Euler systems obtained from congruences between Hilbert modular forms

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Abstract

This paper presents a generalization of the Euler systems considered in [BD2] to the context of Hilbert modular forms. Arithmetic applications are given.

Contents

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1 Introduction

Euler systems can be used to relate the rank of abelian varieties defined over number fields to the behavior of the associated Hasse-Weil L-series. Many definitions of Euler systems have been proposed in the past: see for example [K3] and [Ru]. To fix notations, let A/K be an abelian variety defined over a number field. Let $\mathcal{I}$ be an ideal in the endomorphism ring $\mathrm{End}(A)$ of A and denote by $A[\mathcal{I}]$ the $\mathcal{I}$-torsion of A. For a prime ideal $\mathfrak{q}$ of K of good reduction for A, define the singular cohomology

$$H^1_{\mathrm{sing}}(K_{\mathfrak{q}}, A[\mathcal{I}]) := H^1(I_{\mathfrak{q}}, A[\mathcal{I}])^{\mathrm{Gal}(K_{\mathfrak{q}}^{\mathrm{unr}}/K_{\mathfrak{q}})},$$

where $K_{\mathfrak{q}}$ is the completion of K at $\mathfrak{q}$, $I_{\mathfrak{q}}$ is the inertia subgroup of the absolute Galois group $\mathrm{Gal}(\overline{K}_{\mathfrak{q}}/K_{\mathfrak{q}})$ of $K_{\mathfrak{q}}$ and $K_{\mathfrak{q}}^{\mathrm{unr}}$ is the maximal unramified extension of $K_{\mathfrak{q}}$. Note that, since $\mathfrak{q}$ is a prime of good reduction for A, the kernel of the restriction map $H^1(K_{\mathfrak{q}}, A[\mathcal{I}]) \to H^1_{\mathrm{sing}}(K_{\mathfrak{q}}, A[\mathcal{I}])$ is the image $\mathrm{Im}(\delta_{\mathfrak{q}})$ of the local Kummer map:

$$\delta_{\mathfrak{q}} : A(K_{\mathfrak{q}})/\mathcal{I}A(K_{\mathfrak{q}}) \hookrightarrow H^1(K_{\mathfrak{q}}, A[\mathcal{I}]).$$

For any ideal $\mathfrak{q}$, let $\mathrm{res}_{\mathfrak{q}} : H^1(K, A[\mathcal{I}]) \to H^1(K_{\mathfrak{q}}, A[\mathcal{I}])$ be the restriction map in cohomology. For our purposes, an Euler system relative to A/K and $\mathcal{I}$ will consist, roughly speaking, of a collection of cohomology classes $\{\kappa_\ell : \ell \in L\} \subseteq H^1(K, A[\mathcal{I}])$ indexed by a set L of prime ideals of K of good reduction for A, satisfying the following 3 conditions:

1. The set L controls the $\mathcal{I}$-Selmer group $\mathrm{Sel}_{\mathcal{I}}(A/K)$ of A/K in the sense that for any non-zero element $s \in \mathrm{Sel}_{\mathcal{I}}(A/K)$, there exists $\ell \in L$ such that $\mathrm{res}_\ell(s) \neq 0$. The definition of Selmer group is recalled, at least when $\mathcal{I}$ is a power of a prime ideal, in Section 4.1.

2. If $\mathfrak{q} \neq \ell$ is a prime ideal of K, then $\mathrm{res}_{\mathfrak{q}}(\kappa_\ell) \in \mathrm{Im}(\delta_{\mathfrak{q}})$.

3. The image of $\mathrm{res}_\ell(\kappa_\ell)$ in $H^1_{\mathrm{sing}}(K_{\mathfrak{q}}, A[\mathcal{I}])$ can be expressed in terms of the central critical value of the Hasse-Weil L-series $L(A/K, s)$, divided by a suitable period.

The first example of Euler system is that used by Kolyvagin [K1], [K2] and Kolyvagin-Logachëv [KL1], [KL2] to obtain important results on the rank of modular abelian varieties. These results, combined with the work of Gross [Gr] and Gross-Zagier [GZ], extended by [Z1] and [Z2], lead to the proof of the Birch and Swinnerton-Dyer conjecture for many modular abelian varieties A defined over totally real number fields F when the analytic rank of A over F is at most one. In particular, these results apply to all elliptic curves defined over $\mathbb{Q}$. These Euler systems are constructed by applying the Kolyvagin derivative operator to $\pi(P_n) \in A(K_n)$, where K_n is the ring class field of K of conductor n, positive integer, $\pi : J_X \to A$ is a parametrization of A by the Jacobian variety of a modular curve or a Shimura curve X and $P_n \in J_X(K_n)$ is an Heegner point, that is, a point corresponding via the interpretation of X in terms of moduli space, to an abelian variety with complex multiplication by K_n.

More recently, Bertolini-Darmon [BD2] have used the theory of congruences between modular forms to construct an Euler system which they use to prove one of the two divisibility properties predicted by the Iwasawa's Main Conjecture for elliptic curves defined over $\mathbb{Q}$. In this case, the set L consists of p^n-admissible primes, where p is a rational prime and $n \geq 1$ an integer, whose definition is recalled, in the context of Hilbert modular forms, in Definition 3.1 below. In the case of elliptic curves defined over $\mathbb{Q}$, for a fixed rational prime p and a fixed quadratic imaginary extension $K/\mathbb{Q}$, a prime ℓ of $\mathbb{Q}$ is said to be p^n-admissible if: (i) ℓ it is inert in K; (ii) ℓ does not divide Np, where N is the conductor of E; (iii) p does not divide $\ell^2 - 1$; (iv) p^n divides $\ell + 1 + a_\ell$ or $\ell + 1 - a_\ell$, where a_ℓ is the eigenvalue of the Hecke operator T_ℓ acting on the eigenform associated to E. The local behavior of κ_ℓ at ℓ is encoded in a certain p-adic L-function. In particular, it is possible to prove that if $L_K(E, 1) \neq 0$, then $\mathrm{res}_\ell(\kappa_\ell)$ is not trivial and does not belong to the image of the local Kummer map for all but a finite number of ideals p^n.

The construction of [BD2] can be generalized to the context of Hilbert modular forms of parallel weight 2 with rational eigenvalues under the action of the Hecke algebra. This generalization, performed for p-admissible primes, is used by the author in [L2] to prove the Birch and Swinnerton-Dyer conjecture for modular elliptic curves E with everywhere good reduction, analytic rank zero, without complex multiplication and defined over a totally real number field F of even degree over $\mathbb{Q}$. It is worth to point out that this case can not be treated by Kolyvagin's method because E does not appear as the quotient of the Jacobian variety of a Shimura curve: see [L2, Introduction] for a compete discussion. Observe moreover that the argument in [L2] uses only p-admissible primes because this is enough to bound the Selmer group

$\mathrm{Sel}_p(E/K)$ and obtain a result on the rank of $E(K)$, where K/F is a totally imaginary quadratic extension of a totally real number field F.

The propose of this paper is to provide the construction of Euler systems associated to Hilbert modular newforms ϕ when L is the set of $\wp^n$-admissible primes and $\wp$ is a prime ideal in the ring generated by the eigenvalues of the Hecke algebra acting on ϕ. There are two main applications of this construction:

1. Generalize the argument in [BD2] to the context of the Iwasawa's Main Conjecture for Hilbert modular forms: see [L3].

2. Use the Euler system relative to $\wp$-admissible primes (that is, $n = 1$) to extend the result of [L2] to the context of abelian varieties of GL_2-type: see Section 4.

We now fix the notations and assumptions which will be used throughout the paper. Let $F/\mathbb{Q}$ be totally real of degree $d = [F : \mathbb{Q}]$ and $\mathfrak{n}$ an integral ideal of the ring of integers $\mathcal{O}_F$ of F. Let $\phi \in S_2(\mathfrak{n})$ be a Hilbert modular newform for the $\Gamma_0(\mathfrak{n})$-level structure, trivial central character and parallel weight 2 which is an eigenform for the Hecke algebra $\mathbb{T}_\mathfrak{n}$ generated over $\mathbb{Z}$ by the Hecke operators $\mathrm{T}_\mathfrak{q}$ and the spherical operators $\mathrm{S}_\mathfrak{q}$ for primes $\mathfrak{q} \nmid \mathfrak{n}$ and the Hecke operators $\mathrm{U}_\mathfrak{q}$ for primes $\mathfrak{q} \mid \mathfrak{n}$. See [Z1, Section 3] for precise definitions.

Let K/F be a totally imaginary quadratic extension. If the absolute discriminant $\mathfrak{d}_{K/\mathbb{Q}}$ of K over $\mathbb{Q}$ is prime to $\mathfrak{n}$, there is a factorization

$$\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-,$$

where $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) is divisible only by primes which are split (respectively, inert) in the extension K/F. The extension K/F is required to satisfy the following:

Assumption 1.1 (on K)**.** *The absolute discriminant $\mathfrak{d}_{K/\mathbb{Q}}$ of $K/\mathbb{Q}$ is prime to $\mathfrak{n}$. Moreover, the ideal $\mathfrak{n}^-$ appearing in the above factorization of $\mathfrak{n}$ is square-free and the the number of primes dividing it has the same parity as d.*

Assumption 1.1 has an important consequence: the sign of the functional equation of the Hasse-Weil L-series $L_K(A, s)$ of A/K is $+1$. This is equivalent to say that the order of vanishing of $L_K(A, s)$ at the central critical point $s = 1$ is even. Hence, this condition is compatible with the non vanishing of the special value $L_K(\phi, 1)$ of the Hasse-Weil L-series.

For any Hecke operator $\mathrm{T} \in \mathbb{T}_\mathfrak{n}$, denote by $\theta_\phi(\mathrm{T})$ the eigenvalue of T acting on ϕ. Let $K_\phi = \mathbb{Q}(\theta_\phi(\mathrm{T}), \mathrm{T} \in \mathbb{T}_\mathfrak{n})$ be the finite extension of $\mathbb{Q}$ generated by the eigenvalues of the action of the Hecke algebra $\mathbb{T}_{\mathfrak{n}}$ on ϕ. Denote by $\mathcal{O}_\phi$ its ring of integers. For a prime ideal $\wp$ of $\mathcal{O}_\phi$, denote by $K_{\phi,\wp}$ and $\mathcal{O}_{\phi,\wp}$ the completions of K_ϕ and $\mathcal{O}_\phi$ at $\wp$ and by $\mathbb{F}_{\phi,\wp}$ the residue field of $\mathcal{O}_{\phi,\wp}$ at $\wp$. For any prime $\wp \subseteq \mathcal{O}_\phi$, denote by

$$\rho_{\phi,\wp} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(K_{\phi,\wp})$$

the $\wp$-adic representation associated to ϕ by [Ca], [Wi] and [Ta]. Since the image of $\rho_{\phi,\wp}$ is compact, it is possible to choose a $\mathrm{Gal}(\overline{F}/F)$-stable lattice

$$T_\phi \simeq \mathcal{O}_{\phi,\wp} \times \mathcal{O}_{\phi,\wp} \subseteq K_{\phi,\wp} \times K_{\phi,\wp}$$

such that $\rho_{\phi,\wp}$ is equivalent to the representation

$$\tilde{\rho}_{\phi,\wp} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}(T_\phi) \simeq \mathrm{GL}_2(\mathcal{O}_{\phi,\wp})$$

of $\mathrm{Gal}(\overline{F}/F)$ on T_ϕ. Let

$$\bar{\rho}_{\phi,\wp} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F}_{\phi,\wp})$$

be the reduction of $\tilde{\rho}_{\phi,\wp}$ modulo $\wp$. Note that $\bar{\rho}_{\phi,\wp}$ depends on the choice of the Galois-stable lattice T_ϕ. Its semisimplification $\overline{\rho}^{\mathrm{ss}}_{\phi,\wp}$ (i.e. the unique semisimple representation with the same Jordan-Hölder factors as $\overline{\rho}_{\phi,\wp}$) does not depend on the choice of T_ϕ. If $\rho_{\phi,\wp}$ is irreducible, then it coincides with its semisimplification $\overline{\rho}^{\mathrm{ss}}_{\phi,\wp}$. Say that $\rho_{\phi,\wp}$ is *residually irreducible* if $\overline{\rho}_{\phi,\wp}$ is irreducible.

Let $\wp$ be a prime ideal of $\mathcal{O}_\phi$ of residue characteristic p. The modular form ϕ is said to be *ordinary* at a prime ideal $\mathfrak{p} \mid p$ of $\mathcal{O}_F$ if there exists a root $\alpha_{\mathfrak{p}}$ of the Hecke polynomial at $\mathfrak{p}$ which is a unit in the completion $\mathcal{O}_{\phi,\wp}$ of $\mathcal{O}_\phi$ at $\wp$.

Say that ϕ is *$\wp$-isolated* if there are no non trivial congruences between ϕ and other forms of the same level and weight. By the finiteness of the space of modular forms of given level and weight, ϕ is $\wp$-isolated for all $\wp$ except, possibly, a finite number of them.

To state the main result of this paper and better explain the above conditions on ϕ to be ordinary al all $\mathfrak{q} \mid p$ and residually irreducible at $\wp$, assume that there exists an abelian variety A defined over F of GL_2-type such that:

1. The arithmetic conductor of A is $\mathfrak{n}$.
2. $\mathrm{End}(A) \simeq \mathcal{O}_\phi$.
3. The representation of $\mathrm{Gal}(\overline{F}/F)$ on the $\wp$-adic Tate module of A is equivalent to $\rho_{\phi,\wp}$, where $\wp$ is a prime ideal of $\mathcal{O}_\phi$.

This type of abelian variety is considered in Section 4. The abelian variety A has good reduction outside $\mathfrak{n}$. The residual irreducibility of $\rho_{\phi,\wp}$ for infinitely many $\wp$ stated in Assumption 1.2 can be translated into the irreducibility of the Galois representation $\bar{\rho}_{A,\wp}$ of $\mathrm{Gal}(\overline{F}/F)$ on the $\wp$-torsion $A[\wp]$ of A. If A has ordinary reduction at the prime ideal $\mathfrak{p} \mid p$, then the associated modular form ϕ is also ordinary at $\mathfrak{p}$. Denote as above by

$$\delta_{\mathfrak{q}} : A(K_{\mathfrak{q}})/\wp^n A(K) \hookrightarrow H^1(K_{\mathfrak{q}}, A[\wp^n])$$

the local Kummer map, where $\mathfrak{q}$ is a prime ideal of $\mathcal{O}_K$.

Let $\mathfrak{q}$ be a prime ideal of residue characteristic q. Then, by a theorem of Mattuck, $A(K_{\mathfrak{q}}) \simeq \mathbb{Z}_q^{\dim(A)[K_{\mathfrak{q}}:\mathbb{Q}_q]} \times H$, where H is a finite group. Hence, $\mathrm{Im}(\delta_{\mathfrak{q}}) = 0$ for all prime ideals $\wp$ and all n, except possibly a finite number of them, i.e. those $\wp$ dividing q and the order of H.

Assumption 1.2 (on $\wp$)**.** *Suppose that the prime ideal $\wp$ of residue characteristic $p \geq 5$ satisfies the following conditions:*

1. *$\rho_{\phi,\wp}$ is residually irreducible.*
2. *ϕ is ordinary at all prime ideals $\mathfrak{p} \mid p$.*
3. *ϕ is $\wp$-isolated.*
4. $\mathrm{Im}(\delta_{\mathfrak{q}}) = 0$ *for all prime ideals $\mathfrak{q} \mid \mathfrak{n}$ but $\mathfrak{q} \nmid p$.*

Remark 1.3. Conditions 3 and 4 in Assumption 1.2 are verified for all prime ideals $\wp$ except possibly a finite number of them. Conditions 1 and 2 are more delicate. In the case of elliptic curves without complex multiplication, the existence of infinitely many prime ideals $\wp$ verifying 1 and 2 is ensured by [Se]. Similar results hold for more general modular abelian varieties. In any case, in the following it will be convenient to assume the existence of infinitely many prime ideals $\wp$ verifying Assumption 1.2. This is done, for example, in the next Corollary 1.7 on the rank of $A(K)$.

Define a prime ideal $\ell \subseteq \mathcal{O}_F$ to be $\wp^n$-admissible (see Definition 3.1) if:

1. ℓ does non divide $\mathfrak{n}p$;
2. ℓ is inert in K/F;
3. $\wp$ does non divide $|\ell|^2 - 1$;
4. $\wp^n$ divides $|\ell| + 1 - \epsilon\theta_\phi(\mathrm{T}_\ell)$, where $\epsilon = \pm 1$.

By [BD2, Theorem 3.2], the set of $\wp^n$-admissible primes controls the Selmer group $\mathrm{Sel}_{\wp^n}(A/K)$ in the above sense: for any $s \in \mathrm{Sel}_{\wp^n}(A/K)$, $s \neq 0$, there exists a $\wp^n$-admissible prime ℓ such that $\mathrm{res}_\ell(s) \neq 0$.

To state the main result, denote by $\mathcal{L}_K(\phi)$ the algebraic part of the special value $L_K(A,1) = L_K(\phi,1)$ of the Hasse-Weil L-series $L_K(A,s) = L_K(\phi,s)$. This is an element of $\mathcal{O}_\phi$ whose definition is given in Definition 3.6, such that:

$$\mathcal{L}_K(\phi) \neq 0 \text{ if and only if } L_K(\phi,1) \neq 0.$$

Then the main result of this paper can be expressed as follows:

Theorem 1.4. *Suppose that Assumption 1.1 on K is verified. Let $\wp$ be a prime ideal verifying Assumption 1.2. Then for any $\wp^n$-admissible prime ℓ there exists a class $\kappa_\ell \in H^1(K, A[\wp^n])$ with the following properties:*

1. *For primes $\mathfrak{q} \neq \ell$, $\mathrm{res}_\mathfrak{q}(\kappa_\ell) \in \mathrm{Im}(\delta_\mathfrak{q})$.*
2. *The image of $\mathrm{res}_\ell(\kappa_\ell)$ in $H^1_{\mathrm{sing}}(K_\ell, A[\wp^n]) \simeq \mathcal{O}_\phi/\wp^n$ is equal to $\mathcal{L}_K(\phi)$ $\pmod{\wp^n}$, up to multiplication by invertible elements in $\mathcal{O}_\phi/\wp^n$.*

Remark 1.5. As observed above, the condition $L_K(A,1) \neq 0$ in Theorem 1.4 is consistent with Assumption 1.1 because, as a consequence of this assumption, the sign of the functional equation of the Hasse-Weil L-function is $+1$. Suppose that $L_K(A,1) \neq 0$. Then $\mathcal{L}_K(\phi) \neq 0$. Let $\wp$ be a prime ideal of $\mathcal{O}_\phi$ satisfying Assumption 1.2 and such that $\wp \nmid \mathcal{L}_K(\phi)$. Then the image of $\mathrm{res}_\ell(\kappa_\ell)$ in $H^1_{\mathrm{sing}}(K_\ell, A[\wp])$ is not trivial, hence do not belong to the image of the local Kummer map δ_ℓ. For each of these primes $\wp$, it is possible to prove that $\mathrm{Sel}_\wp(A/K) = 0$. This result follows by applying standard techniques involving the Euler system relative to A/K and $\wp$, which exists by Theorem 1.4, the non-vanishing of $\mathrm{res}_\ell(\kappa_\ell)$ in the singular cohomology and the non-degeneracy of the local Tate pairing. For a proof and more details, see Theorem 4.4 in the text.

Remark 1.6. An other way to state Theorem 1.4 is the following: Suppose that Assumption 1.1 is verified. Then for all prime ideals ideals $\wp$ verifying Assumption 1.2 and for any n there exists an Euler system relative to A/K and $\wp^n$.

Theorem 1.4 corresponds in the text to Theorem 4.3. This result can be obtained combining Theorem 3.10 with the discussion in Section 4 on abelian varieties. It is worth to point out that Theorem 3.10 is stated in a purely theoretical Galois representation setting. In other words, the Euler system may be attached to a modular form ϕ without any reference to abelian varieties.

The following arithmetic application of Theorem 1.4 to the Birch and Swinnerton-Dyer conjecture is explained in Section 4.2:

Corollary 1.7. *Suppose that Assumptions 1.1 is verified and that there are infinitely many prime ideals $\wp$ verifying Assumption 1.2. If the special value $L_K(A, 1)$ of the Hasse-Weil L-series $L_K(A, s)$ of A over K is non-zero, then the rank of $A(K)$ is zero.*

Corollary 1.7 correspond in the text to Corollary 4.6. The proof of this result is based on the possibility of choosing (among infinitely many) one prime ideal $\wp$ verifying Assumption 1.2 and such that $\wp \nmid \mathcal{L}_K(\phi)$. In this case, as observer in Remark 1.5, the $\wp$-Selmer group $\mathrm{Sel}_\wp(A/K)$ is trivial and the result on the rank of $A(K)$ follows from the injectivity of the global Kummer map

$$\delta : A(K)/\wp A(K) \hookrightarrow H^1(K, A[\wp]).$$

The existence of infinitely many such primes $\wp$ is ensured by the existence of infinitely many prime ideals verifying Assumption 1.2 and the non vanishing of $L_K(A, 1)$, which implies the non vanishing of $\mathcal{L}_K(\phi)$.

Remark 1.8. Since to prove Corollary 1.7 only the Euler system relative to the chosen prime $\wp$ is involved, the hypothesis in Corollary 1.7 regarding Assumption 1.2 can be slightly relaxed by requiring the existence of at least one prime ideal $\wp$ not dividing $\mathcal{L}_K(\phi)$ and verifying Assumption 1.2. For details, see Corollary 4.5.

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2 Shimura curves

2.1 Basic definitions

Let $\mathcal{B}$ be a quaternion algebra defined over F which is split at exactly one of the archimedean places, say μ. Denote by $\mathfrak{c}^-$ its discriminant, that is, the product of the finite places $\mathfrak{q}$ where the completion $\mathcal{B}_{\mathfrak{q}} = \mathcal{B} \otimes_F F_{\mathfrak{q}}$ of $\mathcal{B}$ at $\mathfrak{q}$ is a non-commutative field. For any $\mathbb{Z}$-algebra E, denote by $\widehat{E} := E \otimes_{\mathbb{Z}} \prod_q \mathbb{Z}_q$ the profinite completion of E, where q ranges over the set of prime ideals of $\mathbb{Z}$. For any open subgroup $\mathcal{U} \subseteq \widehat{\mathcal{B}}^\times$ which is compact modulo $\widehat{F}^\times$, denote by $X_{\mathcal{U}}$ the model over F of the Shimura curve whose complex points are given by

$$X_{\mathcal{U}}(\mathbb{C}) = \mathcal{U} \backslash (\mathcal{B}^\times \times \mathcal{H}^\pm)/\mathcal{B}^\times,$$

where $\mathcal{H}^\pm = \mathbb{C} - \mathbb{R}$. The curve $X_{\mathcal{U}}$ is connected but not geometrically connected; denote by $J_{\mathcal{U}}$ the connected component subgroup of $\mathrm{Pic}(X_{\mathcal{U}})$ over F. Finally, let $\mathcal{X}_{\mathcal{U}} \to \mathrm{Spec}(\mathcal{O}_F)$ be the integral model of $X_{\mathcal{U}}$ and denote by $\mathcal{J}_{\mathcal{U}} \to \mathrm{Spec}(\mathcal{O}_F)$ the Néron model of $J_{\mathcal{U}}$. For more details on these definitions, see [Z1, Section 1].

For any ideal $\mathfrak{c}^+ \subseteq \mathcal{O}_F$ prime to $\mathfrak{c}^-$, choose an Eichler order $\mathcal{R}(\mathfrak{c}^+) \subseteq \mathcal{B}$ of level $\mathfrak{c}^+$ and denote $X_{\widehat{F}^\times \widehat{\mathcal{R}}(\mathfrak{c}^+)}$ simply by $X_{\mathfrak{c}^+,\mathfrak{c}^-}$. Note that $X_{\mathfrak{c}^+,\mathfrak{c}^-}$ does not depend on the choice of $\mathcal{R}_{\mathfrak{c}^+}$. Adopt the same convention for the Jacobian variety, its Néron model and the integral model of $X_{\mathfrak{c}^+,\mathfrak{c}^-}$.

2.2 Admissible curves

This paragraph collects the basic facts on admissible curves in the sense of [JL]. Let R be a discrete valuation ring with fraction field K and residue field $\mathbb{F}$ perfect of characteristic $p > 0$. Let X_K be smooth proper geometrically connected curve over K and suppose that X is a *nodal model* of X_K over R. More precisely, X is a proper flat R-scheme whose generic fiber is X_K, the only singularities of the special fiber $X_{\mathbb{F}}$ are ordinary double points and the multiplicities of the irreducible components of $X_{\mathbb{F}}$ are one. Following [JL], such a curve X_K is called *admissible*. Let $\mathcal{G}$ be the dual graph associated to $X_{\mathbb{F}}$: the set of vertices $\mathcal{V}$ is the set of irreducible components of $X_{\bar{\mathbb{F}}} := X_{\mathbb{F}} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$, the set of edges $\mathcal{E}(\mathcal{G})$ is the set of singular points of $X_{\bar{\mathbb{F}}}$ and two edges e, e' meet at the vertex v if and only if the corresponding components intersect at v. Here $\bar{\mathbb{F}}$ denotes an algebraic closure of $\mathbb{F}$. Denote by π an uniformizer of R and let $e \in \mathcal{E}$ be a singular point; then locally for the étale topology e is given by an equation as $uv = \pi^{n(e)}$ for some positive integer $n(e)$. Define a pairing

$$\langle , \rangle \ : \mathbb{Z}[\mathcal{E}] \times \mathbb{Z}[\mathcal{E}] \to \mathbb{Z}$$

extending by $\mathbb{Z}$-linearity the following rule: for any pair $(e, e') \in \mathcal{E}^2$, set

$$\langle e, e' \rangle \ := n(e)\delta_{e,e'},$$

where $\delta_{e,e'}$ is the Kronecker symbol. This pairing induces an embedding $j_0 : \mathbb{Z}[\mathcal{E}] \to \mathbb{Z}[\mathcal{E}]^\vee$, where the superscript $\vee$ denotes the $\mathbb{Z}$-dual. Let J_K be the jacobian variety of X_K and let J be the Néron model of J_K over R. Denote by $J_{\mathbb{F}}$ its special fiber and by $J^0_{\mathbb{F}}$ the connected component of $J_{\mathbb{F}}$ containing the origin. Finally denote by Φ the group of connected components $J_{\mathbb{F}}/J^0_{\mathbb{F}}$, by T the maximal torus of $J_{\mathbb{F}}$ and by $\mathbb{X} = \mathrm{Hom}(T, \mathbb{G}_m)$ its character group. Fix an orientation s, t of $\mathcal{E}$, that is a pair of maps $s, t : \mathcal{E} \to \mathcal{V}$ such that for any edge e, $s(e)$ and $t(e)$ are the vertices joined by e. The character group fits into the following exact sequence:

$$0 \to \mathbb{X} \to \mathbb{Z}[\mathcal{E}] \xrightarrow{\partial_*} \mathbb{Z}^0[\mathcal{V}] \to 0, \tag{1}$$

where ∂_* is the map defined by $\partial_*(e) := t(e) - s(e)$ for any edge e. The previous exact sequence gives rise by restriction the so called *monodromy pairing*

$$\langle , \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{Z}.$$

Denote by $j : \mathbb{X} \to \mathbb{X}^\vee$ the embedding induced by this pairing. By [Gt, Theorems 11.5 and 12.5] there is an exact sequence:

$$0 \to \mathbb{X} \xrightarrow{j} \mathbb{X}^\vee \to \Phi \to 0. \tag{2}$$

There exists a natural non-trivial map

$$\omega : \mathbb{Z}^0[\mathcal{V}] \to \Phi \tag{3}$$

defined as follows. Taking $\mathbb{Z}$-duals of the exact sequence:

$$0 \to \mathbb{X} \xrightarrow{i} \mathbb{Z}[\mathcal{E}] \xrightarrow{\partial_*} \mathbb{Z}[\mathcal{V}] \to \mathbb{Z} \to 0$$

yields:

$$0 \to \mathbb{Z} \to \mathbb{Z}[\mathcal{V}] \xrightarrow{\partial^*} \mathbb{Z}[\mathcal{E}] \xrightarrow{i^\vee} \mathbb{X}^\vee \to 0,$$

where the group rings are identified with their $\mathbb{Z}$-duals and ∂^* is defined by $\partial^*(v) := \sum_{t(e)=v} e$ is v is odd and $\partial^*(v) := -\sum_{t(e)=v} e$ is v is even. Define j_0 to be the map induced on $\mathbb{Z}[\mathcal{E}]$ by the pairing $\langle , \rangle$; then there is a commutative diagram:

$$\begin{array}{ccccccccc}
 & & \mathbb{Z}^0[\mathcal{V}] & & & & & & \\
 & & \uparrow \partial_* & & & & & & \\
 & & \mathbb{Z}[\mathcal{E}] & \xrightarrow{j_0} & \mathbb{Z}[\mathcal{E}]^\vee & & & & \\
 & & \uparrow i & & \downarrow i^\vee & & & & \\
0 & \longrightarrow & \mathbb{X} & \xrightarrow{j} & \mathbb{X}^\vee & \xrightarrow{\tau_{\mathrm{I}}} & \Phi & \longrightarrow & 0.
\end{array}$$

Choose an element $x \in \mathbb{Z}^0[\mathcal{V}]$ and choose y such that $\partial_*(y) = x$; then define

$$\omega(x) := (\tau_{\mathrm{I}} \circ i^\vee \circ j_0)(x).$$

It is immediate that ω is well-defined and non-trivial. For more details on this map, see [BD2, Corollary 5.12] or [BD1, Section 1 of Appendix].

Let $\mathrm{Div}(X)$ be the group of divisors of $X_K = X(K)$ with $\mathbb{Z}$ coefficients and $\mathrm{Div}^0(X)$ be the subgroup of $\mathrm{Div}(X)$ consisting of divisors which have degree zero on each connected component of X. Denote by $r : X(K) \to \mathcal{E} \cup \mathcal{V}$ the reduction map defined by sending a point P to the connected component containing its image in $X_{\mathbb{F}}$ if P does not reduces to a singular point and to the image of P in $X_{\mathbb{F}}$ otherwise. Fix a divisor $D = \sum n_P P \in \mathrm{Div}^0(X)$ such that $r(D) \in \mathbb{Z}^0[\mathcal{V}]$. Denote by ∂_ℓ the specialization map $\partial_\ell : J(K) \to \Phi$. The basic relation between ω_ℓ, ∂_ℓ and r is the following equality in Φ:

$$\partial_\ell([D]) = \omega_\ell(r_\ell(D)), \tag{4}$$

where $[D]$ denotes the image of D in $J(K)$. This result follows from Edixhoven's description [BD1, Section 2 of Appendix] of the map ∂_ℓ.

2.3 The Čerednik-Drinfeld Theorem

Let $\mathfrak{l} \mid \mathfrak{c}^-$ be a prime ideal and denote by $\mathcal{X}_{\mathcal{U},\mathfrak{l}}$ the fiber of $\mathcal{X}_{\mathcal{U}}$ over $\mathrm{Spec}(\mathcal{O}_{F,\mathfrak{l}})$. Define the formal group $\widehat{\mathcal{X}}_{\mathcal{U},\mathfrak{l}}$ over $\mathcal{O}_{F,\mathfrak{l}}$ to be the completion of $\mathcal{X}_{\mathcal{U},\mathfrak{l}}$ along its special fiber. Let B/F be the quaternion algebra ramified at all archimedean places and whose discriminant is $\mathfrak{c}^-/\mathfrak{l}$. The quaternion algebra B is said to be obtained from $\mathcal{B}$ by interchanging the invariants $\mathfrak{l}$ and μ. If E is a $\mathcal{O}_F$-algebra, the notation $\widehat{E}^{(\mathfrak{q})}$ means that the $\mathfrak{q}$-component in $\widehat{E}$ has been removed. Fix an isomorphism

$$\varphi : \widehat{B} \xrightarrow{\sim} \widehat{\mathcal{B}}^{(\mathfrak{l})} \mathrm{M}_2(F_{\mathfrak{l}})$$

and choose Eichler orders R and $R_{\mathfrak{l}}$ of B of level $\mathfrak{c}^+$ and $\mathfrak{c}^+\mathfrak{l}$ such that $R \supseteq R_{\mathfrak{l}}$ and, under the above isomorphism, $\widehat{\mathcal{R}}^{(\mathfrak{l})}$ corresponds to $\widehat{R}^{(\mathfrak{l})} = \widehat{R}_{\mathfrak{l}}^{(\mathfrak{l})}$. Finally, set:

$$U := \widehat{F}^\times \cdot \varphi(\mathcal{U}^{(\mathfrak{l})}) \cdot (R^\times \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{l}}) \text{ and } U_{\mathfrak{l}} := \widehat{F}^\times \cdot \varphi(\mathcal{U}^{(\mathfrak{l})}) \cdot (R_{\mathfrak{l}}^\times \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{l}}).$$

Denote by $\mathbb{C}_{\mathfrak{l}}$ the completion of an algebraic closure of $F_{\mathfrak{l}}$ and let $\widehat{\mathcal{H}}_{\mathfrak{l}}$ be the Deligne's formal scheme over $\mathrm{Spec}(\mathcal{O}_{F,\mathfrak{l}})$ obtained by blowing-up the projective line over $\mathrm{Spec}(\mathcal{O}_{F,\mathfrak{l}})$ along its rational points in the special fiber over the residue field $\mathbb{F}_{\mathfrak{l}}$ of $\mathcal{O}_{F,\mathfrak{l}}$. The generic fiber of $\widehat{\mathcal{H}}_{\mathfrak{l}}$ is a rigid analytic space whose $\mathbb{C}_{\mathfrak{l}}$-points are $\mathcal{H}_{\mathfrak{l}} := \mathbb{P}^1(\mathbb{C}_{\mathfrak{l}}) - \mathbb{P}^1(F_{\mathfrak{l}})$. For more details, see [BC, Chapitre I]. Finally, let $\mathrm{Frob}_{\mathfrak{l}}$ be the Frobenius automorphism of $\mathrm{Gal}(\mathcal{O}_{F,\mathfrak{l}}^{\mathrm{unr}}/\mathcal{O}_{F,\mathfrak{l}})$. The Čerednik-Drinfeld Theorem states that there exists an isomorphism of formal schemes over $\mathrm{Spec}(\mathcal{O}_{F,\mathfrak{l}})$:

$$\widehat{\mathcal{X}}_{\mathfrak{l}} \simeq U^{(\mathfrak{l})} \backslash (\widehat{\mathcal{H}}_{\mathfrak{l}} \widehat{\otimes}_{\mathcal{O}_{F,\mathfrak{l}}} \mathcal{O}_{F,\mathfrak{l}}^{\mathrm{unr}} \times \widehat{B}^{(\mathfrak{l})\times}) / B^\times,$$

where $b \in \mathcal{B}^\times$ acts on $\mathcal{O}_{F,\mathfrak{l}}^{\mathrm{unr}}$ by $\mathrm{Frob}_{\mathfrak{l}}^{-\mathrm{val}_{\mathfrak{l}}(b)}$. This result can be obtained by combining Čerednik's description [Ce] of $\mathcal{X}_{\mathfrak{l}}$ as moduli space for certain formal groups with the Drinfeld's description [Dr] of $\mathcal{H}_{\mathfrak{l}}$: see [Z1] for more details.

Fix representatives $g_1, \dots, g_h$ of $U^{(\mathfrak{l})} \backslash \widehat{B}^{(\mathfrak{l})\times}/B^\times$ and define the following subgroups for each $j = 1, \dots, h$:

$$\widetilde{\Gamma}_{j,0,\mathfrak{l}} := g_j^{-1}(U^{(\mathfrak{l})}\mathrm{GL}_2(F_{\mathfrak{l}}))g_j \cap B^\times; \qquad \Gamma_{j,0,\mathfrak{l}} := \widetilde{\Gamma}_{j,0,\mathfrak{l}}/\left(\widetilde{\Gamma}_{j,0,\mathfrak{l}} \cap F^\times\right);$$

$$\tilde{\Gamma}_{j,+,\mathfrak{l}} := (\tilde{\Gamma}_{j,0,\mathfrak{l}})_e; \quad \Gamma_{j,+,\mathfrak{l}} := (\Gamma_{j,0,\mathfrak{l}})_e,$$

where the subscript e means elements whose norm has even $\mathfrak{l}$-adic valuation. Denote by $\mathcal{O}_{F,\mathfrak{l}^2}$ is the ring of integers of the quadratic unramified extension $F_{\mathfrak{l}^2}$ of $F_{\mathfrak{l}}$. Define $\mathcal{X}_{\mathcal{U},\mathfrak{l}^2} := \mathcal{X}_{\mathcal{U},\mathfrak{l}} \otimes_{\mathcal{O}_{F,\mathfrak{l}}} \mathcal{O}_{F,\mathfrak{l}^2}$, where $\mathcal{X}_{\mathcal{U},\ell}$ is the fiber of $\mathcal{X}_{\mathcal{U}}$ at $\mathfrak{l}$. The Čerednik-Drinfeld Theorem can be combined with [JL, Section 4] to deduce that there exists an isomorphism of rigid analytic schemes over $\mathrm{Spec}(\mathcal{O}_{F,\mathfrak{l}^2})$:

$$\mathcal{X}_{\mathcal{U},\mathfrak{l}^2}(\mathbb{C}_{\mathfrak{l}}) \simeq \coprod_{j=1}^{h} \mathcal{H}_{\mathfrak{l}}/\Gamma_{j,+,\mathfrak{l}},$$

where the arithmetic subgroups $\Gamma_{j,+,\mathfrak{l}} \subseteq \mathrm{PGL}_2(F_{\mathfrak{l}})$ (for $j = 1, \dots, h$) act on $\mathcal{H}_{\mathfrak{l}}$ by fractional linear transformations. For details, see [L2, Section 4.2]. From this description it follows that $\mathcal{X}_{\mathcal{U},\mathfrak{l}^2}$ is a disjoint union of admissible curves. Set $\mathcal{X}_j := \mathcal{H}_{\mathfrak{l}}/\Gamma_{j,+,\mathfrak{l}}$ and denote by $\mathcal{G}_j$, (respectively, $\mathbb{X}_j$ and Φ_j) the arithmetic graph (respectively, the character group and the group of connected components) associated to $\mathcal{X}_j$ as above. Moreover, let $\mathcal{V}_j$ and $\mathcal{E}_j$ be respectively the set of vertices and edges of $\mathcal{G}_j$.

By the results in [L2, Section 4.3] it is known that $\mathcal{G}_j \simeq \mathcal{T}_{\mathfrak{l}}/\Gamma_{j,+,\mathfrak{l}}$, where $\mathcal{T}_{\mathfrak{l}}$ is the Bruhat-Tits tree of $\mathrm{PGL}_2(F_{\mathfrak{l}})$. Choose the following orientation in $\mathcal{T}_{\mathfrak{l}}$: define v_0 to be the vertex corresponding to the maximal order $\mathrm{M}_2(\mathcal{O}_{F,\mathfrak{l}})$ and say that a vertex v is even (respectively, odd) if its distance from v_0 is even (respectively, odd). Since the determinant of the elements of $\Gamma_{j,+,\mathfrak{l}}$ have even $\mathfrak{l}$-adic valuation, the choice of this orientation in $\mathcal{T}_{\mathfrak{l}}$ induces in a natural way an orientation in $\mathcal{G}_j$ again denoted by s, t. The exact sequences (1) and (2) can then be rewritten respectively as

$$0 \to \mathbb{X}_j \to \mathbb{Z}[\mathcal{E}_j] \xrightarrow{\partial_{j,*}} \mathbb{Z}^0[\mathcal{V}_j] \to 0 \tag{5}$$

and

$$0 \to \mathbb{X}_j \to \mathbb{X}_j^\vee \to \Phi_j \to 0. \tag{6}$$

Define the the arithmetic graph $\mathcal{G}_{\mathfrak{l}} := \coprod_{j=1}^h \mathcal{G}_j$, the group of connected components $\Phi_{\mathfrak{l}} := \prod_{j=1}^h \Phi_j$ and the character group $\mathbb{X}_{\mathfrak{l}} := \prod_{j=1}^h \mathbb{X}_j$ associated to $\mathcal{X}_{\mathcal{U},\mathfrak{l}^2}$. Make analogous definitions for $\mathcal{V}_\mathfrak{l} := \coprod_{j=1}^h \mathcal{V}_j$ and $\mathcal{E}_\mathfrak{l} := \coprod_{j=1}^h \mathcal{E}_j$. Then there are exact sequences:

$$0 \to \mathbb{X}_\mathfrak{l} \xrightarrow{i} \mathbb{Z}[\mathcal{E}_\mathfrak{l}] \xrightarrow{\partial_*} \prod_{j=1}^{h} \mathbb{Z}^0[\mathcal{V}_j] \to 0 \tag{7}$$

and

$$0 \to \mathbb{X}_\mathfrak{l} \xrightarrow{j} \mathbb{X}_\mathfrak{l}^\vee \xrightarrow{\tau_\mathfrak{l}} \Phi_\mathfrak{l} \to 0, \tag{8}$$

where the maps are obtained by taking the product for $j = 1, \dots, h$ of the corresponding maps in the exact sequences (5) and (6), so that, for example, $\partial_\mathfrak{l} = \prod_{j=1}^h \partial_{j,*}$. Taking the product of the maps ω_j associated to each X_j as in Equation (5) yields a natural non-trivial map

$$\omega_\mathfrak{l} : \prod_{j=1}^{h} \mathbb{Z}^0[\mathcal{V}_j] \to \Phi_\mathfrak{l}. \tag{9}$$

2.4 Hecke modules

All the objects defined above are variously endowed with a Hecke module structure.

Fix an ideal $\mathfrak{c} \subseteq \mathcal{O}_F$. Denote by $\mathrm{T}_\mathfrak{q}$ (respectively, $\mathrm{U}_\mathfrak{q}$) the Hecke operator corresponding to the prime $\mathfrak{q} \nmid \mathfrak{c}$ (respectively, $\mathfrak{q} \mid \mathfrak{c}$). Denote also by $\mathrm{S}_\mathfrak{q}$ the spherical operator at the prime ideal $\mathfrak{q}$. Let $\mathbb{T}_\mathfrak{c}$ denote the Hecke algebra generated over $\mathbb{Z}$ by the Hecke operators $\mathrm{T}_\mathfrak{q}$ and $\mathrm{S}_\mathfrak{q}$ for $\mathfrak{q} \nmid \mathfrak{c}$ and $\mathrm{U}_\mathfrak{q}$ for $\mathfrak{q} \mid \mathfrak{c}$. The Hecke algebra $\mathbb{T}_\mathfrak{c}$ acts on the space of Hilbert modular forms $S_2(\mathfrak{c})$ for the $\Gamma_0(\mathfrak{c})$-level structure of parallel weight 2 and trivial central character. Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be two pairwise coprime ideals such that $\mathfrak{c} = \mathfrak{c}_1\mathfrak{c}_2$. Let $\mathbb{T}_{\mathfrak{c}_1,\mathfrak{c}_2}$ be the quotient of $\mathbb{T}_\mathfrak{c}$ acting faithfully on the $\mathbb{C}$-subspace $S_2^{\mathrm{new}}(\mathfrak{c}_1, \mathfrak{c}_2)$ of $S_2(\mathfrak{c})$ consisting of forms which are new at primes dividing $\mathfrak{c}_2$. Finally, for a square-free ideal $\mathfrak{s}$, denote by $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}_1,\mathfrak{c}_2}$ the subalgebra of $\mathbb{T}_{\mathfrak{c}_1,\mathfrak{c}_2}$ generated by the Hecke operators $\mathrm{T}_\mathfrak{q}$, $\mathrm{U}_\mathfrak{q}$ and $\mathrm{S}_\mathfrak{q}$ for $\mathfrak{q} \nmid \mathfrak{s}$. Write simply $\mathbb{T}^{(\mathfrak{s})}_\mathfrak{c}$ for $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c},1}$.

Let $\mathfrak{c}^+$ and $\mathfrak{c}^-$ be as above and define $\mathfrak{c} := \mathfrak{c}^+\mathfrak{c}^-$. The Jacobian variety $J_{\mathfrak{c}^+,\mathfrak{c}^-}$ of the Shimura curve $X_{\mathfrak{c}^+,\mathfrak{c}^-}$ has a structure of $\mathbb{T}_{\mathfrak{c}^+,\mathfrak{c}^-}$-module. The action of Hecke operators is defined as usual via double coset decomposition. Moreover, let $\mathfrak{s} \subseteq \mathcal{O}_F$ be a square-free ideal prime to $\mathfrak{c}$; if the local component $\mathcal{U}_\mathfrak{q}$ of $\mathcal{U}$ at $\mathfrak{q}$ is isomorphic to the local component at $\mathfrak{q}$ of $\widehat{F}^\times \widehat{\mathcal{R}}(\mathfrak{c}^+)^\times$ for all prime ideals $\mathfrak{q} \nmid \mathfrak{s}$, then the Jacobian variety $J_\mathcal{U}$ of the Shimura curve $X_\mathcal{U}$ has a natural structure of $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+,\mathfrak{c}^-}$-module.

The strong approximation theorem [Vi, page 60] yields the following identifications:

$$\mathcal{V}_\mathfrak{l} \simeq (U\backslash \widehat{B}^\times/B^\times) \times \{0,1\} \quad \text{and} \quad \mathcal{E}_\mathfrak{l} \simeq U_\mathfrak{l}\backslash \widehat{B}^\times/B^\times.$$

If the local components of U are equal to those of $\widehat{F}^\times \widehat{R}_{\mathfrak{c}^+}$ for primes $\mathfrak{q}$ not dividing the ideal $\mathfrak{s}$ prime to $\mathfrak{c}$, then there is a natural action of $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+,\mathfrak{c}^-/\mathfrak{l}}$ (respectively, $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+\mathfrak{l},\mathfrak{c}^-/\mathfrak{l}}$) on $\mathbb{Z}[U\backslash\widehat{B}^\times/B^\times]$ (respectively, $\mathbb{Z}[U_\mathfrak{l}\backslash\widehat{B}^\times/B^\times]$) defined by double coset decompositions. See [L2, Section 3.2] for precise definitions. Once chosen a set of representatives of the double coset space $U\backslash\widehat{B}^\times/B^\times$ or $U_\mathfrak{l}\backslash\widehat{B}^\times/B^\times$, this action is described by the generalized Brandt matrices. See [L2, Section 3.2] for details. It follows that $\mathbb{Z}[\mathcal{V}_\mathfrak{l}]$ (respectively, $\mathbb{Z}[\mathcal{E}_\mathfrak{l}]$) has a structure of $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+,\mathfrak{c}^-/\mathfrak{l}}$-module (respectively, $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+\mathfrak{l},\mathfrak{c}^-/\mathfrak{l}}$-module).

2.5 Eisenstein modules

Let $\phi \in \mathcal{S}_2(\mathfrak{c})$ be a normalized Hilbert modular eigenform over F of parallel weight 2, trivial central character and $\Gamma_0(\mathfrak{c})$-level structure. Let K_ϕ be the finite extension of $\mathbb{Q}$ generated by the eigenvalues of the action of the Hecke algebra $\mathbb{T}_\mathfrak{c}$ on ϕ and let $\mathcal{O}_\phi$ be its ring of integers. Denote by

$$\theta_\phi : \mathbb{T}_\mathfrak{c} \to \mathcal{O}_\phi$$

the morphism associated to ϕ such that $\mathrm{T}(\phi) = \theta_\phi(\mathrm{T})\phi$ for all $\mathrm{T} \in \mathbb{T}_\mathfrak{c}$. For any prime ideal $\wp$, let $\mathcal{O}_{\phi,\wp}$ be the completion of $\mathcal{O}_\phi$ at $\wp$ and denote by

$$\rho_{\phi,\wp} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(K_{\phi,\wp})$$

the Galois representation attached to ϕ. Choose a Galois-stable lattice

$$T_\phi \simeq \mathcal{O}_{\phi,\wp} \times \mathcal{O}_{\phi,\wp} \subseteq K_{\phi,\wp} \times K_{\phi,\wp}$$

and denote by

$$\tilde{\rho}_{\phi,\wp} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathcal{O}_{\phi,\wp})$$

the representation on T_ϕ. Then $\rho_{\phi,\wp}$ is equivalent to $\tilde{\rho}_{\phi,\wp}$. Reduction modulo $\wp$ defines a morphism

$$\bar{\theta}_\phi : \mathbb{T}_\mathfrak{c} \to \mathbb{F}_{\phi,\wp} := \mathcal{O}_\phi/\wp$$

and a residual representation

$$\bar{\rho}_{\phi,\wp} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F}_{\phi,\wp})$$

which depends on the choice of T_ϕ. Denote by $\mathfrak{m}_{\phi,\wp}$ the kernel of $\bar{\theta}_\phi$ and for any square-free ideal $\mathfrak{s} \subseteq \mathcal{O}_F$, let $\mathfrak{m}^{(\mathfrak{s})}_{\phi,\wp} = \mathfrak{m}_{\phi,\wp} \cap \mathbb{T}^{(\mathfrak{s})}_\mathfrak{c}$.

Definition 2.1. *A $\mathbb{T}^{(\mathfrak{s})}_\mathfrak{c}$-module $\mathcal{E}$ is said to be* Eisenstein *if its completion $\mathcal{E}_{\mathfrak{m}^{(\mathfrak{s})}_{\phi,\wp}}$ is zero for any maximal ideal $\mathfrak{m}^{(\mathfrak{s})}_{\phi,\mathfrak{p}}$ such that the residual representation $\bar{\rho}_{\phi,\wp}$ is irreducible.*

There is a characterization of Eisenstein modules introduced by [DT] for $F = \mathbb{Q}$ and generalized by [J3]. An ideal $\mathcal{I}$ of $\mathbb{T}_{\mathfrak{c}}^{(\mathfrak{s})}$ is said to be *Eisenstein* if there exists some integral ideal $\mathfrak{f}$ of $\mathcal{O}_F$ so that for all but finitely many prime ideals $\mathfrak{q}$ of $\mathcal{O}_F$ which are trivial in the narrow ray class group $\mathrm{Cl}(\mathfrak{f})^+$ of $\mathfrak{f}$ the following relations hold:

$$\mathrm{T}_{\mathfrak{q}} \equiv 2 \pmod{\mathcal{I}} \quad \text{and} \quad \mathrm{S}_{\mathfrak{q}} \equiv 1 \pmod{\mathcal{I}}.$$

A $\mathbb{T}_{\mathfrak{c}}^{(\mathfrak{s})}$-module $\mathcal{E}$ is Eisenstein if and only if all maximal ideals in its support are Eisenstein. For more details and the proof of the last assertion, see [J3, Section 3] and [DT, Proposition 2]. These results are in fact generalizations of [Ri, Theorem 5.2, part c].

Let now $J_{\mathcal{U}}$ be a $\mathbb{T}_{\mathfrak{c}^+,\mathfrak{c}^-}^{(\mathfrak{s})}$-module for some ideal $\mathfrak{s}$ prime to $\mathfrak{c} = \mathfrak{c}^+\mathfrak{c}^-$. Define the *Hodge class* of $\mathcal{X}_{\mathcal{U}}$ to be the unique element $\xi \in \mathrm{Pic}(\mathcal{X}_{\mathcal{U}})$ such that ξ has degree one on each connected component of $\mathcal{X}_{\mathcal{U}}$ and for any prime ideal $\mathfrak{q} \subseteq \mathcal{O}_F$ so that $\mathfrak{q} \nmid \mathfrak{c}\mathfrak{s}$, the action of $\mathrm{T}_{\mathfrak{q}} \in \mathbb{T}_{\mathfrak{c}^+,\mathfrak{c}^-}^{(\mathfrak{s})}$ on ξ is given by multiplication by $|\mathfrak{q}| + 1$. For existence and uniqueness of this class, see [Z1, Section 4.1]. Denote by $\mathrm{Pic}^{\mathrm{Eis}}(\mathcal{X}_{\mathcal{U}})$ the subgroup of $\mathrm{Pic}(\mathcal{X}_{\mathcal{U}})$ consisting of those elements whose restriction to any connected component of $\mathcal{X}_{\mathcal{U}}$ is a multiple of ξ. By [Z2, Section 6.1]

$$\mathrm{Pic}(\mathcal{X}_{\mathcal{U}}) = \mathrm{Pic}^{\mathrm{Eis}}(\mathcal{X}_{\mathcal{U}}) \oplus \mathrm{Pic}^0(\mathcal{X}_{\mathcal{U}}).$$

By [Ri, Theorem 5.2, part c] and its generalization [J3, Section 3], $\mathrm{Pic}^{\mathrm{Eis}}(\mathcal{X}_{\mathcal{U}})$ is Eisenstein. It follows that if $\mathcal{I} \subseteq \mathbb{T}_{\mathfrak{c}^+,\mathfrak{c}^-}^{(\mathfrak{s})}$ is an Eisenstein ideal, than the canonical inclusion $\mathrm{Pic}^0(\mathcal{X}_{\mathcal{U}}) \subseteq \mathrm{Pic}(\mathcal{X}_{\mathcal{U}})$ yields an isomorphism:

$$\mathrm{Pic}^0(\mathcal{X}_{\mathcal{U}})/\mathcal{I} \simeq \mathrm{Pic}(\mathcal{X}_{\mathcal{U}})/\mathcal{I}. \tag{10}$$

2.6 Eisenstein pairs

Let $S \subseteq \widehat{B}^\times$ be U or $U_{\mathfrak{l}}$ defined above. Choose a basis $g_1, \dots, g_h$ of $S \backslash \widehat{B}^\times / B^\times$ (which is finite set by a compactness argument: see [Vi, V.2]) and define

$$W(S) := \{\#(\Gamma_{g_i} \cap \Gamma_{g_j})/\mathcal{O}_F^\times : i, j \in \{1, \dots, h\}\},$$

where $\Gamma_g = g^{-1} S g \cap B$.

Definition 2.2. *The pair (F, S) is said to be* Eisenstein *if at least one of the following conditions is verified:*

(*i*) *The class number of F is one and for any $n \in W(S)$ defined as above, the norm map from the ideals of $F(\zeta_n)$ to the ideals of F is injective, where ζ_n is a primitive n-root of unity.*

(*ii*) $W(S) = \{1\}$.

Denote by $\delta_{j,*}$ the restriction of $\partial_{j,*}$ to $\mathbb{Z}^0[\mathcal{E}_j]$ and define $\delta_* := \prod_{j=1}^h \delta_{j,*}$. Let U and $U_\mathfrak{l}$ be defined as above from a open subgroup $\mathcal{U} \subseteq \widehat{B}^\times$ which is compact modulo $\widehat{F}^\times$. Assume as above that $J_\mathcal{U}$ is a $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+,\mathfrak{c}^-}$-module for some ideal $\mathfrak{s}$ prime to $\mathfrak{c}$, where $\mathfrak{c} := \mathfrak{c}^+\mathfrak{c}^-$. Note that $\mathrm{Im}(\delta_*)$ is a priori a $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+,\mathfrak{c}^-/\mathfrak{l}}$-module because it is a submodule of $\mathbb{Z}^0[U\backslash\widehat{B}^\times/B^\times]$. On the other hand, the source of $\omega_\mathfrak{l}$ is a $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+\mathfrak{l},\mathfrak{c}^-/\mathfrak{l}}$-module because it is contained in $\mathbb{Z}^0[U_\mathfrak{l}\backslash\widehat{B}^\times/B^\times]$. From this it follows that $\mathrm{Im}(\delta_*)$ is also a $\mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+\mathfrak{l},\mathfrak{c}^-/\mathfrak{l}}$-module and the relation between the operators $\mathrm{T}_\mathfrak{l} \in \mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+,\mathfrak{c}^-/\mathfrak{l}}$ and $\mathrm{U}_\mathfrak{l} \in \mathbb{T}^{(\mathfrak{s})}_{\mathfrak{c}^+\mathfrak{l},\mathfrak{c}^-/\mathfrak{l}}$ can be explicitly computed: see [L2, Equation (9)]. In the following proposition, $\mathrm{Im}(\delta_*)$ is endowed with this structure of $\mathbb{T}^{(\mathfrak{s})}_\mathfrak{c}$-module.

Proposition 2.3. *Let $\omega_\mathfrak{l}$ be the map defined in (9).*

1. *The restriction of $\omega_\mathfrak{l}$ to $\mathrm{Im}(\delta_*)$ induces a $\mathbb{T}^{(\mathfrak{s})}_\mathfrak{c}$-equivariant map*

$$\overline{\omega}_\mathfrak{l} : \mathrm{Im}(\delta_*)/(\mathrm{U}_\mathfrak{l}^2 - 1) \to \Phi_\mathfrak{l}.$$

2. *If (F, U) (respectively, $(F, U_\mathfrak{l})$) is Eisenstein, then the kernel (respectively, the cokernel) of $\overline{\omega}_\mathfrak{l}$ is Eisenstein.*

3. *If $d = [F : \mathbb{Q}]$ is odd or d is even and $\mathfrak{c}^-/\mathfrak{l} \neq \mathcal{O}_F$, then the kernel and the cokernel of $\omega_\mathfrak{l}$ are Eisenstein.*

Proof. The first part follows from a calculation as in [BD2, Proposition 5.13]. The second part needs generalizations of [Ri, Proposition 3.12] and [Ri, Theorem 5.2, part c] which can be performed when U or $U_\mathfrak{l}$ are Eisenstein. The third part follows form [Ra, Corollary 4]. For details, see [L2, Proposition 4.4]. □

3 Construction of the Euler system

3.1 Congruences between Hilbert modular forms

Let ϕ be a parallel weight 2 Hilbert modular newform with trivial central character and $\Gamma_0(\mathfrak{n})$-level structure which is an eigenform for the Hecke algebra $\mathbb{T}_\mathfrak{n}$. Fix a prime $\wp$ of the ring of integers $\mathcal{O}_\phi$ of the field K_ϕ generated by the eigenvalues of the action of $\mathbb{T}_\mathfrak{n}$ on ϕ. Let $\mathbb{F}_\wp := \mathcal{O}_{\phi,\wp}/\wp$ be of characteristic $p \geq 5$. Denote by

$$f : \mathbb{T}_\mathfrak{n} \to \mathcal{O}_\phi/\wp^n$$

the reduction modulo $\wp^n$ of the morphism $\theta_\phi : \mathbb{T}_{\mathfrak{n}} \to \mathcal{O}_\phi$ associated to ϕ. Let $\mathcal{I}_f$ be the kernel of f and $\mathfrak{m}_f$ be the maximal ideal which contains $\mathcal{I}_f$.

Let K/F be a quadratic totally imaginary extension. Assume from now to the end of Section 3 that

$$K \text{ satisfies Assumption 1.1.}$$

For any ideal $\mathfrak{r} \subseteq \mathcal{O}_F$, denote by $|\mathfrak{r}|$ its norm.

Definition 3.1. *Define a prime ideal ℓ of $\mathcal{O}_F$ to be $\wp^n$-*admissible prime *if:*

1. *ℓ does non divide $\mathfrak{n}p$;*
2. *ℓ is inert in K/F;*
3. *$\wp$ does non divide $|\ell|^2 - 1$;*
4. *$\wp^n$ divides $|\ell| + 1 - \epsilon f(\mathrm{T}_\ell)$, where $\epsilon = \pm 1$.*

Fix an integer $n \geq 1$ and let ℓ be a $\wp^n$-admissible prime. Set $\mathbb{T} := \mathbb{T}_{\mathfrak{n}^+,\mathfrak{n}^-}$ and $\mathbb{T}_\ell := \mathbb{T}_{\mathfrak{n}^+,\mathfrak{n}^-\ell}$. Recall the factorization $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$ associated to K as in the Introduction: $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) is divisible only by prime ideals which are split (respectively, inert) in K. By Assumption 1.1, $\mathfrak{n}^-$ is square free and the number of prime ideals dividing it has the same parity as the degree d of F over $\mathbb{Q}$. It follows that $\mathfrak{n}^-\ell$ is again square free and the number of primes dividing it and d have opposite parity. Hence, it is possible to define the Shimura curve

$$X^{(\ell)} := X_{\mathfrak{n}^+,\mathfrak{n}^-\ell}$$

as in Section 2.1. Let Φ_ℓ and $\mathrm{Im}(\delta_*)$ be the objects defined as in Section 2 relatively to $X^{(\ell)}$.

By the Jacquet-Langlands correspondence [JL], ϕ can be associated to a modular form of level $\mathfrak{n}^+$ on the quaternion algebra B defined in Section 2; in other words, there is a function

$$\widehat{F}^\times \widehat{R}^\times_{\mathfrak{n}^+} \backslash \widehat{B}^\times / B^\times \to \mathcal{O}_\phi$$

which does not factor through the adelization of the norm map and has the same eigenvalues of ϕ under the action of the Hecke algebra $\mathbb{T}$ defined via double coset decomposition.

Definition 3.2. *ϕ is said to be $\wp$-*isolated *if the completion of the group ring of $\widehat{R}^\times_{\mathfrak{n}^+} \backslash \widehat{B}^\times / B^\times$ at $\mathfrak{m}_f$ is free of rank one over $\mathcal{O}_\phi$.*

The condition in the above definition simply asserts that there are no non-trivial congruences between ϕ and other forms of level $\mathfrak{n}$ which are new at $\mathfrak{n}^-$. If ϕ is $\wp$-isolated, it follows as in [BD2, Theorem 5.15] from Definition 3.1 ($\wp^n$-admissible primes) that

$$\mathrm{Im}(\delta_*)/\langle \mathcal{I}_f, \mathrm{U}_\ell^2 - 1\rangle \simeq \mathcal{O}_\phi/\wp^n. \tag{11}$$

Let T (respectively, T′) denote Hecke operators in $\mathbb{T}$ (respectively, in $\mathbb{T}_\ell$).

Theorem 3.3. *Assume that ϕ is $\wp$-isolated and the residual representation $\bar{\rho}_{\phi,\wp}$ associated to ϕ and a choice of Galois stable lattice T_ϕ as in Section 2.5 is irreducible. Then there exists a morphism*

$$f_\ell : \mathbb{T}_\ell \to \mathcal{O}_\phi/\wp^n$$

such that:

1. *Primes $\mathfrak{q} \nmid \mathfrak{n}\ell$: $f_\ell(\mathrm{T}'_\mathfrak{q}) = f(\mathrm{T}_\mathfrak{q})$;*
2. *Primes $\mathfrak{q} \mid \mathfrak{n}$: $f_\ell(\mathrm{U}'_\mathfrak{q}) = f(\mathrm{U}_\mathfrak{q})$;*
3. *$f_\ell(\mathrm{U}'_\ell) = \epsilon$.*

Proof. If $\mathfrak{n} = \mathcal{O}_F$ and d is even, this result is contained in [Ta, Theorem 1]. In the other cases, including the cases when both (F, U) and (F, U_ℓ) are Eisenstein, the proof is a variation of [Ri, Section 7] based on Proposition 2.3, Equation (11) and the discussion of the Hecke operators in Section 2.6. More precisely, recall that $\mathrm{Im}(\delta_*)/\langle \mathcal{I}_f, \mathrm{U}_\ell^2 - 1\rangle$ is a $\mathbb{T}_{\mathfrak{n}^+\ell,\mathfrak{n}^-}$-module by the discussion in Section 2.6. So, the action of the Hecke algebra on it is via a surjective homomorphism

$$f'_\ell : \mathbb{T}_{\mathfrak{n}^+\ell,\mathfrak{n}^-} \to \mathcal{O}_\phi/\wp^n$$

by Equation (11). Denote by $\mathcal{I}_{f'_\ell}$ the kernel of f'_ℓ. Since kernel and cokernel of $\overline{\omega}_\ell$ are Eisenstein by Proposition 2.3, there is an isomorphism

$$\mathcal{O}_\phi/\wp^n \simeq \mathrm{Im}(\delta_*)/\langle \mathcal{I}_f, \mathrm{U}_\ell^2 - 1\rangle \simeq \Phi_\ell/\mathcal{I}_{f'_\ell}. \tag{12}$$

On the other hand, the action of $\mathbb{T}_{\mathfrak{n}\ell}$ on $\Phi_\ell/\mathcal{I}_{f'_\ell}$ is via its ℓ-new part by the discussion in Section 2.5; it follows that f'_ℓ factors through $\mathbb{T}_\ell$ giving the character

$$f_\ell : \mathbb{T}_\ell \to \mathcal{O}_\phi/\wp^n.$$

For more details (when $n = 1$ but the general case is completely analogous) see [Ra, Theorem 3 and Corollary 4]. □

Assume from now on that the assumptions in Theorem 3.3 hold. Let $R := R_{\mathfrak{n}^+}$ and $R_\ell := R_{\mathfrak{n}^+\ell}$ be Eichler orders in B of level, respectively, $\mathfrak{n}^+$ and $\mathfrak{n}^+\ell$. Let $\mathcal{I}_{f_\ell}$ be the kernel of the map f_ℓ defined in Theorem 3.3. Equation (12) and the final step in the proof of Theorem 3.3 imply that in many cases

$$\Phi_\ell/\mathcal{I}_{f_\ell} \simeq \mathrm{Im}(\delta_*)/\langle \mathcal{I}_f, \mathrm{U}_\ell^2 - 1\rangle \simeq \mathcal{O}_\phi/\wp^n$$

where the first isomorphism is induced by the map $\overline{\omega}_\ell$. The next proposition provides in the remaining cases a $\mathcal{O}_\phi/\wp^n$-module free of rank one inside $\Phi_\ell/\mathcal{I}_{f_\ell}$ which is isomorphic via $\overline{\omega}_\ell$ to $\mathrm{Im}(\delta_*)/\langle \mathcal{I}_f, \mathrm{U}_\ell^2 - 1\rangle$.

Proposition 3.4. *If $\mathfrak{n}^- = \mathcal{O}_F$ and $[F : \mathbb{Q}]$ is even, then there is a component $\mathcal{C}_\ell \simeq \mathcal{O}_\phi/\wp^n \hookrightarrow \Phi_\ell/\mathcal{I}_{f_\ell}$ and an isomorphism induced by $\overline{\omega}_\ell$ between $\mathrm{Im}(\delta_*)/\langle \mathcal{I}_f, \mathrm{U}_\ell^2 - 1\rangle$ and $\mathcal{C}_\ell$.*

Proof. For any prime ideal $\mathfrak{q}_0 \nmid \mathfrak{n}\ell$ denote by $\mathcal{U}(\mathfrak{q}_0)$ the subgroup of $\widehat{\mathcal{B}}^\times$:

$$\mathcal{U}(\mathfrak{q}_0) := \mathcal{U}^{(\mathfrak{q}_0)}\Gamma_1(\mathfrak{q}_0),$$

where $\Gamma_1(\mathfrak{q}_0)$ is the subgroup of the matrixes $A \in \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}_0})$ so that

$$A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathfrak{q}_0}.$$

Denote by $X(\mathfrak{q}_0) \to \mathrm{Spec}(F)$ the Shimura curve of level $\mathcal{U}(\mathfrak{q}_0)$ and let $\mathcal{X}(\mathfrak{q}_0) \to \mathrm{Spec}(\mathcal{O}_F)$ be its integral model. Choose $\mathfrak{q}_0$ so that:

(i) There are no congruences between forms of level $\mathfrak{n}$ and forms of level $\mathcal{U}(\mathfrak{q}_0)$ which are new at $\mathfrak{q}_0$.

(ii) The integral model $\mathcal{X}(\mathfrak{q}_0)$ is regular.

This is possible by [J2, Section 12]. See also [J1, Section 6]. Note that ϕ, viewed as a modular form of level $\mathcal{U}(\mathfrak{q}_0)$, is a modulo $\wp^n$ eigenform for $\mathbb{T}^{(\mathfrak{q}_0)}$ with eigenvalues in $\mathcal{O}_\phi/\wp^n$; denote by

$$f^{(\mathfrak{q}_0)} : \mathbb{T}^{(\mathfrak{q}_0)} \to \mathcal{O}_\phi/\wp^n$$

the associated morphism and by $\mathcal{I}_f^{(\mathfrak{q}_0)}$ its kernel. Let $\mathcal{G}_\ell(\mathfrak{q}_0) = \coprod_{j=1}^h \mathcal{G}_j(\mathfrak{q}_0)$ be the dual graph of the special fiber at ℓ of $\mathcal{X}(\mathfrak{q}_0)$. Denote by

$$\mathcal{V}(\mathcal{G}_\ell(\mathfrak{q}_0)) = \coprod_{j=1}^h \mathcal{V}_j(\mathfrak{q}_0) \text{ and } \mathcal{E}(\mathcal{G}_\ell(\mathfrak{q}_0)) = \coprod_{j=1}^h \mathcal{E}_j(\mathfrak{q}_0),$$

respectively, the vertexes and the edges of $\mathcal{G}_\ell(\mathfrak{q}_0)$. Define as above a map $\delta_*(\mathfrak{q}_0)$:

$$\prod_{j=1}^{h} \mathbb{Z}^0[\mathcal{E}(\mathcal{G}_j(\mathfrak{q}_0)] \xrightarrow{\delta_*(\mathfrak{q}_0)} \prod_{j=1}^{h} \mathbb{Z}^0[\mathcal{V}(\mathcal{G}_j(\mathfrak{q}_0))]$$

and denote by

$$\overline{\omega}_\ell(\mathfrak{q}_0) : \mathrm{Im}(\delta_*(\mathfrak{q}_0)) \to \Phi_\ell(\mathfrak{q}_0)$$

the resulting map, where $\Phi_\ell(\mathfrak{q}_0)$ is the group of connected components of the Jacobian of $\mathcal{X}(\mathfrak{q}_0)$ at ℓ. Since $\mathcal{X}(\mathfrak{q}_0)$ is regular, the weights of the singular points are all equal to one, so kernel and cokernel of $\overline{\omega}_\ell$ are Eisenstein and, as in the proof of Theorem 3.3, there is an isomorphism:

$$\mathrm{Im}(\delta_*(\mathfrak{q}_0))/\langle \mathcal{I}_f^{(\mathfrak{q}_0)}, \mathrm{U}_\ell^2 - 1\rangle \to \Phi_\ell(\mathfrak{q}_0)/\mathcal{I}_{f_\ell}^{(\mathfrak{q}_0)}. \tag{13}$$

There are maps:

$$\mathrm{Im}(\delta_*) \times \mathrm{Im}(\delta_*) \to \mathrm{Im}(\delta_*(\mathfrak{q}_0)) \qquad \text{and} \qquad \Phi_\ell \times \Phi_\ell \to \Phi_\ell(\mathfrak{q}_0);$$

denote by $\mathrm{Im}(\delta_*(\mathfrak{q}_0))^{\mathrm{old}}$ and $\Phi_\ell(\mathfrak{q}_0)^{\mathrm{old}}$ the respective images. Since there are no congruences between forms of level $\mathfrak{n}$ and forms of level $\mathcal{U}(\mathfrak{q}_0)$ which are new at $\mathfrak{q}_0$, there are isomorphisms:

$$\mathrm{Im}(\delta_*(\mathfrak{q}_0))^{\mathrm{old}}/\langle \mathcal{I}_f^{(\mathfrak{q}_0)}, \mathrm{U}_\ell^2 - 1\rangle \simeq \mathrm{Im}(\delta_*(\mathfrak{q}_0))/\langle \mathcal{I}_f^{(\mathfrak{q}_0)}, \mathrm{U}_\ell^2 - 1\rangle$$

and

$$\Phi_\ell(\mathfrak{q}_0)^{\mathrm{old}}/\mathcal{I}_{f_\ell}^{(\mathfrak{q}_0)} \simeq \Phi_\ell(\mathfrak{q}_0)/\mathcal{I}_{f_\ell}^{(\mathfrak{q}_0)}.$$

It follows that the map (13) yields an isomorphism:

$$\mathrm{Im}(\delta_*(\mathfrak{q}_0)^{\mathrm{old}})/\langle \mathcal{I}_f^{(\mathfrak{q}_0)}, \mathrm{U}_\ell^2 - 1\rangle \xrightarrow{\overline{\omega}_\ell(\mathfrak{q}_0)} \Phi_\ell(\mathfrak{q}_0)^{\mathrm{old}}/\mathcal{I}_{f_\ell}^{(\mathfrak{q}_0)}.$$

The following diagram, whose vertical arrows are surjections:

$$\begin{array}{ccc} \mathrm{Im}(\delta_*)/\langle \mathcal{I}_f^{(\mathfrak{q}_0)}, \mathrm{U}_\ell^2 - 1\rangle \times \mathrm{Im}(\delta_*)/\langle \mathcal{I}_f^{(\mathfrak{q}_0)}, \mathrm{U}_\ell^2 - 1\rangle & \xrightarrow{\overline{\omega}_\ell^2} & \Phi_\ell/\mathcal{I}_{f_\ell}^{(\mathfrak{q}_0)} \times \Phi_\ell/\mathcal{I}_{f_\ell}^{(\mathfrak{q}_0)} \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathrm{Im}(\delta_*(\mathfrak{q}_0)^{\mathrm{old}})/\langle \mathcal{I}_f^{(\mathfrak{q}_0)}, \mathrm{U}_\ell^2 - 1\rangle & \xrightarrow{\overline{\omega}_\ell(\mathfrak{q}_0)} & \Phi_\ell(\mathfrak{q}_0)^{\mathrm{old}}/\mathcal{I}_{f_\ell}^{(\mathfrak{q}_0)}. \end{array}$$

implies the result. Indeed, choose a generator P of $\Phi_\ell(\mathfrak{q}_0)^{\mathrm{old}}/\mathcal{I}_{f_\ell}^{(\mathfrak{q}_0)} \simeq \mathcal{O}_\phi/\wp^n$. Then there is

$$(Q_1, Q_2) \in \mathrm{Im}(\delta_*)/\langle \mathcal{I}_f^{(\mathfrak{q}_0)}, \mathrm{U}_\ell^2 - 1\rangle \times \mathrm{Im}(\delta_*)/\langle \mathcal{I}_f^{(\mathfrak{q}_0)}, \mathrm{U}_\ell^2 - 1\rangle$$

such that $[\overline{\omega}_\ell(\mathfrak{q}_0) \circ \pi_1](Q_1, Q_2) = P$. Then also $[\pi_2 \circ \overline{\omega}_\ell^2](Q_1, Q_2) = P$; since P is a generator of $\mathcal{O}_\phi/\wp^n$, it follows that the $\mathcal{O}_\phi/\wp^n$-submodule generated by one of the Q_j's, say Q_1, is isomorphic to $\mathcal{O}_\phi/\wp^n$. The desired component can be defined to be $\mathcal{C}_\ell := \langle \overline{\omega}_\ell(Q_1)\rangle$. □

3.2 Galois representations

Let $J^{(\ell)}$ be the jacobian of $X^{(\ell)}$ and $\mathbb{X}_\ell$ the character group associated to $X^{(\ell)}$ as in Section 2.1. Denote by $\mathrm{Ta}_p(J^{(\ell)})$ the p-adic Tate module of J.

Since $X^{(\ell)}$ is a disjoint union of admissible curves, it is possible to use the Mumford-Kurihara theory of ℓ-adic uniformization (see [GP]) and produce an exact sequence:

$$0 \to \mathbb{X}_\ell \xrightarrow{j} \mathbb{X}_\ell^\vee \otimes_{F_{\ell^2}} \overline{F}_{\ell^2} \to J^{(\ell)}(K_\ell) \to 0, \tag{14}$$

where j is the injection $\mathbb{X}_\ell \hookrightarrow \mathbb{X}_\ell^\vee$ induced by the monodromy pairing, F_{ℓ^2} is the unramified quadratic extension of F_ℓ and $\overline{F}_{\ell^2}$ is its algebraic closure. By the same argument as [BD2, Section 5.6], taking cohomology and tensoring by $\mathbb{T}_\ell/\mathcal{I}_{f_\ell}$ yields an exact sequence:

$$\Phi_\ell/\mathcal{I}_{f_\ell} \to H^1(F_{\ell^2}, \mathrm{Ta}_p(J^{(\ell)})/\mathcal{I}_{f_\ell}) \to H^1_{\mathrm{unr}}(F_{\ell^2}, \mathbb{X}_\ell/\mathcal{I}_{f_\ell}). \tag{15}$$

Fix a choice of Galois-stable lattice $T_\phi \simeq \mathcal{O}^2_{\phi,\wp}$ relative to the $\wp$-adic representation $\rho_{\phi,\wp} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathcal{O}_{\phi,\wp})$. For any integer $n \geq 1$, denote by $T_{\phi,n} \simeq \mathcal{O}_{\phi,\wp}/\wp^n$ the module obtained by reducing T_ϕ modulo $\wp^n$. Let $\mathfrak{m}_{f_\ell}$ be the maximal ideal containing $\mathcal{I}_{f_\ell}$ and K/F a quadratic imaginary extension as in Section 3.1. Define the singular (or ramified) part of the cohomology group $H^1(K, T_{\phi,n})$ at ℓ to be:

$$H^1_{\mathrm{sing}}(K_\ell, T_{\phi,n}) := H^1(K_\ell^{\mathrm{unr}}, T_{\phi,n})^{\mathrm{Gal}(K_\ell^{\mathrm{unr}}/K_\ell)},$$

where for a field k and a $\mathrm{Gal}(\bar{k}/k)$-module M, the group $H^1(k, M)$ is the usual continuous cohomology of $\mathrm{Gal}(\bar{k}/k)$ with values in M. Using that $T_{\phi,n}$ is unramified at ℓ and the fact that the eigenvalues of the absolute Frobenius $\mathrm{Frob}_F(\ell)$ of F at ℓ acting on $T_{\phi,n}$ are $\pm|\ell|$ and ± 1 because ℓ is $\wp^n$-admissible, it is possible to show as in [BD2, Lemma 2.6] that

$$H^1_{\mathrm{sing}}(K_\ell, T_{\phi,n}) \simeq \mathcal{O}_\phi/\wp^n.$$

Proposition 3.5. *Assume that ϕ is $\wp$-isolated and that $\rho_{\phi,\wp}$ is residually irreducible. There exists an integer $k \geq 1$ such that $\mathrm{Ta}_p(J^{(\ell)})/\mathfrak{m}_{f_\ell} \simeq T^k_{\phi,1}$ as Galois modules. Moreover, there is a direct summand $\mathcal{D}_\ell \simeq T_{\phi,n} \subseteq \mathrm{Ta}_p(J^{(\ell)})/\mathcal{I}_{f_\ell}$ such that the natural image of $\mathcal{C}_\ell$ in $H^1(K_\ell, \mathcal{D}_\ell)$ via the map (15) is not trivial and isomorphic to $H^1_{\mathrm{sing}}(K_\ell, T_{\phi,n})$. Finally, if both the pairs $(F, \widehat{R}^\times)$ and $(F, \widehat{R}_\ell^\times)$ are geometric, then $k = 1$, so that $\mathrm{Ta}_p(J^{(\ell)})/\mathcal{I}_{f_\ell} \simeq T_{\phi,n}$.*

Proof. Since $\rho_{\phi,\wp}$ is residually irreducible, the description of the structure of $\mathrm{Ta}_p(J^{(\ell)})/\mathfrak{m}_{f_\ell}$ follows from [BLR] combined with the Eichler-Shimura relation, the Cebotarev density Theorem and the Brauer-Nesbitt Theorem. For the structure of $\mathrm{Ta}_p(J^{(\ell)})/\mathcal{I}_{f_\ell}$, first note that there is a natural $\mathrm{Gal}(\overline{F}/F)$-equivariant projection map:

$$\pi : \mathrm{Ta}_p(J^{(\ell)})/\mathcal{I}_{f_\ell} \to \mathrm{Ta}_p(J^{(\ell)})/\mathfrak{m}_{f_\ell} \simeq T_{\phi,1}^k.$$

Since the $\mathrm{Gal}(\overline{F}/F)$-module $T_{\phi,1}$ is irreducible, for each $j = 1, \dots, k$ it is possible to choose elements $\bar{t}_j \in T_{\phi,1}$ and $g_j \in \mathrm{Gal}(\overline{F}/F)$ so that $\{\bar{t}_j, g_j(\bar{t}_j)\}$ form a $\mathbb{F}_\wp$-basis for $T_{\phi,1}$. For any j, choose $t_j \in \mathrm{Ta}_p(J^{(\ell)})/\mathcal{I}_{f_\ell}$ so that $\pi(t_j) = \bar{t}_j$. By Nakayama's lemma, $\mathrm{Ta}_p(J^{(\ell)})/\mathcal{I}_{f_\ell}$ is generated over $\mathcal{O}_\phi/\wp^n$ by $\{t_j, g_j(t_j), j = 1 \dots, k\}$. Define $\mathcal{A}_j$ to be the $\mathcal{O}_\phi/\wp^n[\mathrm{Gal}(\overline{F}/F)]$-module generated by t_j for $j = 1, \dots, k$. Note that π induces for any $j = 1, \dots, k$ a natural projection map:

$$\pi_j : \mathcal{A}_j \to T_{\phi,1}.$$

Again by Nakayama's lemma, $\mathcal{A}_j$ is generated by t_j and $g(t_j)$. Since $\overline{\rho}_{\phi,\wp}$ is irreducible, the $\mathcal{O}_{\phi,\wp}/\wp^n$-submodules generated by t_j and $g(t_j)$ have trivial intersection for any $j = 1, \dots, k$. This implies that $\mathcal{A}_j \simeq (\mathcal{O}_\phi/\wp^{n_j})^2$ for some $n_j \leq n$, so $\mathcal{A}_j \simeq T_{\phi,n_j}$. Finally, since two distinct copies of $T_{\phi,1}$ in $\mathrm{Ta}_p(J^{(\ell)})/\mathfrak{m}_{f_\ell}$ have trivial intersection, it follows that $\mathcal{A}_i \cap \mathcal{A}_j = 0$ for $i \neq j$, so

$$\mathrm{Ta}(J^{(\ell)})/\mathcal{I}_{f_\ell} \simeq \oplus_{j=1}^k T_{\phi,n_j}.$$

By the same argument as in [BD2, Lemma 5.16], the generator of $\mathcal{C}_\ell$ can be lifted to a non-zero element $t \in \mathrm{Ta}_p(J^{(\ell)})/\mathcal{I}_{f_\ell}$, so that at least one of the n_j is n. Choose one of them, say $\bar{j}$ and define $\mathcal{D}_\ell \simeq T_{\phi,n}$ to be the component on the above decomposition corresponding to $\bar{j}$. Since $\mathcal{C}_\ell$ corresponds to $\langle t \rangle \simeq \mathcal{O}_\phi/\wp^n$, the exact sequence (15) shows that the natural projection of $\mathcal{C}_\ell$ in $H^1(K_\ell, \mathcal{D}_\ell)$ corresponds to the ramified cohomology $H^1_{\mathrm{sing}}(K_\ell, T_{\phi,n})$.

The last part of the proposition is a direct generalization of [BD2, Theorem 5.17], since in this case $\Phi_\ell/\mathcal{I}_{f_\ell}$ is isomorphic to $\mathcal{O}_\phi/\wp^n$ and $k = 1$. □

3.3 The Gross-Zhang formula

Keep the same notations as above. Let ϕ be an eigenform of level $\mathfrak{n}$. Recall from Section 3.1 the factorization $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$ associated with the totally imaginary extension K/F. Form the L-series $L_K(\phi, s)$ attached to ϕ and K. If ϵ is the unique non-trivial quadratic character associated with the extension K/F, then $L_K(\phi, s)$ is defined to be the base change

$$L_K(\phi, s) = L(\phi, s)L(\phi, \epsilon, s),$$

where $L(\phi, s)$ is the usual L-series attached to ϕ and $L(\phi, \epsilon, s)$ is its twist. By [S1], $L_K(\phi, s)$ can be continued to an entire function and has a functional equation. More precisely, define

$$\Lambda_K(\phi, s) = (D_{F/\mathbb{Q}}^2 |D_{K/F} \cdot \mathfrak{n}|)^{s-1} \left(\frac{\Gamma(s)}{(2\pi)^{s-1}} \right)^{2d} L_K(\phi, s),$$

where $D_{F/\mathbb{Q}}$ (respectively, $D_{K/F}$) is the discriminant of F over $\mathbb{Q}$ (respectively, of K over F), $\Gamma(s)$ is the usual complex Γ-function and $|\cdot|$ is the norm map on ideals. The the functional equation is the following:

$$\Lambda_K(\phi, s) = (-1)^d \epsilon(\mathfrak{n}) \Lambda_K(\phi, 2 - s).$$

For more details, see [Z1, Sections 3 and 6]. The sign

$$\epsilon_{\mathfrak{n},K}(\phi) := (-1)^d \epsilon(\mathfrak{n}) = \pm 1$$

of the functional equation is related to the order of vanishing of $L_K(\phi, s)$ at its central point $s = 1$; more precisely, this order is even if $\epsilon_{\mathfrak{n},K}(\phi) = 1$ and is odd otherwise. Since the number of primes dividing $\mathfrak{n}^-$ and d have the same parity by Assumption 1.1, then

$$\epsilon_{\mathfrak{n},K}(\phi) = 1. \tag{16}$$

Let B be the quaternion algebra over F which is ramified at all the archimedean places of F and at all the primes dividing $\mathfrak{n}^-$. Fix an Eichler order R of level $\mathfrak{n}^+$ in B. By the Jacquet-Langlands correspondence, there exists (unique up to multiples) a modular form

$$f : \widehat{F}^\times \widehat{R}^\times \backslash \widehat{B}^\times / B^\times \to \mathcal{O}_\phi$$

with the same eigenvalues as ϕ. Since all primes dividing $\mathfrak{n}^-$ are inert in K, it follows by [Vi, III.3.8], that there exists a monomorphism $\Psi : K \to B$. Assume that Ψ is an optimal embedding of the integers $\mathcal{O}_K$ of K into the Eichler order R, that is, $\Psi(\mathcal{O}_K) = \Psi(K) \cap R$. Adelization $\widehat{\Psi} : \widehat{K} \to \widehat{B}$ yields a map, denoted by the same symbol,

$$\widehat{\Psi} : \widehat{F}^\times \widehat{\mathcal{O}}_K^\times \backslash \widehat{K}^\times / K^\times \to \widehat{F}^\times \widehat{R}^\times \backslash \widehat{B}^\times / B^\times.$$

Definition 3.6. *The* algebraic part $\mathcal{L}_K(\phi)$ *of* $L_K(\phi, s)$ *is:*

$$\mathcal{L}_K(\phi) := \sum_{\mathfrak{a}} (f \circ \widehat{\Psi})(\mathfrak{a}),$$

where the sum is extended over a set of representatives $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_K)$.

Theorem 3.7. $L_K(\phi,1)\neq 0$ *if and only if* $\mathcal{L}_K(\phi)\neq 0$.

Proof. For a reference of this result in this form, see [Va, Theorem 6.4]. For a more precise statement, see [Z2, Theorem 1.3.2], which is a generalization of [Gr, Proposition 7.7]. □

Remark 3.8. By Equation (16), the sign of the functional equation of $L_K(f,1)$ is $+1$, so this condition is compatible with the non vanishing of $L_K(\phi,1)$. Moreover, if $L(\phi,1)\neq 0$, then by [Wa] it is possible to find K satisfying Assumption 1.1 and such that $L_K(\phi,1)\neq 0$.

3.4 Heegner points on Shimura curves

Recall the complex analytic description from Section 2.1 of the Shimura curve $X^{(\ell)}$ considered in Section 3.1:

$$X^{(\ell)}(\mathbb{C})=\widehat{F}^\times\widehat{\mathcal{R}}^\times\backslash\widehat{\mathcal{B}}^\times\times\mathcal{H}^\pm/\mathcal{B}^\times,$$

where $\mathcal{B}$ is a quaternion algebra over F of discriminant $\mathfrak{n}^-\ell$ which is split in precisely one of the archimedean places of F, say μ, and $\mathcal{R}$ is an Eichler order of level $\mathfrak{n}^+$. The existence of a factorization such as $\mathfrak{n}=\mathfrak{n}^+\mathfrak{n}^-$ implies that the set of F-homomorphisms $\mathrm{Hom}(K,\mathcal{B})$ is not empty. Each point $P=(g,\psi)$ of the double coset space

$$\mathbf{X}(K):=\widehat{F}^\times\widehat{\mathcal{R}}^\times\backslash\widehat{\mathcal{B}}^\times\times\mathrm{Hom}(K,\mathcal{B})/\mathcal{B}^\times$$

defines naturally a point $P\in X(K_1)$ by [S2, Chapter 9, Theorem 9.6], where K_1 is the Hilbert class field of K. Define an *Heegner point by* $\mathcal{O}_K$ to be a point $(g,z_\psi)\in X(K_1)$ defined by the previous construction from a pair

$$(g,\psi)\in\widehat{F}^\times\widehat{\mathcal{R}}^\times\backslash(\widehat{\mathcal{B}}^\times\times\mathrm{Hom}(K,\mathcal{B}))/\mathcal{B}^\times,$$

where ψ is an optimal embedding of $\mathcal{O}_K$ into $\mathcal{R}_g:=g^{-1}\widehat{\mathcal{R}}g\cap\mathcal{B}$.

Define the action of $\mathrm{Pic}(\mathcal{O}_K)/\mathrm{Pic}(\mathcal{O}_F)$ on $\mathbf{X}(K)$ as follows. For any element $\mathfrak{a}$ of

$$\widehat{F}^\times\widehat{\mathcal{O}}_K^\times\backslash\widehat{K}^\times/K^\times\simeq\mathrm{Pic}(\mathcal{O}_K)/\mathrm{Pic}(\mathcal{O}_F)$$

and any point $(g,\psi)\in\mathbf{X}(K)$, define:

$$\mathfrak{a}(g,\psi):=(g\psi(\mathfrak{a}),\psi).$$

By [S2, Theorem 9.6], the action of $\mathrm{Pic}(\mathcal{O}_K)/\mathrm{Pic}(\mathcal{O}_F)$ on Heegner points is free and corresponds via class field theory to the Galois action of $\mathrm{Gal}(K_1/K)$.

Let P be an Heegner point by $\mathcal{O}_K$ and define the *Heegner divisor*

$$D_K := \sum_{\sigma} P^{\sigma} \in \mathrm{Div}(X)(K_1),$$

where the sum is extended over all $\sigma \in \mathrm{Gal}(K_1/K)$. By the isomorphism induced by the inclusion (10), it follows that D_K defined a point

$$P_K \in J^{(\ell)}(K)/\mathcal{I}_{f_\ell}.$$

3.5 The reciprocity law

To simplify notations, denote by $\mathcal{C}_\ell$ (respectively, $\mathcal{D}_\ell$):

- the component group $\Phi_\ell/\mathcal{I}_{f_\ell}$ (respectively, the Galois module $T_{\phi,n}$) if $[F:\mathbb{Q}]$ is odd or $[F:\mathbb{Q}]$ is even and $\mathfrak{n}^- \neq \mathcal{O}_F$,
- the component in Proposition 3.4 (respectively, in Proposition 3.5) otherwise.

Since the completion of K_1 at a prime ideal above ℓ is isomorphic to K_ℓ because ℓ is inert in K, it follows that D_K can be viewed as an element in $\mathrm{Div}(\mathcal{X}_\ell^{(\ell)}(K_\ell))$, where $\mathcal{X}_\ell^{(\ell)}$ is the fiber at ℓ of $\mathcal{X}^{(\ell)}$. By the same reason, P_K defines a point in $J^{(\ell)}(K_\ell)/\mathcal{I}_{f_\ell}$.

Define $\mathcal{X}_{\ell^2}^{(\ell)} := \mathcal{X}_\ell^{(\ell)} \otimes_{\mathcal{O}_{F,\ell}} \mathcal{O}_{F,\ell^2}$. (Recall the definition given in Section 2.3). Using the ℓ-adic description of Heegner points obtained from the Cereknik-Drinfeld Theorem, it is possible to show that an Heegner point $P \in X(K_1)$ reduces to a non singular point of the special fiber $\mathcal{X}_{\mathbb{F}_{\ell^2}}$ of $\mathcal{X}_{\ell^2}^{(\ell)}$. It follows that D_K defines a divisor $v_K \in \mathrm{Im}(\delta_*)/\langle \mathcal{I}_f, \mathrm{U}_\ell^2 - 1\rangle$.

Denote by ∂_ℓ the reduction map $J(K_\ell) \to \Phi_\ell$. Combining Equation (4) and the isomorphism coming from the inclusion (10) yields the following relation in $\mathcal{C}_\ell$

$$\partial_\ell(P_K) = \overline{\omega}_\ell(v_K). \tag{17}$$

(Recall that the image of $\overline{\omega}_\ell$ is always contained in $\mathcal{C}_\ell$). The following proposition is the *reciprocity law* connecting Heegner points and the special value of L-series.

Proposition 3.9. *The equality*

$$\partial_\ell(P_K) = \mathcal{L}_K(\phi)$$

holds in $\mathcal{C}_\ell \simeq \mathcal{O}_\phi/\wp^n$ up to multiplication by invertible elements.

Proof. The ℓ-adic description of Heegner points recalled in [L2, Section 5.2] implies that the action of $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)/\mathrm{Pic}(\mathcal{O}_F)$ on Heegner points corresponds to right multiplication by $\mathfrak{a}$. The result follows by comparing the right hand side of Equation (17) with the definition of $\mathcal{L}_K(\phi)$. □

3.6 The Euler system

From now on assume that $\rho_{\phi,\wp}$ is residually irreducible and that ϕ is $\wp$-isolated. Assume moreover that p is prime to $\mathfrak{n}$ and the absolute discriminant of K. Taking tensor product by $\mathcal{O}_\phi$ and quotients by $\mathcal{I}_{f_\ell}$, the Kummer map $\delta : J^{(\ell)}(K)/p^n J^{(\ell)}(K) \to H^1(K, J^{(\ell)}[p^n])$ yields a map:

$$d : J^{(\ell)}(K)/\mathcal{I}_{f_\ell} \to H^1(K, \mathrm{Ta}_p(J^{(\ell)}))/\mathcal{I}_{f_\ell}) \simeq \prod_{j=1}^{k} H^1(K, T_{\phi,n_j}).$$

Let

$$\pi_\ell : \prod_{j=1}^{k} H^1(K, T_{\phi,n_j}) \to H^1(K_\ell, \mathcal{D}_\ell) \simeq H^1(K_\ell, T_{\phi,n})$$

be the projection map to $\mathcal{D}_\ell$. Define $\kappa_\ell := \pi_\ell(d(P_K))$.

To describe the behavior of κ_ℓ and show that it has the property of Euler systems described in Section 1, it is necessary to study the local structures associated to $T_{\phi,n}$. Denote as above by $T_\phi \simeq \mathcal{O}^2_{\phi,\wp}$ the $\mathrm{Gal}(\overline{F}/F)$-stable lattice associated to the representation $\rho_{\phi,\wp}$ and define $V_\phi := T_\phi \otimes K_\phi \simeq K^2_{\phi,\wp}$. Define as above

$$T_{\phi,n} := T_\phi/\wp^n T_\phi \simeq (\mathcal{O}_\phi/\wp^n)^2$$

so that the multiplication by $\wp$: $T_{\phi,n} \to T_{\phi,n-1}$ yields a projective system and $T_\phi = \lim_{\leftarrow n} T_{\phi,n}$. Define:

$$A_\phi := V_\phi/T_\phi \simeq (K_{\phi,\wp}/\mathcal{O}_{\phi,\wp})^2$$

and denote by $A_{\phi,n} \simeq (\mathcal{O}_\phi/\wp^n)^2$ its $\wp^n$-torsion. The natural inclusion

$$A_{\phi,n} \hookrightarrow A_{\phi,n+1}$$

yields an inductive system and $A_\phi = \lim_{\rightarrow n} A_{\phi,n}$. Note that the Galois modules $T_{\phi,n}$ and $A_{\phi,n}$ are isomorphic and they correspond to the Galois module associated to the reduction modulo $\wp^n$ of the representation $\rho_{\phi,\wp}$.

For a $\mathrm{Gal}(\overline{F}/F)$-module M and a prime ideal $\mathfrak{q}$ of $\mathcal{O}_F$, let $H^1(K_\mathfrak{q}, M)$ denote the direct sum $\oplus_{\mathfrak{q}'|\mathfrak{q}} H^1(K_{\mathfrak{q}'}, M)$ of the local cohomology groups al all prime ideals $\mathfrak{q}'$ of K dividing $\mathfrak{q}$.

Define the following finite/singular structures, where $M = A_{\phi,n}$ or $T_{\phi,n}$: Let $\mathfrak{q} \subseteq \mathcal{O}_F$ be a prime ideal such that $\mathfrak{q} \nmid \mathfrak{n}p$. The *singular part* of $H^1(K_\mathfrak{q}, M)$ is

$$H^1_{\mathrm{sing}}(K_\mathfrak{q}, M) := \oplus_{\mathfrak{q}'|\mathfrak{q}} H^1(I_{\mathfrak{q}'}, M)^{\mathrm{Gal}(K^{\mathrm{unr}}_{\mathfrak{q}'}/K_{\mathfrak{q}'})},$$

where the sum is extended over the primes $\mathfrak{q}' \subseteq \mathcal{O}_K$ dividing $\mathfrak{q}$, $G_{\mathfrak{q}'} \subseteq \mathrm{Gal}(\overline{K}/K)$ is the choice of a decomposition subgroup at $\mathfrak{q}'$ and $I_{\mathfrak{q}'} \subseteq G_{\mathfrak{q}'}$

denotes the inertia subgroup. The *finite part* is defined by the exactness of the following sequence:

$$0 \to H^1_{\mathrm{fin}}(K_{\mathfrak{q}}, M) \to H^1(K_{\mathfrak{q}}, M) \xrightarrow{\partial_{\mathfrak{q}}} H^1_{\mathrm{sing}}(K_{\mathfrak{q}}, M).$$

The groups $H^1_{\mathrm{fin}}(K_{\mathfrak{q}}, A_{\phi,n})$ and $H^1_{\mathrm{sing}}(K_{\mathfrak{q}}, T_{\phi,n})$ are annihilators of each other under the local Tate pairing $\langle , \rangle_{\mathfrak{q}}$. See [Mi, Chapter I] for the definition of the local Tate pairing and its properties.

Let $\mathfrak{p} \mid p$ be a prime ideal of $\mathcal{O}_K$ and suppose that ϕ is ordinary at $\mathfrak{p}$. Then there is an exact sequence of $I_{\mathfrak{p}}$-modules (where $I_{\mathfrak{p}}$ is the inertia subgroup of $G_{\mathfrak{q}}$):

$$0 \to A^{(\mathfrak{p})}_{\phi} \to A_{\phi} \to A^{(1)}_{\phi} \to 0$$

such that $I_{\mathfrak{p}}$ acts on the free of rank one $\mathcal{O}_{\phi,\wp}$-module $A^{(\mathfrak{p})}_{\phi}$ by the cyclotomic character ϵ and it acts trivially on the quotient $A^{(1)}_{\phi}$, which is also free of rank one over $\mathcal{O}_{\phi,\wp}$. Let

$$\lambda_{\mathfrak{p}} : \oplus_{\mathfrak{p}'|\mathfrak{p}} H^1(K_{\mathfrak{p}'}, A^{(\mathfrak{p})}_{\phi}) \to \oplus_{\mathfrak{p}'|\mathfrak{p}} H^1(K_{\mathfrak{p}'}, A_{\phi})$$

be the map of cohomology groups induced by the inclusion $A^{(\mathfrak{p})}_{\phi} \subseteq A_{\phi}$, where the sum is over all prime ideals $\mathfrak{p}'$ of K dividing $\mathfrak{p}$. Define the *ordinary part* $H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, A_{\phi})$ of $H^1(K_{\mathfrak{p}}, A_{\phi})$ to be the maximal divisible subgroup of $\mathrm{Image}(\lambda_{\mathfrak{p}})$. Define

$$\mathcal{H}(\mathfrak{p}, n) := H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, A_{\phi}) \cap H^1(K_{\mathfrak{p}}, A_{\phi,n}).$$

For any subgroup $\mathcal{H} \subseteq H^1(K_{\mathfrak{p}}, A_{\phi,n})$, use the isomorphism $A_{\phi,n} \simeq T_{\phi,n}$ to define a subgroup $\mathcal{H}^* \subseteq H^1(K_{\mathfrak{p}}, T_{\phi,n})$ such that $\mathcal{H} \simeq \mathcal{H}^*$. Then define $H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, A_{\phi,n})$ to be the maximal subgroup of $H^1(K_{\mathfrak{p}}, A_{\phi,n})$ containing $\mathcal{H}(\mathfrak{p}, n)$ and such that $H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, A_{\phi,n})$ and $H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, A_{\phi,n})^*$ are the exact annihilators of each other under the local Tate pairing at $\mathfrak{p}$.

For any prime ideal $\mathfrak{q} \subseteq \mathcal{O}_F$, let $\mathrm{res}_{\mathfrak{q}} : H^1(K, M_{\phi,n}) \to H^1(K_{\mathfrak{q}}, M_{\phi,n})$ be the restriction map. For primes $\mathfrak{q} \nmid \mathfrak{n}p$, denote $\partial_{\mathfrak{q}}(\mathrm{res}_{\mathfrak{q}}(\kappa))$ simply by $\partial_{\mathfrak{q}}(\kappa)$, where $\kappa \in H^1(K, M_{\phi,n})$.

Theorem 3.10. *Suppose that ϕ is $\wp$-isolated, $\rho_{\phi,\wp}$ is residually irreducible and ϕ is ordinary at every prime ideal $\mathfrak{p}$ dividing the residue characteristic p of $\wp$. Then the class $\kappa_{\ell} \in H^1(K, T_{\phi,n})$ satisfies the following properties.*

1. *For $\mathfrak{q} \nmid \mathfrak{n}p\ell$, $\partial_{\mathfrak{q}}(\kappa_{\ell}) = 0$.*

2. *For $\mathfrak{p} \mid p$, $\mathrm{res}_{\mathfrak{p}}(\kappa_{\ell}) \in H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, T_{\phi,n})$.*

3. *The equality $\partial_\ell(\kappa_\ell) = \mathcal{L}_\phi(K)$ holds in $\mathcal{O}_\phi/\wp^n$ up to multiplication by invertible elements.*

Proof. For primes not dividing $\mathfrak{n}p\ell$, this follows by [Mi, Chapter I, Section 3]. The second part follows form the description of the Kummer map given by [CG, Proposition 4.5]. For the last part, first note that by Proposition 3.5, $\mathcal{C}_\ell \simeq H^1_{\text{sing}}(K_\ell, T_{\phi,n}) \simeq \mathcal{O}_\phi/\wp^n$. By Equation (17), the image of κ_ℓ is then contained in the singular cohomology group. The result follows by Proposition 3.9. $\square$

4 Arithmetic applications

4.1 Modular abelian varieties

Let A/F be an abelian variety of GL_2-type, that is, $[\mathrm{End}_{\mathbb{Q}}(A) : \mathbb{Q}] = \dim(A)$, where $\mathrm{End}_{\mathbb{Q}}(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Set $E := \mathrm{End}_{\mathbb{Q}}(A)$ and assume that $\mathrm{End}(A) \simeq \mathcal{O}_E$, where $\mathcal{O}_E$ is the ring of integers of E. For any prime ideal $\wp \subseteq \mathcal{O}_E$, denote by $A[\wp^n]$ the $\wp^n$-torsion in A and by $T_\wp(A)$ its $\wp$-adic Tate module. Denote by

$$\rho_{A,\wp} : G_F \to \mathrm{Aut}(T_\wp(A)) \simeq \mathrm{GL}_2(\mathcal{O}_{E,\wp})$$

the associated representation. Finally, denote by $\mathfrak{n} \subseteq \mathcal{O}_F$ the arithmetic conductor of A/F.

Definition 4.1. *An abelian variety A/F is said to be* modular *if there exists a newform $\phi \in S_2(\mathfrak{n})$, which is an eigenform for the Hecke algebra $\mathbb{T}_\mathfrak{n}$, such that $E = K_\phi$ and $\rho_{A,\wp}$ is equivalent to $\rho_{\phi,\wp}$, where $\wp$ is a prime ideal of $\mathcal{O}_\phi$.*

Let K/F be a quadratic totally imaginary extension and assume A/F be modular. Fix a prime ideal $\wp$ of $\mathcal{O}_\phi$ of residue characteristic p. The Selmer group $\mathrm{Sel}_{\wp^n}(A/K)$ is defined by the exactness of the following sequence:

$$0 \to \mathrm{Sel}_{\wp^n}(A/K) \to H^1(K, A[\wp^n]) \to \oplus_\mathfrak{q} H^1(K_\mathfrak{q}, A[\wp^n])/\mathrm{Im}(\delta_\mathfrak{q}),$$

where the direct sum is over the set of prime ideals $\mathfrak{q}$ of $\mathcal{O}_F$ and $\mathrm{Im}(\delta_\mathfrak{q})$ is the image of the local Kummer map

$$\delta_\mathfrak{q} : A(K_\mathfrak{q})/\wp^n A(K_\mathfrak{q}) \hookrightarrow H^1(K_\mathfrak{q}, A[\wp^n]).$$

Note that the image $\mathrm{Im}(\delta_\mathfrak{q})$ of the Kummer map can be described in terms of the representation $A_{\phi,n}$ as follows:

1. Primes $\mathfrak{q} \nmid \mathfrak{n}p$: $\mathrm{Im}(\delta_{\mathfrak{q}}) = H^1_{\mathrm{fin}}(K_{\mathfrak{q}}, A_{\phi,n})$. This follows from [Mi, Chapter I, Section 3].

2. Primes $\mathfrak{p}$ dividing p and such that A/K has ordinary reduction at $\mathfrak{p}$: $\mathrm{Im}(\delta_{\mathfrak{p}}) = H^1_{\mathrm{ord}}(K_{\mathfrak{p}}, A_{\phi,n})$. This follows from [CG, Proposition 4.5].

Finally, define the $\wp$-primary part $\mathrm{Sel}_{\wp^\infty}(A/K)$ of the Selmer group to be:

$$\mathrm{Sel}_{\wp^\infty}(A/K) := \lim_{\to n} \mathrm{Sel}_{\wp^n}(A/K),$$

where the direct limit is computed by means of the inclusion maps

$$A[\wp^n] \hookrightarrow A[\wp^{n+1}].$$

4.2 Bounding Selmer groups

Let A/K be a modular abelian variety and let ϕ its associated Hilbert modular form. Define $L(A,s) := L(\phi,s)$ and $L_K(A,s) := L_K(\phi,s)$ to be the Hasse-Weil L series of A over F and K. Fix a positive integer n and denote as above by $\delta_{\mathfrak{q}} : A(K_{\mathfrak{q}})/\wp^n A(K_{\mathfrak{q}}) \hookrightarrow H^1(K_{\mathfrak{q}}, A[\wp^n])$ the local Kummer map.

Recall the conditions on the prime ideal $\wp$ of residue characteristic p stated in Assumption 1.2. For convenience, these conditions are restated and briefly commented in the following:

Assumption 1.2, Condition 1. *$\rho_{\phi,\wp}$ is residually irreducible.* This is equivalent to require that the representation of $\mathrm{Gal}(\overline{F}/F)$ on the $\wp$-torsion $A[\wp]$ of A is irreducible. If A is an elliptic curve without complex multiplication, thanks to [Se] it is known that $A[p]$ is irreducible for all rational primes p except possibly a finite number of them.

Assumption 1.2, Condition 2. *ϕ is ordinary at all prime ideals $\mathfrak{p} \mid p$.* If the modular abelian variety A is ordinary at $\mathfrak{p} \mid p$, then the associated Hilbert modular form ϕ is also ordinary at $\mathfrak{p}$. For a proof, see for example [Go, Chapter 3, §6.2].

Assumption 1.2, Condition 3. *ϕ is $\wp$-isolated.* Since the dimension of the space of Hilbert modular forms of given weight and level is finite, this condition is verified by all prime ideals $\wp$ except, possibly, a finite number of them.

Assumption 1.2, Condition 4. $\mathrm{Im}(\delta_{\mathfrak{q}}) = 0$ *for primes $\mathfrak{q} \mid \mathfrak{n}$ and $\mathfrak{q} \nmid p$, where p is the residue characteristic of $\wp$.* This condition is verified by all prime ideals $\wp$ except, possibly, a finite number of them. This follows from a theorem of Mattuck which states that, for an abelian variety $\mathcal{A}$ of dimension g defined over a finite extension $\mathcal{K}$ of $\mathbb{Q}_l$, where l is a prime of $\mathbb{Q}$, there is an isomorphism $\mathcal{A}(\mathcal{K}) \simeq \mathbb{Z}_l^{g[\mathcal{K}:\mathbb{Q}_l]} \times H$, where H is a finite group. For references, see for example [CG, Section 4].

Remark 4.2. Suppose that Assumptions 1.1 is verified and that the special value $L_K(\phi, 1)$ is not zero. Suppose also that there are infinitely many prime ideals $\wp$ verifying Assumption 1.2. Then there are infinitely many prime ideals $\wp$ verifying Assumption 1.2 and such that $\wp \nmid \mathcal{L}_K(\phi)$.

The following theorem corresponds to Theorem 1.4 of the Introduction and proves the existence of Euler systems relative to A/K and $\wp^n$, where $\wp$ satisfies Assumption 1.2.

Theorem 4.3. *Suppose that K/F satisfies Assumption 1.1. If $\wp$ satisfies Assumption 1.2, then for every positive integer n the set*

$$\{\kappa_\ell : \ell \text{ is a } \wp^n\text{-admissible prime}\}$$

is an Euler system relative to A/K and $\wp^n$. More precisely:

1. *The set of $\wp^n$-admissible primes controls the Selmer group* $\mathrm{Sel}_{\wp^n}(A/K)$.
2. *Primes $\mathfrak{q} \neq \ell$:* $\mathrm{res}_{\mathfrak{q}}(\kappa_\ell) \in \mathrm{Im}(\delta_{\mathfrak{q}})$.
3. *The equality $\partial_\ell(\kappa_\ell) = \mathcal{L}_\phi(K)$ holds in $H^1_{\mathrm{sing}}(K_\ell, A[\wp^n]) \simeq \mathcal{O}_\phi/\wp^n$ up to multiplication by invertible elements in $\mathcal{O}_\phi/\wp^n$.*

Proof. By the generalization of [BD2, Theorem 3.2], the set of $\wp^n$-admissible primes controls the Selmer group $\mathrm{Sel}_{\wp^n}(A/K)$. The equality of $\partial_\ell(\kappa_\ell)$ and $\mathcal{L}_K(\phi)$ up to invertible elements follows from Theorem 3.10, statement 3. The description of $\mathrm{res}_{\mathfrak{q}}(\kappa_\ell)$ for primes $\mathfrak{q} \nmid \mathfrak{n}$ and primes $\mathfrak{p} \mid p$ comes from Theorem 3.10, statements 1 and 2, combined with Condition 2 of Assumption 1.2 enjoyed by $\wp$. Since, by Condition 4 in Assumption 1.2, the image of the Kummer map at primes dividing $\mathfrak{n}$ but not p is trivial, the result follows. □

The following theorem explains how to use the Euler system of Theorem 4.3 to control the Selmer group of A/K.

Theorem 4.4. *Suppose that K/F satisfies Assumption 1.1 and the central critical value $L_K(A, 1)$ of $L_K(A, s)$ is not zero. If $\wp$ satisfies Assumption 1.2 and $\wp \nmid \mathcal{L}_K(\phi)$, then $\mathrm{Sel}_{\wp^n}(A/K) = 0$ for every n. It follows that the $\wp$-primary part $\mathrm{Sel}_{\wp^\infty}(A/K)$ of the Selmer group is trivial too.*

Proof. Assume that $\mathrm{Sel}_{\wp^n}(A/K) \neq 0$ and fix $s \in \mathrm{Sel}_{\wp^n}(A/K)$ a non-zero element. As in [BD2, Theorem 3.2], choose, among infinitely many, a $\wp^n$-admissible prime ℓ so that $\mathrm{res}_\ell(s) \neq 0$ and consider the class κ_ℓ built in Section 3.6. By the choice of $\wp$ and Theorem 4.3, it follows that $\mathrm{res}_{\mathfrak{q}}(\kappa_\ell)$ is orthogonal to $\mathrm{res}_{\mathfrak{q}}(s)$ with respect to the local Tate pairing $\langle , \rangle_{\mathfrak{q}}$ for all prime ideals $\mathfrak{q} \neq \ell$. By the global reciprocity law of class field theory, it follows that

$$\langle \partial_\ell(\kappa_\ell), \mathrm{res}_\ell(s) \rangle_\ell = 0. \tag{18}$$

Since ℓ is $\wp^n$-admissible, it follows as recalled above that

$$H^1_{\mathrm{sing}}(K_\ell, T_{\phi,n}) \simeq \mathcal{O}_\phi/\wp^n$$

and, by the same argument as in [BD2, Lemma 2.6], its orthogonal complement $H^1_{\mathrm{fin}}(K_\ell, A_{\phi,n})$ is isomorphic to $\mathcal{O}_\phi/\wp^n$ too. Since $\partial_\ell(\kappa_\ell) \neq 0$ by Theorem 4.3 and $\mathrm{res}_\ell(s) \neq 0$, Equation (18) contradicts the non degeneracy of the local Tate pairing. It follows that the Selmer group $\mathrm{Sel}_{\wp^n}(A/K)$ is trivial. The last statement follows from the definition of $\mathrm{Sel}_{\wp^\infty}(A/K)$. □

Corollary 4.5. *Suppose that K/F satisfies Assumption 1.1 and the central critical value $L_K(A,1)$ of $L_K(A,s)$ is not zero. Suppose also that there exists at least one prime ideal $\wp$ verifying Assumption 1.2 and such that $\wp \nmid \mathcal{L}_K(\phi)$. Then the rank of $A(K)$ is zero.*

Proof. Let $\wp$ be as in the statement. Then, by Theorem 4.4, the Selmer group $\mathrm{Sel}_\wp(A/K)$ is trivial. Since the Kummer map

$$A(K)/\wp A(K) \hookrightarrow \mathrm{Sel}_\wp(A/K)$$

is injective, the rank of $A(K)$ is zero. □

Under stronger hypotheses on A such that the existence of infinitely many prime ideals $\wp$ verifying Assumption 1.2, it is possible to prove the following more compact result. The first statement corresponds to Corollary 1.7 of the Introduction.

Corollary 4.6. *Suppose that Assumption 1.1 is verified. Suppose also that there exists infinitely many prime ideals $\wp$ satisfying Assumption 1.2.*

1. *If $L_K(A,1) \neq 0$, then the rank of $A(K)$ is zero.*

2. *If $L(A,1) \neq 0$, then the rank of $A(F)$ is zero.*

Proof. To prove the first statement about the rank of $A(K)$, choose, among infinitely many, a prime ideal $\wp$ satisfying Assumption 1.2 and such that $\wp \nmid \mathcal{L}_K(\phi)$. This is possible by Remark 4.2. Then apply Corollary 4.5.

To prove the second statement on the rank of $A(F)$, choose, by Remark 3.8, a quadratic totally imaginary extension K/F verifying Assumption 1.1 and such that the special value $L_K(A,1)$ is not zero. Then apply the first part of this Corollary and use that $A(F) \subseteq A(K)$. □

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