THE RATIONALITY OF QUATERNIONIC DARMON POINTS OVER GENUS FIELDS OF REAL QUADRATIC FIELDS

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Abstract. Darmon points on p-adic tori and Jacobians of Shimura curves over $\mathbb{Q}$ were introduced in joint articles with Rotger as generalizations of Darmon's Stark–Heegner points. In this article we study the algebraicity over extensions of a real quadratic field K of the projections of Darmon points to elliptic curves, which coincide with the points on elliptic curves previously defined by M. Greenberg. More precisely, we prove that linear combinations of Darmon points on elliptic curves weighted by certain genus characters of K are rational over the predicted genus fields of K. This extends to an arbitrary quaternionic setting the main theorem on the rationality of Stark–Heegner points obtained by Bertolini and Darmon, and at the same time gives evidence for the rationality conjectures formulated in a joint paper with Rotger and by Greenberg in his article on Stark–Heegner points. In light of this result, quaternionic Darmon points represent the first instance of a systema tic supply of points of Stark–Heegner type other than Darmon's original ones for which explicit rationality results are known.

1. Introduction

Given an elliptic curve E over $\mathbb{Q}$ of conductor N, a prime number p dividing N exactly and a real quadratic field K in which p is inert and all primes dividing N/p split (Stark–Heegner hypothesis), Darmon introduced in [10] the notion of Stark–Heegner points on E. These local points live in $E(K_p)$, where K_p is the completion of K at p, but are conjectured to be rational over narrow ring class fields of K and to satisfy a Shimura reciprocity law under Galois actions. In fact, the arithmetic properties of Stark–Heegner points are expected to be as rich as those enjoyed by classical Heegner points, which are defined via the theory of complex multiplication and are known to be rational over ring class fields of imaginary quadratic fields.

Stark–Heegner points were later lifted to certain quotients of classical modular Jacobians by Dasgupta in [12]; this was done by proving a rigid analytic uniformization result for modular Jacobians. This uniformization result boils down to an equality of $\mathcal{L}$-invariants and turns out to be a strong form of a theorem of Greenberg and Stevens ([19]). It is important to observe that both Darmon's and Dasgupta's strategies rely heavily on the theory of modular symbols (and so on the presence of cusps on classical modular curves), and thus they do not extend to more general situations where the Stark–Heegner condition is not satisfied but considerations on the signs of the functional equations of the relevant L-functions predict the existence of points of Stark–Heegner type.

To overcome this difficulty, in [17] M. Greenberg replaced modular symbols by a general cohomological study of Shimura varieties over totally real fields and reinterpreted Darmon's theory in terms of group cohomology. The outcome of his work is, in the simplest case where the base field is $\mathbb{Q}$, a conjectural construction of local Stark–Heegner points on E which are expected to be rational over appropriate ring class fields of a real quadratic field K. In

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this context, the field K satisfies a modified Stark–Heegner hypothesis forcing the number of primes dividing N/p and inert in K to be even.

In [24] we gave an explicit rigid analytic uniformization of the maximal toric quotient of the Jacobian of a Shimura curve over $\mathbb{Q}$ at a prime dividing exactly the level, a result that can be viewed as complementary to the classical theorem of Čerednik and Drinfeld which provides rigid analytic uniformizations at primes dividing the discriminant ([8]). More precisely, denote by $J_0^D(Mp)$ the Jacobian variety of the Shimura curve of discriminant $D > 1$ and level Mp. Combining techniques in group cohomology and p-adic integration in the spirit of the works of Dasgupta and Greenberg, we exhibited a p-adic torus which is isogenous to the product of two copies of the p-new quotient of $J_0^D(Mp)$. As a corollary, we offered a proof of the isogeny conjecture formulated in [17, Conjecture 2] (an alternative proof of which has been given by Dasgupta and Greenberg in [13]): this made Greenberg's construction of local points on elliptic curves over $\mathbb{Q}$ unconditional, but left the rationality conjecture [17, Conjecture 3] wide open.

As an application of our work on rigid analytic uniformizations, in [25] we introduced a new family of Stark–Heegner type points on p-adic tori and Jacobians of Shimura curves, which we called *Darmon points*. These are, at the same time, lifts of Greenberg's points on elliptic curves and generalizations of Dasgupta's Stark–Heegner points on modular Jacobians. We formulated conjectures about their rationality over narrow ring class fields of real quadratic fields and their Galois properties. Moreover, assuming these rationality conjectures and applying the methods of [26], we proved the conjecture of Birch and Swinnerton-Dyer for elliptic curves over ring class fields of real quadratic fields in the case of analytic rank 0.

At present, the central open problems in the theories of Stark–Heegner and Darmon points are the rationality conjectures. The first result towards the rationality of Stark–Heegner points was obtained by Bertolini and Darmon in [7], where they proved that linear combinations of Stark–Heegner points weighted by genus characters of K are rational over the predicted genus fields of K. To achieve this, the authors applied their previous results on Hida families of modular forms ([6]) and showed that linear combinations of this kind can be expressed in terms of Heegner points.

In this paper we prove the analogue of the results of [7] for projections to elliptic curves of the quaternionic Darmon points introduced in [25], that is, for the points on elliptic curves over $\mathbb{Q}$ defined by Greenberg in [17]. In other words, we show that linear combinations with genus character coefficients of Darmon points on elliptic curves are rational over the fields that were predicted in [25]. In light of this feature, quaternionic Darmon points represent the first instance of a systematic supply of points of Stark–Heegner type other than Darmon's original ones for which explicit rationality results are known. Thus our main theorem provides some evidence for both the rationality conjecture formulated in [25] and the conjecture proposed by Greenberg in [17, Conjecture 3] when the totally real base field is $\mathbb{Q}$. The present article should be viewed as a natural step further in the line of research begun in [24] and carried on in [25]. Related results for Heegner (and Stark–Heegner) cycles in the spirit of [6] and [7] are being independently investigated by Greenberg, Seveso and Shahabi ([18]).

While the reader familiar with the work of Bertolini and Darmon will notice that our strategy follows [7] closely (e.g., the main result of [6] plays a crucial role in the last part of our arguments), the methods we use are of a different nature and build on the constructions of [24] in an essential way. In fact, in this article we develop a series of new techniques and obtain various auxiliary results which may be of independent interest. Finally, we remark that, as in [7], the scheme of proof presents formal analogies with that of Kronecker's solution to Pell's equation (see [7, Introduction] for an illustration of this parallelism).

Now we describe our results more in detail. Fix an elliptic curve E over $\mathbb{Q}$ of square-free conductor N and write f for the normalized newform of weight 2 and level N attached to E

by modularity. Fix also a real quadratic field K of discriminant d_K prime to N, whose ring of integers we denote $\mathcal{O}_K$. Suppose that there is a factorization $N = MDp$ such that

- all the primes dividing M (respectively, Dp) are split (respectively, inert) in K;
- the number of primes dividing D is *even* and ≥ 2.

Let B denote the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant D and choose Eichler orders $R_0^D(Mp) \subset R_0^D(M)$ of B of level Mp and M, respectively. Write $\Gamma_0^D(Mp)$ and $\Gamma_0^D(M)$ for the groups of elements of norm 1 in $R_0^D(Mp)$ and $R_0^D(M)$, respectively, and consider the Ihara group

$$\Gamma := \Big\{\gamma \in R_0^D(M) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \;\Big|\; \mathrm{norm}(\gamma) = 1\Big\}.$$

For any sign $\pm$ fix an isomorphism $H^1(E(\mathbb{C}), \mathbb{Z})^{\pm} \simeq \mathbb{Z}$, where the superscript denotes the $\pm$-eigenspace for complex conjugation. Following [24, §7.7], we define a canonical element

$$\boldsymbol{\mu}_f^{\pm} \in H^1\big(\Gamma, \mathcal{M}_0(\mathbb{Q})\big)$$

where $\mathcal{M}_0(\mathbb{Q})$ is the $\mathbb{Q}$-vector space of $\mathbb{Q}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ with total mass 0. Its image in $H^1\big(\Gamma_0^D(Mp), \mathbb{Q}\big)$, obtained by applying Shapiro's lemma, spans the one-dimensional $\mathbb{Q}$-vector space on which the archimedean involution W_∞ acts via $\pm$ and the Hecke algebra of level N acts via the character associated with f (see §2.2 for details on these actions).

Fix a sign $\pm$. Using techniques introduced in [24] and [25], in §2.4 we use $\boldsymbol{\mu}_f^{\pm}$ to attach a Darmon point $P_\psi^{\pm} \in E(K_p)$ to (the $\Gamma_0^D(M)$-conjugacy class of) each optimal embedding ψ of $\mathcal{O}_K$ into $R_0^D(M)$: these points are the object of our investigation in the present paper.

Let G_K^+ be the narrow class group of K (i.e., the Galois group over K of the narrow Hilbert class field of K). For a genus character χ of K (i.e., an unramified quadratic character of G_K^+) we define the point

$$P_\chi := \sum_{\sigma \in G_K^+} P_{\psi_\sigma}^{\epsilon} \in E(K_p),$$

where the Galois action $\psi \mapsto \psi_\sigma$ on (conjugacy classes of) optimal embeddings is described in §2.5 and the choice of $\epsilon \in \{\pm\}$ depends on χ.

The main result of this paper, which is Theorem 5.1 in the text, is the following

Theorem 1.1. *Let χ be a genus character of K associated with a pair (χ_1, χ_2) of Dirichlet characters such that $\chi_i(-MD) = -w_{MD}$ for $i = 1, 2$, where w_{MD} is the eigenvalue of the Atkin–Lehner involution W_{MD} acting on f.*

1. *There exists $n \in \mathbb{Z}$ such that $nP_\chi \in E(H_\chi)$ where H_χ is the genus field of K cut out by χ.*
2. *The point nP_χ is of infinite order if and only if $L'(E/K, \chi, 1) \neq 0$.*

Now we outline the strategy of proof of this theorem. In doing this, we state two results (Theorems 1.2 and 1.4) which may be of independent interest.

Embed $\mathbb{Z}$ into the weight space

$$\mathcal{X} := \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times) \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times (1 + p\mathbb{Z}_p) \tag{1}$$

by sending k to the map $\big(x \mapsto x^{k-2}\big)$. Hida theory associates with f a neighbourhood $\mathbb{U}$ of 2 in $\mathcal{X}$ (which we assume small enough so that all the results we will state below hold for $\mathbb{U}$) and a formal q-expansion

$$f_\infty = \sum_{n=1}^{\infty} a_n(\kappa) q^n \tag{2}$$

where $a_1 = 1$ and a_n is a rigid analytic function on $\mathbb{U}$ such that $\sum_{n=1}^{\infty} a_n(k) q^n$ for an even integer $k \geq 2$ is the q-expansion of a p-stabilized weight k eigenform f_k on $\Gamma_0(N)$ and $f_2 = f$.

The form f_k is associated, via the Jacquet–Langlands correspondence, with a modular form f_k^{JL} of weight k and level structure $\Gamma_0^D(Mp)$ (so f_k^{JL} is well defined only up to scalars). Thanks to a result of Matsushima and Shimura ([29]), f_k^{JL} corresponds to an element $\boldsymbol{\phi}_k$ in the first cohomology group of $\Gamma_0^D(Mp)$ with values in the dual of the $\mathbb{C}$-vector space of homogeneous polynomials of degree $k-2$.

Write $\mathbb{D}$ for the module of $\mathbb{Z}_p$-valued measures on $\mathbb{Y} := \mathbb{Z}_p^2$ which are supported on the subset $\mathbb{X}$ of primitive elements (i.e., those vectors in $\mathbb{Y}$ which are not divisible by p). Moreover, let $\Lambda \simeq \mathbb{Z}_p[\![T]\!]$ be the Iwasawa algebra and let $\mathcal{R}$ denote the integral closure of Λ in the primitive component to which f belongs, in the sense of [27, §5.3]. The ring $\mathcal{R}$ is a complete local noetherian domain, finitely generated as a Λ-module. In Section 3 we exploit a control theorem for Hida families in the quaternionic setting (analogous to the one proved by Greenberg and Stevens in [19]) to construct a canonical element

$$\tilde{\boldsymbol{\mu}}_f^{\pm} \in \Big(H^1\big(\Gamma_0^D(M), \mathbb{D}\big) \otimes_\Lambda \mathcal{R}\Big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

such that the image of $\tilde{\boldsymbol{\mu}}_f^{\pm}$ via the map induced by the map $\pi : \mathbb{X} \to \mathbb{P}^1(\mathbb{Q}_p)$ taking (x,y) to x/y coincides with $\boldsymbol{\mu}_f^{\pm}$. The property characterizing $\tilde{\boldsymbol{\mu}}_f^{\pm}$ asserts that integrating homogeneous polynomials of degree $k-2$ against $\tilde{\mu}_{f,\gamma}^{\pm}$ over $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ (with $\gamma \in \Gamma_0^D(Mp)$) defines a multiple of the projection of $\boldsymbol{\phi}_k$ to the $\pm$-eigenspace for W_∞ (see §3.3 for details). We fix a representative $\tilde{\mu}_f^{\pm}$ of this class, which can be chosen so that $\pi_*(\tilde{\mu}_f^{\pm}) = \mu_f^{\pm}$.

Now let

$$\Phi_{\mathrm{Tate}} : \bar{K}_p^\times / q^{\mathbb{Z}} \longrightarrow E(\bar{K}_p) \tag{3}$$

denote Tate's p-adic uniformization, where $q \in p\mathbb{Z}_p$ is Tate's period for E at p and $\bar{K}_p$ is an algebraic closure of K_p. Let $\log_q$ be the branch of the p-adic logarithm satisfying $\log_q(q) = 0$ and define $\log_E(P) := \log_q\big(\Phi_{\mathrm{Tate}}^{-1}(P)\big)$ for all $P \in E(K_p)$.

As a piece of notation, let z_ψ denote the fixed point of $\psi(K^\times)$ acting on $\mathbb{P}^1(\bar{\mathbb{Q}}_p)$ such that $\psi(\alpha)(z_\psi, 1) = \alpha(z_\psi, 1)$ for all $\alpha \in K^\times$ (see §2.3 for precise definitions). Let also ε_K be a generator of the group of norm 1 elements in $\mathcal{O}_K^\times$ such that $\varepsilon_K > 1$ with respect to a fixed embedding $K \hookrightarrow \mathbb{R}$ and define $\gamma_\psi := \psi(\varepsilon_K)$.

The first auxiliary result proved in the paper is the following

Theorem 1.2. *There is an integer $m \neq 0$ such that*

$$\log_E(P_\psi^{\pm}) = m \cdot \int_{\mathbb{X}} \log_q(x - z_\psi y) d\tilde{\mu}_{f,\gamma_\psi}^{\pm}.$$

This theorem, which corresponds to Corollary 3.5, allows us to explicitly compute (at least in principle) Darmon points on elliptic curves. Actually, in Theorem 3.3 we prove a more general version for Darmon points on the quotient T/L by means of which we obtained in [24] a parametrization of E (cf. §2.7).

Remark 1.3. The integer m appearing in Theorem 1.2 can be made explicit in terms of the exponent of the abelianization of Γ and the degree of an isogeny between T/L and the elliptic curve E (see §2.7).

In Section 4 we use the cohomology class $\tilde{\boldsymbol{\mu}}_f^{\pm}$ to introduce a p-adic L-function $L_p(f_\infty/K, \chi, k)$ attached to f_∞ and a genus character χ of K, whose p-adic variable k ranges in $\mathbb{U}$. The main result proved by Popa in [34] can then be combined with the interpolation properties of $\tilde{\boldsymbol{\mu}}_f^{\pm}$. For a classical modular form f_k of even weight $k \geq 4$ and trivial character in the Hida family, we prove that the value of $L_p(f_\infty/K, \chi, k)$ is equal, up to multiplication by an explicit non-zero constant, to the special value of the complex L-function of $f_k^\sharp$ twisted by χ, where the form $f_k^\sharp$ is the p-stabilization of f_k (see Theorem 4.19).

Now recall that a genus character χ of K corresponds to an unordered pair of quadratic Dirichlet characters χ_1 and χ_2 of discriminants d_1 and d_2, respectively, such that $d_K = d_1 d_2$, and for $j = 1, 2$ let $L_p(f_\infty, \chi_j, k, s)$ denote the Mazur–Kitagawa two-variable p-adic L-function associated with f_∞ and χ_j.

Our last result we highlight in this introduction is the following factorization formula, corresponding to Theorem 4.33 in the main body of the article.

Theorem 1.4. *Let χ be a genus character of K and let (χ_1, χ_2) be the associated pair of Dirichlet characters. There exists a p-adic analytic function η on $\mathbb{U}$ such that $\eta(k) \neq 0$ and*

$$L_p(f_\infty/K, \chi, k) = \eta(k) L_p(f_\infty, \chi_1, k, k/2) L_p(f_\infty, \chi_2, k, k/2)$$

for all $k \in \mathbb{U}$. Moreover, $\eta(2) \in (\mathbb{Q}^\times)^2$.

Here is a brief explaination of how Theorems 1.2 and 1.4 are used to prove Theorem 1.1.

(1) Theorem 1.4 provides a link between the values at $k = 2$ of the second derivatives of $L_p(f_\infty/K, \chi, k)$ and of the restriction of $L_p(f_\infty, \chi_1, k, s)$ to the line $s = k/2$ (here we need to order the pair (χ_1, χ_2) so that the sign of the functional equation of the L-function $L(E, \chi_1, s)$ is -1, which can be done thanks to our assumptions).
(2) On the one hand, the second derivative of $L_p(f_\infty/K, \chi, k)$ evaluated at $k = 2$ is essentially computed by the integrals appearing on the right hand side of Theorem 1.2, thus providing a link with Darmon points (this is proved in Theorem 4.31).
(3) On the other hand, [6, Theorem 5.4] shows that the value at $k = 2$ of the second derivative of $L_p(f_\infty, \chi_1, k, k/2)$ encodes the logarithm of certain linear combinations of classical Heegner points.

The three observations above establish a relation between Heegner points and Darmon points, yielding the desired rationality result for P_χ.

Remark 1.5. The hypothesis that M be square-free is imposed only to be able to apply the results of [34], which are proved under this assumption. Granting an extension of [34] to more general situations, our arguments work equally well under the weaker condition that M be a product of odd powers of distinct primes (and this restriction comes into play exclusively in §4.6 for some calculations with local Hilbert symbols).

2. Darmon points on p-adic tori and elliptic curves

2.1. **Quaternion algebras and Shimura curves.** Let $D > 1$ be a square-free product of an *even* number of primes and let $M \geq 1$ be a square-free integer prime to D. Write B for the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant D. Fix a prime number p not dividing $6MD$. Throughout the paper fix also algebraic closures $\bar{\mathbb{Q}}$, $\bar{\mathbb{Q}}_p$ and field embeddings

$$\bar{\mathbb{Q}} \longrightarrow \bar{\mathbb{Q}}_p, \qquad \bar{\mathbb{Q}}_p \longrightarrow \mathbb{C}. \tag{4}$$

Let $K = \mathbb{Q}(\sqrt{d_K})$ be a real quadratic field with fundamental discriminant d_K such that all the primes dividing Dp are inert in K and all the primes dividing M split in K, and fix an embedding $K \hookrightarrow \mathbb{R}$. For the rest of the paper, fix an auxiliary real quadratic field F such that all the primes dividing D are inert in F while p splits in F. In particular, $F \neq K$. The condition on the primes dividing D implies that F is a splitting field for B, so we can (and do) choose an isomorphism $i_F : B \otimes_{\mathbb{Q}} F \xrightarrow{\simeq} \mathrm{M}_2(F)$ of F-algebras. Since p splits in F, the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ fixed in (4) and the isomorphism i_F induce an isomorphism $i_p : B \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{Q}_p)$ of $\mathbb{Q}_p$-algebras. Moreover, (4) fixes an embedding $F \hookrightarrow \mathbb{R}$; this induces via i_F an isomorphism $i_\infty : B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{R})$ of $\mathbb{R}$-algebras. Finally, for all primes $\ell \nmid Dp$ fix isomorphisms $i_\ell : B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{Q}_\ell)$ of $\mathbb{Q}_\ell$-algebras.

Remark 2.1. When describing local arguments at p, the fact that p splits in F allows us to work with $\mathbb{Q}_p$ and not with quadratic extensions of it, which simplifies the computations.

Let $R^{\max}$ be the maximal order of B satisfying the following condition: if ℓ is a prime such that $\ell \nmid D$ then $i_\ell(R^{\max} \otimes \mathbb{Z}_\ell) = \mathrm{M}_2(\mathbb{Z}_\ell)$. We fix an Eichler order $R_0^D(M)$ of level M contained in $R^{\max}$ by requiring that for all primes $\ell | M$ the order $i_\ell\big(R_0^D(M) \otimes \mathbb{Z}_\ell\big)$ is equal to the order $R_0^{\mathrm{loc}}(\ell)$ of $\mathrm{M}_2(\mathbb{Z}_\ell)$ consisting of those matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \equiv 0 \pmod{\ell}$. Furthermore, we also fix an Eichler order $R_0^D(Mp) \subset R_0^D(M)$ of level Mp by requiring that, in addition, $i_p\big(R_0^D(Mp) \otimes \mathbb{Z}_p\big)$ is equal to the order $R_0^{\mathrm{loc}}(p)$ of $\mathrm{M}_2(\mathbb{Z}_p)$ consisting of those matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \equiv 0 \pmod{p}$. We denote by $\Gamma_0^D(M)$ and $\Gamma_0^D(Mp)$ the groups of elements in $R_0^D(M)$ and $R_0^D(Mp)$, respectively, of reduced norm 1.

Finally, let $X_0^D(M)$ (respectively, $X_0^D(Mp)$) denote the (compact) Shimura curve whose corresponding Riemann surface is equal to the analytic quotient $\Gamma_0^D(M)\backslash\mathcal{H}$ (respectively, $\Gamma_0^D(Mp)\backslash\mathcal{H}$), where $\mathcal{H}$ is the complex upper half plane and the elements of $B^\times$ with positive norm act on it by Möbius (i.e., fractional linear) transformations via i_∞.

2.2. **Hecke algebras.** For any subgroup G of $B^\times$ consisting of elements having reduced norm 1 and any subsemigroup S of $B^\times$ such that (G, S) is a Hecke pair in the sense of [1, §1.1] we denote by $\mathcal{H}(G, S)$ the Hecke algebra (over $\mathbb{Z}$) of the pair (G, S) whose elements are combinations with integer coefficients of double cosets $T(s) := GsG$ for $s \in S$.

Let norm be the reduced norm of B, let $g \mapsto g^* := \mathrm{norm}(g)g^{-1}$ be the main involution of $B^\times$ and for any subset S of $B^\times$ let S^* denote the image of S under the main involution.

If M is a left $\mathbb{Z}[S^*]$-module, the group $H^r(G, M)$ has a natural right action of $\mathcal{H}(G, S)$ defined at the level of cochains $c \in Z^r(G, M)$ as follows: for $s \in S$ set

$$\big(c|T(s)\big)(\gamma_1, \dots, \gamma_r) := \sum_i s_i^* \cdot c\big(t_i(\gamma_1), \dots, t_i(\gamma_r)\big)$$

where $GsG = \coprod_i Gs_i$ (finite disjoint decomposition) and $t_i : G \to G$ is defined by the equations $Gs_i\gamma = Gs_j$ (for some j) and $s_i\gamma = t_i(\gamma)s_j$. We also define a right action of $\mathcal{H}(G, S)$ on $H_r(G, M)$ by the formula

$$c|T(s) := \sum_{i,j} s_i(m_j) \otimes \big[t_i(\gamma_{1,j})| \dots |t_i(\gamma_{r,j})\big]$$

for $c = \sum_j m_j \otimes [\gamma_{1,j}| \dots |\gamma_{r,j}] \in Z_r(G, M)$.

Remark 2.2. The above formalism of Hecke operators slightly differs from the one adopted in [24, §2.1], where we considered left actions on homology and cohomology groups instead of right actions.

For primes ℓ let $\Sigma_\ell^{\mathrm{loc}}$ denote the semigroup of elements in $R^{\max} \otimes \mathbb{Z}_\ell$ with non-zero norm. For every prime $\ell | Mp$ let $\Sigma_0^{\mathrm{loc}}(\ell) \subset \Sigma_\ell^{\mathrm{loc}}$ be the inverse image under i_ℓ of the semigroup of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_\ell) \cap \mathrm{M}_2(\mathbb{Z}_\ell)$ with $a \in \mathbb{Z}_\ell^\times$ and $c \equiv 0 \pmod{\ell}$. Then define the semigroup

$$S_0^D(Mp) := B^+ \cap \left(\prod_{\ell | Mp} \Sigma_0^{\mathrm{loc}}(\ell) \times \prod_{\ell \nmid Mp} \Sigma_\ell^{\mathrm{loc}} \right)$$

where B^+ is the subgroup of elements in $B^\times$ of positive norm. Then we can form the commutative Hecke algebra $\mathcal{H}(Mp) = \mathcal{H}\big(\Gamma_0^D(Mp), S_0^D(Mp)\big)$ generated over $\mathbb{Z}$ by the standard elements $T_n = \sum T(\alpha)$ for $n \geq 1$ (the sum being over all the elements $\alpha \in S$ of norm n) and $T_{n,n} = T(n)$ for $n \geq 1$, $n \nmid MDp$. Analogously, we can define the Hecke algebra $\mathcal{H}(M)$.

As in [24], consider the Ihara group

$$\Gamma := \Big\{ \gamma \in R_0^D(M) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \;\Big|\; \mathrm{norm}(\gamma) = 1 \Big\} \longleftrightarrow \mathrm{SL}_2(\mathbb{Q}_p)$$

where norm : $B \to \mathbb{Q}$ denotes the reduced norm map and the injection is induced by the composition of the canonical injection $B \hookrightarrow B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ with the isomorphism i_p fixed in (4). If

$$S_p := B^+ \cap \left(\mathrm{M}_2(\mathbb{Q}_p) \times \prod_{\ell | M} \Sigma_0^{\mathrm{loc}}(\ell) \times \prod_{\ell \nmid Mp} \Sigma_\ell^{\mathrm{loc}} \right)$$

then we may also consider the Hecke algebra $\mathcal{H}(\Gamma, S_p)$. See [24, §2.2 and §2.3] for details.

Now we introduce Atkin–Lehner involutions. For every prime $\ell | MDp$ choose an element $\omega_\ell \in R_0^D(Mp)$ of norm ℓ; if $\ell | Mp$ then we can (and do) choose ω_ℓ in such a way that $i_\ell(\omega_\ell) = \left(\begin{smallmatrix} 0 & -1 \\ \ell & 0 \end{smallmatrix}\right)$ up to an element of $R_0^{\mathrm{loc}}(\ell)$ of norm 1. It is known that ω_ℓ normalizes both $\Gamma_0^D(Mp)$ and $\Gamma_0^D(M)$ when $\ell | MD$ (cf. [33, §2]), and it turns out that the same is true of ω_p. Finally, choose $\omega_\infty \in R_0^D(Mp)$ of norm -1; of course, ω_∞ normalizes both $\Gamma_0^D(Mp)$ and $\Gamma_0^D(M)$.

Suppose that a semigroup $S \subset B^\times$ contains ω_∞ and ω_ℓ for all primes $\ell | MDp$. Then define $W_\infty := T(\omega_\infty)$ and $W_\ell := T(\omega_\ell)$ in $\mathcal{H}(G, S)$; further, for every integer $m | MDp$ set $W_m := \prod_{\ell | m} W_\ell$. Note that $W_m = T(\omega_m)$ where $\omega_m := \prod_{\ell | m} \omega_\ell$. In particular, W_p belongs to $\mathcal{H}(\Gamma, S_p)$ and, in fact, $W_p = U_p$ in this Hecke algebra (see [24, §2.3] for details).

2.3. **Quadratic forms.** Let τ denote the generator of $\mathrm{Gal}\big(F(\sqrt{d_K})/F\big)$. If $\psi : K \hookrightarrow B$ is an embedding of $\mathbb{Q}$-algebras then we may consider the quadratic form $Q_\psi(x, y)$ associated with ψ, which is defined by

$$Q_\psi(x, y) := cx^2 - 2axy - by^2 \qquad \text{where } i_F\big(\psi(\sqrt{d_K})\big) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

We explicitly observe that $c \neq 0$. If fact, if $c = 0$ then $d_K = a^2$, whence $K \subset F$, a possibility that is ruled out by our choice of F.

We can factor the quadratic form $Q_\psi(x, y)$ as

$$Q_\psi(x, y) = c(x - z_\psi y)(x - \bar{z}_\psi y)$$

where $z_\psi, \bar{z}_\psi \in F(\sqrt{d_K}) - F$ are the roots of the equation $cz^2 - 2az - b = 0$ and $\tau(z_\psi) = \bar{z}_\psi$. The two roots z_ψ and $\bar{z}_\psi$ are the only fixed points for the action of $\psi(K^\times)$ on $\mathbb{P}^1\big(F(\sqrt{d_K})\big)$ by Möbius transformations via i_F. We may also order z_ψ and $\bar{z}_\psi$ by requiring that

$$\psi(\alpha)\begin{pmatrix} z_\psi \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} z_\psi \\ 1 \end{pmatrix}, \qquad \psi(\alpha)\begin{pmatrix} \bar{z}_\psi \\ 1 \end{pmatrix} = \tau(\alpha)\begin{pmatrix} \bar{z}_\psi \\ 1 \end{pmatrix}$$

for all $\alpha \in K$.

Let $\mathcal{H}_p := \mathbb{C}_p - \mathbb{Q}_p$ denote Drinfeld's p-adic half plane. The completion K_p of K at the prime (p) is the (unique, up to isomorphism) unramified quadratic extension of $\mathbb{Q}_p$. Since p splits in F, the completion of $F(\sqrt{d_K})$ with respect to the valuation induced by the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ chosen in (4) is again K_p. Thus z_ψ and $\bar{z}_\psi$ can also be seen as points in $K_p - \mathbb{Q}_p$. Since there are canonical isomorphisms

$$\mathrm{Gal}\big(F(\sqrt{d_K})/F\big) \simeq \mathrm{Gal}(K/\mathbb{Q}) \simeq \mathrm{Gal}(K_p/\mathbb{Q}_p),$$

in the rest of the article we shall view τ as the generator of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$ as well.

2.4. **Darmon points on tori.** For any abelian group A denote by $\mathcal{M}(A)$ the group of A-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ and by $\mathcal{M}_0(A)$ those measures in $\mathcal{M}(A)$ with total mass 0. There is a canonical left action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathcal{M}(A)$ and $\mathcal{M}_0(A)$ defined by the integration formula

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \varphi(t) d(\gamma \cdot \nu)(t) := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \varphi\left(\frac{at + b}{ct + d}\right) d\nu$$

for all step functions $\varphi : \mathbb{P}^1(\mathbb{Q}_p) \to \mathbb{C}_p$ and all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_p)$. Then $B^\times$ acts on $\mathcal{M}(A)$ and $\mathcal{M}_0(A)$ via i_p and the embedding $B \hookrightarrow B \otimes \mathbb{Q}_p$.

The group $H^1\big(\Gamma,\mathcal{M}_0(\mathbb{Q})\big)$ is endowed with actions of the Hecke algebra $\mathcal{H}(\Gamma,S_p)$ considered in §2.2 and of the involutions W_m and W_∞. As in the introduction, let E be an elliptic curve over $\mathbb{Q}$ of conductor $N=MDp$ and let f be the normalized newform of weight 2 and level N attached to E. For any choice of sign $\pm$ define $H^1\big(\Gamma,\mathcal{M}_0(\mathbb{Q})\big)^{f,\pm}$ to be the subspace of $H^1\big(\Gamma,\mathcal{M}_0(\mathbb{Q})\big)$ consisting of those ξ such that $\xi|T=\theta_f(T)\xi$ for all $T\in\mathcal{H}(Mp)$ and $\xi|W_\infty=\pm\xi$ where $\theta_f:\mathcal{H}(Mp)\to\mathbb{Z}$ is the morphism associated with f. By [17, Proposition 25], the $\mathbb{Q}$-vector space $H^1\big(\Gamma,\mathcal{M}_0(\mathbb{Q})\big)^{f,\pm}$ has dimension 1. In [24, Section 4] we constructed an explicit generator for this vector space. Since we will need this description in the following computations, we briefly review setting and results of [24].

Recall the identification between $H^1\big(\Gamma,\mathcal{M}_0(A)\big)$ with $H^1\big(\Gamma,\mathcal{F}_{\mathrm{har}}(A)\big)$ where, for any abelian group A, the symbol $\mathcal{F}_{\mathrm{har}}(A)$ stands for the abelian group of A-valued harmonic cocycles (see, e.g., [17, Lemma 27] and [24, §4.1]). Write $\mathcal{T}$ for the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$ (see [35, Ch. II, §1]), whose set of vertices (respectively, oriented edges) will be denoted by $\mathcal{V}$ (respectively, $\mathcal{E}$). The groups $B^\times$ and $(B\otimes\mathbb{Q}_p)^\times$ act on the left on $\mathcal{T}$ via i_p. Let $v_*\in\mathcal{V}$ be the distinguished vertex corresponding to the maximal order $\mathrm{M}_2(\mathbb{Z}_p)$ of $\mathrm{M}_2(\mathbb{Q}_p)$, and say that a vertex $v\in\mathcal{V}$ is *even* (respectively, *odd*) if its distance from v_* is even (respectively, odd). Moreover, denote by $\hat v_*$ the vertex corresponding to $\left(\begin{smallmatrix}\mathbb{Z}_p & p^{-1}\mathbb{Z}_p\\ p\mathbb{Z}_p & \mathbb{Z}_p\end{smallmatrix}\right)$, fix the edge $e_*=(v_*,\hat v_*)\in\mathcal{E}$ and say that an edge $e=(v_1,v_2)\in\mathcal{E}$ is *even* (respectively, *odd*) if v_1 is even (respectively, odd). Write $\mathcal{E}^+$ for the set of even vertices of $\mathcal{T}$. Finally, if $e=(v_1,v_2)$ write $\bar e$ for the reversed edge (v_2,v_1).

Choose a harmonic (in the sense of [24, Definition 4.6]) system of representatives $\mathcal{Y}_{\mathrm{har}}$ for the cosets in $\Gamma_0^D(Mp)\backslash\Gamma$, which can be written as $\mathcal{Y}_{\mathrm{har}}=\{\gamma_e\}_{e\in\mathcal{E}^+}$ with $\gamma_e\in\Gamma$ such that $\gamma_e(e)=e_*$. Set $H_E:=H_1\big(E(\mathbb{C}),\mathbb{Z}\big)$ and write $H_E^\pm$ for the $\pm$-eigenspace for complex conjugation acting on H_E. Fix an isomorphism $H_E^\pm\simeq\mathbb{Z}$. Keeping in mind that $H_E^\pm$ is (canonically isomorphic to) a quotient of $\Gamma_0^D(Mp)$, define, as in [24, Definition 4.2], the *universal* 1-*cochain* $\mu_{\mathrm{univ}}^{\mathcal{Y}}:\Gamma\to\mathcal{F}_{\mathrm{har}}(\mathbb{Z})$ associated with $\mathcal{Y}$ by the following rules:

- if $\gamma\in\Gamma$ and $e\in\mathcal{E}^+$ let $g_{\gamma,e}\in\Gamma_0^D(Mp)$ be defined by the equation $\gamma_e\gamma=g_{\gamma,e}\gamma_{\gamma^{-1}(e)}$, then set $\mu_{\mathrm{univ}}^{\mathcal{Y}}(\gamma)(e):=[g_{\gamma,e}]\in H_E^\pm\simeq\mathbb{Z}$;
- if $\gamma\in\Gamma$ and $e\notin\mathcal{E}^+$ then set $\mu_{\mathrm{univ}}^{\mathcal{Y}}(\gamma)(e):=-\mu_{\mathrm{univ}}^{\mathcal{Y}}(\gamma)(\bar e)$.

Harmonic systems always exist, by [24, Proposition 4.8]. The canonical generator $\boldsymbol{\mu}_f^\pm$ of $H^1\big(\Gamma,\mathcal{M}_0(\mathbb{Q})\big)$ is obtained by fixing a prime number $r\nmid MDp$, applying the Hecke operator $t_r:=T_r-(r+1)$ to the class of $\mu_{\mathrm{univ}}^{\mathcal{Y}}$ in $H^1\big(\Gamma,\mathcal{F}_{\mathrm{har}}(\mathbb{Z})\big)$ and then using the isomorphism

$$H^1\big(\Gamma,\mathcal{M}_0(\mathbb{Q})\big)\simeq H^1\big(\Gamma,\mathcal{M}_0(\mathbb{Z})\big)\otimes_{\mathbb{Z}}\mathbb{Q}.$$

As shown in [24, Lemma 4.11], the resulting class $\boldsymbol{\mu}_f^\pm$ independent of the choice of $\mathcal{Y}_{\mathrm{har}}$ as above. We also fix a representative $\mu_f^\pm$ of $\boldsymbol{\mu}_f^\pm$.

Let $T:=\mathbb{G}_m$ denote the multiplicative group (viewed as a functor on commutative $\mathbb{Q}$-algebras). As explained in [17, Section 6], the measure $\boldsymbol{\mu}_f^\pm$ can be used to define a lattice L in $T(\mathbb{C}_p)$ as the image of the composition

$$H_2(\Gamma,\mathbb{Z})\longrightarrow H_1\big(\Gamma,\mathrm{Div}^0(\mathcal{H}_p)\big)\longrightarrow T(\mathbb{C}_p). \tag{5}$$

Here the first map is extracted from the long exact sequence in homology associated with the short exact sequence

$$0\longrightarrow\mathrm{Div}^0(\mathcal{H}_p)\longrightarrow\mathrm{Div}(\mathcal{H}_p)\xrightarrow{\deg}\mathbb{Z}\longrightarrow 0$$

and the second is the integration map described in [24, §5.1]. In fact, the lattice L is contained in $T(\mathbb{Q}_p)$ and is Hecke-stable ([24, Proposition 6.1]). Fix $z\in K_p-\mathbb{Q}_p$ and let

$\boldsymbol{d}_z \in H^2\big(\Gamma, T(\mathbb{C}_p)\big)$ be the cohomology class represented by the 2-cocycle

$$d_z : \Gamma \times \Gamma \longrightarrow T(K_p), \qquad (\gamma_1, \gamma_2) \longmapsto \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{s - \gamma_1^{-1}(z)}{s - z} d\mu^{\pm}_{f,\gamma_2}(s). \tag{6}$$

(See [24, §5.1] for the definition of the multiplicative integral.) The class $\boldsymbol{d}_z$ does not depend on the choice of the representative $\mu_f^{\pm}$ of $\boldsymbol{\mu}_f^{\pm}$. Write $\bar{d}_z$ for the composition of d_z with the canonical projection onto $T(K_p)/L$. By construction, L is the smallest subgroup of $T(\mathbb{Q}_p)$ such that $\boldsymbol{d}_z$ becomes trivial in $H^2\big(\Gamma, T(\mathbb{C}_p)/L\big)$, so there exists

$$\beta_z : \Gamma \longrightarrow T/L$$

such that

$$\bar{d}_z(\gamma_1, \gamma_2) = \beta_z(\gamma_1\gamma_2) \cdot \beta_z(\gamma_1)^{-1} \cdot \beta_z(\gamma_2)^{-1}$$

for all $\gamma_1, \gamma_2 \in \Gamma$ (the action of Γ on $H_E^{\pm}$ used in [24] is the trivial one). Note that β_z is well defined only up to elements in $\mathrm{Hom}(\Gamma, T/L)$. To deal with this ambiguity, recall that Γ^{ab} is a finite group ([25, Proposition 2.1]). Thus, if t is the exponent of Γ^{ab} then $t \cdot \beta_z$ is well defined (i.e., it depends only on the choice of a representative for $\boldsymbol{\mu}_f^{\pm}$).

Let H_K^+ denote the narrow Hilbert class field of K and set $G_K^+ := \mathrm{Gal}(H_K^+/K)$. Write $\mathrm{Pic}^+(\mathcal{O}_K)$ for the narrow class group of the ring of integers $\mathcal{O}_K$ of K. The reciprocity map of global class field theory induces an isomorphism

$$\mathrm{rec} : \mathrm{Pic}^+(\mathcal{O}_K) \xrightarrow{\simeq} G_K^+. \tag{7}$$

Let $\psi : K \hookrightarrow B$ be an optimal embedding of $\mathcal{O}_K$ into $R_0^D(M)$, i.e. an embedding of K into B such that $\psi(\mathcal{O}_K) = \psi(K) \cap R_0^D(M)$, and write $\mathrm{Emb}\big(\mathcal{O}_K, R_0^D(M)\big)$ for the set of such embeddings. The group $\Gamma_0^D(M)$ acts on $\mathrm{Emb}\big(\mathcal{O}_K, R_0^D(M)\big)$ by conjugation.

Recall the points z_ψ and $\bar{z}_\psi$ defined in §2.3. By Dirichlet's unit theorem, the abelian group of units in $\mathcal{O}_K^\times$ of norm 1 is free of rank one. Choose a generator ε_K of this group such that $\varepsilon_K > 1$ with respect to the fixed embedding $K \hookrightarrow \mathbb{R}$ of §2.1 and set $\gamma_\psi := \psi(\varepsilon_K) \in \Gamma_0^D(M)$.

Definition 2.3. The *Darmon points* on $T(K_p)/L$ are the points

$$\mathcal{P}_\psi^{\pm} := t \cdot \beta_{z_\psi}(\gamma_\psi) \in T(K_p)/L$$

where ψ varies in $\mathrm{Emb}\big(\mathcal{O}_K, R_0^D(M)\big)$.

To simplify our notation, in the following we will drop the dependence on the sign $\pm$ and write $\mathcal{P}_\psi = \mathcal{P}_\psi^{\pm}$.

Let ψ be as before. While, thanks to the multiplicative factor t, the point $\mathcal{P}_\psi$ does not depend on the choice of a map β_{z_ψ} splitting the 2-cocycle d_{z_ψ} modulo L, it might *a priori* depend on the choice of the representative $\mu_f^{\pm}$ of $\boldsymbol{\mu}_f^{\pm}$. In fact, this is not the case: a direct calculation explained in [25] shows that the point $\mathcal{P}_\psi$ is independent of the choice of a representative of $\boldsymbol{\mu}_f^{\pm}$ and depends only on the $\Gamma_0^D(M)$-conjugacy class of ψ. Although, in light of this result, the symbol $\mathcal{P}_{[\psi]}$ would be more appropriate, in order not to burden our notation we will continue to write $\mathcal{P}_\psi$ for the Darmon points. However, the reader should always bear in mind that $\mathcal{P}_\psi = \mathcal{P}_{\psi'}$ whenever ψ and ψ' are $\Gamma_0^D(M)$-conjugate.

2.5. **Galois action on optimal embeddings.** As in [25, §4.2], for every prime ℓ dividing MD fix orientations of $R_0^D(M)$ and $\mathcal{O}_K$ at ℓ, i.e., ring homomorphisms

$$\mathfrak{O}_\ell : R_0^D(M) \longrightarrow \mathbb{F}_{\ell^\delta}, \qquad \mathfrak{o}_\ell : \mathcal{O}_K \longrightarrow \mathbb{F}_{\ell^\delta}$$

where $\delta = 1$ if $\ell | M$ and $\delta = 2$ if $\ell | D$. There are exactly two orientations of $\mathcal{O}_K$ at every $\ell | MD$.

Definition 2.4. 1) Two embeddings $\psi, \psi' \in \mathrm{Emb}\big(\mathcal{O}_K, R_0^D(M)\big)$ are said to have *the same orientation* at a prime $\ell | MD$ if $\mathfrak{O}_\ell \circ (\psi|_{\mathcal{O}_K}) = \mathfrak{O}_\ell \circ (\psi'|_{\mathcal{O}_K})$ and are said to have *opposite orientations* at ℓ otherwise.

2) An embedding $\psi \in \mathrm{Emb}\big(\mathcal{O}_K, R_0^D(M)\big)$ is said to be *oriented* if $\mathfrak{O}_\ell \circ (\psi|_{\mathcal{O}_K}) = \mathfrak{o}_\ell$ for all primes $\ell | MD$.

We denote the set of oriented optimal embeddings of $\mathcal{O}_K$ into $R_0^D(M)$ by $\mathcal{E}\big(\mathcal{O}_K, R_0^D(M)\big)$. Note that the action of $\Gamma_0^D(M)$ on $\mathrm{Emb}\big(\mathcal{O}_K, R_0^D(M)\big)$ by conjugation restricts to an action on $\mathcal{E}\big(\mathcal{O}_K, R_0^D(M)\big)$. Moreover, if $\psi \in \mathcal{E}\big(\mathcal{O}_K, R_0^D(M)\big)$ then define

$$\psi^* := (\cdot)^* \circ \psi,$$

where $(\cdot)^*$ is the canonical involution on B.

The next lemma, the proof of which we omit, collects some basic properties of (oriented) optimal embeddings.

Lemma 2.5. *Take* $\psi, \psi' \in \mathrm{Emb}\big(\mathcal{O}_K, R_0^D(M)\big)$.

(1) *If ψ and ψ' have opposite orientations at a prime dividing MD then at most one of them is oriented in the sense of Definition 2.4.*
(2) *If ℓ is a prime dividing MD then ψ and $\omega_\ell \psi \omega_\ell^{-1}$ have opposite orientations at ℓ and the same orientation at all the other primes dividing MD.*
(3) *The embeddings ψ and $\psi \circ \tau$ have opposite orientations at all the primes dividing MD.*
(4) $\psi^* = \omega_\infty \psi \omega_\infty^{-1}$ *and* $\psi^* \in \mathcal{E}\big(\mathcal{O}_K, R_0^D(M)\big)$.

The principal ideal $(\sqrt{d_K})$ is a proper $\mathcal{O}_K$-ideal of K, so we can consider its class $\mathfrak{D}_K$ in $\mathrm{Pic}^+(\mathcal{O}_K)$; define $\sigma_K := \mathrm{rec}(\mathfrak{D}_K) \in G_K^+$. From here to the end of this subsection, for notational convenience set $R := R_0^D(M)$. We want to recall the actions of $\mathrm{Pic}^+(\mathcal{O}_K)$ and G_K^+ on the sets of $\Gamma_0^D(M)$-conjugacy classes of optimal and oriented optimal embeddings of $\mathcal{O}_K$ into R. For details, see [25, Proposition 4.2]. Let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal representing a class $[\mathfrak{a}] \in \mathrm{Pic}^+(\mathcal{O}_K)$ and let $\psi \in \mathrm{Emb}(\mathcal{O}_K, R)$. Since the quaternion algebra B is indefinite, the left R-ideal $R\psi(\mathfrak{a})$ is principal; let $a \in R$ be a generator of this ideal with positive reduced norm, which is unique up to elements in $\Gamma_0^D(M)$. The right action of $\psi(\mathcal{O}_K)$ on $R\psi(\mathfrak{a})$ shows that $\psi(\mathcal{O}_K)$ is contained in the right order of $R\psi(\mathfrak{a})$, which is equal to $a^{-1}Ra$. It can be checked that this recipe induces a well-defined action of $\mathrm{Pic}^+(\mathcal{O}_K)$ on conjugacy classes of embeddings given by

$$[\mathfrak{a}] \star [\psi] := \big[a\psi a^{-1}\big] \in \mathrm{Emb}(\mathcal{O}_K, R)/\Gamma_0^D(M).$$

Notice that if $\mathfrak{a} = (\sqrt{d_K})$ then we can take $a = \omega_\infty \cdot \psi(\sqrt{d_K})$, whence

$$\mathfrak{D}_K \star [\psi] = \big[\omega_\infty \psi \omega_\infty^{-1}\big] = [\psi^*]. \tag{8}$$

Recalling the reciprocity map of (7), for all $\sigma \in G_c^+$ and $[\psi] \in \mathrm{Emb}(\mathcal{O}_K, R)/\Gamma_0^D(M)$ define

$$\sigma \star [\psi] := \mathrm{rec}^{-1}(\sigma) \star [\psi].$$

In particular, it follows from (8) that $\sigma_K \star [\psi] = [\psi^*]$ for all $\psi \in \mathrm{Emb}(\mathcal{O}_K, R)$. Finally, for every $\sigma \in G_c^+$ and every $\psi \in \mathrm{Emb}(\mathcal{O}_K, R)$ choose an element

$$\sigma \star \psi \in \sigma \star [\psi].$$

If now ψ is an oriented optimal embedding (with respect to the orientations of $\mathcal{O}_K$ and R fixed above) then, with notation as before, the Eichler order $a^{-1}Ra$ inherits an orientation from the one of R. Actually, it can be checked that we get an induced action of $\mathrm{Pic}^+(\mathcal{O}_K)$ (and G_K^+) on the set $\mathcal{E}(\mathcal{O}_K, R)/\Gamma_0^D(M)$, and this action turns out to be free and transitive. In light of this, it is convenient to describe a (non-canonical) bijection between $\mathcal{E}(\mathcal{O}_K, R)/\Gamma_0^D(M)$

and G_K^+. To this end, fix once and for all an auxiliary embedding $\psi_0 \in \mathcal{E}(\mathcal{O}_K, R)$ and define $E : G_K^+ \to \mathcal{E}(\mathcal{O}_K, R)/\Gamma_0^D(M)$ by $E(\sigma) := \sigma \star [\psi_0]$. Finally, set

$$G := E^{-1} : \mathcal{E}(\mathcal{O}_K, R)/\Gamma_0^D(M) \longrightarrow G_K^+,$$

which is a bijection satisfying

$$G([\psi^*]) = \sigma_K \cdot G([\psi]) \tag{9}$$

for all $\psi \in \mathcal{E}(\mathcal{O}_K, R)$. Now choose for every $\sigma \in G_K^+$ an embedding $\psi_\sigma \in E(\sigma)$, so that the family $\{\psi_\sigma\}_{\sigma \in G_K^+}$ is a set of representatives of the $\Gamma_0^D(M)$-conjugacy classes of oriented optimal embeddings of $\mathcal{O}_K$ into R. If $\gamma, \gamma' \in R$ write $\gamma \sim \gamma'$ to indicate that γ and γ' are in the same $\Gamma_0^D(M)$-conjugacy class, and adopt a similar notation for (oriented) optimal embeddings of $\mathcal{O}_K$ into R. For all $\sigma, \sigma' \in G_K^+$ one has

$$\sigma \star [\psi_{\sigma'}] = \sigma \star E(\sigma') = \sigma \star \bigl(\sigma' \star [\psi_0]\bigr) = (\sigma\sigma') \star [\psi_0] = E(\sigma\sigma'),$$

which means that

$$\sigma \star \psi_{\sigma'} \sim \psi_{\sigma\sigma'}.$$

Furthermore, since $G([\psi_\sigma^*]) = \sigma \cdot G([\psi_\sigma]) = \sigma_K \sigma$, by equality (9), we deduce that

$$\psi_\sigma^* \sim \psi_{\sigma_K \sigma}$$

for all $\sigma \in G_K^+$.

To conclude this subsection, we prove a lemma which will be useful later on. As above, for every prime ℓ dividing MD fix an orientation $\mathfrak{O}_\ell : R \longrightarrow \mathbb{F}_{\ell^\delta}$ of the Eichler order R at ℓ.

Lemma 2.6. *Let $\psi, \psi' \in \mathrm{Emb}(\mathcal{O}_K, R)$. If ψ and ψ' have the same orientation at all primes dividing MD then there exists a unique $\sigma \in G_K^+$ such that $\psi \sim \sigma \star \psi'$.*

Proof. For every prime $\ell | MD$ choose the unique orientation of $\mathcal{O}_c$ at ℓ such that ψ and ψ' become *oriented* optimal embeddings. Since the action of G_K^+ on $\Gamma_0^D(M)$-conjugacy classes of oriented optimal embeddings is free and transitive, there exists a unique $\sigma \in G_K^+$ such that $[\psi] = \sigma \star [\psi']$, and the claim follows. □

Remark 2.7. The element $\sigma \in G_K^+$ whose existence is established in Lemma 2.6 depends on the chosen orientations $\mathfrak{O}_\ell$ of R.

2.6. **Real conjugation on Darmon points.** In the sequel we will need to understand the action of the local Galois group $\mathrm{Gal}(K_p/\mathbb{Q}_p)$ on Darmon points. As in §2.3, denote by τ the generator of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$. We start with the following

Lemma 2.8. $\mathcal{P}_{\psi\circ\tau} = \tau(\mathcal{P}_\psi)^{-1}$.

Proof. First recall that $\bar{z}_\psi$ satisfies the condition $\psi(\alpha)\binom{\bar{z}_\psi}{1} = \bar{\alpha}\binom{\bar{z}_\psi}{1}$, so $z_{\psi\circ\tau} = \bar{z}_\psi$. Since $(\psi \circ \tau)(u) = \psi(u^{-1}) = \gamma_\psi^{-1}$, it follows that $\mathcal{P}_{\psi\circ\tau} = t \cdot \beta_{\bar{z}_\psi}(\gamma_\psi^{-1})$. Observe that the lattice L, being contained in $T(\mathbb{Q}_p)$, inherits the trivial action of τ. With notation as before, if β_z splits $\bar{d}_z$ then $\tau \circ \beta_z$ splits $\bar{d}_{\tau(z)}$, so it follows that

$$\mathcal{P}_{\psi\circ\tau} = t \cdot \beta_{\bar{z}_\psi}(\gamma_\psi^{-1}) = t \cdot \tau\bigl(\beta_{z_\psi}(\gamma_\psi^{-1})\bigr) = \tau\bigl(t \cdot \beta_{z_\psi}(\gamma_\psi^{-1})\bigr). \tag{10}$$

It remains to show that $\beta_{z_\psi}(\gamma_\psi^{-1}) = \beta_{z_\psi}(\gamma_\psi)^{-1}$. To prove this equality, note that, since $\gamma_\psi(z_\psi) = z_\psi$ and $\mu_{f,\gamma}^\pm$ has total mass 0 for all $\gamma \in \Gamma$, one has

$$d_{z_\psi}(\gamma_\psi, \gamma) = \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{s - \gamma_\psi^{-1}(z_\psi)}{s - z_\psi} d\mu_{f,\gamma}^\pm = 1$$

for all $\gamma \in \Gamma$. Thus we obtain that

$$1 = d_{z_\psi}(\gamma_\psi, \gamma_\psi^{-1}) = \beta_{z_\psi}(1) \cdot \beta_{z_\psi}(\gamma_\psi)^{-1} \cdot \beta_{z_\psi}\bigl(\gamma_\psi^{-1}\bigr)^{-1}. \tag{11}$$

On the other hand, one has

$$1 = d_{z_\psi}(\gamma_\psi, 1) = \beta_{z_\psi}(\gamma_\psi) \cdot \beta_{z_\psi}(\gamma_\psi)^{-1} \cdot \beta_{z_\psi}(1)^{-1},$$

from which we also get $\beta_{z_\psi}(1) = 1$. Substituting in (11) gives $\beta_{z_\psi}(\gamma_\psi^{-1}) = \beta_{z_\psi}(\gamma_\psi)^{-1}$, as claimed. Equality (10) then becomes

$$\mathcal{P}_{\psi\circ\tau} = \tau\big(t \cdot \beta_{z_\psi}(\gamma_\psi)^{-1}\big) = \tau(\mathcal{P}_\psi)^{-1},$$

completing the proof. □

By part (3) of Lemma 2.5, if $\psi \in \mathrm{Emb}\big(\mathcal{O}_K, R_0^D(M)\big)$ then ψ and $\psi \circ \tau$ have opposite orientations at every prime dividing MD. Since conjugation by ω_{MD} reverses the orientation at every prime (by (2) of the same lemma) and ω_{MD} normalizes $\Gamma_0^D(M)$, Lemma 2.6 ensures that there exists a unique $\sigma_\psi \in G_K^+$ such that

$$\psi \circ \tau \sim \omega_{MD}(\sigma_\psi \star \psi)\omega_{MD}^{-1}. \tag{12}$$

Proposition 2.9. *Let w_{MD} denote the eigenvalue of W_{MD} on $H_E^\pm$. Then*

$$\tau(\mathcal{P}_\psi) = \mathcal{P}_{\sigma\star\psi}^{-w_{MD}}$$

where $\sigma = \sigma_\psi$ is the element of G_K^+ appearing in (12).

Proof. We first notice that the integration map appearing in (5) is equivariant for the action of W_{MD}. This can be shown as in [24, Proposition 5.1], where the same result is proved for W_p and W_∞, so we just sketch the arguments in this case. Observe that $\omega_{MD} \in R_0^D(Mp)$ satisfies $\Gamma\omega_{MD} = \omega_{MD}\Gamma$ and $\Gamma_0^D(Mp)\omega_{MD} = \omega_{MD}\Gamma_0^D(Mp)$. Let $\mathcal{Y}_{\mathrm{rad}}$ be a fixed radial system of representatives for $\Gamma_0^D(Mp)\backslash\Gamma$ used to compute a representative $\mu_f^\pm$ of $\boldsymbol{\mu}_f^\pm$ (so $\mu_{f,\gamma}^\pm = t_r\mu_{\mathrm{univ},\gamma}^{\mathcal{Y}_{\mathrm{rad}}}$). Introduce the system

$$\mathcal{Y}'_{\mathrm{rad}} := \big\{\omega_{MD}^{-1}\gamma_{\omega(e)}\omega_{MD}\big\}_{e\in\mathcal{E}^+}.$$

As noticed in [24, §5.2], $\mathcal{Y}'_{\mathrm{rad}}$ is again radial and thus can be used to define a representative of $\boldsymbol{\mu}_f^\pm$. Since the Hecke action is defined at the cochain level, one easily verifies that

$$\times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{t - \omega_{MD}\gamma_1^{-1}(z_\psi)}{t - \omega_{MD}z_\psi} dt_r\mu_{\mathrm{univ},\omega_{MD}\gamma_2\omega_{MD}^{-1}}^{\mathcal{Y}_{\mathrm{rad}}} = W_{MD} \cdot \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{t - \gamma_1^{-1}(z_\psi)}{t - z_\psi} dt_r\mu_{\mathrm{univ},\gamma_2}^{\mathcal{Y}'_{\mathrm{rad}}} \tag{13}$$

for all $\gamma_1, \gamma_2 \in \Gamma$. To simplify notations, set $\mu'_\gamma := t_r\mu_{\mathrm{univ},\gamma}^{\mathcal{Y}'_{\mathrm{rad}}}$ and $\psi' := \omega_{MD}(\sigma \star \psi)\omega_{MD}^{-1}$, with $\sigma = \sigma_\psi$ as in (12). Then $z_{\psi'} = \omega_{MD}(z_\psi)$ and $\bar{z}_{\psi'} = \omega_{MD}(\bar{z}_\psi)$. By (13), if β'_{z_ψ} splits the 2-cocycle

$$(\gamma_1, \gamma_2) \longmapsto \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{t - \gamma_1^{-1}(z_\psi)}{t - z_\psi} d\mu'_{\gamma_2}$$

then $\gamma \mapsto w_{MD}\beta'_{z_\psi}(\omega_{MD}^{-1}\gamma\omega_{MD})$ splits the 2-cocycle $d_{\omega_{MD}z_\psi} = d_{z_{\psi'}}$ defined in (6) (here use the fact that W_{MD} acts on T as multiplication by w_{MD}). If follows that

$$w_{MD} \cdot \beta'_{z_\psi}(\omega_{MD}^{-1}\gamma\omega_{MD}) = \beta_{z_{\psi'}}(\gamma) + \varphi(\gamma)$$

where $\varphi : \Gamma \to T(K_p)/L_T$ is a homomorphism. Therefore we get

$$\mathcal{P}_{\psi'} = t \cdot \beta_{z_{\psi'}}(\gamma_{\psi'}) = w_{MD}t \cdot \beta'_{z_\psi}(\omega_{MD}^{-1}\gamma_{\psi'}\omega_{MD}) = w_{MD}t \cdot \beta'_{z_\psi}(\gamma_{\sigma\star\psi}). \tag{14}$$

As remarked in §2.4, $t \cdot \beta'_{z_\psi}(\gamma_{\sigma\star\psi}) = t \cdot \beta_{z_\psi}(\gamma_{\sigma\star\psi})$, hence (14) implies that

$$\mathcal{P}_{\psi'} = \mathcal{P}_{\sigma\star\psi}^{w_{MD}}. \tag{15}$$

From Lemma 2.8 and (12) we obtain

$$\tau(\mathcal{P}_\psi) = \mathcal{P}_{\psi'}^{-1}, \tag{16}$$

and then combining (15) with (16) yields the result. □

2.7. **Darmon points on elliptic curves.** Define a p-adic logarithm on the quotient $T(\mathbb{C}_p)/L$ as follows. As in the introduction, let $q \in p\mathbb{Z}_p$ be Tate's period for E at p and let $\log_q$ be the branch of the p-adic logarithm such that $\log_q(q) = 0$. Since $T(\mathbb{C}_p) \simeq \mathbb{C}_p^\times$, the fact that $\log_q$ induces a map on $T(\mathbb{C}_p)/L$ is an immediate consequence of the following

Lemma 2.10. $\log_q(L) = 0$.

Proof. By [24, Theorem 7.16], the lattices L and $\langle q\rangle$ are homothetic, i.e., $L \cap \langle q\rangle$ has finite index both in L and in $\langle q\rangle$. If $n := [L : L \cap \langle q\rangle]$ then $L^n \subset \langle q\rangle$, so $\log_q(L^n) = 0$ by the choice of $\log_q$. Since $\log_q$ is a homomorphism from $\mathbb{C}_p^\times$ to $\mathbb{C}_p$, the lemma is proved. □

¿From now on, by an abuse of notation we will use the symbol $\log_q$ for both maps induced by $\log_q$ on $T(\bar{K}_p)/L$ and $\bar{K}_p^\times/\langle q\rangle$; these two maps are K_p-valued on the K_p-rational points.

As in the proof of the lemma above, set $n := [L : L \cap \langle q\rangle]$. Raising to the n-th power induces a Galois-equivariant isogeny of p-adic tori

$$T(\bar{K}_p)/L \longrightarrow \bar{K}_p^\times/\langle q\rangle \tag{17}$$

defined over K_p, and then composing (17) with Tate's analytic uniformization Φ_{Tate} introduced in (3) yields a Galois-equivariant isogeny

$$\varphi_E : T(\bar{K}_p)/L \longrightarrow E(\bar{K}_p).$$

Recall the logarithm $\log_E : E(K_p) \to K_p$ defined in the introduction. It follows that

$$\log_E\bigl(\varphi_E(x)\bigr) = n \log_q(x) \qquad \text{for all } x \in T(K_p)/L. \tag{18}$$

For every $\psi \in \mathrm{Emb}\bigl(\mathcal{O}_c, R_0^D(M)\bigr)$ define

$$P_\psi^\pm := \varphi_E\bigl(\mathcal{P}_\psi\bigr) \in E(K_p).$$

Remark 2.11. The point $P_\psi^\pm$ is independent of the representative of $\boldsymbol{\mu}_f^\pm$, and $P_\psi^\pm = P_{\psi'}^\pm$ if ψ and ψ' are $\Gamma_0^D(M)$-conjugate, since the same is true for $\mathcal{P}_\psi$,

3. Explicit formulas for Darmon points

The aim of this section is to use a control theorem of Greenberg–Stevens type in the quaternionic setting (Theorem 3.1) to obtain a canonical lift $\tilde{\boldsymbol{\mu}}_f^\pm$ of $\boldsymbol{\mu}_f^\pm$. The role of this explicit lift is twofold: on the one hand, it is used to perform computations with Darmon points (cf. Corollary 3.5); on the other hand, in Section 4 it allows us to define a p-adic L-function which, thanks to Popa's work ([34]), encodes (for suitable integers k) special values of the L-functions of the Hida forms f_k over K twisted by genus characters of K.

3.1. **Modular forms on quaternion algebras.** For any ring R and any integer $n \geq 0$ let $P_n(R) := \mathrm{Sym}^n(R^2)$ denote the R-module of homogeneous polynomials $P(x, y)$ in two variables x, y of degree $n \geq 0$ with coefficients in R. It is equipped with a right action of the group $\mathrm{GL}_2(R)$ given by the rule

$$(P|\gamma)(x, y) := P(ax + by, cx + dy) \qquad \text{for } \gamma = \bigl(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\bigr).$$

The R-linear dual $V_n(R)$ of $P_n(R)$ is then endowed with a left action of $\mathrm{GL}_2(R)$ by the formula

$$(\gamma \cdot \phi)(P) := \phi(P|\gamma).$$

Finally, if R is a field extension of F then $P_n(R)$ (respectively, $V_n(R)$) is equipped with a right (respectively, left) action of $B^\times$ via the map i_F and the rules above. With an abuse of notations, we will write $V_n(R)$ also for the locally constant sheaf on $X_0^D(Mp)$ associated with $V_n(R)$.

Let g be a cusp form of level $\Gamma_0^D(Mp)$ and weight k, suppose that g is an eigenform for all the Hecke operators in the usual Hecke algebra $\mathfrak{h}_k$ and write $\lambda : \mathfrak{h}_k \to \mathbb{C}$ for the corresponding morphism (introductions to modular forms on quaternion algebras can be found in [11, §4.2] and [21, Section 2]). Let L be a subfield of $\mathbb{C}$ containing the image of λ. If ι denotes complex conjugation, define $H^1\big(X_0^D(Mp), V_{k-2}(L)\big)^{g,\pm}$ to be the L-subspace of $H^1\big(X_0^D(Mp), V_{k-2}(L)\big)$ consisting of the elements ξ such that $\xi|T = \lambda(T)\xi$ for all $T \in \mathfrak{h}_k$ and $\xi|\iota = \pm\xi$. By [29], we know that

$$\dim_L\Big(H^1\big(X_0^D(Mp), V_{k-2}(L)\big)^{g,\pm}\Big) = 1.$$

Recall that there is a canonical isomorphism

$$H^1\big(X_0^D(Mp), V_{k-2}(\mathbb{C})\big) \simeq H^1\big(\Gamma_0^D(Mp), V_{k-2}(\mathbb{C})\big)$$

which is equivariant for the action of ι. Then the $\mathbb{C}$-vector space $H^1\big(\Gamma_0^D(Mp), V_{k-2}(\mathbb{C})\big)^{g,\pm}$ is spanned by the projection on the $\pm$-eigenspace for ι of the class represented by the 1-cocycle $\gamma \mapsto v(g)_\gamma$ with

$$v(g)_\gamma(P) := \int_\tau^{\gamma(\tau)} g(z)P(z,1)dz$$

(this class does not depend on the choice of $\tau \in \mathcal{H}$). For details, see [36, §8.2].

3.2. **Hida families.** Recall the group $\mathcal{X}$ in (1) and the Hida family $f_\infty = \sum_{n=1}^\infty a_n(\kappa)q^n$ of f in (2), where the p-adic analytic functions a_n are all defined in a p-adic neighbourhood $\mathbb{U}_f$ of 2 in $\mathcal{X}$. For a fixed even integer $k \geq 2$ in $\mathbb{U}_f$ write f_k for the normalized eigenform of weight k on $\Gamma_0(MDp)$ whose q-expansion is obtained by setting $\kappa = k$ in f_∞; in particular, $f_2 = f$ (see [6, §1.2] for details). Shrinking it if necessary, we can assume that $\mathbb{U}_f$ is contained in the residue class of 2 modulo $p-1$. For any even integer $k \in \mathbb{U}_f$ denote by F_k the finite extension of $\mathbb{Q}_p$ generated by the Fourier coefficients of f_k. In the notations of §3.1, define the one-dimensional F_k-vector space

$$\mathbb{W}_{f_k}^\pm := H^1\big(\Gamma_0^D(Mp), V_{k-2}(F_k)\big)^{f_k,\pm},$$

then fix a generator $\boldsymbol{\phi}_k^\pm$ of $\mathbb{W}_{f_k}^\pm$ as well as a representative $\phi_k^\pm$ of $\boldsymbol{\phi}_k^\pm$.

3.3. **The Control Theorem.** Let $\mathbb{X} := (\mathbb{Z}_p^2)'$ denote the set of primitive vectors in $\mathbb{Y} := \mathbb{Z}_p^2$. For any $\mathbb{Z}_p$-module G let $\tilde{\mathbb{D}}(G)$ be the group of G-valued measures on $\mathbb{Y}$ and let $\mathbb{D}(G)$ be the subgroup consisting of those measures which are supported on $\mathbb{X}$. The left action of $\Sigma := \mathrm{GL}_2(\mathbb{Q}_p) \cap \mathrm{M}_2(\mathbb{Z}_p)$ on $\tilde{\mathbb{D}}(G)$ considered in [24] is given by the integration formula

$$\int_{\mathbb{Y}} \varphi(x,y)d(\gamma\cdot\nu)(x,y) = \int_{\mathbb{Y}} \varphi\big(\gamma\cdot(x,y)\big)d\mu(x,y)$$

with $\gamma\cdot(x,y) := (ax+by, cx+dy)$ for all step functions $\varphi : \mathbb{Y} \to G$, all measures $\nu \in \tilde{\mathbb{D}}(G)$ and all matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma$. This action also induces an action of Σ on $\mathbb{D}(G)$ (see [24, Lemma 7.4]). Denote by $\pi : \mathbb{X} \to \mathbb{P}^1(\mathbb{Q}_p)$ the fibration defined by $(a,b) \mapsto a/b$, so that we obtain a canonical map

$$\pi_* : H^1\big(\Gamma_0^D(M), \mathbb{D}(\mathbb{Q}_p)\big) \longrightarrow H^1\big(\Gamma_0^D(M), \mathcal{M}(\mathbb{Q}_p)\big).$$

To simplify the notation, write $\mathbb{D} := \mathbb{D}(\mathbb{Z}_p)$. The group $H^1\big(\Gamma_0^D(M), \mathbb{D}\big)$ comes equipped with an action of the Hecke algebra $\mathcal{H}(M)$, which allows us to define its *ordinary submodule* as

$$\mathbb{W} := \bigcap_{n=1}^\infty H^1\big(\Gamma_0^D(M), \mathbb{D}\big)\big|T_p^n$$

where $T_p \in \mathcal{H}(M)$ is the p-th Hecke operator. One can show that T_p acts invertibly on $\mathbb{W}$ and that $H^1\big(\Gamma_0^D(M), \mathbb{D}\big)$ decomposes as a direct sum of $\mathbb{W}$ and a submodule on which T_p acts topologically nilpotently. Finally, let

$$\Lambda := \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!] \simeq \mathbb{Z}_p[\![T]\!]$$

be the Iwasawa algebra. Following the constructions in [19, Sections 1 and 5], one endows $\mathbb{W}$ with a structure of (finitely generated) Λ-module.

The following Control Theorem is probably known to the experts. To state this result, let us introduce a minimal set of notations. For details, proofs, and a dictionary between classical Hida families and their quaternionic counterparts, see [27, Sections 5 and 6].

As in [27, §5.3], let $\mathcal{R}$ be the integral closure of Λ in the primitive component to which f belongs. The ring $\mathcal{R}$ is a complete local noetherian domain, finitely generated as a Λ-module and equipped with a canonical map $\eta : \mathfrak{h}_\infty^{\mathrm{ord}} \to \mathcal{R}$ of Λ-algebras (where $\mathfrak{h}_\infty^{\mathrm{ord}}$ is Hida's ordinary Hecke algebra; see [27, §5.1]). If $k \in \mathbb{U}_f$ is an even positive integer there exists a continuous morphism $\vartheta_k : \mathcal{R} \to \bar{\mathbb{Q}}_p$ whose restriction to group-like elements in Λ is the character $x \mapsto x^{k-2}$ and such that $f_k = \sum_{n=1}^{\infty}(\vartheta_k \circ \eta)(T_n)q^n$, where $T_n \in \mathfrak{h}_\infty^{\mathrm{ord}}$ is the n-th Hecke operator. (It is well-known how to relate opens in $\mathcal{R}$ and in $\mathcal{X}$; see for example [23, Remark 3.9].) Set $\mathfrak{p}_k := \ker(\vartheta_k)$; if $\mathcal{R}_{\mathfrak{p}_k}$ is the localization of $\mathcal{R}$ at $\mathfrak{p}_k$, we have $\mathcal{R}_{\mathfrak{p}_k}/\mathfrak{p}_k\mathcal{R}_{\mathfrak{p}_k} \simeq F_k$. Define

$$\mathbb{W}^{\pm}_{\mathcal{R}} := \mathbb{W}^{\pm} \otimes_\Lambda \mathcal{R}, \qquad \mathbb{W}^{\pm}_{\mathfrak{p}_k} := \mathbb{W}^{\pm} \otimes_\Lambda \mathcal{R}_{\mathfrak{p}_k} = \mathbb{W}^{\pm}_{\mathcal{R}} \otimes_{\mathcal{R}} \mathcal{R}_{\mathfrak{p}_k}$$

where $\mathbb{W}^{\pm}$ denotes the $\pm$-eigenmodule for the action of W_∞ on $\mathbb{W}$.

Let $\boldsymbol{\nu} \in \mathbb{W}$, fix a representative ν of $\boldsymbol{\nu}$ and a field extension L of $\mathbb{Q}_p$. The function

$$\gamma \longmapsto \rho_k(\nu_\gamma) := \left(P(x,y) \mapsto \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x,y)d\nu_\gamma\right)$$

defined on $\Gamma_0^D(Mp)$ with values in $V_{k-2}(L)$ is a 1-cocycle. Its class in $H^1\big(\Gamma_0^D(Mp), V_{k-2}(L)\big)$, denoted $\rho_k(\boldsymbol{\nu})$, does not depend on the choice of ν. We use the same symbol for the extensions of ρ_k to $\mathbb{W}^{\pm}_{\mathcal{R}}$ and $\mathbb{W}^{\pm}_{\mathfrak{p}_k}$ defined by sending $\boldsymbol{x} \otimes r$ to $\rho_k(\boldsymbol{x}) \cdot \vartheta_k(r)$. Finally, let $h_k : \mathfrak{h}_\infty^{\mathrm{ord}} \to \mathcal{R}_{\mathfrak{p}_k}$ denote the map obtained by composing η with the localization map and define $\mathbb{W}^{\pm}_{h_k}$ to be the h_k-eigenmodule of $\mathbb{W}^{\pm}_{\mathfrak{p}_k}$.

Theorem 3.1. *Let $k \in \mathbb{U}_f$ be an even positive integer. For any choice of sign $\pm$, the $\mathcal{R}_{\mathfrak{p}_k}$-module $\mathbb{W}^{\pm}_{h_k}$ is free of rank 1 and ρ_k induces an isomorphism $\rho_k : \mathbb{W}^{\pm}_{h_k}/\mathfrak{p}_k\mathbb{W}^{\pm}_{h_k} \xrightarrow{\simeq} \mathbb{W}^{\pm}_{f_k}$.*

A proof of Theorem 3.1, which follows the arguments in [19, §5 and §6] closely, can be found in [28]. Notice that the articles [4] and [13] provide other generalizations of the results of [19] to the quaternionic setting we are working in (see also [2] and [3] for a more general approach to this kind of questions).

To ease the notations, for $\boldsymbol{x} = \sum_i \boldsymbol{x}_i \otimes r_i \in \mathbb{W}^{\pm}_{\mathcal{R}}$ or $\mathbb{W}^{\pm}_{\mathfrak{p}_k}$ we set

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x,y)dx_\gamma := \sum_i \vartheta_k(r_i) \cdot \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x,y)dx_{i,\gamma}$$

where the x_i are representatives of the classes $\boldsymbol{x}_i$, and similarly for other open subsets of $\mathbb{X}$. Furthermore, for a *representative* x of such an $\boldsymbol{x}$ we mean the choice of representatives x_i of $\boldsymbol{x}_i$ for all i. We adopt similar conventions for the $\mathcal{R}$-linear and the $\mathcal{R}_{\mathfrak{p}_2}$-linear extensions of π_*. More precisely, we write $\pi_*(\boldsymbol{x}) := \sum_i \pi_*(\boldsymbol{x}_i) \cdot \vartheta_2(r_i)$, so that $\pi_* = \rho_2$.

Proposition 3.2. *There exists $\tilde{\boldsymbol{\mu}}^{\pm}_f \in \mathbb{W}^{\pm}_{\mathcal{R}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the following properties:*

1. $\pi_*(\tilde{\boldsymbol{\mu}}^{\pm}_f) = \mathrm{res}^{\Gamma}_{\Gamma_0^D(M)}(\boldsymbol{\mu}^{\pm}_f)$ *in* $H^1\big(\Gamma_0^D(M), \mathcal{M}_0(\mathbb{Q}_p)\big)$*;*
2. *at the cost of replacing $\mathbb{U}_f$ with a smaller neighbourhood, for every $k \in \mathbb{U}_f$ there exists $\lambda^{\pm}_B(k) \in F_k^\times$ such that $\lambda^{\pm}_B(k)\boldsymbol{\phi}^{\pm}_k = \rho_k(\tilde{\boldsymbol{\mu}}^{\pm}_f)$ in $\mathbb{W}^{\pm}_{f_k}$.*

Proof. In light of Theorem 3.1, we can (and do) choose $\boldsymbol{\Phi}^{\pm} \in \mathbb{W}^{\pm}_{\mathcal{R}}$ such that $\rho_2(\boldsymbol{\Phi}^{\pm}) = \lambda^{\pm}_1\boldsymbol{\phi}^{\pm}_2$ with $\lambda^{\pm}_1 \in \mathbb{Q}_p^\times$. Fix a representative $\Phi^{\pm}$ of $\boldsymbol{\Phi}^{\pm}$ and note that $\rho_2(\boldsymbol{\Phi}^{\pm})$ is represented by the cocycle $\gamma \mapsto (\pi_*(\Phi^\pm_\gamma))(\mathbb{Z}_p)$ on $\Gamma_0^D(Mp)$. By [17, Proposition 25], there is a canonical isomorphism

$$\mathcal{S} : H^1\big(\Gamma, \mathcal{M}_0(\mathbb{Q}_p)\big)^{f,\pm} \overset{\simeq}{\longrightarrow} \mathbb{W}_f^\pm,$$

and thus there exists $\lambda_2^\pm \in \mathbb{Q}_p^\times$ such that $\mathcal{S}(\boldsymbol{\mu}_f^\pm) = \lambda_2^\pm \boldsymbol{\phi}_2^\pm$. Moreover, observe that $\mathcal{S}(\boldsymbol{\mu}_f^\pm)$ is represented by the cocycle $\gamma \mapsto \mu^\pm_{f,\gamma}(\mathbb{Z}_p)$ on $\Gamma_0^D(Mp)$.

Let $\lambda^\pm := \lambda_2^\pm/\lambda_1^\pm$, define $c_\gamma := \lambda^\pm \pi_*(\Phi^\pm_\gamma) - \mu^\pm_{f,\gamma}$ for $\gamma \in \Gamma_0^D(M)$ and set $\boldsymbol{c} := [c]$ in $H^1\big(\Gamma_0^D(M), \mathcal{M}(\mathbb{Q}_p)\big)$. Using the surjectivity of π_* proved in [24, Proposition 7.7], choose $\tilde{\boldsymbol{c}} \in \mathbb{W}^\pm \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\pi_*(\tilde{\boldsymbol{c}}) = \boldsymbol{c}$. (Note that one can choose $\tilde{\boldsymbol{c}}$ in the ordinary submodule because the arrows between Tate modules at the end of the proof of [24, Proposition 7.7] respect ordinary summands; the same remark applies to the $\pm$-eigenmodules for W_∞). Since $[\gamma \mapsto c_\gamma(\mathbb{Z}_p)] = 0$ in $\mathbb{W}_f^\pm$, one has $\rho_2(\tilde{\boldsymbol{c}}) = 0$. Therefore the class $\tilde{\boldsymbol{\mu}}_f^\pm := \lambda^\pm \boldsymbol{\Phi}^\pm - \tilde{\boldsymbol{c}}$ satisfies $\rho_2(\tilde{\boldsymbol{\mu}}_f^\pm) = \mathcal{S}(\boldsymbol{\mu}_f^\pm)$ and $\pi_*(\tilde{\boldsymbol{\mu}}_f^\pm) = \mathrm{res}^\Gamma_{\Gamma_0^D(M)}(\boldsymbol{\mu}_f^\pm)$, which proves part (1).

As for part (2), by Theorem 3.1 it is enough to observe that, since the intersection of infinitely many prime ideals of height 1 of $\mathcal{R}$ is trivial, $\rho_k(\tilde{\boldsymbol{\mu}}_f^\pm)$ is 0 only for finitely many integers k, among which the integer 2 does not appear because $\rho_2(\tilde{\boldsymbol{\mu}}_f^\pm) = \mathcal{S}(\boldsymbol{\mu}_f^\pm)$. □

3.4. **Explicit formulas for Darmon points.** As explained in [24, §7.3], we can choose a representative $\tilde{\mu}_f^\pm$ of $\tilde{\boldsymbol{\mu}}_f^\pm$ in Proposition 3.2 so that $\pi_*(\tilde{\mu}^\pm_{f,\gamma}) = \mu^\pm_{f,\gamma}$ for all $\gamma \in \Gamma_0^D(M)$ (here the map π_* is viewed as defined on cocycles). From now on we fix such a choice of $\tilde{\mu}_f^\pm$.

Let $\log : \mathbb{C}_p^\times \to \mathbb{C}_p$ be any branch of the p-adic logarithm; such a branch is determined by choosing $\theta \in \mathbb{C}_p^\times$ with $|\theta|_p < 1$ (where $|\cdot|_p$ is the p-adic absolute value on $\mathbb{C}_p$ normalized so that $|p|_p = 1/p$), setting $\log(\theta) := 0$ and requiring that log be a homomorphism. In particular, since $x - z_\psi y \in (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times$ when $(x, y) \in \mathbb{X}$, the value $\log(x - z_\psi y)$ for $(x, y) \in \mathbb{X}$ does not depend on the choice of log (see also the discussion in the proof of [7, Theorem 2.5]).

To simplify the notations, let $G_0 := \Gamma_0^D(Mp)$ and $G_1 := \Gamma_0^D(M)$. Define $G_2 = \hat{\Gamma}_0^D(M) := \omega_p^{-1} G_1 \omega_p$. Recall that Γ is the amalgamated product $\Gamma = G_1 *_{G_0} G_2$ with respect to the natural inclusion $G_0 \subset G_1$ and the injection

$$\varsigma : G_0 \longhookrightarrow G_2 \tag{19}$$

sending x to $\omega_p x \omega_p^{-1}$ (cf. [24, §4.1]).

Theorem 3.3. $\log(\mathcal{P}_\psi) = -t \cdot \displaystyle\int_{\mathbb{X}} \log(x - z_\psi y) d\tilde{\mu}^\pm_{f,\gamma_\psi} \pmod{\log(L)}$.

Proof. Define the 1-cochain $\rho_\gamma := -\int_{\mathbb{X}} \log(x - z_\psi y) d\tilde{\mu}^\pm_{f,\gamma}$ for $\gamma \in G_1$ and for $\gamma \in G_2$ let $\hat{\rho}_\gamma$ be defined from ρ_γ as in [24, §7.5]. Recall that $\pi_*(\tilde{\mu}_f^\pm) = \mu_f^\pm$. A combination of this equality with the calculations in [24, §7.6] shows that $\log(\boldsymbol{d}_z) = \Delta(\boldsymbol{\rho} - \hat{\boldsymbol{\rho}})$, where Δ is the connecting map appearing in the Mayer–Vietoris exact sequence and $\boldsymbol{\rho} - \hat{\boldsymbol{\rho}} \in H^1(G_0, K_p)$ is represented by $\rho - \hat{\rho}$. We already know that, modulo $\log(L)$, the 2-cocycle $\log(\boldsymbol{d}_z)$ is split by $\log(\beta_z)$. Hence

$$\delta\big(\log(\beta_z)\big) = \Delta(\rho - \hat{\rho}) \pmod{\log(L)},$$

where δ is the connecting map on cochains. Since t annihilates Γ^{ab}, the claim is then a consequence of the following lemma. □

Lemma 3.4. *Let A be an abelian group with trivial Γ-action and consider the portion of the Mayer–Vietoris exact sequence given by*

$$H^1(G_1, A) \oplus H^1(G_2, A) \longrightarrow H^1(G_0, A) \overset{\Delta}{\longrightarrow} H^2(\Gamma, A) \longrightarrow H^2(G_1, A) \oplus H^2(G_2, A).$$

Assume that

- *$c = \Delta(\rho)$ for some $\rho \in H^1(G_0, A)$, so there are 1-cochains $\theta_1 \in C^1(G_1, A)$ and $\theta_2 \in C^1(G_2, A)$ such that $c_{|G_1} = \delta(\theta_1)$ and $c_{|G_2} = \delta(\theta_2)$, where δ is the connecting map and $\rho = \theta_{1|G_0} - \theta_{2|G_0}$;*
- *c is trivial in $H^2(\Gamma, A)$, so $c = \delta(b)$ for some 1-cochain $b \in C^1(\Gamma, A)$.*

Then $b_{|G_1} = \theta_1 + \varphi_{1|G_1}$ and $b_{|G_2} = \theta_2 + \varphi_{2|G_2}$ where $\varphi_1, \varphi_2 : \Gamma \to A$ are group homomorphisms.

Proof. We only prove the existence of φ_1, since the result for φ_2 can be shown in the same way up to the non-trivial permutation of the set of indices $\{1, 2\}$. First of all, since $c_{|G_1} = \delta(\theta_1) = \delta(b_{|G_1})$ and all the groups involved act trivially on A, it is immediate to deduce that

$$b_{|G_1} = \theta_1 + \psi_1$$

where $\psi_1 : G_1 \to A$ is a group homomorphism. We want to prove that ψ_1 extends to a homomorphism $\varphi_1 : \Gamma \to A$. To this end, consider the natural isomorphism $\theta : G_1 \xrightarrow{\simeq} G_2$ given by $x \mapsto \omega_p x \omega_p^{-1}$ and define the homomorphism

$$\tilde{\psi}_1 := \psi_1 \circ \theta^{-1} : G_2 \longrightarrow A$$

It follows that $\tilde{\psi}_1 \circ \varsigma = \psi_{1|G_0}$, where ς is the injection (19). But then the universal property of the amalgamated product (see [35, Ch. I, §1]) ensures that there exists a homomorphism $\varphi_1 : \Gamma \to A$ such that $\varphi_{1|G_1} = \psi_1$, as was to be shown. □

Now recall the notation in §2.7.

Corollary 3.5. $\log_E(P_\psi^\pm) = -nt \cdot \int_{\mathbb{X}} \log_q(x - z_\psi y) d\tilde{\mu}^\pm_{f,\gamma_\psi}$.

Proof. Immediate from Theorem 3.3 and formula (18). □

4. Special values of L-functions

In this section we use the class $\tilde{\boldsymbol{\mu}}_f^\pm$ to define a p-adic L-function $L_p(f_\infty/K, \chi, \kappa)$ associated with f_∞, K and a genus character χ of K. This L-function is defined on the open set $\mathbb{U}_f$ attached to f (cf. Proposition 3.2). Using Popa's results in [34], we establish a link between the values of $L_p(f_\infty/K, \chi, \kappa)$ at even positive integers $\kappa = k$ and the special values of the complex L-functions of the forms f_k over K twisted by χ.

4.1. Modular forms and 1-cocycles.

First, we extend the notation employed in §3.1. For any cusp form g of level $\Gamma_0^D(S)$ with $S = M$ or $S = Mp$ and weight $k \geq 2$, any point $\tau \in \mathcal{H}$ and any element $\gamma \in B^\times$ of positive norm define $I(g, \tau, \gamma) \in V_{k-2}(\mathbb{C})$ by the formula

$$I(g, \tau, \gamma)\big(P(x, y)\big) := \int_\tau^{\gamma(\tau)} g(z) P(z, 1) dz$$

for all $P \in P_{k-2}(\mathbb{C})$. As in §3.1, let $B^\times$ act on the left on $V_n(\mathbb{C})$ via i_F. A direct computation shows that

$$\big(b \cdot I(g|b, \tau, \gamma)\big)(P) = \det\big(i_F(b)\big)^{k-2} I\big(g, b(\tau), b\gamma b^{-1}\big)(P) \tag{20}$$

for all g and τ as above, all $b, \gamma \in B^\times$ of positive norm and all $P \in P_{k-2}(\mathbb{C})$. The cohomology class in $H^1\big(\Gamma_0^D(Mp), V_{k-2}(\mathbb{C})\big)$ represented by the cocycle $\gamma \mapsto I(g, \tau, \gamma)$ does not depend on the choice of τ. Denote $\gamma \mapsto I^\pm(g, \tau, \gamma)$ the $\pm$-component of $\gamma \mapsto I(g, \tau, \gamma)$ under the action of ι; the class represented by this cocycle is again independent of the choice of τ.

If f_k^{JL} is a form on $\Gamma_0^D(Mp)$ associated with f_k by the Jacquet–Langlands correspondence, it follows that the cohomology class represented by $\gamma \mapsto I^\pm(f_k^{\mathrm{JL}}, \tau, \gamma)$ is a multiple of the generator $\boldsymbol{\phi}_k^\pm$ introduced in §3.3. From now on, for a fixed sign $\pm$ we normalize the choice of $f_k^{\mathrm{JL}} = f_{k,\pm}^{\mathrm{JL}}$ in such a way that

$$\big[\gamma \mapsto I^\pm\big(f_k^{\mathrm{JL}}, \tau, \gamma\big)\big] = \boldsymbol{\phi}_k^\pm \tag{21}$$

in $H^1\big(\Gamma_0^D(Mp),V_{k-2}(\mathbb{C})\big)^{f_k,\pm}$, where we identify $\boldsymbol{\phi}_k^\pm\in\mathbb{W}_k^\pm$ with its image in this cohomology group via the embedding in (4).

Let $L=\mathbb{Q}(\sqrt{d_L})$ be a real quadratic field with discriminant d_L such that

- all the primes dividing M are split in L;
- all the primes dividing D are inert in L;
- the prime p is unramified in L.

As before, let S denote either M or Mp and let $\psi:L\hookrightarrow B$ be an optimal embedding of the maximal order $\mathcal{O}_L$ of L into $R_0^D(S)$ (such an embedding exists unless $S=Mp$ and p is inert in L). As done in §2.3 for $L=K$, we may consider the element $i_F\big(\psi(\sqrt{d_L})\big)=\left(\begin{smallmatrix}a&b\\c&-a\end{smallmatrix}\right)$ and the quadratic form

$$Q_\psi(x,y)=cx^2-2axy-by^2.$$

In order to uniformize our notation, we assume that $L\neq F$, so that $c\neq 0$. Moreover, for every even integer $k\geq 2$ define

$$Q_\psi^{(k)}(x,y):=Q_\psi(x,y)^{\frac{k-2}{2}}.$$

As before, for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Sigma:=\mathrm{GL}_2(\mathbb{Q}_p)\cap\mathrm{M}_2(\mathbb{Z}_p)$ write $\gamma(x,y):=(ax+by,cx+dy)$. Then

$$Q_{\gamma\psi\gamma^{-1}}(x,y)=\det(\gamma)\cdot Q_\psi\big(\gamma^{-1}(x,y)\big)\tag{22}$$

for all $\gamma\in\Sigma$.

Fix an embedding $L\hookrightarrow\mathbb{R}$ (coinciding with the one fixed in §2.1 when $L=K$) and a fundamental unit ε_L of L such that $\varepsilon_L>1$ under this embedding, then set $\gamma_\psi:=\psi(\varepsilon_L)$. An easy computation shows that if $k\geq 2$ is the weight of g then

$$I(g,\psi):=I(g,\tau,\gamma_\psi)\Big(Q_\psi^{(k)}(x,y)\Big)\tag{23}$$

does not depend on τ, justifying the notation. Moreover, one checks that $I(g,\psi)$ depends only on the $\Gamma_0^D(S)$-conjugacy class of ψ. We define $I^\pm(g,\psi)$ in an analogous manner. Finally, recall that

$$\big(I^\pm(g,\tau,\gamma)|W_p\big)(P)=I^\pm(g,\tau,\omega_p\gamma\omega_p^{-1})(P|\omega_p^*)$$

so we also have

$$\big(I^\pm(g,\tau,\gamma)|W_p\big)(P)=p^{k-2}I^\pm(g,\tau,\omega_p\gamma\omega_p^{-1})(P|\omega_p^{-1})$$

and, combining with (20),

$$I^\pm(g|W_p,\tau,\gamma)(P)=p^{k-2}I^\pm(g,\omega_p\tau,\omega_p\gamma\omega_p^{-1})(P|\omega_p^{-1})=\big(I^\pm(g,\omega_p\tau,\gamma)|W_p\big)(P).$$

Since the last class does not depend on the choice of the base point, we finally obtain that

(24) the classes represented by $\gamma\mapsto I^\pm(g|W_p,\tau,\gamma)$ and $\gamma\mapsto I^\pm(g,\tau,\gamma)|W_p$ are the same.

4.2. **Preliminary calculations.** In this subsection we collect a number of computations concerning the restrictions of (the "values" of) $\tilde{\boldsymbol{\mu}}_f^\pm$ to various open subsets of $\mathbb{X}$.

If ν is a measure on $\mathbb{Y}$ and $\mathbb{U}\subset\mathbb{Y}$ is a compact open subset then let $\nu|_{\mathbb{U}}$ denote the restriction of ν to $\mathbb{U}$. Define the compact open subsets

$$\mathbb{X}_\infty:=\mathbb{Z}_p^\times\times p\mathbb{Z}_p,\qquad\mathbb{X}_{\mathrm{aff}}:=\mathbb{Z}_p\times\mathbb{Z}_p^\times$$

of $\mathbb{X}$, so that $\mathbb{X}=\mathbb{X}_{\mathrm{aff}}\coprod\mathbb{X}_\infty$. Let $\Gamma_0^{\mathrm{loc}}(p)$ denote the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ consisting of the matrices that are upper triangular modulo p. Write

$$U_p=\coprod_{i=0}^{p-1}\Gamma_0^D(Mp)g_i$$

with g_i such that $i_p(g_i) = u_i\left(\begin{smallmatrix} 1 & a_i \\ 0 & p \end{smallmatrix}\right)$ for some $u_i \in \Gamma_0^{\mathrm{loc}}(p)$ and $a_0, \ldots, a_{p-1} \in \mathbb{Z}$ reducing modulo p to a complete system of representatives of $\mathbb{Z}/p\mathbb{Z}$. Let us fix an element ω_p such that $i_p(\omega_p) = u_p\left(\begin{smallmatrix} 0 & 1 \\ -p & 0 \end{smallmatrix}\right)$ with $u_p \in \Gamma_0^{\mathrm{loc}}(p)$. Then $i_p(\omega_p^{-1}g_i) = u_i'\left(\begin{smallmatrix} 0 & -1 \\ 1 & a_i \end{smallmatrix}\right)$ for some $u_i' \in \Gamma_0^{\mathrm{loc}}(p)$. Define $\alpha_i := \omega_p^{-1}g_i$ and

$$\mathbb{X}_i := \alpha_i^* \cdot \mathbb{X}_\infty \tag{25}$$

for $i = 0, \ldots, p-1$. The set $\{\alpha_1^*, \ldots, \alpha_{p-1}^*\}$ is a complete system of representatives for the cosets of $\Gamma_0^D(Mp)$ in $\Gamma_0^D(M)$ and there is a decomposition

$$\mathbb{X}_{\mathrm{aff}} = \coprod_{i=0}^{p-1} \mathbb{X}_i. \tag{26}$$

To simplify our notation in the next few lines, set

$$\mathcal{W} := \mathbb{W}_{\mathcal{R}}^{\pm} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

and for any open subset $U \subset \mathbb{X}$ denote by $\mathcal{W}_U$ the subset of $\mathcal{W}$ consisting of cohomology classes that are supported on U. Therefore for any $\boldsymbol{\nu} \in \mathcal{W}$ and any open subset $U \subset \mathbb{X}$ one may consider the restriction $\boldsymbol{\nu}|_U$ of $\boldsymbol{\nu}$ to U, which is an element of $\mathcal{W}_U$. In light of the decompositions (25) and (26) and the above discussion, applying the operator $W_p^{-1}U_p$ to $\boldsymbol{\nu}|_{\mathbb{X}_\infty} \in \mathcal{W}_{\mathbb{X}_\infty}$ gives an element in $\mathcal{W}_{\mathbb{X}_{\mathrm{aff}}}$. On the other hand, also $\boldsymbol{\nu}|_{\mathbb{X}_{\mathrm{aff}}}$ belongs to $\mathcal{W}_{\mathbb{X}_{\mathrm{aff}}}$. The next lemma compares these two elements for $\boldsymbol{\nu} = \mathrm{res}(\tilde{\boldsymbol{\mu}}_f^{\pm})$, the restriction of $\tilde{\boldsymbol{\mu}}_f^{\pm}$ to $\Gamma_0^D(Mp)$.

Lemma 4.1. $\left(\mathrm{res}(\tilde{\boldsymbol{\mu}}_f^{\pm})|_{\mathbb{X}_\infty}\right)\Big|(W_p^{-1}U_p) = \mathrm{res}(\tilde{\boldsymbol{\mu}}_f^{\pm})|_{\mathbb{X}_{\mathrm{aff}}}$ *in* $\mathcal{W}_{\mathbb{X}_{\mathrm{aff}}}$.

Proof. A variation on the proof of [24, Lemma 7.11], where a different convention for the Hecke action is adopted. For the convenience of the reader, we briefly explain the details. An easy formal calculation shows that for all $\gamma \in \Gamma_0^D(Mp)$ we have

$$\alpha_i^* \cdot \tilde{\mu}^{\pm}_{f,\alpha_i\gamma\alpha^*_{j(i)}} = \tilde{\mu}^{\pm}_{f,\gamma} - \tilde{\mu}^{\pm}_{f,\alpha_i^*} + \gamma \cdot \tilde{\mu}^{\pm}_{f,\alpha^*_{j(i)}},$$

where $i \mapsto j(i)$ is the permutation of indices such that $\alpha_i\gamma\alpha_{j(i)}^{-1} \in \Gamma_0^D(Mp)$. Define $\gamma \mapsto n_\gamma$ to be the restriction $\gamma \mapsto \tilde{\mu}^{\pm}_{f,\gamma}|_{\mathbb{X}_\infty}$. Now $\alpha_i^* \cdot n_{\alpha_i\gamma\alpha^*_{j(i)}}$ is supported on $\mathbb{X}_i$, hence

$$\alpha_i^* \cdot n_{\alpha_i\gamma\alpha^*_{j(i)}} = \tilde{\mu}^{\pm}_{f,\gamma}|_{\mathbb{X}_i} - \tilde{\mu}_{f,\alpha_i^*}|_{\mathbb{X}_i} + \left(\gamma \cdot \tilde{\mu}^{\pm}_{f,\alpha^*_{j(i)}}\right)|_{\mathbb{X}_i}.$$

Since $\gamma \in \alpha_i^{-1}\Gamma_0^D(Mp)\alpha_{j(i)}$, we have $\left(\gamma \cdot \tilde{\mu}^{\pm}_{f,\alpha^*_{j(i)}}\right)|_{\mathbb{X}_i} = \gamma \cdot \left(\nu_{f,\alpha^*_{j(i)}}|_{\mathbb{X}_{j(i)}}\right)$ (both measures are supported on $\mathbb{X}_i$ and are in fact the restriction of $\tilde{\mu}^{\pm}_{f,\alpha^*_{j(i)}}$ to this compact open subset). Setting $m := \sum_{i=0}^{p-1} \tilde{\mu}^{\pm}_{f,\alpha_i}|_{\mathbb{X}_i}$, it follows that

$$\sum_{i=0}^{p-1} \alpha_i^* \cdot n_{\alpha_i\gamma\alpha^*_{j(i)}} = \sum_{i=0}^{p-1} n_\gamma|_{\mathbb{X}_i} + \gamma \cdot m - m.$$

Since $\alpha_i = \omega_p^{-1}g_i$, by definition the first term is $n|(W_p^{-1}U_p)$, while the cohomology class represented by the cocycle on the right hand side is $\mathrm{res}(\tilde{\boldsymbol{\mu}}_f^{\pm})|_{\mathbb{X}_{\mathrm{aff}}}$. Therefore

$$\left(\mathrm{res}(\tilde{\boldsymbol{\mu}}_f^{\pm})|_{\mathbb{X}_\infty}\right)\Big|(W_p^{-1}U_p) = \mathrm{res}(\tilde{\boldsymbol{\mu}}_f^{\pm})|_{\mathbb{X}_{\mathrm{aff}}},$$

as was to be shown. □

Notation 4.2. In the following, we often let $[\gamma \mapsto c_\gamma]$ or even $[c_\gamma]$ denote the class represented in $H^1(G, M)$ by a cocycle $c \in Z^1(G, M)$. Moreover, we write $c \equiv c'$ or (by abuse of notation) $c_\gamma \equiv c'_\gamma$ if the two cocycles c and c' are cohomologous, i.e., differ by a coboundary.

Recall the open set $\mathbb{U}_f$ introduced in Proposition 3.2. In light of the normalization (21), we can restate Proposition 3.2 as follows.

Proposition 4.3. *For any choice of $\tau \in \mathcal{H}$ and every $k \in \mathbb{U}_f$ the equality*

$$\left[P \longmapsto \int_{\mathbb{X}_{\mathrm{aff}}} P(x,y)d\tilde{\mu}^{\pm}_{f,\gamma}\right] = \left[\lambda_B^{\pm}(k)I^{\pm}\big(f_k^{\mathrm{JL}},\tau,\gamma\big)\right]$$

holds in $H^1\big(\Gamma_0^D(Mp),V_{k-2}(\mathbb{C})\big)$.

Lemma 4.4. *For every $k \in \mathbb{U}_f$, one has*

$$\left[P \longmapsto \int_{\mathbb{X}_\infty} P(x,y)d\tilde{\mu}^{\pm}_{f,\gamma}\right] = \left[P \longmapsto a_p(k)^{-1}\int_{\mathbb{X}_{\mathrm{aff}}} P(x,y)d(\tilde{\mu}^{\pm}_{f,\gamma}|W_p)\right]$$

in $H^1\big(\Gamma_0^D(Mp),V_{k-2}(F_k)\big)$.

Proof. By Lemma 4.1, there exists a $\mathbb{Z}_p$-valued measure m on $\mathbb{X}_\infty$ such that

$$\int_{\mathbb{X}_\infty} P(x,y)d\tilde{\mu}^{\pm}_{f,\gamma} = \int_{\mathbb{X}_{\mathrm{aff}}} P(x,y)d(\tilde{\mu}^{\pm}_{f,\gamma}|U_p^{-1}W_p) + \int_{\mathbb{X}_\infty} P(x,y)d(\gamma m - m).$$

Since $\tilde{\mu}^{\pm}_f|U_p^{-1} \equiv a_p(k)^{-1}\tilde{\mu}^{\pm}_f$, there exists also an F_k-valued measure m' on $\mathbb{X}_{\mathrm{aff}}$ such that

$$\int_{\mathbb{X}_{\mathrm{aff}}} P(x,y)d(\tilde{\mu}^{\pm}_{f,\gamma}|U_p^{-1}W_p) = a_p(k)^{-1}\int_{\mathbb{X}_{\mathrm{aff}}} P(x,y)d(\tilde{\mu}^{\pm}_{f,\gamma}|W_p) + \int_{\mathbb{X}_{\mathrm{aff}}} (P|\omega_p^*)(x,y)d(\gamma m' - m')$$

where $\omega_p^* := p\omega_p^{-1}$. Therefore, setting $v(P) := \int_{\mathbb{X}_\infty} Pdm + \int_{\mathbb{X}_{\mathrm{aff}}}(P|\omega_p^*)dm'$ we get the equality

$$\int_{\mathbb{X}_\infty} P(x,y)d\tilde{\mu}^{\pm}_{f,\gamma} = a_p(k)^{-1}\int_{\mathbb{X}_{\mathrm{aff}}} P(x,y)d(\tilde{\mu}^{\pm}_{f,\gamma}|W_p) + (\gamma v - v)(P),$$

which completes the proof. □

Lemma 4.5. *For any choice of $\tau \in \mathcal{H}$ and every $k \in \mathbb{U}_f$, the equality*

$$\left[P \longmapsto \int_{\mathbb{X}_{\mathrm{aff}}} P(x,y)d(\tilde{\mu}^{\pm}_{f,\gamma}|W_p)\right] = \left[\lambda_B^{\pm}(k)I^{\pm}\big(f_k^{\mathrm{JL}},\tau,\gamma\big)\big|W_p\right]$$

holds in $H^1\big(\Gamma_0^D(Mp),V_{k-2}(\mathbb{C})\big)$.

Proof. Suppose that $b \in \mathrm{GL}_2(\mathbb{Q}_p)$ normalizes $i_p\big(\Gamma_0^D(Mp)\big)$. A direct computation (essentially amounting to an application of the change of variables formula in the relevant integrals) shows that the equality of Proposition 4.3 can be rewritten as

$$\left[P(x,y) \mapsto \int_{b^{-1}\mathbb{X}_{\mathrm{aff}}} P(x,y)d(b^{-1}\tilde{\mu}^{\pm}_{f,b\gamma b^{-1}})\right] = \left[\lambda_B^{\pm}(\kappa)b^{-1}\cdot I^{\pm}(f_\kappa^{\mathrm{JL}},\tau,b\gamma b^{-1})\right]. \tag{27}$$

The lemma then follows from (27) upon taking $b^{-1} = \omega_p^*$, where ω_p^* is as in the proof of Lemma 4.4. □

Lemma 4.6. *For any choice of $\tau \in \mathcal{H}$ and every $k \in \mathbb{U}_f$, the equality*

$$\left[P \longmapsto \int_{\mathbb{X}_\infty} P(x,y)d\tilde{\mu}^{\pm}_{f,\gamma}\right] = \left[P \longmapsto \frac{\lambda_B^{\pm}(k)}{a_p(k)}\big(I^{\pm}(f_k^{\mathrm{JL}},\tau,\gamma)|W_p\big)(P)\right]$$

holds in $H^1\big(\Gamma_0^D(Mp),V_{k-2}(\mathbb{C})\big)$.

Proof. Follows by combining Lemmas 4.4 and 4.5. □

Let $k \geq 4$ be an even integer and let $f_k^\sharp$ be the modular form on $\Gamma_0(MD)$ whose p-stabilization is f_k, as defined, e.g., in [19, p. 410]. Then $f_k^\sharp$ corresponds by Jacquet–Langlands to a form $f_k^{\mathrm{JL},\sharp}$, which can be chosen in such a way that

$$f_k^{\mathrm{JL}} = f_k^{\mathrm{JL},\sharp} - a_p(k)^{-1} f_k^{\mathrm{JL},\sharp}|W_p. \tag{28}$$

Taking $\pm$-eigenspaces in the modular symbols, write

$$I^\pm\big(f_k^{\mathrm{JL}},\tau,\gamma\big)(P) = I^\pm\big(f_k^{\mathrm{JL},\sharp},\tau,\gamma\big)(P) - a_p(k)^{-1} I^\pm(f_k^{\mathrm{JL},\sharp}|W_p,\tau,\gamma)(P). \tag{29}$$

Proposition 4.7. *Let $k \geq 4$ be an even integer in $\mathbb{U}_f$ and let $\tau \in \mathcal{H}$. Then*

$$\left[P \longmapsto \int_{\mathbb{X}} P(x,y)d\tilde{\mu}^\pm_{f,\gamma}\right] = \left[\lambda_B^\pm(k)\left(1-\frac{p^{k-2}}{a_p(k)^2}\right) I^\pm\big(f_k^{\mathrm{JL},\sharp},\tau,\gamma\big)\right]$$

in $H^1\big(\Gamma_0^D(Mp), V_{k-2}\mathbb{C})\big)$.

Proof. Split the integral on the left hand side as

$$\int_{\mathbb{X}} P(x,y)d\tilde{\mu}^\pm_{f,\gamma} = \int_{\mathbb{X}_{\mathrm{aff}}} P(x,y)d\tilde{\mu}^\pm_{f,\gamma} + \int_{\mathbb{X}_\infty} P(x,y)d\tilde{\mu}^\pm_{f,\gamma}. \tag{30}$$

We begin by evaluating the first integral on the right hand side of (30). Combining Proposition 4.3, (24) and (29), we see that this is equivalent to

$$\gamma \longmapsto \Big(P \mapsto \lambda_B^\pm(k) I^\pm\big(f_k^{\mathrm{JL},\sharp},\tau,\gamma\big)(P) - \lambda_B^\pm(k) a_p(k)^{-1}\big(I^\pm(f_k^{\mathrm{JL},\sharp},\tau,\gamma)|W_p\big)(P)\Big). \tag{31}$$

Now we compute the second term on the right of (30). First, note that combining (24) and (29), we obtain an equality of classes

$$[I^\pm(f_k^{\mathrm{JL}},\tau,\gamma)|W_p] = [I^\pm(f_k^{\mathrm{JL},\sharp},\tau,\gamma)|W_p] - [a_p(k)^{-1} I^\pm(f_k^{\mathrm{JL},\sharp},\omega_p(\tau),\gamma)|W_p^2].$$

To study the second term on the right hand side of the above equation, we apply (20), the equality $f_k^{\mathrm{JL},\sharp}|W_p^2 = p^{k-2} f_k^{\mathrm{JL},\sharp}$ and the fact that the class of $\gamma \mapsto I^\pm\big(f_k^{\mathrm{JL},\sharp},\tau,\gamma\big)$ does not depend on the choice of $\tau \in \mathcal{H}$. Combining this observation with Lemma 4.6, we obtain that $P \mapsto \int_{\mathbb{X}_\infty} P(x,y)d\tilde{\mu}^\pm_{f,\gamma}$ is equivalent to

$$P \longmapsto \lambda_B^\pm(k) a_p(k)^{-1}\Big(\big(I^\pm(f_k^{\mathrm{JL},\sharp},\tau,\gamma)|W_p\big)(P) - a_p(k)^{-1} p^{k-2} I^\pm(f_k^{\mathrm{JL},\sharp},\tau,\gamma)(P)\Big). \tag{32}$$

Combining (31) and (32) yields the desired result. $\square$

Corollary 4.8. *Let $k \geq 4$ be an even integer in $\mathbb{U}_f$ and let $\tau \in \mathcal{H}$. Then*

$$\left[P \longmapsto \int_{\mathbb{X}} P(x,y)d\tilde{\mu}^\pm_{f,\gamma}\right] = \left[\lambda_B^\pm(k)\left(1-\frac{p^{k-2}}{a_p(k)^2}\right) I^\pm\big(f_k^{\mathrm{JL},\sharp},\tau,\gamma\big)\right]$$

in $H^1\big(\Gamma_0^D(M), V_{k-2}(\mathbb{C})\big)$.

Proof. The left hand side and the right hand side represent classes in $H^1\big(\Gamma_0^D(M), V_{k-2}(\mathbb{C})\big)$ that, to simplify, we denote by c_1 and c_2. Moreover, we write

$$\mathrm{res}: H^1\big(\Gamma_0^D(M), V_{k-2}(\mathbb{C})\big) \longrightarrow H^1\big(\Gamma_0^D(Mp), V_{k-2}(\mathbb{C})\big)$$

and

$$\mathrm{cores}: H^1\big(\Gamma_0^D(Mp), V_{k-2}(\mathbb{C})\big) \longrightarrow H^1\big(\Gamma_0^D(M), V_{k-2}(\mathbb{C})\big)$$

for the restriction and the corestriction maps, respectively. By Proposition 4.7, $\mathrm{res}(c_1) = \mathrm{res}(c_2)$, i.e., $c_1 - c_2 \in \ker(\mathrm{res})$. However, this kernel is trivial because $\Gamma_0^D(Mp)$ has finite index in $\Gamma_0^D(M)$. More precisely, $\ker(\mathrm{res}) \subset \ker(\mathrm{cores}\circ\mathrm{res})$ and $\mathrm{cores}\circ\mathrm{res}$ is multiplication by $[\Gamma_0^D(M) : \Gamma_0^D(Mp)]$, which acts invertibly on $V_{k-2}(\mathbb{C})$. $\square$

4.3. **Popa's work.** Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and set $B_{\mathbb{A}} := B \otimes_{\mathbb{Q}} \mathbb{A}$ for the adelization of our quaternion algebra B. Let g be a weight k normalized newform of level $\Gamma_0(SD)$ where S is a square-free positive integer prime to D. Let $R_0^D(S)$ be an Eichler order of B of level S and let $\Gamma_0^D(S)$ denote the group of its elements of norm 1. Let g^{JL} denote a modular form on $\Gamma_0^D(S)$ corresponding to g via the Jacquet–Langlands correspondence; notice that g^{JL} is well defined up to multiplication by non-zero constants. Let φ_g denote the automorphic form on $\mathrm{GL}_2(\mathbb{A})$ associated with g, whose definitions can be found, e.g., in [14, §3], and let φ_g^{JL} denote the automorphic form on $B_{\mathbb{A}}^\times$ attached to g^{JL} as in [34, Proposition 5.3.6] (see also [14, §10]). We just recall, because it will be used in the following computations, that φ_g^{JL} is defined in *loc. cit.* using the decomposition

$$B_{\mathbb{A}}^\times = B^\times (B_\infty^\times)^0 K_0^D(S) \tag{33}$$

where $K_0^D(S) := \big(R_0^D(S) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}\big)^\times$ ($\hat{\mathbb{Z}}$ denotes, as usual, the profinite completion of $\mathbb{Z}$) and $(B_\infty^\times)^0$ is the connected component of the identity in $B_\infty^\times := (B \otimes_{\mathbb{Q}} \mathbb{R})^\times$ (cf. [14, Lemma 10.3] for details).

Fix a real quadratic field $L = \mathbb{Q}(\sqrt{d_L})$ such that all the primes dividing S (respectively, D) are split (respectively, inert) in L. For simplicity, we assume that $L \neq F$ (actually, the case $L = F$ is easier but requires different conventions: note, for example, that one may diagonally embed F into $\mathrm{M}_2(F)$ and that the quadratic form associated with this embedding has $c = b = 0$, in the notations of §4.1). Set $G_L^+ := \mathrm{Gal}(H_L^+/L)$ where H_L^+ is the narrow Hilbert class field of L. Fix an unramified character $\chi : G_L^+ \to \mathbb{C}^\times$. By class field theory, χ can be identified with a finite-order Hecke character $\chi : \mathbb{A}^\times L^\times \backslash \mathbb{A}_L^\times \to \mathbb{C}^\times$, where $\mathbb{A}_L = L \otimes_{\mathbb{Q}} \mathbb{A}$ is the adele ring of L. Denote by π_χ the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ attached to χ via the Jacquet–Langlands correspondence.

Fix an embedding $L \hookrightarrow \mathbb{R}$ and choose the generator ε_L of the group of norm 1 elements in $\mathcal{O}_L$ so that $\varepsilon_L > 1$ with respect to this embedding. In the special case where $L = K$ fix the embedding $K \hookrightarrow \mathbb{R}$ as in §2.1, so that ε_L is the element denoted by ε_1 in §2.4. Write dx for the Haar measure on $L^\times \mathbb{A}^\times \backslash \mathbb{A}_L^\times$ defined in [34, p. 840], which is normalized in such a way that the total mass is equal to $h_L^+ \ln \varepsilon_L$, where h_L^+ is the cardinality of $\mathrm{Pic}^+(\mathcal{O}_L)$. As above, write $\mathrm{Emb}\big(\mathcal{O}_L, R_0^D(S)\big)$ for the set of optimal embeddings of $\mathcal{O}_L$ into $R_0^D(S)$. Fix also an embedding $\psi_0 \in \mathrm{Emb}\big(\mathcal{O}_L, R_0^D(S)\big)$ (if $L = K$ choose ψ_0 as in §2.6) and write $\psi_{0,\mathbb{A}}$ for its adelization. Also, write $\hat{\psi}_0$ and $\psi_{0,\infty}$ for the finite and the infinite part of $\psi_{0,\mathbb{A}}$, respectively. At the archimedean places of L we may also consider, as in [34, (5.3.1)], the diagonal embedding $\psi_{\mathrm{diag}} : L \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow \mathrm{M}_2(\mathbb{R})$ obtained by extending by $\mathbb{R}$-linearity the embedding $L \hookrightarrow \mathrm{M}_2(\mathbb{R})$ given by $x \mapsto \big(\begin{smallmatrix} x & 0 \\ 0 & \bar{x} \end{smallmatrix}\big)$ where $x \mapsto \bar{x}$ is the involution of L. Let $i_F\big(\psi_0(\sqrt{d_L})\big) = \big(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\big)$ and define $\gamma_\infty := \big(\begin{smallmatrix} \frac{a+\sqrt{d_L}}{c} & \frac{a-\sqrt{d_L}}{c} \\ 1 & 1 \end{smallmatrix}\big) \in \mathrm{GL}_2(\mathbb{R})$ with $c > 0$ (note that, since $L \neq F$, we always have $c \neq 0$). Then $\psi_{0,\infty} = \gamma_\infty \psi_{\mathrm{diag}} \gamma_\infty^{-1}$. Define

$$l(\varphi_g^{\mathrm{JL}}, \chi) := \int_{\mathbb{A}^\times L^\times \backslash \mathbb{A}_L^\times} \varphi_g^{\mathrm{JL}}\big(\psi_{0,\mathbb{A}}(x)\gamma_\infty\big)\chi^{-1}(x)dx.$$

By [34, Theorem 5.4.1], there is the formula

$$L(1/2, \pi_g \times \pi_\chi) = \frac{4}{\sqrt{d_L}} \cdot \prod_{\ell \mid D} \frac{\ell+1}{\ell-1} \cdot \frac{\|\varphi_g\|^2}{\|\varphi_g^{\mathrm{JL}}\|^2} \cdot |l(\varphi_g^{\mathrm{JL}}, \chi)|^2.$$

Although $l(\varphi_g^{\mathrm{JL}}, \chi)$ depends on the choice of g^{JL} and φ_g^{JL}, the right hand side of the above expression does not depend on it.

Now we compute the term $l(\varphi_g^{\mathrm{JL}},\chi)$ more explicitly. Set $\tau_0 := i_F(\gamma_\infty)(i)$. Following [34, §6.1], one shows that

$$l(\varphi_g^{\mathrm{JL}},\chi) = \frac{1}{(2i)^{k/2}\sqrt{d_L}^{(k-2)/2}} \sum_{[\mathfrak{a}]\in\mathrm{Pic}^+(\mathcal{O}_L)} \chi^{-1}([\mathfrak{a}])M([\mathfrak{a}])$$

with

$$M([\mathfrak{a}]) := \det(\alpha_{[\mathfrak{a}]})^{\frac{2-k}{2}} \int_{\tau_{[\mathfrak{a}]}}^{(\alpha_{[\mathfrak{a}]}^{-1}\psi_0\alpha_{[\mathfrak{a}]})(\varepsilon_L)\tau_{[\mathfrak{a}]}} f(z)\big(Q_{\psi_0}^{(k)}|\alpha_{[\mathfrak{a}]}\big)(z,1)dz. \tag{34}$$

The term $\alpha_{[\mathfrak{a}]}\in(B_\infty^\times)^0$ is defined, using decomposition (33), by the equation $\hat{\psi}_0(\hat{\mathfrak{a}}) = b\alpha_{[\mathfrak{a}]}^{-1}\boldsymbol{k}$ with $b\in B^\times$, $\alpha_{[\mathfrak{a}]}^{-1}\in(B_\infty^\times)^0$, $\boldsymbol{k}\in K_0^D(S)$ and $\tau_{[\mathfrak{a}]} := \alpha_{[\mathfrak{a}]}^{-1}(\tau_0)$; here $\hat{\mathfrak{a}}$ is any finite idele that corresponds to an ideal $\mathfrak{a}$ representing the class $[\mathfrak{a}]$. Although $\alpha_{[\mathfrak{a}]}$ is not unique, the results will not depend on the actual choice. To simplify notations, we also identify $\alpha_{[\mathfrak{a}]}$ with its image via i_∞ and view $\alpha_{[\mathfrak{a}]}$ as an element of $\mathrm{GL}_2^+(\mathbb{R})$.

Let h_L be the class number of L, so that $h_L^+/h_L = 1$ or 2, let $\mathrm{Pic}(\mathcal{O}_L)$ be the ideal class group of L and write H_L for the Hilbert class field of L (hence $\mathrm{Pic}(\mathcal{O}_L)\simeq\mathrm{Gal}(H_L/L)$ via the reciprocity map of class field theory). Let $\mathfrak{D}_L$ denote the class of $(\sqrt{d_L})$ in $\mathrm{Pic}^+(\mathcal{O}_L)$. If for every $\sigma\in\mathrm{Gal}(H_L/L)$ we write $[\mathfrak{a}_\sigma] = \mathrm{rec}^{-1}(\sigma)$ for an ideal $\mathfrak{a}_\sigma$ of $\mathcal{O}_L$ then

$$\mathrm{Pic}(\mathcal{O}_L) = \big\{[\mathfrak{a}_\sigma]\big\}_{\sigma\in\mathrm{Gal}(H_L/L)}$$

and

$$\mathrm{Pic}^+(\mathcal{O}_L) = \begin{cases} \big\{[\mathfrak{a}_\sigma]\big\}_{\sigma\in\mathrm{Gal}(H_L/L)} & \text{if } h_L^+ = h_L,\\ \big\{[\mathfrak{a}_\sigma]\big\}_{\sigma\in\mathrm{Gal}(H_L/L)} \cup \big\{\mathfrak{D}_L\cdot[\mathfrak{a}_\sigma]\big\}_{\sigma\in\mathrm{Gal}(H_L/L)} & \text{otherwise.}\end{cases}$$

Here, with a slight abuse of notation, we have adopted the symbol $[\star]$ to denote ideal classes both in $\mathrm{Pic}(\mathcal{O}_L)$ and in $\mathrm{Pic}^+(\mathcal{O}_L)$.

Write $\sigma_L\in G_L^+$ for the image of $\mathfrak{D}_L$ under the reciprocity map. For every $\sigma\in G_L^+$ let $[\mathfrak{a}_\sigma]$ be the corresponding element in $\mathrm{Pic}^+(\mathcal{O}_L)$ via the Artin map, represented by an ideal $\mathfrak{a}_\sigma\subset\mathcal{O}_L$. We extend in the obvious way the notation and results of §2.5 about oriented optimal embeddings to the more general case needed here. In particular, for an oriented optimal embedding $\psi\in\mathcal{E}\big(\mathcal{O}_L,R_0^D(S)\big)$ define

$$[\mathfrak{a}_\sigma]\star[\psi] := \big[a_\sigma\psi a_\sigma^{-1}\big]\in\mathcal{E}\big(\mathcal{O}_L,R_0^D(S)\big)/\Gamma_0^D(S) \tag{35}$$

where $a_\sigma\in R_0^D(S)$ has positive reduced norm and satisfies $R_0^D(S)\psi(\mathfrak{a}_\sigma) = R_0^D(S)a_\sigma$. Fixing ψ, the rule $\sigma\mapsto[\psi_\sigma]$ with $\psi_\sigma := a_\sigma\psi a_\sigma^{-1}$ is the inverse of a bijection G between $\mathcal{E}\big(\mathcal{O}_L,R_0^D(S)\big)/\Gamma_0^D(S)$ and G_L^+. The map G satisfies the equation $G([\psi^*]) = \sigma_L\cdot G([\psi])$ for all $\psi\in\mathcal{E}\big(\mathcal{O}_L,R_0^D(S)\big)$, where $\psi^* := \omega_\infty\psi\omega_\infty^{-1}$. The family $\{\psi_\sigma\}_{\sigma\in G_L^+}$ is a set of representatives of the $\Gamma_0^D(S)$-conjugacy classes of oriented optimal embeddings of $\mathcal{O}_L$ into $R_0^D(S)$. Finally, for every $\sigma\in G_L^+$ set $\gamma_\sigma := \psi_\sigma(\varepsilon_L)\in\Gamma_0^D(S)$.

If the subscript ∞ indicates that an element of $B^\times$ of positive norm is to be viewed as belonging to $(B_\infty^\times)^0$, it follows that $\hat{\psi}_0(\hat{\mathfrak{a}}_\sigma) = a_\sigma(a_\sigma^{-1})_\infty\boldsymbol{k}$, with respect to decomposition (33), for a suitable $\boldsymbol{k}\in K_0^D(S)$. Hence for every $\sigma\in G_L^+$ we can take $\alpha_{[\mathfrak{a}_\sigma]} = a_\sigma$, in the above notation. Moreover, if $h_L^+\neq h_L$ then for every $\sigma\in G_L^+$ we can also take $\alpha_{\mathfrak{D}_L\cdot[\mathfrak{a}_\sigma]}$ to be equal to $\omega_\infty a_\sigma\psi_0(\sqrt{d_L})$.

In light of the expression of the integrals $M([\mathfrak{a}_\sigma])$ given in (34), we summarize the above calculations in the following

Theorem 4.9 (Popa). *Let χ be a complex-valued character of G_L^+. Then*

$$L(1/2,\pi_g\times\pi_\chi)=\frac{4}{(2i)^k\sqrt{d_L}^{k-1}}\cdot\frac{\|\varphi_g\|^2}{\|\varphi_g^{\mathrm{JL}}\|^2}\cdot\prod_{\ell|D}\frac{\ell+1}{\ell-1}\cdot\left|\sum_{\sigma\in G_L^+}\chi^{-1}(\sigma)\int_{\tau_0}^{\gamma_\sigma(\tau_0)}g^{\mathrm{JL}}(z)Q_{\psi_\sigma}^{(k)}(z,1)dz\right|^2.$$

With notation as in (23), the next task is to rewrite more explicitly the integral

$$\Theta_\psi:=\int_{\tau_0}^{\gamma_\psi(\tau_0)}g^{\mathrm{JL}}(z)Q_\psi^{(k)}(z,1)dz=I\big(g^{\mathrm{JL}},\psi\big)$$

for any of the $\psi=\psi_\sigma$ and $\gamma_\psi=\gamma_\sigma$. Define

$$(g^{\mathrm{JL}}|\omega_\infty)(z):=(Cz+D)^{-k}\cdot\overline{g^{\mathrm{JL}}(\omega_\infty\bar z)},$$

where $i_\infty(\omega_\infty)=\left(\begin{smallmatrix}*&*\\C&D\end{smallmatrix}\right)$.

Lemma 4.10. *Up to rescaling g^{JL}, we may assume that $g^{\mathrm{JL}}|\omega_\infty=g^{\mathrm{JL}}$.*

Proof. A simple computation shows that $g^{\mathrm{JL}}|\omega_\infty$ has the same weight and level as g^{JL}. Since the Hecke eigenvalues of g are real because g is a newform with trivial character, one sees that the same is true of g^{JL}. This can be used to show that $g^{\mathrm{JL}}|\omega_\infty$ has the same eigenvalues as g^{JL} and so, by multiplicity one, there exists a complex number λ such that $g^{\mathrm{JL}}|\omega_\infty=\lambda g^{\mathrm{JL}}$. Since $\omega_\infty^2\in\Gamma_0^D(M)$, we have that $\big(g^{\mathrm{JL}}|\omega_\infty\big)|\omega_\infty=g^{\mathrm{JL}}$. (Thus, in particular, $|\lambda|=1$, but we will not need this here). If $\lambda=1$ then g^{JL} verifies the claimed condition without rescaling, while if $\lambda=-1$ then ig^{JL} does the job. In the other cases, the modular form $g^{\mathrm{JL}}+g^{\mathrm{JL}}|\omega_\infty=(1+\lambda)g^{\mathrm{JL}}$ has the required property. □

¿From now on assume that g^{JL} verifies the condition in Lemma 4.10. Using this, a simple computation shows that

$$\overline{\Theta_\psi}=\int_{\omega_\infty\bar\tau_0}^{\gamma_{\psi^*}(\omega_\infty\bar\tau_0)}g^{\mathrm{JL}}(z)Q_{\psi^*}^{(k)}(z,1)dz=I\big(g^{\mathrm{JL}},\psi^*\big)=\Theta_{\psi^*}$$

(recall that $\psi^*:=\omega_\infty\psi\omega_\infty^{-1}$). By the above discussion, the embeddings ψ_σ^* and $\psi_{\sigma_L\sigma}$ are $\Gamma_0^D(S)$-conjugate. Since Θ_ψ depends only on the $\Gamma_0^D(S)$-conjugacy class of ψ, we obtain

$$\overline{\Theta_{\psi_\sigma}}=\Theta_{\psi_{\sigma_L\sigma}}.\tag{36}$$

Let χ be a *genus character* of L, i.e., an unramified quadratic character of G_L^+. Thanks to (36), we have

$$\overline{\sum_{\sigma\in G_L^+}\chi(\sigma)\Theta_{\psi_\sigma}}=\sum_{\sigma\in G_L^+}\chi(\sigma)\overline{\Theta_{\psi_\sigma}}=\sum_{\sigma\in G_L^+}\chi(\sigma)\Theta_{\psi_{\sigma_L\sigma}}=\chi(\sigma_L)\cdot\Bigg(\sum_{\sigma\in G_L^+}\chi(\sigma)\Theta_{\psi_\sigma}\Bigg).$$

Definition 4.11. The *sign* of χ is the sign of $\chi(\sigma_L)$.

A straightforward calculation shows that

$$\overline{\int_{\tau_0}^{\gamma(\tau_0)}g(z)P(z,1)dz}=-\int_{\omega_\infty\bar\tau_0}^{\omega_\infty\gamma\omega_\infty^{-1}(\omega_\infty\bar\tau_0)}g(z)(P|\omega_\infty^*)(z,1)dz.$$

In other words, the action of complex conjugation on $I(g,\tau,\gamma)(P)$ coincides, up to a change of sign, with the action of W_∞ (for a more detailed discussion of this relation in the case of elliptic modular forms, see [20, p. 588]). Hence if χ has sign $-\epsilon\in\{\pm\}$ then

$$\sum_{\sigma\in G_L^+}\chi(\sigma)\Theta_{\psi_\sigma}=\sum_{\sigma\in G_L^+}\chi(\sigma)I^\epsilon\big(g^{\mathrm{JL}},\psi_\sigma\big).$$

Summing up, Theorem 4.9 admits the following reformulation for genus characters.

Corollary 4.12. *Let χ be a genus character of L of sign $-\epsilon$. Then*

$$L(1/2,\pi_g\times\pi_\chi)=\frac{4\epsilon}{(2i)^k\sqrt{d_L}^{k-1}}\cdot\frac{\|\varphi_g\|^2}{\|\varphi_g^{\mathrm{JL}}\|^2}\cdot\prod_{\ell\mid D}\frac{\ell+1}{\ell-1}\cdot\left(\sum_{\sigma\in G_L^+}\chi(\sigma)I^\epsilon\big(g^{\mathrm{JL}},\psi_\sigma\big)\right)^2.$$

4.4. **p-adic L-functions over real quadratic fields: the inert case.** Before studying p-adic L-functions, let us observe a simple fact. Let L be a real quadratic field and let $\psi: L\hookrightarrow B$ be an optimal embedding of $\mathcal{O}_L$ into $R_0^D(N)$, where N denotes either M or Mp. Fix a subset $\mathbb{U}$ of $\mathbb{X}$ such that $\gamma_\psi\mathbb{U}=\mathbb{U}$. Let $k\geq 2$ be an *even* integer, fix $\boldsymbol{\nu}\in\mathbb{W}$ and choose a representative ν of $\boldsymbol{\nu}$.

Lemma 4.13. *Let the notation be as above.*

1. *The value $\int_{\mathbb{U}}Q_\psi^{(k)}(x,y)d\nu_{\gamma_\psi}$ does not depend on the choice of a representative ν of $\boldsymbol{\nu}$.*
2. *If $\psi'=\alpha\psi\alpha^{-1}$ for some $\alpha\in\Gamma_0^D(N)$ then $\int_{\mathbb{U}}Q_\psi^{(k)}(x,y)d\nu_{\gamma_\psi}=\int_{\alpha\mathbb{U}}Q_{\psi'}^{(k)}(x,y)d\nu_{\gamma_\psi}$.*
3. *If $\gamma\mathbb{U}=\mathbb{U}$ for all $\gamma\in\Gamma_0^D(N)$ then $\int_{\mathbb{U}}Q_\psi^{(k)}(x,y)d\nu_{\gamma_\psi}$ depends only on the $\Gamma_0^D(N)$-conjugacy class of ψ.*

Proof. The proof follows from a direct calculation using the equality

$$Q_\psi\big(\gamma_\psi(x,y)\big)=Q_\psi(x,y),\tag{37}$$

which is a consequence of (22). Details are left to the reader. □

Recall now the fixed real quadratic field K where p is inert. Recall also the open set $\mathbb{U}_f$ introduced in Proposition 3.2. Let $\langle x\rangle$ denote the principal unit attached to $x\in\mathbb{Q}_p^\times$, defined as the unique element of $1+p\mathbb{Z}_p$ such that $x=p^{\mathrm{ord}_p(x)}\zeta_x\langle x\rangle$ where ζ_x is a $(p-1)$-st root of unity. For $k\in\mathbb{U}_f$ and $(x,y)\in\mathbb{X}$ define

$$Q_\psi^{(k)}(x,y):=Q_\psi(x,y)^{\frac{k-2}{2}}=\exp_p\left(\frac{k-2}{2}\log\big(\langle Q_\psi(x,y)\rangle\big)\right)$$

where $\exp_p$ is the p-adic exponential and log is *any* branch of the p-adic logarithm. Since $Q_\psi(x,y)=\langle Q_\psi(x,y)\rangle$ for $(x,y)\in\mathbb{X}$, this definition does not depend on the choice of log.

Definition 4.14. Let ψ be an optimal embedding of $\mathcal{O}_K$ into $R_0^D(M)$. The *partial square root p-adic L-function* attached to f_∞, K and ψ is

$$\mathcal{L}_p^{\pm}(f_\infty,\psi,k):=\int_{\mathbb{X}}Q_\psi^{(k)}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_\psi}(x,y)$$

where k varies in $\mathbb{U}_f$.

Remark 4.15. Since $Q_\psi(x,y)=\langle Q_\psi(x,y)\rangle$ on $\mathbb{X}$, we have

$$\mathcal{L}_p^{\pm}(f_\infty,\psi,k)=\int_{\mathbb{X}}\langle Q_\psi(x,y)\rangle^{\frac{k-2}{2}}d\tilde{\mu}^{\pm}_{f,\gamma_\psi}(x,y).$$

Moreover, if $k\geq 2$ is an integer in $\mathbb{U}_f$ then $Q_\psi^{(k)}(x,y)$ is equal to the $(k-2)/2$-fold self product of $Q_\psi(x,y)$. This justifies the restriction of $\mathbb{U}_f$ to the residue class of 2 modulo $p-1$ in $\mathcal{X}$.

Proposition 4.16. *$\mathcal{L}_p^{\pm}(f_\infty,\psi,k)$ is independent of the choice of a representative $\tilde{\mu}_f^{\pm}$ of $\tilde{\boldsymbol{\mu}}_f^{\pm}$ and only depends on the $\Gamma_0^D(M)$-conjugacy class of ψ.*

Proof. Lemma 4.13 shows that the restriction of $\mathcal{L}_p^{\pm}(f_\infty,\psi,k)$ to the subset of $\mathbb{U}_f$ consisting of points with trivial character has the required properties. The result then follows from a density argument. □

Proposition 4.17. *For every even integer $k \geq 4$ in $\mathbb{U}_f$ one has*

$$\mathcal{L}_p^{\pm}(f_\infty, \psi, k) = \lambda_B^{\pm}(k)\left(1 - \frac{p^{k-2}}{a_p(k)^2}\right) I^{\pm}\big(f_k^{\mathrm{JL},\sharp}, \psi\big).$$

Proof. Corollary 4.8 ensures the existence of $v \in V_{k-2}(\mathbb{C})$ such that

$$\int_{\mathbb{X}} Q_\psi^{(k)}(x,y) d\tilde{\mu}_{f,\gamma_\psi}^{\pm}(x,y) = \lambda_B^{\pm}(k)\left(1 - \frac{p^{k-2}}{a_p(k)^2}\right) I^{\pm}\big(f_k^{\mathrm{JL},\sharp}, \psi\big) + (\gamma v - v)\Big(Q_\psi^{(k)}(x,y)\Big)$$

for all $\gamma \in \Gamma_0^D(Mp)$. We conclude by evaluating this expression at $\gamma = \gamma_\psi$ and using (37). □

Definition 4.18. Let χ be a quadratic character of G_K^+ of sign $-\epsilon$ and let k vary in $\mathbb{U}_f$.

(1) The *square root p-adic L-function* attached to f_∞ and χ is

$$\mathcal{L}_p(f_\infty/K, \chi, k) := \sum_{\sigma \in G_K^+} \chi(\sigma) \mathcal{L}_p^{\epsilon}(f_\infty, \psi_\sigma, k).$$

(2) The *p-adic L-function* attached to f_∞ and χ is

$$L_p(f_\infty/K, \chi, k) := \mathcal{L}_p(f_\infty/K, \chi, k)^2.$$

Theorem 4.19. *Let χ be a genus character of K of sign $-\epsilon$. Then for all even integers $k \geq 4$ in $\mathbb{U}_f$ we have*

$$L_p(f_\infty/K, \chi, k) = \lambda_B^{\epsilon}(k)^2\left(1 - \frac{p^{k-2}}{a_p(k)^2}\right)^2 \prod_{\ell | D} \frac{\ell - 1}{\ell + 1} \cdot \frac{(2i)^k \sqrt{d_K}^{k-1}}{\epsilon 4} \cdot \frac{\|\varphi_{f_k}^{\mathrm{JL},\sharp}\|^2}{\|\varphi_{f_k^\sharp}\|^2} \cdot L_K\big(1/2, \pi_{f_k^\sharp} \times \pi_\chi\big).$$

Proof. If $k \geq 2$ then $f_k^\sharp$ is a newform on $\Gamma_0^D(M)$, so the result follows immediately from Corollary 4.12 and Proposition 4.17. □

Remark 4.20. The above result holds also for $k = 2$ if we set $f_2^\sharp := 0$. Indeed, in this case

$$\mathcal{L}_p(f_\infty/K, \chi, 2) = \int_{\mathbb{X}} d\tilde{\mu}_{f,\gamma_\psi} = \int_{\mathbb{P}^1(\mathbb{Q}_p)} d\pi_*(\mu_{f,\gamma_\psi}) = 0, \tag{38}$$

and the equality in the statement is trivially verified.

4.5. **p-adic L-functions over real quadratic fields: the split case.** We will also need an interpolation formula similar to that of Theorem 4.19 when p splits in a real quadratic field K'. We only develop the theory strictly needed for the proof of Theorem 4.33; for more general results, see [18, Section 5].

Let K' be a real quadratic field with ring of integers $\mathcal{O}_{K'}$ in which all the primes dividing D are inert and all the primes dividing Mp split. Choose a prime $\mathfrak{p}$ of K' above p and let $\sigma_{\mathfrak{p}} \in G_{K'}^+$ be its associated Frobenius element. Let K'_0 be the maximal subfield of $H_{K'}^+$ in which $\mathfrak{p}$ splits completely; the Galois group of K'_0 over K' is identified with the quotient

$$\Delta := G_{K'}^+/\langle \sigma_{\mathfrak{p}} \rangle.$$

Let t be the order of $\sigma_{\mathfrak{p}}$ and n the cardinality of Δ, so that $h_{K'} = tn$ is the narrow class number of K'. Fix an optimal embedding $\psi : K' \hookrightarrow B$ of $\mathcal{O}_{K'}$ into $R_0^D(Mp)$. Set

$$K'_p := K' \otimes_{\mathbb{Q}} \mathbb{Q}_p = K'_{\mathfrak{p}} \oplus K'_{\bar{\mathfrak{p}}} \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p$$

where $p\mathcal{O}_{K'} = \mathfrak{p}\bar{\mathfrak{p}}$ and $K'_{\mathfrak{p}}$, $K'_{\bar{\mathfrak{p}}}$ are the completions of K' at $\mathfrak{p}$ and $\mathfrak{p}'$, respectively. For any embedding $\psi : B' \hookrightarrow B$ let $\psi_p : K'_p \hookrightarrow B_p$ denote its the local component at p, where $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Recall that $\Gamma_0^{\mathrm{loc}}(p)$ is the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ consisting of upper triangular matrices modulo p. There are two $\Gamma_0^{\mathrm{loc}}(p)$-equivalence classes of optimal embeddings $\varphi : K'_p \hookrightarrow B_p$ of the split quadratic order $\mathcal{O}_{K',p} := \mathcal{O}_{K'} \otimes \mathbb{Z}_p$ into the Eichler order $R_0^D(Mp) \otimes \mathbb{Z}_p$. Note that the latter is isomorphic to the order R_p of $\mathrm{M}_2(\mathbb{Q}_p)$ consisting of the matrices in $\mathrm{M}_2(\mathbb{Z}_p)$ that are upper triangular modulo p. The two equivalence classes are switched by the action of the Atkin–Lehner involution w_p and can be characterized in terms of *local orientation at* p, as in [5, §2.2] (see also [30, §2.1]). Let φ and $\varphi' := w_p\varphi w_p^{-1}$ be representatives of the two $\Gamma_0^{\mathrm{loc}}(p)$-equivalence classes chosen in such a way that φ is the diagonal embedding sending $(a,b) \in K'_p$ to $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in R_p$ under the above isomorphisms. Then every local optimal embedding of $\mathcal{O}_{K',p}$ into R_p is $\Gamma_0^{\mathrm{loc}}(p)$-equivalent to one of these two and if two local optimal embeddings as above have the same orientation at p then they are $\Gamma_0^{\mathrm{loc}}(p)$-equivalent. In particular, if $\psi : K' \hookrightarrow B$ is a global optimal embedding of $\mathcal{O}_{K'}$ into $R_0^D(Mp)$ that, locally at p, is equivalent to φ and σ is an element of $G_{K'}^+$ then ψ^σ is again locally equivalent to φ because the Galois action preserves orientations (see [5, §2.3]).

Fix $\psi : K' \hookrightarrow B$ optimal embedding of $\mathcal{O}_{K'}$ into $R_0^D(Mp)$ such that ψ_p is locally equivalent to φ. This means that there exists $\gamma \in \Gamma_0^{\mathrm{loc}}(p)$ such that $\gamma\psi_p\gamma^{-1} = \varphi$. Put $\delta := \left(\begin{smallmatrix} \det(\gamma) & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $g := \delta^{-1}\gamma$ is a matrix in $\Gamma_0^{\mathrm{loc}}(p)$ of determinant 1 and, since $\delta^{-1}\varphi\delta = \varphi$, the embedding $g\psi_p g^{-1}$ is still diagonal. By the approximation theorem, we can find an element $u \in \Gamma_0^D(Mp)$ such that $i_p(u) \equiv g \pmod p$. Therefore, up to replacing ψ with $u\psi u^{-1}$, we can assume that

(39) ψ is an optimal embedding of $\mathcal{O}_{K'}$ into $R_0^D(Mp)$ such that ψ_p is diagonal modulo p.

Since $\psi \equiv \varphi \pmod p$, we see that $Q_\psi(x,y) \equiv Q_\varphi(x,y) = 2\sqrt{d_{K'}}xy \pmod p$, where $d_{K'}$ is the discriminant of K'. Therefore the p-adic valuation of $Q_\psi(x,y)$ is identically equal to 0 on $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$.

Choose an element $\pi \in K'$ such that $\mathrm{ord}_{\mathfrak{p}}(\pi) = 1$; then π is a uniformizer of $K'_{\mathfrak{p}}$. Since ψ_p is equal modulo p to the diagonal embedding φ and $\varphi(\pi) = \left(\begin{smallmatrix} pu & 0 \\ 0 & w \end{smallmatrix}\right)$ for some $u, w \in \mathbb{Z}_p^\times$, it follows that $\psi_p(\pi) \equiv \left(\begin{smallmatrix} 0 & 0 \\ 0 & w \end{smallmatrix}\right) \pmod p$ for some $w \in (\mathbb{Z}/p\mathbb{Z})^\times$.

Before moving on, we need some technical lemmas.

Lemma 4.21. *There exists $\alpha \in \Gamma_0^D(M)$ such that $\psi_p(\pi) = \alpha\omega_p$ and $i_p(\alpha) \in \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Gamma_0^{\mathrm{loc}}(p)$.*

Proof. We use the notation of Lemma 4.1 and write γ for $i_p(\gamma)$ if $\gamma \in \Gamma_0^D(Mp)$. The element $\psi_p(\pi)^*$ obtained by applying the involution $x \mapsto x^*$ to $\psi_p(\pi)$ has norm p and lies in $R_0^D(Mp)$, hence it can be written as γg_i for some $\gamma \in \Gamma_0^D(Mp)$. Therefore $\psi_p(\pi) = g_i^*\gamma^*$. Multiplying by ω_p^{-1}, and using the fact that this element normalizes $\Gamma_0^D(Mp)$, gives $\psi_p(\pi)\omega_p^{-1} = g_i^*\omega_p^{-1}\gamma'$ for some $\gamma' \in \Gamma_0^D(Mp)$. Now $\alpha_i = g_i^*\omega_p^{-1}$ belongs to $\Gamma_0^D(M) \cap \left(\begin{smallmatrix} i & 1 \\ -1 & 0 \end{smallmatrix}\right)\Gamma_0^{\mathrm{loc}}(p)$ and satisfies the relation $\psi_p(\pi) = \alpha_i\omega_p\gamma''$ for some $\gamma'' \in \Gamma_0^D(Mp)$. Now write $\gamma'' = \left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right)$. Then $\alpha_i\omega_p\gamma'' = \left(\begin{smallmatrix} pi(a-c) & i(pb-d) \\ pc & d \end{smallmatrix}\right)$ and, since $\psi_p(\pi) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & w \end{smallmatrix}\right)$ with $w \in \mathbb{Z}_p^\times$, we find $i = 0$. □

Lemma 4.22. *For any embedding ψ satisfying condition* (39) *and any integer $k \geq 2$ we have*

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} Q_\psi^{(k)}(x,y)d\tilde{\mu}^\pm_{f,\gamma_\psi} = \int_{\mathbb{X}_{\mathrm{aff}}} Q_\psi^{(k)}(x,y)d\tilde{\mu}^\pm_{f,\gamma_\psi} - \frac{p^{(k-2)/2}}{a_p(k)}\int_{\mathbb{X}_{\mathrm{aff}}} Q_{\psi^{\sigma_{\mathfrak{p}}}}^{(k)}(x,y)d\tilde{\mu}^\pm_{f,\gamma_{\psi^{\sigma_{\mathfrak{p}}}}}.$$

Proof. Fix an integer $k \geq 2$ and an embedding ψ satisfying (39). Using the decomposition

$$\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times = \mathbb{X}_{\mathrm{aff}} - (p\mathbb{Z}_p \times \mathbb{Z}_p^\times),$$

write

$$\int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} Q_\psi^{(k)}(x,y)d\tilde{\mu}^\pm_{f,\gamma_\psi} = \int_{\mathbb{X}_{\mathrm{aff}}} Q_\psi^{(k)}(x,y)d\tilde{\mu}^\pm_{f,\gamma_\psi} - \int_{p\mathbb{Z}_p \times \mathbb{Z}_p^\times} Q_\psi^{(k)}(x,y)d\tilde{\mu}^\pm_{f,\gamma_\psi}.$$

Choose $\alpha \in \Gamma_0^D(M)$ as in Lemma 4.21, so that $p\mathbb{Z}_p \times \mathbb{Z}_p^\times = \alpha\mathbb{X}_\infty$. There are equalities

$$\int_{p\mathbb{Z}_p \times \mathbb{Z}_p^\times} Q_\psi^{(k)}(x,y)d\tilde{\mu}^\pm_{f,\gamma_\psi} = \int_{\alpha\mathbb{X}_\infty} Q_\psi^{(k)}(x,y)d\tilde{\mu}^\pm_{f,\gamma_\psi} = \int_{\mathbb{X}_\infty} (Q_\psi^{(k)}|\alpha)(x,y)d\alpha^{-1}\tilde{\mu}^\pm_{f,\gamma_\psi}.$$

Since
$$\mu_{\alpha^{-1}\gamma_\psi\alpha}=\alpha^{-1}\mu_{\gamma_\psi}+\alpha^{-1}\gamma_\psi\mu_\alpha-\alpha^{-1}\mu_\alpha,$$
it follows that
$$\begin{aligned}&\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\alpha^{-1}\tilde{\mu}^{\pm}_{f,\gamma_\psi}\\ &=\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\tilde{\mu}^{\pm}_{f,\alpha^{-1}\gamma_\psi\alpha}+\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\alpha^{-1}(\tilde{\mu}^{\pm}_{f,\alpha}-\gamma_\psi\tilde{\mu}^{\pm}_{f,\alpha}).\end{aligned}\tag{40}$$
Using the fact that $\mathbb{X}_\infty$ is stable under the action of $\Gamma_0^D(Mp)$, an easy computation shows that
$$\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\alpha^{-1}\gamma_\psi\tilde{\mu}^{\pm}_{f,\alpha}=\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\gamma_\psi\alpha)(x,y)d\alpha^{-1}\tilde{\mu}^{\pm}_{f,\alpha}.$$
Since $Q_\psi^{(k)}|\gamma_\psi=Q_\psi^{(k)}$, one has
$$\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\alpha^{-1}(\tilde{\mu}^{\pm}_{f,\alpha}-\gamma_\psi\tilde{\mu}^{\pm}_{f,\alpha})=0,$$
and therefore (40) gives
$$\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\alpha^{-1}\tilde{\mu}^{\pm}_{f,\gamma_\psi}=\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\tilde{\mu}^{\pm}_{f,\alpha^{-1}\gamma_\psi\alpha}.$$
Using Lemma 4.4 and the equality $Q_\psi^{(k)}|\gamma_\psi=Q_\psi^{(k)}$ as above, one has
$$\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\alpha^{-1}\tilde{\mu}^{\pm}_{f,\gamma_\psi}=a_p(k)^{-1}\int_{\mathbb{X}_{\mathrm{aff}}}(Q_\psi^{(k)}|\alpha\omega_p)(x,y)d\tilde{\mu}^{\pm}_{f,(\alpha\omega_p)^{-1}\gamma_\psi(\alpha\omega_p)}.$$
(To justify this, note that Lemma 4.4 actually tells us that the two expressions differ by a coboundary. So, *a priori*, there exists $v\in V_{k-2}(F_k)$ such that the two expressions differ by the quantity $\gamma_\psi v\big(Q_\psi^{(k)}\big)-v\big(Q_\psi^{(k)}\big)$. However, since $Q_\psi^{(k)}|\gamma_\psi=Q_\psi^{(k)}$, this contribution vanishes and we obtain the above formula.) Since
$$(\alpha\omega_p)^{-1}\gamma_\psi(\alpha\omega_p)=\gamma_{(\alpha\omega_p)^{-1}\psi(\alpha\omega_p)},$$
using (22) and setting $\psi':=(\alpha\omega_p)^{-1}\psi(\alpha\omega_p)$ we obtain the relation
$$\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\alpha^{-1}\tilde{\mu}^{\pm}_{f,\gamma_\psi}=\frac{p^{(k-2)/2}}{a_p(k)}\int_{\mathbb{X}_{\mathrm{aff}}}Q_{\psi'}^{(k)}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi'}}.$$
Now Lemma 4.21 shows that ψ' is equivalent to $\psi^{\sigma_\mathfrak{p}}$ and so, by Lemma 4.13, one has
$$\int_{\mathbb{X}_\infty}(Q_\psi^{(k)}|\alpha)(x,y)d\alpha^{-1}\tilde{\mu}^{\pm}_{f,\gamma_\psi}=\frac{p^{(k-2)/2}}{a_p(k)}\int_{\mathbb{X}_{\mathrm{aff}}}Q_{\psi^{\sigma_\mathfrak{p}}}^{(k)}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi^{\sigma_\mathfrak{p}}}},$$
which completes the proof. □

Lemma 4.23. *Let ψ and ψ' be $\Gamma_0^D(Mp)$-equivalent optimal embeddings and suppose that ψ' satisfies* (39)*. Then*
$$\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}Q_{\psi'}^{(k)}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi'}}=\int_{\mathbb{X}_{\mathrm{aff}}}Q_\psi^{(k)}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_\psi}-\frac{p^{(k-2)/2}}{a_p(k)}\int_{\mathbb{X}_{\mathrm{aff}}}Q_{\psi^{\sigma_\mathfrak{p}}}^{(k)}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi^{\sigma_\mathfrak{p}}}}.$$

Proof. Since ψ and ψ' are $\Gamma_0^D(Mp)$-equivalent, the same is true for $\psi^{\sigma_\mathfrak{p}}$ and $\psi'^{\sigma_\mathfrak{p}}$. By part (3) of Lemma 4.13, the integrals $\int_{\mathbb{X}_{\mathrm{aff}}}Q_\psi^{(k)}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_\psi}$ and $\int_{\mathbb{X}_{\mathrm{aff}}}Q_{\psi^{\sigma_\mathfrak{p}}}^{(k)}d\tilde{\mu}^{\pm}_{f,\gamma_{\psi^{\sigma_\mathfrak{p}}}}$ do not depend on the choice of representatives of the $\Gamma_0^D(Mp)$-conjugacy classes $[\psi]$ and $[\psi^{\sigma_\mathfrak{p}}]$, hence
$$\int_{\mathbb{X}_{\mathrm{aff}}}Q_\psi^{(k)}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_\psi}=\int_{\mathbb{X}_{\mathrm{aff}}}Q_{\psi'}^{(k)}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi'}}$$

and

$$\int_{\mathbb{X}_{\mathrm{aff}}} Q^{(k)}_{\psi^{\sigma_{\mathfrak{p}}}}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi^{\sigma_{\mathfrak{p}}}}} = \int_{\mathbb{X}_{\mathrm{aff}}} Q^{(k)}_{\psi'^{\sigma_{\mathfrak{p}}}}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi'^{\sigma_{\mathfrak{p}}}}}.$$

The result then follows from Lemma 4.22. $\square$

Fix an optimal embedding ψ_0 satisfying condition (39) and for every $j \geq 0$ set $\psi_j := \psi_0^{\sigma_{\mathfrak{p}}^j}$. The optimal embeddings ψ_j may fail to satisfy (39); however, since ψ_0 and ψ_j have the same orientation at p, we can replace the ψ_j with $\Gamma_0^D(Mp)$-equivalent embeddings ψ'_j that satisfy this condition, as explained above.

Proposition 4.24. *Let χ' be a genus character of K'.*

(1) *For every integer $k \geq 2$ one has*

$$\sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} Q^{(k)}_{\psi'_j}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi'_j}} = \lambda_B^{\pm}(k)\left(1 - \frac{\chi'(\sigma_{\mathfrak{p}})p^{(k-2)/2}}{a_p(k)}\right) \times \sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) I^{\pm}\big(f_k^{\mathrm{JL}}, \tau, \gamma_{\psi_j}\big)(Q_{\psi_j}).$$

(2) *For every integer $k \geq 4$ one has*

$$\sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} Q^{(k)}_{\psi'_j}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi'_j}} = \lambda_B^{\pm}(k)\left(1 - \frac{\chi'(\sigma_{\mathfrak{p}})p^{(k-2)/2}}{a_p(k)}\right)^2 \times \sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) I^{\pm}\big(f_k^{\mathrm{JL},\sharp}, \tau, \gamma_{\psi_j}\big)(Q_{\psi_j}).$$

Proof. By Lemma 4.23, we have

$$\begin{aligned}\sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} Q^{(k)}_{\psi'_j}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi'_j}} &= \sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) \int_{\mathbb{X}_{\mathrm{aff}}} Q^{(k)}_{\psi_j}(x,y)d\tilde{\mu}^{\pm}_{f,\gamma_{\psi_j}} \\ &\quad - \frac{p^{(k-2)/2}}{a_p(k)} \sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) \int_{\mathbb{X}_{\mathrm{aff}}} Q^{(k)}_{\psi_{j+1}} d\tilde{\mu}^{\pm}_{f,\gamma_{\psi_{j+1}}} \\ &= \left(1 - \frac{\chi'(\sigma_{\mathfrak{p}})p^{(k-2)/2}}{a_p(k)}\right) \sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) \int_{\mathbb{X}_{\mathrm{aff}}} Q^{(k)}_{\psi_{j+1}} d\tilde{\mu}^{\pm}_{f,\gamma_{\psi_{j+1}}} \\ &= \lambda_B^{\pm}(k)\left(1 - \frac{\chi'(\sigma_{\mathfrak{p}})p^{(p-2)/2}}{a_p(k)}\right) \\ &\quad \times \sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) I^{\pm}\big(f_k^{\mathrm{JL}}, \tau, \gamma_{\psi_j}\big)(Q_{\psi_j}),\end{aligned}$$

where the first equality follows from Lemma 4.23, the second from the fact that $\sigma_{\mathfrak{p}}$ has order t and the third from Lemma 4.5. This proves part (1).

As for part (2), there is an equality

$$\begin{aligned}\sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) I^{\pm}\big(f_k^{\mathrm{JL}}, \tau, \gamma_{\psi_j}\big)(Q_{\psi_j}) &= \sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) I^{\pm}\big(f_k^{\mathrm{JL},\sharp}, \tau, \gamma_{\psi_j}\big)(Q_{\psi_j}) \\ &\quad - \sum_{j=0}^{t-1} \chi'(\sigma_{\mathfrak{p}}^j) \frac{p^{(k-2)/2}}{a_p(k)} I^{\pm}\big(f_k^{\mathrm{JL},\sharp}, \omega_p\tau, \gamma_{\omega_p\psi_j\omega_p^{-1}}\big)(Q_{\omega_p\psi_j\omega_p^{-1}}).\end{aligned}$$

But $\omega_p\psi_j\omega_p^{-1}$ is $\Gamma_0^D(M)$-equivalent to ψ_{j-1}, and the result follows. $\square$

Choose representatives $\delta_1,\dots,\delta_n$ of Δ in $G^+_{K'}$. Fix an embedding ψ_0 satisfying condition (39), then for each $j=0,\dots,t$ replace $\psi_0^{\delta_i}$ with a $\Gamma_0^D(Mp)$-equivalent embedding $\psi_0^{(i)}$ satisfying (39) and $(\psi_0^{(i)})^{\sigma_{\mathfrak{p}}^j}$ with an equivalent embedding $\psi_j^{(i)}$ satisfying (39) (this is possible because ψ_0^σ and ψ_0 have the same orientation at p for all $\sigma\in G^+_{K'}$). In this way we obtain a collection of representatives

$$\mathcal{S}:=\big\{\psi_j^{(i)}\mid i=1,\dots,n,\ \ j=0,\dots,t-1\big\}$$

for the set $\{\psi_0^\sigma\mid\sigma\in G^+_{K'}\}$ such that every element of $\mathcal{S}$ satisfies (39). We also denote by σ_ψ the element of $G^+_{K'}$ such that $\psi_0^{\sigma_\psi}$ is equivalent to ψ.

Let χ' be a quadratic character of $G^+_{K'}$ of sign $-\epsilon$ and let k vary in $\mathbb{U}_f$. Define

$$L_p(f_\infty/K',\chi',k):=\left(\sum_{\psi\in\mathcal{S}}\chi(\sigma_\psi)\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}Q_\psi^{(k)}(x,y)d\tilde\mu^\pm_{f,\gamma_\psi}\right)^2.$$

The function $L_p(f_\infty/K',\chi',k)$ can be interpreted as a p-adic L-function attached to the Hida family f_∞, the real quadratic field K' and the character χ'. More precisely, we observe that

1) the choice of ψ that we made in Lemma 4.21 shows that the set $\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times$ is stable under the action of the elements γ_{ψ_σ} for all $\psi\in\mathcal{S}$, hence, thanks to Lemma 4.13 and the density argument in the proof of Proposition 4.16, the function $L_p(f_\infty/K',\chi',k)$ is well defined independently of the choices of the representative $\tilde\mu_f^\epsilon$;

2) as in Remark 4.15, since the p-adic valuation of $Q_\psi(x,y)$ is constant on $\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times$, one can show that for all $k\in\mathbb{U}_f$ there is an equality

$$\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}Q_\psi^{(k)}(x,y)d\tilde\mu^\pm_{f,\gamma_\psi}=\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}\langle Q_\psi(x,y)\rangle^{\frac{k-2}{2}}d\tilde\mu^\pm_{f,\gamma_\psi}.$$

Theorem 4.25. *Let χ' be a genus character of K of sign $-\epsilon$.*

(1) *For $k=2$ one has*

$$L_p(f_\infty/K',\chi',2)=\lambda_B^\epsilon(2)^2\left(1-\frac{\chi'(\sigma_{\mathfrak{p}})}{a_p^2}\right)^2\prod_{\ell|D}\frac{\ell-1}{\ell+1}\cdot\frac{(2i)^2\sqrt{d_{K'}}}{4\epsilon}\times\frac{\|\varphi_f^{\mathrm{JL}}\|^2}{\|\varphi_f\|^2}\cdot L_{K'}\big(1/2,\pi_f\times\pi_{\chi'}\big).$$

(2) *For every integer $k\geq 4$ in $\mathbb{U}_f$ one has*

$$L_p(f_\infty/K',\chi',k)=\lambda_B^\epsilon(k)^2\left(1-\frac{p^{k-2}\chi'(\sigma_{\mathfrak{p}})}{a_p(k)^2}\right)^4\prod_{\ell|D}\frac{\ell-1}{\ell+1}\cdot\frac{(2i)^k\sqrt{d_{K'}}^{k-1}}{4\epsilon}\times\frac{\|\varphi_{f_k}^{\mathrm{JL},\sharp}\|^2}{\|\varphi_{f_k^\sharp}\|^2}\cdot L_{K'}\big(1/2,\pi_{f_k^\sharp}\times\pi_{\chi'}\big).$$

Proof. We can rewrite the sum in the definition of $L_p(f_\infty/K',\chi',k)$ as

$$\sum_{\psi\in\mathcal{S}}\chi'(\sigma_\psi)\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}Q_\psi^{(k)}(x,y)d\tilde\mu^\pm_{f,\gamma_\psi}=\sum_{i=1}^n\chi'(\delta_i)\sum_{j=0}^{t-1}\chi'(\sigma_{\mathfrak{p}}^j)\int_{\mathbb{Z}_p^\times\times\mathbb{Z}_p^\times}Q^{(k)}_{\psi_j^{(i)}}(x,y)d\tilde\mu^\pm_{f,\gamma_{\psi_j^{(i)}}}.$$

The theorem follows from a combination of Corollary 4.12 and Proposition 4.24. $\square$

4.6. **Atkin–Lehner involutions and genus characters.** A genus character χ of K cuts out the genus field H_χ of K given by

$$H_\chi = \mathbb{Q}\big(\sqrt{d_1}, \sqrt{d_2}\big)$$

where $d_K = d_1 d_2$. The extension $H_\chi/\mathbb{Q}$ is biquadratic unless χ is trivial (in which case, of course, $H_\chi = K$). Let χ_1, χ_2 and ϵ_K be the characters associated with the quadratic fields $\mathbb{Q}(\sqrt{d_1})$, $\mathbb{Q}(\sqrt{d_2})$ and K, respectively, so that $\epsilon_K = \chi_1 \cdot \chi_2$. In fact, the genus characters of K are in bijection with the factorizations of d into a product of relatively prime discriminants d_1 and d_2 or, equivalently, with the unordered pairs (χ_1, χ_2) of primitive quadratic Dirichlet characters of coprime conductors satisfying $\epsilon_K = \chi_1 \cdot \chi_2$ (the trivial character corresponds to the factorization $d = 1 \cdot d$). For more details see, e.g., [9, Ch. 14, §G].

Let now ψ be an oriented optimal embedding of $\mathcal{O}_K$ into $R_0^D(M)$ and recall the unique element $\sigma_\psi \in G_K^+$ introduced in (12) such that

$$\psi \circ \tau \sim \omega_{MD}(\sigma_\psi \star \psi)\omega_{MD}^{-1}. \tag{41}$$

To ease the writing, from now until the end of this subsection set $G := G_K^+$. Denote by G^2 the subgroup of G consisting of the squares of the elements of G.

Lemma 4.26. *The class of σ_ψ in G/G^2 does not depend on ψ.*

In light of this result, write $\bar{\sigma}$ for the class of any σ_ψ in G/G^2.

Proof. To begin with, recall that σ_ψ is characterized as the unique element of G such that

$$\big[\omega_{MD}^{-1}(\psi \circ \tau)\omega_{MD}\big] = \sigma_\psi \star [\psi].$$

Suppose that ψ' is another oriented optimal embedding of $\mathcal{O}_K$ into $R_0^D(M)$ and write $[\psi'] = \sigma \star [\psi]$ for a suitable $\sigma \in G$. Then

$$\begin{aligned}\big(\sigma_\psi\sigma^{-2}\big) \star [\psi'] &= \big(\sigma_\psi\sigma^{-2}\big) \star (\sigma \star [\psi]) = \sigma_\psi \star \big(\sigma^{-1} \star [\psi]\big)\\ &= \sigma^{-1} \star (\sigma_\psi \star [\psi]) = \sigma^{-1} \star \big[\omega_{MD}^{-1}(\psi \circ \tau)\omega_{MD}\big].\end{aligned} \tag{42}$$

On the other hand, $\sigma_{\psi'}$ is characterized as the unique element of G such that

$$\big[\omega_{MD}^{-1}\big((\sigma \star \psi) \circ \tau\big)\omega_{MD}\big] = \big[\omega_{MD}^{-1}(\psi' \circ \tau)\omega_{MD}\big] = \sigma_{\psi'} \star [\psi']. \tag{43}$$

If $\mathfrak{a} \subset \mathcal{O}_K$ is an ideal whose class $[\mathfrak{a}] \in \mathrm{Pic}^+(\mathcal{O}_K)$ corresponds to σ under the reciprocity map then σ^{-1} is represented by $\bar{\mathfrak{a}} := \tau(\mathfrak{a})$. With notation as in §2.5, it follows that

$$R_0^D(M)(\psi \circ \tau)(\bar{\mathfrak{a}}) = R_0^D(M)\psi(\mathfrak{a}) = R_0^D(M)a,$$

and of course $\big(a\psi a^{-1}\big) \circ \tau = a(\psi \circ \tau)a^{-1}$. Hence

$$\sigma^{-1} \star \big[\omega_{MD}^{-1}(\psi \circ \tau)\omega_{MD}\big] = \big[\omega_{MD}^{-1}\big((\sigma \star \psi) \circ \tau\big)\omega_{MD}\big], \tag{44}$$

and by comparing (42) and (43) we conclude that

$$\sigma_{\psi'} = \sigma_\psi\sigma^{-2}. \tag{45}$$

The lemma is proved. □

Remark 4.27. Equality (44) is a manifestation of the fact that the Galois extension $H_K^+/\mathbb{Q}$ is generalized dihedral, which is expressed by the formula $\tau\sigma\tau = \sigma^{-1}$ for all $\sigma \in G$.

Now we want to give a description of the quaternion algebra B in terms of the Galois element σ_ψ introduced in (41) (or, rather, of $\sigma_K\sigma_\psi$). This represents the analogue in our real quadratic setting of results for CM points on Shimura curves obtained by Jordan in [22] (see [16, Lemma 5.10] for a detailed proof of a generalization of Jordan's results).

Recall the element $\omega_\infty \in R_0^D(M)$ of norm -1 and consider

$$\mathfrak{D}_K \star [\psi \circ \tau] = \big[\omega_\infty(\psi \circ \tau)\omega_\infty^{-1}\big] = [(\psi \circ \tau)^*].$$

Since $\omega_\infty(\psi\circ\tau)\omega_\infty^{-1}$ has the same orientation as $\psi\circ\tau$ at all the primes dividing MD, it follows from Lemma 2.6 that there exists $\sigma_{\mathfrak{a}}\in G$ corresponding to (the class of) an ideal $\mathfrak{a}$ of $\mathcal{O}_K$ via the reciprocity map of class field theory such that

$$[(\psi\circ\tau)^*]=\big[\omega_{MD}(\sigma_{\mathfrak{a}}\star\psi)\omega_{MD}^{-1}\big]. \tag{46}$$

Proposition 4.28. *There is an isomorphism*

$$B\simeq\left(\frac{d_K,MD\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})}{\mathbb{Q}}\right) \tag{47}$$

of quaternion algebras over $\mathbb{Q}$. Moreover, if $\mathfrak{b}$ is an ideal of $\mathcal{O}_K$ satisfying (47) and $\sigma_{\mathfrak{b}}\in G$ corresponds to the class of $\mathfrak{b}$ in $\mathrm{Pic}^+(\mathcal{O}_K)$ then the images of $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ in G/G^2 are equal.

Proof. Set $i:=\psi(\sqrt{d_K})\in R_0^D(M)$. We first show the following

Claim. *There exists $j\in R_0^D(M)$ such that $j^2=MD\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})$ and $ij=-ji$.*

Proof of Claim. As in §2.5, let $a\in R_0^D(M)$ be an element of positive reduced norm such that $R_0^D(M)\psi(\mathfrak{a})=R_0^D(M)a$. In particular, the reduced norm of a is equal to $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})$. Then equality (46) means that there exists $\alpha\in\Gamma_0^D(M)$ such that

$$\alpha\omega_\infty(\psi\circ\tau)\omega_\infty^{-1}\alpha^{-1}=\omega_{MD}a\psi a^{-1}\omega_{MD}^{-1}.$$

Define $j:=\omega_\infty^{-1}\alpha^{-1}\omega_{MD}a\in R_0^D(M)$, so that

$$\psi\circ\tau=j\psi j^{-1}. \tag{48}$$

Now we want to prove that $j^2\in\mathbb{Q}^\times$ and $ij=-ji$. To begin with, the relation $ij=-ji$ follows immediately from (48) and the fact that $(\psi\circ\tau)(\sqrt{d_K})=-\psi(\sqrt{d_K})$. As for the first assertion, we use the canonical injection $B\hookrightarrow\mathrm{M}_2(F)$ and denote by the same symbol ψ the injection $K\hookrightarrow\mathrm{M}_2(F)$ obtained by composing ψ with this map. Recall that, in our notation, $\psi(\sqrt{d_K})=\left(\begin{smallmatrix}a&b\\c&-a\end{smallmatrix}\right)\in\mathrm{M}_2(F)$. Write $j=\left(\begin{smallmatrix}x&y\\z&w\end{smallmatrix}\right)\in\mathrm{M}_2(F)$. Since $ij=-ji$, we see that $w=-x$, and then it follows that $j^2=\left(\begin{smallmatrix}-\mathrm{norm}(j)&0\\0&-\mathrm{norm}(j)\end{smallmatrix}\right)$. Thus $j^2\in\mathbb{Q}^\times$ because $\mathrm{norm}(j)\in\mathbb{Q}$. Finally, since $\mathrm{norm}(j)=-MD\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})\in\mathbb{Z}$ we conclude that $j^2=MD\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})$. □

Since $i,j\in R_0^D(M)$, in light of the above Claim we conclude that

$$B\simeq\mathbb{Q}\oplus\mathbb{Q}i\oplus\mathbb{Q}j\oplus\mathbb{Q}ij=\left(\frac{d_K,MD\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})}{\mathbb{Q}}\right).$$

To complete the proof of the proposition, one can proceed as in [16, Remark 5.11] and note that, by genus theory, if the ideal $\mathfrak{b}$ of $\mathcal{O}_K$ satisfies (47) then $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$ have the same image in G/G^2. □

With notation as before, it follows that

$$\sigma_\psi=\sigma_K\sigma_{\mathfrak{a}}. \tag{49}$$

Now let λ be a prime number that splits in K and is such that $\lambda\equiv MD\pmod{d_K}$. A calculation with local Hilbert symbols analogous to the one in [34, §5.1] shows that there is an isomorphism

$$B\simeq\left(\frac{d_K,MD\lambda}{\mathbb{Q}}\right) \tag{50}$$

of quaternion algebras over $\mathbb{Q}$ (here we use the fact that M is square-free, although the condition that M be a product of odd powers of distinct primes would suffice). Choose an ideal $\mathfrak{b}$ of $\mathcal{O}_K$ such that $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b})=\lambda$ (such a $\mathfrak{b}$ exists because λ splits in K). Combining isomorphism (50) with Proposition 4.28 and equality (49), it follows that the class of $\sigma_K\sigma_{\mathfrak{b}}$ in G/G^2 coincides with the class $\bar{\sigma}$ of σ_ψ.

Suppose that the genus character χ is associated with the pair (χ_1, χ_2) of quadratic characters. It is now straightforward to prove the main result of this subsection.

Proposition 4.29. $\chi(\bar{\sigma}) = \chi_1(-MD)$.

Proof. With notation as before, the congruence $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}) \equiv MD \pmod{d_K}$ implies that $\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}) \equiv MD \pmod{d_1}$. Hence, since $\chi(\sigma_K) = \mathrm{sign}(d_1) = \chi_1(-1)$ (cf. [9, Remark 14.47]), we get that

$$\chi(\bar{\sigma}) = \chi(\sigma_K) \cdot \chi(\sigma_{\mathfrak{b}}) = \chi_1(-MD),$$

as was to be shown. □

Remark 4.30. Throughout this article we assume that $D > 1$, i.e., that B is *not* isomorphic to the split algebra $\mathrm{M}_2(\mathbb{Q})$. However, a moment's thought reveals that the arguments in this subsection remain valid also when $D = 1$. With this in mind, it turns out that if $D = 1$ then Proposition 4.29 reduces to [7, Proposition 1.8], and so our arguments provide an alternative (and more conceptual) proof of this result.

4.7. **Derivatives of p-adic L-functions and Darmon points.** For every embedding $\psi \in \mathrm{Emb}\big(\mathcal{O}_K, R_0^D(M)\big)$ set $J_\psi^\pm := \log_E(P_\psi^\pm)$, then for a genus character χ of K of sign $-\epsilon \in \{\pm\}$ define

$$J_\chi := \sum_{\sigma \in G_K^+} \chi(\sigma) J_{\psi_\sigma}^\epsilon = \log_E(P_\chi)$$

with $P_\chi := \sum_{\sigma \in G_K^+} \chi(\sigma) P_{\psi_\sigma}^\epsilon \in E(K_p)$. Finally, recall the integers n and t in Corollary 3.5.

Theorem 4.31. *With notation as before, there is an equality*

$$\frac{d}{dk}\mathcal{L}_p(f_\infty/K, \chi, k)_{k=2} = \frac{w_{MD}\chi_1(-MD) - 1}{2nt} J_\chi.$$

Proof. Recall the choice $\log_q$ of the branch of the logarithm made in §2.7. The integrals below do not depend on this choice, as observed before Definition 4.14. However, we will write $\log_q$ instead of a generic log to directly obtain the result to be shown. First, we have

$$\frac{d}{dk}\mathcal{L}_p^\epsilon(f_\infty, \psi, k)_{k=2} = \int_{\mathbb{X}} \log_q\big(Q_\psi(x,y)\big) d\tilde{\mu}_{f,\gamma_\psi}^\epsilon.$$

Since $Q_\psi(x,y) = c(x - z_\psi y)(x - \bar{z}_\psi y)$, we can write

$$\log_q\big(Q_\psi(x,y)\big) = \log_q(c) + \log_q(x - z_\psi y) + \log_q(x - \bar{z}_\psi y).$$

Since

$$\int_{\mathbb{X}} \log_q(c) d\tilde{\mu}_{f,\gamma_\psi}^\epsilon = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_q(c) d\pi_*(\tilde{\mu}_f^\epsilon)_{\gamma_\psi} = 0$$

(because $\pi_*(\tilde{\mu}_f)$ is cohomologous to μ_f and thus has total mass 0 at every γ), we find that

$$\frac{d}{dk}\mathcal{L}_p^\epsilon(f_\infty, \psi, k)_{k=2} = \frac{1}{2}\int_{\mathbb{X}} \log_q(x - z_\psi y) d\mu_{f,\gamma_\psi}^\epsilon + \frac{1}{2}\int_{\mathbb{X}} \log_q(x - \bar{z}_\psi y) d\mu_{f,\gamma_\psi}^\epsilon.$$

Thanks to Corollary 3.5, the first term on the right hand side is equal to $-J_\psi^\epsilon/2nt$. By the same result, the second term on the right hand side is equal to $-\tau(J_\psi^\epsilon)/2nt$. Proposition 2.9 then shows that

$$\frac{d}{dk}\mathcal{L}_p^\epsilon(f_\infty, \psi, k)_{k=2} = \frac{w_{MD} J_{\sigma_\psi \star \psi}^\epsilon - J_\psi^\epsilon}{2nt} \tag{51}$$

with σ_ψ as in (12). Using formula (45), it is easy to check that the map from G_K^+ to itself sending σ to $\sigma_{\psi_\sigma}\sigma$ is a bijection. In light of this fact and Proposition 4.29, specializing (51) to $\psi = \psi_\sigma$ and then summing over all $\sigma \in G_K^+$ yields the result. □

Corollary 4.32. *There is an equality*

$$\frac{d^2}{d^2k}L_p(f_\infty/K,\chi,k)_{k=2}=\begin{cases}J_\chi^2/(nt)^2 & \text{if } \chi_1(-MD)=-w_{MD};\\ 0 & \text{if } \chi_1(-MD)=w_{MD}.\end{cases}$$

Proof. The second derivative of $L_p(f_\infty/K,\chi,k)$ evaluated at $k=2$ is equal to the sum

$$2\left(\frac{d}{dk}\mathcal{L}_p(f_\infty/K,\chi,k)_{k=2}\right)^2+2\mathcal{L}_p(f_\infty/K,\chi,2)\frac{d^2}{d^2k}\mathcal{L}_p(f_\infty/K,\chi,k)_{k=2}.$$

Since $\mathcal{L}_p(f_\infty/K,\chi,2)=0$ by (38), the result follows from Theorem 4.31. □

4.8. **Factorization formulas for p-adic L-functions.** For $j=1,2$ let $L_p(f_\infty,\chi_j,k,s)$ be the Mazur–Kitagawa p-adic L-function associated with f_∞ and χ_j; we refer to [6, §1.4] for its definition and main properties. In particular, recall that, by [7, Corollary 3.2], if $\chi_j(-1)=(-1)^{(k-2)/2}\tilde{\epsilon}$ with $\tilde{\epsilon}=\pm1$ (since K is real, $\epsilon_K(-1)=1$, so $\chi_1(-1)=\chi_2(-1)$) we have

$$L_p(f_\infty,\chi_j,k,k/2)=\lambda^{\tilde{\epsilon}}(k)\left(1-\frac{\chi_j(p)p^{(k-2)/2}}{a_p(k)}\right)^2\cdot\frac{\Gamma(k/2)\tau(\chi_j)}{(-2\pi i)^{(k-2)/2}}\cdot\frac{L(f_k^\sharp,\chi_j,k/2)}{\Omega^{\tilde{\epsilon}}_{f_k^\sharp}}\tag{52}$$

where $\tau(\chi_j):=\sum_{a=1}^{d_j}\chi_j(a)e^{2\pi ia/d_j}$ is the Gauss sum attached to χ_j, $\Omega^{\tilde{\epsilon}}_{f_k}$ is the complex period of $f_k^\sharp$ defined as in [6, Proposition 1.1] and $\lambda^{\tilde{\epsilon}}(k)\in\mathbb{C}_p$ is defined as in [6, Theorem 1.5]. Finally, recall that the sign of the functional equation of the L-function $L(f,\delta,s)$ of f twisted by a quadratic Dirichlet character δ is equal to $-w_{MDp}\delta(-MDp)$ and that the p-adic L-function $L_p(f,\delta,s)$ has an exceptional zero at $s=1$ if and only if $\delta(p)=-w_p$.

The following factorization result will play a crucial role in our subsequent arguments.

Theorem 4.33. *There exist a neighbourhood $\mathbb{U}\subset\mathbb{U}_f$ of 2 and a p-adic analytic function η on $\mathbb{U}$ such that*

(1) *$\eta(k)\neq0$ for all $k\in\mathbb{U}$ and $\eta(2)\in(\mathbb{Q}^\times)^2$;*
(2) *for all $k\in\mathbb{U}$ there is a factorization*

$$L_p(f_\infty/K,\chi,k)=\eta(k)L_p(f_\infty,\chi_1,k,k/2)L_p(f_\infty,\chi_2,k,k/2).$$

Proof. By comparing Euler factors, we first notice that there is a factorization of complex L-functions

$$L(s,\pi_{f_k}\times\pi_\chi)=L(s,\pi_{f_k}\times\pi_{\chi_1})L(s,\pi_{f_k}\times\pi_{\chi_2}).$$

On the other hand (see, e.g., [15, p. 202]), for $j=1,2$ one has

$$L\big(s,\pi_{f_k^\sharp}\times\pi_{\chi_j}\big)=\frac{\Gamma(s+(k-1)/2)}{(2\pi)^{s+(k-1)/2}}L\big(f_k^\sharp,\chi_j,s+(k-1)/2\big).$$

Let the sign of χ be $-\epsilon$. Then it follows from Theorem 4.19 that for all even integers $k\geq4$ in $\mathbb{U}_f$ there is an equality

$$\begin{aligned}L_p(f_\infty/K,\chi,k)=\lambda_B^\epsilon(k)^2\left(1-\frac{p^{k-2}}{a_p(k)^2}\right)^2\prod_{\ell\mid D}\frac{\ell-1}{\ell+1}\cdot\frac{\sqrt{d_K}^{k-1}}{4\epsilon}\cdot\frac{\|\varphi^{\mathrm{JL}}_{f_k^\sharp}\|^2}{\|\varphi_{f_k^\sharp}\|^2}\cdot\frac{\Gamma(k/2)^2}{(-i\pi)^k}\\ \times L\big(f_k^\sharp,\chi_1,k/2\big)L\big(f_k^\sharp,\chi_2,k/2\big).\end{aligned}\tag{53}$$

Since p is inert in K, we have $\chi_1(p)=-\chi_2(p)$, and then formulas (52) and (53) imply that

$$L_p(f_\infty/K,\chi,k)=\eta(k)L_p(f_\infty,\chi_1,k,k/2)L_p(f_\infty,\chi_2,k,k/2)\tag{54}$$

for all even integers $k\geq4$, where $\eta(k)\in\mathbb{C}_p$ can be made explicit by comparing (52) and (53).

Now we show that η can be extended to an analytic function with the prescribed properties. For this we need an argument similar to the one in the proof of [6, Proposition 5.2]. Consider the set of primitive Dirichlet characters χ_1' of conductor prime to MDp such that

- $\chi_1'(-1) = \chi_1(-1) = \chi_2(-1)$;
- $\chi_1'(-MD) = -w_{MD}$;
- $\chi_1'(p) = -a_p = w_p$.

The condition $\chi_1'(-MD) = -w_{MD}$ implies that $L(f, \chi_1', s)$ vanishes of even order at $s = 1$, while the condition $\chi_1'(p) = -a_p = w_p$ implies that $L_p(f, \chi_1', 1)$ does not have an exceptional zero, so that $L_p(f, \chi_1', 1) \in \mathbb{Q}^\times$. Fix a character χ_1' as above such that $L(f, \chi_1', 1) \neq 0$ (by the main result of [31], there are infinitely many).

Consider now the set of primitive Dirichlet characters χ_2' of conductor prime to MDp and the conductor of χ_1' satisfying

- $\chi_2'(-1) = \chi_1'(-1)$;
- $\chi_2'(\ell) = \chi_1'(\ell)$ for $\ell | pM$;
- $\chi_2'(\ell) = -\chi_1'(\ell)$ for $\ell | D$.

Since D has an even number of prime factors, $\chi_2'(-MD) = \chi_1'(-MD)$ and $L(f, \chi_2', s)$ also vanishes to even order at $s = 1$. Also, since $\chi_2'(p) = -a_p$, the p-adic L-function $L_p(f, \chi_2', s)$ does not have an exceptional zero. Hence, again by [31], there are infinitely many characters χ_2' as above such that $L(f, \chi_2', 1) \neq 0$ and $L_p(f, \chi_2', 1) \neq 0$. Choose a pair (χ_1', χ_2') as prescribed above and denote by d_1' and d_2' the conductors of χ_1' and χ_2', respectively. Let $d_{K'} := d_1' \cdot d_2'$ and consider the real quadratic field $K' := \mathbb{Q}(\sqrt{d_{K'}})$ and the genus character χ' of K' corresponding to the pair (χ_1', χ_2'). In particular, by definition, the sign of χ' is equal to the sign of χ. Recall that $G_{K'}^+ := \mathrm{Gal}(H_{K'}^+/K')$. Note that

$$\chi'(\sigma_{\mathfrak{p}}) = \chi_1'(p) = w_p$$

and therefore from Theorem 4.25 (2) for all even $k \geq 4$ we obtain the interpolation formula

$$L_p(f_\infty/K', \chi', k) = \lambda_B^\epsilon(k)^2 \left(1 - \frac{w_p p^{(k-2)/2}}{a_p(k)}\right)^4 \prod_{\ell | D} \frac{\ell - 1}{\ell + 1} \cdot \frac{(2i)^k \sqrt{d_K}^{k-1}}{4\epsilon}$$
$$\times \frac{\|\varphi_{f_k}^{\mathrm{JL},\sharp}\|^2}{\|\varphi_{f_k^\sharp}\|^2} \cdot L_K(1/2, \pi_{f_k^\sharp} \times \pi_\chi).$$

Therefore we get the factorization

$$L_p(f_\infty/K', \chi', k) = \eta(k) L_p(f_\infty, \chi_1', k, k/2) L_p(f_\infty, \chi_2', k, k/2)$$

where η is the function appearing in (54).

With these choices in force, $L_p(f_\infty/K', \chi', k)$, $L_p(f_\infty, \chi_1', k, k/2)$ and $L_p(f_\infty, \chi_2', k, k/2)$ do not vanish at $k = 2$ (note that $L_p(f_\infty/K', \chi', 2) \neq 0$ is a consequence of Theorem 4.25 (1)) so there exists a neighbourhood $\mathbb{U}$ of 2 such that they do not vanish at any point of $\mathbb{U}$. Recall that $\lambda^{\tilde{\epsilon}}(2)$ is chosen to be equal to 1. Thus the quotient

$$\eta(k, \chi_1', \chi_2') := \frac{L_p(f_\infty/K', \chi', k)}{L_p(f_\infty, \chi_1', k, k/2) L_p(f_\infty, \chi_2', k, k/2)}$$

is an analytic function on $\mathbb{U}$ which does not vanish at any point of $\mathbb{U}$, and we define $\eta(k) := \eta(k, \chi_1', \chi_2')$. (Observe that $\eta(k, \chi_1', \chi_2')$ does not depend on the choice of χ_1' and χ_2' made above: any other choice defines a function which coincides with $\eta(k, \chi_1', \chi_2')$ on the dense subset of points $k \in \mathbb{U}$ with trivial character because, for these points, $\eta(k)$ does not depend on χ_1', χ_2'.)

To conclude the proof, we need only show the statement about $\eta(2)$. For this, we use the following argument. Fix a prime $d|D$ and consider the set of Dirichlet characters χ'_3 and χ'_4 of conductor prime to MDp satisfying the following conditions:

- $\chi'_1(-1) = \chi'_2(-1) = -\chi'_3(-1) = \chi'_4(-1)$;
- $\chi'_1(\ell) = \chi'_2(\ell) = \chi'_3(\ell) = \chi'_4(\ell)$ for all primes $\ell|pM$;
- $\chi'_1(d) = -\chi'_2(d) = -\chi'_3(d) = -\chi'_4(d)$;
- $\chi'_1(\ell) = -\chi'_2(\ell) = \chi'_3(\ell) = -\chi'_4(\ell)$ for all primes $\ell|(D/d)$.

To simplify notations, set $\Lambda_j := L_p(f_\infty, \chi'_j, 2, 1)$ for $j = 1, \dots, 4$. Note that $\chi'_3(-MD) = \chi'_4(-MD) = -w_{MD}$ and $\chi'_3(p) = \chi'_4(p) = w_p$. Thus, as above, the order of vanishing of $L(f, \chi'_3, s)$ and $L(f, \chi'_4, s)$ at $s = 1$ is even and $L_p(f, \chi'_3, s)$ and $L_p(f, \chi'_4, s)$ do not have an exceptional zero at $s = 1$. Using [31] again, we choose χ'_3 and χ'_4 so that $\Lambda_3 \neq 0$ and $\Lambda_4 \neq 0$.

For the pairs of indices $(1, 3)$, $(3, 4)$ and $(2, 4)$ we consider quadratic fields $K'_{i,j}$ and genus characters $\chi'_{i,j}$ of $K'_{i,j}$ such that the associated Dirichlet characters are (χ'_i, χ'_j); in particular, if d'_j is the conductor of χ'_j then the discriminant of $K'_{i,j}$ is $d'_i d'_j$. Note that

- $K'_{1,3}$ and $K'_{3,4}$ are imaginary, while $K'_{2,4}$ is real;
- the primes dividing Mp are split in $K'_{1,3}$ and $K'_{3,4}$ and the number of primes dividing D which are inert in these fields is odd;
- the primes dividing MDp are split in $K'_{2,4}$.

Thanks to [6, Proposition 5.1], we have

$$L_p(f_\infty/K'_{i,j}, \chi'_{i,j}, 2) = \langle \phi_2, \phi_2 \rangle \Lambda_i \Lambda_j \qquad \text{for } (i, j) = (1, 3) \text{ and } (i, j) = (3, 4),$$

where we use the notation in [6]. In particular, recall that the p-adic L-function in the equation above is defined in [6, §3.2] and that $\langle \phi_2, \phi_2 \rangle \in \mathbb{Q}^\times$ is defined in [6, p. 395]. Since, by construction, $L_p(f_\infty/K'_{i,j}, \chi'_{i,j}, 2)$ is a square in $\mathbb{Q}^\times$, it follows that $\Lambda_i \Lambda_j \in \langle \phi_2, \phi_2 \rangle^{-1} \cdot (\mathbb{Q}^\times)^2$ for $(i, j) = (1, 3)$ and $(i, j) = (3, 4)$.

By [7, Theorem 3.6], there is also an equality

$$L_p(f_\infty/K'_{2,4}, \chi'_{2,4}, 2) = \Lambda_2 \Lambda_4$$

where, this time, $L_p(f_\infty/K'_{2,4}, \chi'_{2,4}, 2)$ is defined in [7, Definition 3.4]. Again, by construction, $L_p(f_\infty/K'_{2,4}, \chi'_{2,4}, 2)$ is a square in $\mathbb{Q}^\times$, so the same is true of the product $\Lambda_2 \Lambda_4$. Now the factorization

$$\prod_{i=1}^{4} \Lambda_i = (\Lambda_1 \Lambda_2) \cdot (\Lambda_3 \Lambda_4) = (\Lambda_1 \Lambda_3) \cdot (\Lambda_2 \Lambda_4) \in \langle \phi_2, \phi_2 \rangle^{-1} \cdot (\mathbb{Q}^\times)^2$$

shows that $\Lambda_1 \Lambda_2$ is a square in $\mathbb{Q}^\times$. Thanks to the interpolation formula

$$L_p(f_\infty/K', \chi', 2) = \eta(2) \Lambda_1 \Lambda_2$$

proved before and the fact that, by construction, $L_p(f_\infty/K', \chi', 2)$ is a square in $\mathbb{Q}^\times$ (keep in mind the normalization chosen in part (1) of Proposition 3.2), it follows that the same is true of $\eta(2)$, as was to be shown. □

5. Rationality results for Darmon points on elliptic curves

The main result of this paper is the following

Theorem 5.1. *Let χ be a genus character of K corresponding to a pair (χ_1, χ_2) such that $\chi_i(-MD) = -w_{MD}$ for $i = 1, 2$.*

(1) *There exists a point $\mathbf{P}_\chi \in E(H_\chi)^\chi$ and a rational number $c \in \mathbb{Q}^\times$ such that*

$$J_\chi = c \log_E(\mathbf{P}_\chi).$$

(2) *The point $\mathbf{P}_\chi$ is of infinite order if and only if $L'(E/K, \chi, 1) \neq 0$.*

(3) *A suitable integral multiple of* P_χ *belongs to the natural image of* $E(H_\chi)^\chi$ *in* $E(K_p)$. *In particular,* P_χ *coincides with the image of a global point in* $E(K_p) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. As in the proof of [7, Theorem 4.3], order χ_1, χ_2 in such a way that $\mathrm{sign}(E, \chi_1) = -1$. Then $\chi_1(p) = -w_p = a_p$ and, thanks to [6, Theorem 5.4], $L_p(f_\infty, \chi_1, k, k/2)$ vanishes of order at least 2 at $k = 2$. Furthermore, there are a global point $\mathbf{P}_{\chi_1} \in E\big(\mathbb{Q}(\sqrt{d_1})\big)^{\chi_1}$, which is of infinite order if and only if $L'(E/\mathbb{Q}, \chi_1, 1) \neq 0$, and $s \in \mathbb{Q}^\times$ such that

$$\frac{d^2}{dk^2} L_p(f_\infty, \chi_1, k, k/2)_{k=2} = s \log_E(\mathbf{P}_{\chi_1})^2 \tag{55}$$

and

$$s \equiv L^*(f, \psi, 1) \pmod{(\mathbb{Q}^\times)^2} \tag{56}$$

for *any* primitive Dirichlet character ψ for which $L(f, \psi, 1) \neq 0$, $\psi(p) = -\chi(p)$ and $\psi(\ell) = \chi(\ell)$ for all primes $\ell | MD$. Here the *algebraic part* $L^*(f, \psi, 1)$ of $L(f, \psi, 1)$ is as in [7, eq. (24)].

¿From Theorem 4.33 and the fact that $L_p(f_\infty, \chi_1, k, k/2)$ has order of vanishing at $k = 2$ greater than or equal to 2, it follows that

$$\frac{d^2}{dk^2} L_p(f_\infty/K, \chi, k)_{k=2} = \eta(2) \frac{d^2}{dk^2} L_p(f_\infty, \chi_1, k, k/2)_{k=2} L_p(f_\infty, \chi_2, 2, 1). \tag{57}$$

First suppose that $L'(E/K, \chi, 1) \neq 0$. In this case $L_p(f_\infty, \chi_2, 2, 1) = 2L^*(f, \chi_2, 1) \in \mathbb{Q}^\times$. Set $m := L^*(f, \chi_2, 1)$. By (56), there exists $r \in \mathbb{Q}^\times$ such that $r^2 = s/m$. Define $\mathbf{P}_\chi := \mathbf{P}_{\chi_1}$ in this situation. Setting $v := ntrm \in \mathbb{Q}^\times$, a combination of (55), (57) and Corollary 4.32 yields

$$J_\chi^2 = \eta(2)(nt)^2 sm \log_E(\mathbf{P}_{\chi_1})^2 = \big(v\sqrt{\eta(2)}\big)^2 \log_E(\mathbf{P}_\chi)^2,$$

from which we obtain the desired result for $c := \pm v\sqrt{\eta(2)} \in \mathbb{Q}^\times$ (recall that, by Theorem 4.33, $\eta(2)$ is a square in $\mathbb{Q}^\times$).

Now suppose that $L'(E/K, \chi, 1) = 0$. In this case $L_p(f_\infty, \chi_2, 2, 1) = 2L^*(f, \chi_2, 1) = 0$, and the result is an immediate consequence of (57) and Corollary 4.32 on setting $\mathbf{P}_\chi := 0$. This completes the proof of parts (1) and (2).

Finally, since $J_\chi = \log_E(P_\chi)$, if $c = a/b$ with $a, b \in \mathbb{Z}$ then part (2) gives the equality

$$\log_E(bP_\chi) = \log_E(a\mathbf{P}_\chi),$$

which implies part (3) since the kernel of $\log_E$ is the torsion subgroup of $E(K_p)$. □

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