in $\Phi \in \operatorname{Hom}_{\Gamma_0}(F_1, \mathbb{D}(\mathcal{O}))$. Fix an integer $m \geq 1$, choose $\mathbf{\Psi} \in \mathbb{W}(\mathcal{O})^{\mathrm{ord}}$ such that $\mathbf{\Psi}|T_p^m = \mathbf{\Phi}$ (this is possible because T_p induces an isomorphism on $\mathbb{W}(\mathcal{O})^{\mathrm{ord}}$) and represent $\mathbf{\Psi}$ by a cocycle Ψ. Write T_p^m as

$$T_p^m = \coprod_i \Gamma_0 g_{m,i}$$

where the $g_{m,i}$ are suitable products of m elements, not necessarily distinct, chosen in the set $\{g_a\}_{a=0,\dots,p-1,\infty}$ defined in (5). Therefore, for all $f \in F_1$ we have

$$\int_{\mathbb{X}} \psi_{m,\kappa}(x,y) d\Phi(f) = \sum_i \int_{\mathbb{X}} \psi_{m,\kappa}\big(g_{m,i}^*(x,y)\big) d\Psi(f_i),$$

where the f_i are suitable elements in F_1 which can be made explicit using the definition of the Hecke actions given in terms of elements in $Z^1(\Gamma_0, \mathbb{D}(\mathcal{O}))$ and the identification between $Z^1(\Gamma_0, \mathbb{D}(\mathcal{O}))$ and $\mathrm{Hom}_{\Gamma_0}(F_1, \mathbb{D}(\mathcal{O}))$; however, we will not need this description in the following.

Now $\psi_{m,\kappa}\big(g_{m,i}^*(x,y)\big) = 0$ unless $g_{m,i} = \big(\begin{smallmatrix} 1 & 0 \\ 0 & p^m \end{smallmatrix}\big)$. More precisely, since $\Psi(f_i)$ is supported on $\mathbb{X}$ and $\psi_{m,\kappa}$ is supported on the set $\mathbb{U}(m)$ defined in (10), the above integral does not vanish only if $\mathbb{U}(m) \cap \mathbb{X} \neq \emptyset$. An easy calculation shows that $g_{m,i}^* \mathbb{X} \cap \mathbb{X} \neq \emptyset$ if and only if $g_{m,i} = \big(\begin{smallmatrix} 1 & 0 \\ 0 & p^m \end{smallmatrix}\big)$, and in this case one has $\big(\begin{smallmatrix} 1 & 0 \\ 0 & p^m \end{smallmatrix}\big)^* \mathbb{X} \subset \mathbb{U}(m)$. Hence the i-th summand in the above sum is equal to

$$\int_{\mathbb{X}} \psi_{\kappa,m}(p^m x, y) d\Psi(f_i) = \int_{\mathbb{X}} \kappa(y) d\Psi(f_i) = \int_{\mathbb{X}} \epsilon(y) y^n d\Psi(f_i).$$

On the other hand, by (7) there are equalities

$$0 = \rho_{n,\epsilon}^{\mathrm{ord}}(\Phi) = \rho_{n,\epsilon}^{\mathrm{ord}}(\Psi|T_p^m) = \rho_{n,\epsilon}^{\mathrm{ord}}(\Psi)|U_p^m.$$

Since U_p acts invertibly on $H^1(\Gamma_r, V_n(F_f))^{\mathrm{ord}}$, it follows that $\rho_{n,\epsilon}^{\mathrm{ord}}(\Psi) = 0$.

By invoking Lemma 2.13, we choose a representative Ψ of Ψ such that $\rho_{n,\epsilon}(\Psi) = 0$ in $\mathrm{Hom}_{\Gamma_r}(F_1, V_n(F_\kappa))$, and then we conclude that

$$\int_{\mathbb{X}} \epsilon(y) y^n d\Psi(f) = 0$$

for all $f \in F_1$. Define $\Phi_m := \Psi|T_p^m$, which is a representative of $\Phi = \Psi|T_p^m$. Then, since the map $\rho_{n,\epsilon}$ is compatible with the action of T_p, we conclude that $\rho_{n,\epsilon}(\Phi_m) = 0$ in $\mathrm{Hom}_{\Gamma_r}(F_1, \mathbb{D}(\mathcal{O}))$, and so we get

$$\int_{\mathbb{X}} \psi_{m,\kappa}(x,y) d\Phi_m(f) = 0$$

for all $f \in F_1$. Since $\mathrm{Hom}_{\Gamma_0}(F_1, \mathbb{D}(\mathcal{O}))$ is compact, we can assume that the sequence $(\Psi_m)_{m \geq 1}$ has a limit, which we denote Ψ_∞. But the coboundary map is continuous, hence Ψ_∞ is also a cocycle. Since coboundaries form a compact subspace of the group of cocycles, Ψ_∞ still represents Ψ. Finally, since the topology on the space of measures is induced by pointwise convergence on continuous functions, we see that

$$\int_{\mathbb{X}} \psi_{m,\kappa}(x,y) d\Phi_\infty(f) = 0$$

for all $f \in F_1$ and all integers $m \geq 1$. From the equivalence of conditions (4') above and (4) in Lemma 2.5 and the implication (4) $\Rightarrow$ (1) in Lemma 2.5 it follows that $\Phi \in P_\kappa \mathbb{W}(\mathcal{O})^{\mathrm{ord}}$, as was to be shown. $\qquad\square$

Lemma 2.15. *Fix $\kappa \in \mathcal{A}(\tilde{\Lambda})$ of weight k and a character ϵ factoring through $(\mathbb{Z}/p^r\mathbb{Z})^\times$ for some integer $r \geq 1$. Set $n := k - 2$. The map $\rho_{n,\epsilon}^{\mathrm{ord}}$ induces an injective, $\mathfrak{h}_{\mathrm{univ}}^D[\iota]$-equivariant homomorphism of $\tilde{\Lambda}/P_\kappa \tilde{\Lambda}$-modules*

$$\rho_{n,\epsilon}^{\mathrm{ord}} : \mathbb{W}^{\mathrm{ord}}/P_\kappa \mathbb{W}^{\mathrm{ord}} \hookrightarrow H^1\big(\Gamma_r, V_n(\mathcal{O})\big)^{\mathrm{ord}}.$$

Proof. Fix $\mathcal{O}$ sufficiently large so that the above lemma can be applied. The extension $\mathcal{O}_f \subset \mathcal{O}$ is fully faithful. Moreover, there are isomorphisms

$$\mathbb{W} \otimes_{\mathcal{O}_f} \mathcal{O} = H^1(\Gamma_0, \mathbb{D}) \otimes_{\mathcal{O}_f} \mathcal{O} \simeq H^1(\Gamma_0 \backslash \mathcal{H}, \mathbb{D}) \otimes_{\mathcal{O}_f} \mathcal{O} \simeq H^1(\Gamma_0 \backslash \mathcal{H}, \mathbb{D}(\mathcal{O})) \simeq \mathbb{W}(\mathcal{O})$$

and

$$H^1\big(\Gamma_r, V_n(\mathcal{O}_f)\big) \otimes_{\mathcal{O}_f} \mathcal{O} \simeq H^1\big(\Gamma_r, V_n(\mathcal{O})\big)$$

coming from the universal coefficient theorem for cohomology (see [6, Theorem 15.3]) which are compatible with the action of Hecke operators. The result follows. $\square$

For any $\mathcal{O}_f[\mathfrak{h}_{\mathrm{univ}}^D]$-module M and any ring homomorphism $\vartheta : \mathfrak{h}_{\mathrm{univ}}^D \to R$ define

$$M^\vartheta := M \otimes_{\mathfrak{h}_{\mathrm{univ}}^D} R,$$

the tensor product being taken with respect to ϑ. Let $\kappa \in \mathcal{A}(\mathcal{R})$ and define the F_κ-vector space

$$\mathbb{W}_\kappa^{\mathrm{ord}} := \Big(H^1\big(\Gamma_{m_\kappa}, V_{n_\kappa}(F_\kappa)\big)^{\mathrm{ord}} \Big)^{f_\kappa}.$$

Let h_κ denote the composition of $h : \mathfrak{h}_{\mathrm{univ}}^D \to \mathfrak{h}_\infty^{D,\mathrm{ord}}$ with the canonical map $\mathfrak{h}_\infty^{D,\mathrm{ord}} \to \mathcal{R}_{P_\kappa}$. Then we can consider the $\mathcal{R}_{P_\kappa}$-submodule $\mathbb{W}_{h_\kappa}^{\mathrm{ord}}$ of $\mathbb{W}_{\mathcal{R}_{P_\kappa}}^{\mathrm{ord}}$ defined by

$$\mathbb{W}_{h_\kappa}^{\mathrm{ord}} := \big(\mathbb{W}_{\mathcal{R}_{P_\kappa}}^{\mathrm{ord}}\big)^{h_\kappa}.$$

The action of the involution ι on a $\mathbb{Q}_p$-vector space M induces a splitting $M = M^+ \oplus M^-$, where $M^\pm$ are the $\pm$-eigenspaces for ι.

Proposition 2.16. *Let $\kappa \in \mathcal{A}(\mathcal{R})$ have weight $k = k_\kappa$ and character $\epsilon = \epsilon_\kappa$. Set $n := k - 2$. The map $\rho_{n,\epsilon}^{\mathrm{ord}}$ of Lemma 2.14 induces an injective homomorphism of F_κ-vector spaces*

$$\rho_\kappa : \mathbb{W}_{h_\kappa}^{\mathrm{ord},\pm}/P_\kappa \mathbb{W}_{h_\kappa}^{\mathrm{ord},\pm} \hookrightarrow \mathbb{W}_\kappa^{\mathrm{ord},\pm}.$$

Proof. Define $\mathfrak{p}_\kappa := P_\kappa \cap \Lambda$. Using the canonical projection $\tilde{\Lambda} \to \Lambda$ we may view $\mathfrak{p}_\kappa$ as an element in $\mathcal{A}(\tilde{\Lambda})$. We deduce from Lemma 2.14 the existence of an injective homomorphism of $\tilde{\Lambda}/\mathfrak{p}_\kappa\tilde{\Lambda}$-modules

$$\rho_{n,\epsilon}^{\mathrm{ord}} : \mathbb{W}^{\mathrm{ord}}/\mathfrak{p}_\kappa\mathbb{W}^{\mathrm{ord}} \hookrightarrow H^1\big(\Gamma_r, V_n(\mathcal{O}_f)\big)^{\mathrm{ord}}.$$

Now $\tilde{\Lambda}/\mathfrak{p}_\kappa\tilde{\Lambda} \simeq \Lambda/\mathfrak{p}_\kappa\Lambda$, thus we get an injective homomorphism of Λ-modules

$$\rho_{n,\epsilon}^{\mathrm{ord}} : \mathbb{W}_\Lambda^{\mathrm{ord}}/\mathfrak{p}_\kappa\mathbb{W}_\Lambda^{\mathrm{ord}} \hookrightarrow H^1\big(\Gamma_r, V_n(\mathcal{O}_f)\big)^{\mathrm{ord}}.$$

Since $\Lambda_{\mathfrak{p}_\kappa}$ is flat over Λ, we also obtain an injective homomorphism of $\Lambda_{\mathfrak{p}_\kappa}/\mathfrak{p}_\kappa\Lambda_{\mathfrak{p}_\kappa}$-vector spaces

$$\rho_{n,\epsilon}^{\mathrm{ord}} : \mathbb{W}_{\Lambda_{\mathfrak{p}_\kappa}}^{\mathrm{ord}}/\mathfrak{p}_\kappa\mathbb{W}_{\Lambda_{\mathfrak{p}_\kappa}}^{\mathrm{ord}} \hookrightarrow H^1\big(\Gamma_r, V_n(F_f)\big)^{\mathrm{ord}}.$$

Now $\mathcal{R}_{P_\kappa}$ and $\Lambda_{\mathfrak{p}_\kappa}$ are normal domains and $\mathcal{R}_{P_\kappa}$, being unramified over $\Lambda_{\mathfrak{p}_\kappa}$ thanks to [13, Corollary 1.4], is flat over $\Lambda_{\mathfrak{p}_\kappa}$. Using the universal coefficient theorem for cohomology (see [6, Theorem 15.3]) and recalling that $H^1(\Gamma_r, V_n(F))$ is canonically isomorphic to $H^1(\Gamma_r\backslash\mathcal{H}, V_n(F))$, we get an isomorphism

$$H^1\big(\Gamma_r, V_n(F_f)\big) \otimes_{F_f} F_\kappa \simeq H^1\big(\Gamma_r, V_n(F_\kappa)\big),$$

from which we deduce an injective homomorphism of F_κ-vector spaces

$$\rho_\kappa^{\mathrm{ord}} : \mathbb{W}_{\mathcal{R}_{P_\kappa}}^{\mathrm{ord}}/P_\kappa\mathbb{W}_{\mathcal{R}_{P_\kappa}}^{\mathrm{ord}} \hookrightarrow H^1\big(\Gamma_r, V_n(F_\kappa)\big)^{\mathrm{ord}}.$$

Thanks to the $\mathfrak{h}_{\mathrm{univ}}^D[\iota]$-equivariance of $\rho_{n,\epsilon}^{\mathrm{ord}}$, restricting $\rho_\kappa^{\mathrm{ord}}$ to $\mathbb{W}_{h_\kappa}^{\mathrm{ord},\pm}$ gives the searched-for injection. $\square$

2.6.2. *Dimension bounds.* Now we compute the dimensions of the source and the target of the specialization map. Recall that $\mathcal{L}$ (respectively, $\mathcal{K}$) is the fraction field of Λ (respectively, $\mathcal{R}$) and that there is a canonical decomposition

$$\tag{13} \mathfrak{h}_\infty^{D,\mathrm{ord}} \otimes_\Lambda \mathcal{L} \simeq \mathcal{K} \oplus \mathcal{N}$$

where $\mathcal{N}$ is a sum of finitely many fields and of a non-reduced part. Define

$$h_\mathcal{R} : \mathfrak{h}_{\mathrm{univ}}^D \xrightarrow{h} \mathfrak{h}_\infty^D \longrightarrow \mathfrak{h}_\infty^{D,\mathrm{ord}} \xrightarrow{f_\infty} \mathcal{R}.$$

Proposition 2.17. *The module $\mathbb{W}_\mathcal{K}^{\mathrm{ord}}$ is a 2-dimensional vector space over $\mathcal{K}$ and each eigenmodule for ι has dimension 1. Moreover, the action of $\mathfrak{h}_{\mathrm{univ}}^D$ on $\mathbb{W}_\mathcal{K}^{\mathrm{ord}}$ factors through $h_\mathcal{R}$.*

Proof. For every integer $r \geq 1$ let X_r denote the compact Shimura curve $\Gamma_r \backslash \mathcal{H}$. Define the p-divisible abelian group

$$\mathcal{V} := \varinjlim_r H^1(X_r, F_f/\mathcal{O}_f)^{\mathrm{ord}} = \varinjlim_r H^1(\Gamma_r, F_f/\mathcal{O}_f)^{\mathrm{ord}},$$

where the inductive limit is taken with respect to the restriction maps. The Hecke operators T_n, $T_{n,n}$ and the involution ι act naturally on $\mathcal{V}$, since the Hecke action is compatible with the restriction maps (see [14, (2.9 a,b) and (3.5)]). Consider the eigenmodules $\mathcal{V}^\pm$ for ι and define V to be the Pontryagin dual of one of $\mathcal{V}^\pm$. Thanks to [14, Corollary 10.4], we know that V is free of finite rank over Λ.

Replacing [14, eq. (7.6)] with Proposition 2.2, we can mimic the proof of [14, Theorem 12.1] and show that there is an isomorphism $V \otimes_\Lambda \Lambda_{P_\kappa} \simeq \mathfrak{h}_\infty^{D,\mathrm{ord}} \otimes_\Lambda \Lambda_{P_\kappa}$ for all $\kappa \in \mathcal{A}(\Lambda)$. Now the Pontryagin dual of $\mathcal{V}$ is identified with the inverse limit of the cohomology groups $H^1(\Gamma_r, \mathcal{O}_f)^{\mathrm{ord}}$ with respect to the corestriction maps. On the other hand, Shapiro's Lemma (3) is equivariant for the action of $\mathfrak{h}_{\mathrm{univ}}^D$. Therefore $\mathbb{W}^{\mathrm{ord}} \otimes_\Lambda \Lambda_{P_\kappa}$ is free of rank 2 over $\mathfrak{h}_\infty^{D,\mathrm{ord}} \otimes_\Lambda \Lambda_{P_\kappa}$ for all $\kappa \in \mathcal{A}(\Lambda)$, and each eigenmodule for ι is free of rank 1. Hence it follows that $\mathbb{W}_\mathcal{L}^{\mathrm{ord}}$ is free of rank 2 over $\mathfrak{h}_\infty^{D,\mathrm{ord}} \otimes_\Lambda \mathcal{L}$, and each eigenspace for ι is free of rank 1. In light of decomposition (13), the proof is complete. $\square$

2.6.3. *Control Theorem.* We are now ready to state and prove the main result of this section.

Theorem 2.18. *For every $\kappa \in \mathcal{A}(\mathcal{R})$ the map ρ_κ of Proposition 2.16 induces an isomorphism of 1-dimensional F_κ-vector spaces*

$$\rho_\kappa : \mathbb{W}_{h_\kappa}^{\mathrm{ord},\pm}/P_\kappa \mathbb{W}_{h_\kappa}^{\mathrm{ord},\pm} \xrightarrow{\simeq} \mathbb{W}_\kappa^{\mathrm{ord},\pm}.$$

Proof. Thanks to Proposition 2.16, we only need to show that

$$\dim_{F_\kappa}(\mathbb{W}_{h_\kappa}^{\mathrm{ord}}/P_\kappa \mathbb{W}_{h_\kappa}^{\mathrm{ord}}) \geq \dim_{F_\kappa}(\mathbb{W}_\kappa^{\mathrm{ord}}).$$

Since $\dim_{F_\kappa}(\mathbb{W}_\kappa^{\mathrm{ord}}) = 2$ as observed in §2.3, we are reduced to proving that

$$\dim_{F_\kappa}(\mathbb{W}_{h_\kappa}^{\mathrm{ord}}/P_\kappa \mathbb{W}_{h_\kappa}^{\mathrm{ord}}) \geq 2.$$

However, Proposition 2.17 shows that the intersection of $\mathbb{W}_{\mathcal{R}_\kappa}^{\mathrm{ord}}$ with the $h_\mathcal{R}$-eigenmodule of $\mathbb{W}_\mathcal{K}^{\mathrm{ord}}$ is a free $\mathcal{R}_{P_\kappa}$-module of rank 2, and we are done. $\square$

3. The definite case

In this section B is a *definite* quaternion algebra over $\mathbb{Q}$, whose discriminant $D > 1$ is then a square-free product of an *odd* number of primes.

3.1. **Modular forms on quaternion algebras.** We will often use notations and results from Section 2. For all primes $\ell \nmid D$ fix isomorphisms of $\mathbb{Q}_\ell$-agebras

$$i_\ell : B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Q}_\ell)$$

and a maximal order $R^{\max}$ in B such that $i_\ell(R^{\max} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) = \mathrm{M}_2(\mathbb{Z}_\ell)$. For all primes $\ell \nmid D$ and all integers $r \geq 1$ choose Eichler orders $R_0^D(Mp^r) \subset B$ of level Mp^r such that $i_\ell(R_0^D(Mp^r) \otimes \mathbb{Z}_\ell)$ is the order of $\mathrm{M}_2(\mathbb{Z}_\ell)$ consisting of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \equiv 0 \pmod{Mp^r}$. For every integer $r \geq 0$ define the compact open subgroup U_r of $\hat{B}^\times$ as

$$U_r := \left\{ (x_\ell)_\ell \in \hat{R}_0^D(Mp^r)^\times \mid i_\ell(x_\ell) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ with } a \equiv 1 \pmod{\ell^{\mathrm{ord}_\ell(Mp^r)}} \text{ for all } \ell | Mp^r \right\}.$$

We begin by recalling the definition of modular forms on B which can be found, e.g., in [5, Definition 2.1]; references are [5, Section 2], [7, Section 4] and [14, Section 2]. If A is a $\mathbb{Z}_p$-module equipped with a left linear action of $\mathrm{M}_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p)$ and U is a compact open subgroup of $\hat{B}^\times$ then an A-*valued modular form on B of level U* is an element of the A-module $S(U, A)$ of functions

$$s : \hat{B}^\times \longrightarrow A$$

such that

$$g(bgu) = i_p(u_p)^{-1} s(g) \qquad \text{for all } b \in B^\times,\ g \in \hat{B}^\times \text{ and } u \in U,$$

where u_p denotes the p-component of u. Therefore a modular form in $S(U, A)$ is completely determined by its values on the finite set

$$X(U) := B^\times \backslash \hat{B}^\times / U.$$

Finally, for $U = U_r$ set $X_r := X(U_r)$ and $S_k(U_r, A) := S(U_r, V_{k-2}(A))$.

Remark 3.1. As in [5], the above definition works for $\mathbb{Z}_p$-modules endowed with a *left* linear action of $\mathrm{GL}_2(\mathbb{Q}_p) \cap \mathrm{M}_2(\mathbb{Z}_p)$. The definition which can be found in [7] uses, on the contrary, *right* actions on A. Of course, the two definitions are compatible, as one sees by turning the right action in [7] into a left one via the formula $\gamma \cdot a := a | \gamma^*$.

Remark 3.2. The definition in [14] looks different from the ones in in [7] and [5]. The point is that in [14] the weight action on polynomials is concentrated in the archimedean place, while the above definition makes use of the place at p. However, the two notions are equivalent whenever we fix an embedding $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. For details, see [7, Section 4] and the references quoted there.

3.2. **Hecke algebras.** We review the theory of §2.1 in the adelic language, which is more suitable for applications to definite quaternion algebras (and for generalizations of the theory to the case of totally real fields).

For any integer $r \geq 0$ define

$$\Sigma_r := \left\{ (x_\ell)_\ell \in \hat{R}_0^D(Mp^r) \cap \hat{B}^\times \mid i_\ell(x_\ell) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ with } a \equiv 1 \pmod{\ell^{\mathrm{ord}_\ell(Mp^r)}} \text{ for all } \ell | Mp^r \right\}$$

and

$$\Delta_r := \left\{ (x_\ell)_\ell \in \hat{R}^{\max} \cap \hat{B}^\times \mid i_\ell(x_p) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \text{ with } a \equiv 1 \pmod{p^r} \right\}.$$

Now fix an integer $r \geq 0$. For every integer $n \geq 2$ there is a Hecke operator $T_n = \sum_i T(\alpha_i)$ in $\mathcal{H}(U_r, \Sigma_r)$ and $\mathcal{H}(U_r, \Delta_r)$, where the sum is taken over all double cosets $U_r \alpha_i U_r$ with $\alpha_i \in \Sigma_r$ and $\mathrm{norm}(\alpha_i)\hat{\mathbb{Z}} = n\hat{\mathbb{Z}}$ (here norm $: \hat{B} \to \hat{\mathbb{Q}}$ is the adelization of the norm map). There are also Hecke operators $T_{n,n}$ in $\mathcal{H}(U_r, \Sigma_r)$ and $\mathcal{H}(U_r, \Delta_r)$ for integers $n \geq 1$ prime to MDp^r, defined as follows. For any such n choose $\nu \in \hat{\mathbb{Z}} \cap \hat{\mathbb{Q}}^\times$ such that $n\hat{\mathbb{Z}} = \nu\hat{\mathbb{Z}}$ and $\nu - 1 \in Mp^r\hat{\mathbb{Z}}$, then define $T_{n,n} = U_r \nu U_r$. Finally, recall that $\mathcal{H}(U_r, \Sigma_r)$ is the commutative algebra generated over $\mathbb{Z}$ by the operators T_n and $T_{n,n}$.

Fix a finite field extension F of $\mathbb{Q}_p$ and denote $\mathcal{O}$ its ring of integers. An operator $T \in \mathcal{H}(U_r, \Delta_r)$ acts on $S_2(U_r, F)$ as follows. Write $T = \coprod_i U\alpha_i$ and define

$$(s|T)(g) := \sum_i s(g\alpha_i).$$

Let $\mathfrak{h}_r^D$ denote the image of $\mathcal{H}(U_r, \Sigma_r) \otimes_{\mathbb{Z}} \mathcal{O}$ in the endomorphism algebra of $S_2(U_r, F)$. As in §2.1, for $r \geq 1$ let $\mathfrak{h}_r^{D,\mathrm{ord}}$ denote the product of those local rings of $\mathfrak{h}_r^D$ where U_p is invertible and define $\mathfrak{h}_\infty^D$ and $\mathfrak{h}_\infty^{D,\mathrm{ord}}$ to be the inverse limits over $r \geq 1$ of the rings $\mathfrak{h}_r^D$ and $\mathfrak{h}_r^{D,\mathrm{ord}}$, respectively.

For any $a \in \mathbb{Z}'$ choose $\alpha \in \hat{\mathbb{Z}} \cap \hat{\mathbb{Q}}^\times$ such that $a\hat{\mathbb{Z}} = \alpha\hat{\mathbb{Z}}$ and $\alpha - 1 \in p^r\hat{\mathbb{Z}}$, then define $\langle a \rangle = U_r\alpha U_r$ in $\mathcal{H}(U_r, \Delta_r)$. The map $a \mapsto \langle a \rangle$ is multiplicative and thus extends to a ring homomorphism $\mathbb{Z}[\mathbb{Z}'] \to \mathcal{H}(\Gamma_r, \Delta_r)$. Since $\mathbb{Z}[\mathbb{Z}']$ embeds naturally in $\tilde{\Lambda}$, we may form the $\tilde{\Lambda}$-algebras

$$\mathcal{H}(p^r) := \mathcal{H}(U_r, \Delta_r) \otimes_{\mathbb{Z}[\mathbb{Z}']} \tilde{\Lambda}.$$

Now the Hecke pair (U_r, Δ_r) is weakly compatible (in the sense of [1, Definition 2.1]) to (U_0, Δ_0), hence, as explained in [1, §2], there is a canonical surjective $\tilde{\Lambda}$-algebra homomorphism

$$\rho_r : \mathcal{H}(1) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{H}(p^r)$$

for every integer $r \geq 1$.

We define the commutative $\tilde{\Lambda}$-algebra

$$\mathfrak{h}_{\mathrm{univ}}^D := \tilde{\Lambda}\big[\, T_n \text{ for every } n \geq 1 \text{ and } T_{n,n} \text{ for every } n \geq 1 \text{ with } (n, MD) = 1 \,\big] \subset \mathcal{H}(1).$$

The $\tilde{\Lambda}$-algebra $\mathfrak{h}_{\mathrm{univ}}^D$ acts compatibly on the $\mathbb{C}$-vector spaces $S_2(U_r)$, in the sense that the diagram of $\tilde{\Lambda}$-algebras

$$
\begin{array}{ccc}
\mathfrak{h}_{\mathrm{univ}}^D & \xrightarrow{\ \rho_r\ } & \mathcal{H}(p^r) \\
& {\scriptstyle \rho_{r-1}}\searrow & \downarrow \\
& & \mathcal{H}(p^{r-1})
\end{array}
$$

commutes for all $r \geq 1$ (here the vertical arrow is the canonical map arising from the weakly compatibility of the Hecke pairs (U_r, Δ_r) and (U_{r-1}, Δ_{r-1})). For all $r \geq 1$ the image of $\mathfrak{h}_{\mathrm{univ}}^D$ in the endomorphism algebra of $S_2(U_r, \mathbb{C})$ is canonically isomorphic to $\mathfrak{h}_r^D$, hence we obtain a canonical morphism
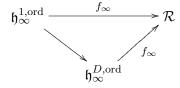
$$h : \mathfrak{h}_{\mathrm{univ}}^D \longrightarrow \mathfrak{h}_\infty^{D,\mathrm{ord}}.$$

3.3. **Hida families.** The Jacquet–Langlands correspondence (which, in this case, can be concretely established via the Eichler trace formula) ensures that $\mathfrak{h}_r^D$ is canonically isomorphic to the quotient of $\mathfrak{h}_r^1$ acting faithfully on the $\mathbb{C}$-vector space $S_2^{D\text{-new}}(\Gamma_1(MDp^r))$ of cusp forms of weight 2 and level $\Gamma_1(MDp^r)$ which are new at all the primes dividing D. We also fix a (non-canonical) isomorphism $S_2^{D\text{-new}}(\Gamma_1(MDp^r)) \simeq S_2(U_r, \mathbb{C})$. Therefore for all $r \geq 1$ there is a canonical projection $\mathfrak{h}_r^1 \to \mathfrak{h}_r^D$, which restricts to the ordinary parts, yielding a canonical map $\mathfrak{h}_\infty^{1,\mathrm{ord}} \to \mathfrak{h}_\infty^{D,\mathrm{ord}}$. As above, there is a splitting

$$\mathfrak{h}_\infty^{D,\mathrm{ord}} \otimes_\Lambda \mathcal{L} \simeq \left(\bigoplus_{j \in J} \mathcal{F}_j\right) \oplus \mathcal{M}$$

where the $\mathcal{F}_j$ are finite field extensions of $\mathcal{L}$ and $\mathcal{M}$ is non-reduced. Since the morphism associated with f factors through $\mathfrak{h}_\infty^{D,\mathrm{ord}}$, it must factor through some $\mathcal{F} \in \{\mathcal{F}_j\}_{j \in J}$ which is

canonically isomorphic to $\mathcal{K}$ (the primitive component through which the morphism associated with f factors). Summing up, we get a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{h}_\infty^{1,\mathrm{ord}} & \xrightarrow{\;\;f_\infty\;\;} & \mathcal{R} \\
& \searrow \qquad \nearrow_{f_\infty} & \\
& \mathfrak{h}_\infty^{D,\mathrm{ord}} &
\end{array}
$$

where we write f_∞ also for the factoring map and the unlabeled arrow is the canonical projection considered before.

3.4. Hecke action on modular forms.
For a field F, the Hecke action on $S_2(U_r, F)$ has been described above. Now we consider the Hecke action on $S_k(U_r, A)$ for general weights k and $\mathbb{Q}_p$-vector spaces A with a left linear $\mathrm{GL}_2(\mathbb{Q}_p) \cap \mathrm{M}_2(\mathbb{Z}_p)$-action.

Suppose that $\eta \in \Sigma_r$ and $s \in S_k(U_r, A)$, then set

$$(\eta \cdot s)(g) := i_p(\eta_p) s(g\eta).$$

Observe that

$$S_k(U_r, F) = \left\{ s : B^\times \backslash \hat{B}^\times \to V_{k-2}(F) \mid \eta \cdot s = s \text{ for all } \eta \in U_r \right\}.$$

Now, if $T = U_r \alpha U_r = \coprod_i \alpha_i U_r$ we define $T \cdot s := \sum_i \alpha_i \cdot s$.

Proposition 3.3. *For any subfield F of $\mathbb{C}$ containing F_f via the fixed embedding $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$ the $\mathfrak{h}_k^D \otimes_{\mathcal{O}_f} F$-module $S_k(U_r, F)$ is free of rank 1 if $k > 2$. If $k = 2$ then the same is true for the quotient of $S_2(U_r, F)$ by the subspace $S_2^{\mathrm{triv}}(U_r, F)$ consisting of those functions factoring through the norm map.*

Proof. In order to have uniform notations, for any field F define $\tilde{S}_k(U_r, F) := S_k(U_r, F)$ if $k > 2$ and $\tilde{S}_2(U_r, F) := S_2(U_r, F)/S_2^{\mathrm{triv}}(U_r, F)$. As before, fix a (non-canonical) isomorphism

$$\tag{14} S_k^{D\text{-new}}(\Gamma_1(MDp^r)) \simeq \tilde{S}_k(U_r, \mathbb{C})$$

(*cf.* [7, Theorem 2]). Thanks to [12, Proposition 3.1], there is a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{C}}\big(\mathfrak{h}_k^1 \otimes_{\mathcal{O}_f} \mathbb{C}, \mathbb{C}\big) \simeq S_k\big(\Gamma_1(MDp^r)\big).$$

Now a homomorphism in the left hand side factors through $\mathfrak{h}_k^D$ precisely when the corresponding cusp form is new at all the primes dividing D, and so, combining this fact with (14), we obtain a non-canonical isomorphism

$$\mathrm{Hom}_{\mathbb{C}}\big(\mathfrak{h}_k^D \otimes_{\mathcal{O}_f} \mathbb{C}, \mathbb{C}\big) \simeq \tilde{S}_k(U_r, \mathbb{C}).$$

Thanks to [13, Corollary 6.5], $\mathfrak{h}_r^1$ is a Frobenius algebra over $\mathbb{C}$ (actually, over any field $K \subset \mathbb{C}$), hence there is a canonical isomorphism of $\mathfrak{h}_r^1 \otimes_{\mathcal{O}_f} \mathbb{C}$-modules

$$\mathfrak{h}_r^1 \otimes_{\mathcal{O}_f} \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{C}}\big(\mathfrak{h}_r^1 \otimes_{\mathcal{O}_f} \mathbb{C}, \mathbb{C}\big).$$

Since $\mathfrak{h}_r^D \otimes_{\mathcal{O}_f} \mathbb{C}$ is a direct factor of $\mathfrak{h}_r^1 \otimes_{\mathcal{O}_f} \mathbb{C}$, one deduces that

$$\mathrm{Hom}_{\mathbb{C}}\big(\mathfrak{h}_r^D \otimes_{\mathcal{O}_f} \mathbb{C}, \mathbb{C}\big) \simeq \mathfrak{h}_r^D \otimes_{\mathcal{O}_f} \mathbb{C}$$

as $\mathfrak{h}_r^D \otimes \mathbb{C}$-modules. Therefore we get isomorphisms

$$\big(\mathfrak{h}_r^D \otimes_{\mathcal{O}_f} F\big) \otimes_F \mathbb{C} \simeq \mathfrak{h}_k^D \otimes_{\mathcal{O}_f} \mathbb{C} \simeq \tilde{S}_k(U_r, \mathbb{C}) \simeq \tilde{S}_k(U_r, F) \otimes_F \mathbb{C},$$

and the result follows because $\mathbb{C}$ is fully faithful over F. $\qquad\square$

3.5. **Measure-valued modular forms.** With notation as in §2.4, in our present context define

$$\mathbb{W} := S_2(U_0, \mathbb{D}).$$

Then $\mathbb{W}$ has a natural action of $\mathfrak{h}_{\mathrm{univ}}^D$ and, since

$$\mathbb{W} \simeq \varprojlim_r S_2(U_r, \mathcal{O}_f),$$

we can define its ordinary part $\mathbb{W}^{\mathrm{ord}}$ as in §2.4. Moreover, for any Λ-algebra R put

$$\mathbb{W}_R := \mathbb{W} \otimes_\Lambda R, \qquad \mathbb{W}_R^{\mathrm{ord}} := \mathbb{W}^{\mathrm{ord}} \otimes_\Lambda R.$$

The operator $T_p \in \mathfrak{h}_{\mathrm{univ}}^D$ gives rise to a coset decomposition

$$(15) \qquad\qquad T_p = \coprod_{a \in \{0,\dots,p-1,\infty\}} U_0 g_a$$

where $i_\ell(g_\infty) = i_\ell(g_i) = 1$ for all $\ell \neq p$ while $i_p(g_\infty) = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $i_p(g_i) = \left(\begin{smallmatrix} 1 & a_i \\ 0 & p \end{smallmatrix}\right)$ and the a_i are integers forming a complete system of representatives of $\mathbb{Z}/p\mathbb{Z}$.

3.6. **Specialization maps.** As in §2.5, for an even integer $n \geq 0$, a character $\epsilon : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ factoring through $(\mathbb{Z}/p^r\mathbb{Z})^\times$ for some integer $r \geq 1$ and a finite field extension $L/\mathbb{Q}_p$ containing the values of ϵ there is a *specialization map*

$$\rho_{n,\epsilon} = \rho_{n,\epsilon,L} : \mathbb{W} \longrightarrow S_k(U_r, L)$$

defined by

$$\rho_{n,\epsilon}(s)(g)(P) := \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} \epsilon(a) P(x,y) ds(g)$$

for all $g \in \hat{B}^\times$ and $P \in P_n(L)$.

Let $\gamma \in B^\times$ and write $i_p(\gamma) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Suppose that $i_p(\gamma) \in \mathrm{GL}_2(\mathbb{Q}_p) \cap \mathrm{M}_2(\mathbb{Z}_p)$ with $a \in \mathbb{Z}_p^\times$ and $c \equiv 0 \pmod{p^r}$. Then for $s \in \mathbb{W}$, $g \in \hat{B}^\times$ and $P \in P_n(L)$ there are equalities

$$\rho_{n,\epsilon}(\gamma \cdot s)(g)(P) = \int_{\mathbb{Y}} \chi_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p}(x,y) \epsilon(a) P(x,y) d(\gamma \cdot s)(g)$$

$$= \int_{\mathbb{Y}} \chi_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p}(x,y) \epsilon(y) P(x,y) di_p(\gamma) s(g\gamma)$$

$$= \int_{\mathbb{Y}} \chi_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p}(ax+by, cx+dy) \epsilon(ax+by) P(ax+by, cx+dy) ds(g\gamma)$$

$$= \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} \epsilon(a) \epsilon(x) P(ax+by, cx+dy) ds(g\gamma)$$

$$= \epsilon(a) \rho_{n,\epsilon}(s)(g\gamma)(P|\gamma) = \epsilon(a)\big(\gamma \cdot \rho_{n,\epsilon}(s)\big)(P).$$

The above computation shows that

$$\rho_{n,\epsilon}(\gamma \cdot s) = \gamma \cdot \rho_{n,\epsilon}(s)$$

for all $\gamma \in \Delta_r$. Note also that $\rho_{n,\epsilon}(T_p s) = U_p \rho_{n,\epsilon}(s)$ because $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)(\mathbb{Z}_p^\times \times p\mathbb{Z}_p) \cap \mathbb{X} = \emptyset$. It follows that the map $\rho_{n,\epsilon}$ is $\mathfrak{h}_{\mathrm{univ}}^D$-equivariant.

Restricting to the ordinary parts, we also get an $\mathfrak{h}_{\mathrm{univ}}^D$-equivariant map

$$\rho_{n,\epsilon}^{\mathrm{ord}} : \mathbb{W}^{\mathrm{ord}} \longrightarrow S_k(U_r, L)^{\mathrm{ord}}.$$

3.7. **The Control Theorem.** For any integer $m \geq 1$ and any character $\chi : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$ define the function $\psi_{m,\chi} : \mathbb{X} \to \mathbb{Z}_p$ by

$$\psi_{m,\chi}((x,y)) := \begin{cases} \chi(x) & \text{if } y \in p^m \mathbb{Z}_p, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\psi_{m,\chi}$ is homogeneous of degree χ for all integers $m \geq 1$.

The next result is the counterpart of Lemma 2.14.

Lemma 3.4. *Fix* $\kappa \in \mathcal{A}(\tilde{\Lambda})$ *of weight* k *and character* ϵ *factoring through* $(\mathbb{Z}/p^r\mathbb{Z})^\times$ *for some integer* $r \geq 1$. *Set* $n := k - 2$. *The map* $\rho_{n,\epsilon}^{\mathrm{ord}}$ *induces an injective,* $\mathfrak{h}_{\mathrm{univ}}^D$-*equivariant homomorphism of* $\tilde{\Lambda}/P_\kappa\tilde{\Lambda}$-*modules*

$$\rho_{n,\epsilon}^{\mathrm{ord}} : \mathbb{W}^{\mathrm{ord}}/P_\kappa\mathbb{W}^{\mathrm{ord}} \hookrightarrow S_k(U_r, F_f)^{\mathrm{ord}}.$$

Proof. First suppose that $s \in P_\kappa\mathbb{W}^{\mathrm{ord}}$, so that $s(g) \in P_\kappa\mathbb{D}$. Since the integrand $\epsilon(y)P(x,y)$ appearing in the expression of $\rho_{n,\epsilon}^{\mathrm{ord}}$ is homogeneous of degree κ, [11, Lemma 6.3] shows that $\rho_{n,\epsilon}(s)(g)(P) = 0$ and thus $s \in \ker(\rho_{n,\epsilon}^{\mathrm{ord}})$.

Now we show the opposite inclusion $\ker(\rho_{n,\epsilon}^{\mathrm{ord}}) \subset P_\kappa\mathbb{W}^{\mathrm{ord}}$. Fix $s \in \ker(\rho_{n,\epsilon}^{\mathrm{ord}})$ and an integer $m \geq 1$, then choose $t \in \mathbb{W}^{\mathrm{ord}}$ such that $T_p^m \cdot t = s$ (this is possible because T_p induces an isomorphism on $\mathbb{W}^{\mathrm{ord}}$). Write T_p^m as

$$T_p^m = \coprod U_0 g_{m,i}$$

where the $g_{m,i}$ are suitable products of m elements, not necessarily distinct, chosen in the set $\{g_a\}_{a=0,\dots,p-1,\infty}$ defined in (15). Therefore we have

$$\int_\mathbb{X} \psi_{m,\kappa}(x,y)ds(g)(x,y) = \sum_i \int_\mathbb{X} \psi_{m,\kappa}\big(g_{m,i}(x,y)\big) dt(gg_{m,i})(x,y).$$

Now $\psi_{m,\kappa}(g_{m,i}(x,y)) = 0$ unless $g_{m,i}$ is the product of elements of the form g_i with $i \neq \infty$. Furthermore, there is a decomposition

$$\mathbb{Z}_p^\times \times p\mathbb{Z}_p = \coprod_{i=0}^{p-1} g_i(\mathbb{Z}_p^\times \times p\mathbb{Z}_p),$$

so the above sum is equal to

$$\sum_{m,i} \int_{g_{m,i}(\mathbb{Z}_p^\times \times p\mathbb{Z}_p)} \kappa(x)dt(gg_{m,i}) = \sum_{m,i} \int_{g_{m,i}(\mathbb{Z}_p^\times \times p\mathbb{Z}_p)} \epsilon(x)x^n dt(gg_{m,i}) = U_p^m \rho_{n,\epsilon}^{\mathrm{ord}}(t)(g)(x^n).$$

Now

(16) $$0 = \rho_{n,\epsilon}^{\mathrm{ord}}(s) = \rho_{n,\epsilon}^{\mathrm{ord}}(T_p^m t) = U_p^m \rho_{n,\epsilon}^{\mathrm{ord}}(t).$$

Since $s \in \mathbb{W}^{\mathrm{ord}}$, the same is true of t and also of $\rho_{n,\epsilon}^{\mathrm{ord}}(t)$. Equation (16) then shows that $t = 0$ because U_p acts invertibly on the ordinary submodule, hence we conclude that

$$\int_\mathbb{X} \psi_{m,\kappa}(x,y)ds(g) = 0$$

for all $g \in \hat{B}^\times$. Finally, from [11, Lemma 6.3] it follows that $s(g) \in P_\kappa\mathbb{D}$ for all $g \in \hat{B}^\times$, and the lemma is proved. $\square$

Let $\kappa \in \mathcal{A}(\mathcal{R})$ and define the F_κ-vector space

$$\mathbb{W}_\kappa^{\mathrm{ord}} := \left(S_k(U_r, F_\kappa)^{\mathrm{ord}} \right)^{f_\kappa}.$$

Since f_κ is either a newform or a p-stabilized newform, $\mathbb{W}_\kappa^{\mathrm{ord}}$ is 1-dimensional over F_κ.

Define $h_\kappa : \mathfrak{h}^D_{\mathrm{univ}} \to \mathcal{R}_{P_\kappa}$ to be the composition of h with the localization map $\mathcal{R} \to \mathcal{R}_{P_\kappa}$ at the kernel P_κ of κ. We can consider the $\mathcal{R}_{P_\kappa}$-submodule $\mathbb{W}^{\mathrm{ord}}_{h_\kappa}$ of $\mathbb{W}^{\mathrm{ord}}_{\mathcal{R}_{P_\kappa}}$ defined by

$$\mathbb{W}^{\mathrm{ord}}_{h_\kappa} := \big(\mathbb{W}^{\mathrm{ord}}_{\mathcal{R}_{P_\kappa}} \big)^{h_\kappa}.$$

Proposition 3.5. *Let $\kappa \in \mathcal{A}(\mathcal{R})$ be of weight $k = k_\kappa$ and character $\epsilon = \epsilon_\kappa$. Set $n := k - 2$. The map $\rho^{\mathrm{ord}}_{n,\epsilon}$ of Lemma 3.4 induces an injective homomorphism of F_κ-vector spaces*

$$\rho_\kappa : \mathbb{W}^{\mathrm{ord}}_{h_\kappa}/P_\kappa \mathbb{W}^{\mathrm{ord}}_{h_\kappa} \hookrightarrow \mathbb{W}^{\mathrm{ord}}_\kappa.$$

Proof. Define $\mathfrak{p}_\kappa := P_\kappa \cap \Lambda$. As in the proof of Proposition 2.16, Lemma 3.4 ensures the existence of an injective homomorphism of $\Lambda_{\mathfrak{p}_\kappa}/\mathfrak{p}_\kappa \Lambda_{\mathfrak{p}_\kappa}$-vector spaces

$$\rho^{\mathrm{ord}}_{n,\epsilon} : \mathbb{W}^{\mathrm{ord}}_{\Lambda_{\mathfrak{p}_\kappa}}/\mathfrak{p}_\kappa \mathbb{W}^{\mathrm{ord}}_{\Lambda_{\mathfrak{p}_\kappa}} \hookrightarrow S_k(U_r, F_f)^{\mathrm{ord}}.$$

Recall that $\mathcal{R}_{P_\kappa}$ and $\Lambda_{\mathfrak{p}_\kappa}$ are normal domains and that $\mathcal{R}_{P_\kappa}$, being unramified over $\Lambda_{\mathfrak{p}_\kappa}$ thanks to [13, Corollary 1.4], is flat over $\Lambda_{\mathfrak{p}_\kappa}$. Now

$$S(U_r, F_f) \otimes_{F_f} F_\kappa \simeq S_k(U_r, F_\kappa),$$

from which we easily deduce the result. $\square$

Now recall that $\mathcal{L}$ (respectively, $\mathcal{K}$) is the fraction field of Λ (respectively, $\mathcal{R}$) and there is a canonical decomposition

(17) $$\mathfrak{h}^{D,\mathrm{ord}}_\infty \otimes_\Lambda \mathcal{L} \simeq \mathcal{K} \oplus \mathcal{N}$$

where $\mathcal{N}$ is a direct sum of finitely many fields plus a non-reduced part. As in §2.6, consider the composition

$$h_\mathcal{R} : \mathfrak{h}^D_{\mathrm{univ}} \xrightarrow{h} \mathfrak{h}^D_\infty \longrightarrow \mathfrak{h}^{D,\mathrm{ord}}_\infty \xrightarrow{f_\infty} \mathcal{R}.$$

Proposition 3.6. *The module $\mathbb{W}^{\mathrm{ord}}_\mathcal{K}$ is a 1-dimensional vector space over $\mathcal{K}$. Furthermore, the action of $\mathfrak{h}^D_{\mathrm{univ}}$ on $\mathbb{W}^{\mathrm{ord}}_\mathcal{K}$ factors through $h_\mathcal{R}$.*

Proof. Define the p-divisible abelian group

$$\mathcal{V} := \varinjlim_r S_2(U_r, F_f/\mathcal{O}_f)^{\mathrm{ord}}$$

where the direct limit is induced by the maps $U_r \subset U_{r-1}$. The Hecke operators T_n, $T_{n,n}$ and the involution ι act naturally on $\mathcal{V}$, since the Hecke action is compatible with the restriction maps (see [14, (2.9 a,b) and (3.5)]). Define V to be the Pontryagin dual of $\mathcal{V}$ and note that

$$V = \varprojlim_r S_2(U_r, \mathcal{O}_f)^{\mathrm{ord}} \simeq \mathbb{W}^{\mathrm{ord}}.$$

Thanks to [14, Corollary 10.4], we know that V is free of finite rank over Λ. Using Proposition 3.3 in place of Proposition 2.2, one can proceed as in the proof of Proposition 2.17 to show that $\mathbb{W}^{\mathrm{ord}}_\mathcal{L}$ is free of rank 1 over $\mathfrak{h}^{D,\mathrm{ord}}_\infty \otimes_\Lambda \mathcal{L}$, and we are done thanks to decomposition (17). $\square$

Now we can prove the analogue of Theorem 2.18 in the definite setting.

Theorem 3.7. *For every $\kappa \in \mathcal{A}(\mathcal{R})$ the map ρ_κ of Proposition 3.5 induces an isomorphism of 1-dimensional F_κ-vector spaces*

$$\rho_\kappa : \mathbb{W}^{\mathrm{ord}}_{h_\kappa}/P_\kappa \mathbb{W}^{\mathrm{ord}}_{h_\kappa} \xrightarrow{\simeq} \mathbb{W}^{\mathrm{ord}}_\kappa.$$

Proof. Thanks to Proposition 3.5, we only need to show the inequality

$$\dim_{F_\kappa}(\mathbb{W}^{\mathrm{ord}}_{h_\kappa}/P_\kappa \mathbb{W}^{\mathrm{ord}}_{h_\kappa}) \geq \dim_{F_\kappa}(\mathbb{W}^{\mathrm{ord}}_\kappa).$$

Since $\dim_{F_\kappa}(\mathbb{W}^{\mathrm{ord}}_\kappa) = 1$, we are reduced to proving that

$$\dim_{F_\kappa}(\mathbb{W}^{\mathrm{ord}}_{h_\kappa}/P_\kappa \mathbb{W}^{\mathrm{ord}}_{h_\kappa}) \geq 1.$$

However, Proposition 3.6 shows that the intersection of $\mathbb{W}^{\mathrm{ord}}_{\mathcal{R}_\kappa}$ with the $h_{\mathcal{R}}$-eigenmodule of $\mathbb{W}^{\mathrm{ord}}_{\mathcal{K}}$ is a free $\mathcal{R}_{P_\kappa}$-module of rank 1, which completes the proof. $\qquad\square$

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