

VANISHING OF SPECIAL VALUES AND CENTRAL DERIVATIVES IN HIDA FAMILIES

MATTEO LONGO AND STEFANO VIGNI

ABSTRACT. The theme of this work is the study of the Nekovář–Selmer group $\tilde{H}_f^1(K, \mathbb{T}^\dagger)$ attached to a twisted Hida family \mathbb{T}^\dagger of Galois representations and a quadratic number field K . The results that we obtain have the following shape: if a twisted L -function of a suitable modular form in the Hida family has order of vanishing $r \leq 1$ at the central critical point then the rank of $\tilde{H}_f^1(K, \mathbb{T}^\dagger)$ as a module over a certain local Hida–Hecke algebra is equal to r . Under the above assumption, we also show that infinitely many twisted L -functions of modular forms in the Hida family have the same order of vanishing at the central critical point. Our theorems extend to more general arithmetic situations results obtained by Howard when K is an imaginary quadratic field and all the primes dividing the tame level of the Hida family split in K .

1. INTRODUCTION

Fix an integer $N \geq 1$, a prime number $p \nmid 6N\phi(N)$ and an ordinary p -stabilized newform

$$f = \sum_{n \geq 1} a_n q^n \in S_k(\Gamma_0(Np), \omega^j)$$

of weight $k \geq 2$ (where ω is the Teichmüller character and $j \equiv k \pmod{2}$) whose associated mod p Galois representation is absolutely irreducible and p -distinguished (see [20, p. 297]). Let $F := \mathbb{Q}_p(a_n \mid n \geq 1)$ be the finite extension of \mathbb{Q}_p generated by the Fourier coefficients of f and let \mathcal{O}_F denote its ring of integers. Write \mathcal{R} for the branch of the Hida family where f lives (see [20, §5.3]); then \mathcal{R} is a complete local noetherian domain, finite and flat over the Iwasawa algebra $\Lambda := \mathcal{O}_F[[1 + p\mathbb{Z}_p]]$, and the Hida family we are considering can be viewed as a formal power series $f_\infty \in \mathcal{R}[[q]]$.

For $\mathfrak{p} \in \text{Spec}(\mathcal{R})$ let $\mathcal{R}_{\mathfrak{p}}$ denote the localization of \mathcal{R} at \mathfrak{p} and set $F_{\mathfrak{p}} := \mathcal{R}_{\mathfrak{p}}/\mathfrak{p}\mathcal{R}_{\mathfrak{p}}$. Hida’s theory ([10], [11], [12]) shows that if $\mathfrak{p} \in \text{Spec}(\mathcal{R})$ is an *arithmetic prime* (in the sense of [20, §5.5]) then $F_{\mathfrak{p}}$ is a finite extension of F and the power series in $F_{\mathfrak{p}}[[q]]$ obtained from f_∞ by composing with the canonical map $\mathcal{R} \rightarrow F_{\mathfrak{p}}$ is the q -expansion of a classical cusp form $f_{\mathfrak{p}}$; in particular, $f = f_{\mathfrak{p}}$ for a certain arithmetic prime \mathfrak{p} of weight k . Furthermore, with f_∞ is associated a $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -representation \mathbb{T} which is free of rank 2 over \mathcal{R} and such that $V_{\mathfrak{p}} := \mathbb{T} \otimes_{\mathcal{R}} F_{\mathfrak{p}}$ is isomorphic to (a Tate twist of) the $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -representation attached to $f_{\mathfrak{p}}$ by Deligne. The choice of a so-called critical character

$$\Theta : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \Lambda^\times$$

allows us to uniformly twist \mathbb{T} and obtain a Galois representation $\mathbb{T}^\dagger := \mathbb{T} \otimes \Theta^{-1}$ which is free of rank 2 over \mathcal{R} and such that for every arithmetic prime \mathfrak{p} of \mathcal{R} the representation $V_{\mathfrak{p}}^\dagger := \mathbb{T}^\dagger \otimes_{\mathcal{R}} F_{\mathfrak{p}}$ is a self-dual twist of $V_{\mathfrak{p}}$.

The general theme of this article is the study of the \mathcal{R} -rank of Nekovář’s Selmer group $\tilde{H}_f^1(K, \mathbb{T}^\dagger)$ attached to \mathbb{T}^\dagger and a quadratic number field K .

We first let K be an imaginary quadratic field K of discriminant prime to Np and define the factorization $N = N^+N^-$ by requiring that the primes dividing N^+ split in K and the

2010 *Mathematics Subject Classification.* 11F11, 11G18.

Key words and phrases. Hida families, quaternionic big Heegner points, L -functions, Selmer groups.

primes dividing N^- are inert in K . We assume throughout that N^- is square-free. Moreover, since the case $N^- = 1$ was studied in [13], we also assume that $N^- > 1$. We say that we are in the *indefinite* (respectively, *definite*) case if N^- is a product of an *even* (respectively, *odd*) number of primes. For every arithmetic prime \mathfrak{p} of \mathcal{R} , the choice of Θ gives an $F_{\mathfrak{p}}^{\times}$ -valued character $\chi_{\mathfrak{p}}$ of \mathbb{A}_K^{\times} whose restriction to $\mathbb{A}_{\mathbb{Q}}^{\times}$ is the inverse of the nebentype of $f_{\mathfrak{p}}$ (here \mathbb{A}_L is the adèle ring of a number field L). Then the Rankin–Selberg L -function $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, s)$ admits a self-dual functional equation whose sign controls the order of vanishing of $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, s)$ at the critical point $s = 1$. We say that the pair $(f_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has *analytic rank* $r \geq 0$ if the order of vanishing of $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, s)$ at $s = 1$ is r .

A simplified version of our main results can be stated as follows.

Theorem 1.1. *Let K be an imaginary quadratic field. If there is an arithmetic prime \mathfrak{p} of weight 2 and non-trivial nebentype such that $(f_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank one (respectively, zero) then $\tilde{H}_f^1(K, \mathbb{T}^{\dagger})$ is an \mathcal{R} -module of rank one (respectively, an \mathcal{R} -torsion module).*

Proposition 5.3 (respectively, Proposition 4.3) implies that if the pair $(f_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank one (respectively, zero) then we are in the indefinite (respectively, definite) case. Under this assumption, we also show that $(f_{\mathfrak{p}'}, \chi_{\mathfrak{p}'})$ has analytic rank one (respectively, zero) for all but finitely many arithmetic primes \mathfrak{p}' of weight 2; see Corollary 4.5 (respectively, Corollary 5.5) for details. These results represent weight 2 analogues (over K) of the conjectures on the generic analytic rank of the forms $f_{\mathfrak{p}}$ formulated by Greenberg in [8], and the reader is suggested to compare them with the statements in [20, §9.4] (in particular, [20, Conjecture 9.13]), where conjectures for forms $f_{\mathfrak{p}}$ of arbitrary weight are proposed. Theorem 1.1 is a combination of Theorems 4.6 and 5.6, where more general formulations (involving the notion of *generic* arithmetic primes of weight 2, cf. Definition 2.1) can be found. In the case of rank one, the corresponding statement in the split (i.e., $N^- = 1$) setting can be proved using the results obtained by Howard in [14, §3].

The proof of Theorem 1.1 given in Sections 4 and 5 relies crucially on the properties of the *big Heegner points* whose construction is described in [20], where we generalized previous work of Howard on the variation of Heegner points in Hida families ([13]). With a slight abuse of terminology (cf. Definition 3.1), for the purposes of this introduction we can say that big Heegner points are certain elements $\mathfrak{Z} \in H^1(K, \mathbb{T}^{\dagger})$ (indefinite case) and $\mathfrak{Z} \in \mathbb{J}$ (definite case) defined in terms of distributions of suitable Heegner (or Gross–Heegner) points on Shimura curves, where \mathbb{J} is an \mathcal{R} -module obtained from the inverse limit of the Picard groups attached to a tower of definite Shimura curves with increasing p -level structures. Since the Galois representation \mathbb{T}^{\dagger} can be introduced using an analogous tower of indefinite (compact) Shimura curves, the construction of the elements \mathfrak{Z} proceeds along similar lines in both cases. These constructions are reviewed in Section 3. It turns out that the image of \mathfrak{Z} in the $F_{\mathfrak{p}}$ -vector space $H^1(K, V_{\mathfrak{p}}^{\dagger})$ (indefinite case) or $\mathbb{J} \otimes_{\mathcal{R}} F_{\mathfrak{p}}$ (definite case) as \mathfrak{p} varies in the set of (weight 2) arithmetic primes controls the rank of $\tilde{H}_f^1(K, \mathbb{T}^{\dagger})$. In fact, in light of Propositions 4.3 and 5.3, Theorem 1.1 tells us that knowledge of the local non-vanishing of the big Heegner point \mathfrak{Z} at a single arithmetic prime \mathfrak{p} of weight 2 and non-trivial nebentype allows us to predict the rank over \mathcal{R} of the big Selmer group $\tilde{H}_f^1(K, \mathbb{T}^{\dagger})$.

Finally, by suitably splitting $\tilde{H}_f^1(K, V_{\mathfrak{p}}^{\dagger})$ and considering twists of Hida families, one can obtain results in the same vein as Theorem 1.1 for general quadratic fields K (not only imaginary ones) from the results (over \mathbb{Q}) proved in [14, Section 4]. This strategy does not make use of the big Heegner points introduced in [20], and the input has to be an arithmetic prime of *trivial* character. We provide details in Section 6 (see Theorems 6.3 and 6.4).

Convention. Throughout the paper we fix algebraic closures $\bar{\mathbb{Q}}$, $\bar{\mathbb{Q}}_p$ and field embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$.

2. HIDA FAMILIES OF MODULAR FORMS

Define $\Gamma := 1 + p\mathbb{Z}_p$ (so $\mathbb{Z}_p^\times = \mu_{p-1} \times \Gamma$ where μ_{p-1} is the group of $(p-1)$ -st roots of unity) and $\Lambda := \mathcal{O}_F[[\Gamma]]$. Let \mathcal{R} denote the integral closure of Λ in the primitive component \mathcal{K} to which f belongs (see [20, §5.3] for precise definitions). Then \mathcal{R} is a complete local noetherian domain, finitely generated and flat as a Λ -module, and if $\mathfrak{h}_\infty^{\text{ord}}$ is Hida's ordinary Hecke algebra over \mathcal{O}_F of tame level $\Gamma_0(N)$, whose construction is recalled in [20, §5.1], then there is a canonical map

$$\phi_\infty : \mathfrak{h}_\infty^{\text{ord}} \longrightarrow \mathcal{R}$$

which we shall sometimes call the *Hida family* of f . For every arithmetic prime \mathfrak{p} of \mathcal{R} (see [20, §5.5] for the definition) set $F_{\mathfrak{p}} := \mathcal{R}_{\mathfrak{p}}/\mathfrak{p}\mathcal{R}_{\mathfrak{p}}$. Then $F_{\mathfrak{p}}$ is a finite field extension of F and we fix an F -algebra embedding $F_{\mathfrak{p}} \hookrightarrow \bar{\mathbb{Q}}_p$, so that from here on we shall view $F_{\mathfrak{p}}$ as a subfield of $\bar{\mathbb{Q}}_p$. Now consider the composition

$$(1) \quad \phi_{\mathfrak{p}} : \mathfrak{h}_\infty^{\text{ord}} \xrightarrow{\phi_\infty} \mathcal{R} \xrightarrow{\pi_{\mathfrak{p}}} F_{\mathfrak{p}},$$

where $\pi_{\mathfrak{p}}$ is the map induced by the canonical projection $\mathcal{R}_{\mathfrak{p}} \twoheadrightarrow F_{\mathfrak{p}}$. For every integer $m \geq 1$ define $\Phi_m := \Gamma_0(N) \cap \Gamma_1(p^m)$. By duality, with $\phi_{\mathfrak{p}}$ is associated a modular form

$$f_{\mathfrak{p}} = \sum_{n \geq 1} a_n(\mathfrak{p}) q^n \in S_{k_{\mathfrak{p}}}(\Phi_{m_{\mathfrak{p}}}, \psi_{\mathfrak{p}} \omega^{k+j-k_{\mathfrak{p}}}, F_{\mathfrak{p}}),$$

of suitable weight $k_{\mathfrak{p}}$, level $\Phi_{m_{\mathfrak{p}}}$ and finite order character $\psi_{\mathfrak{p}} : \Gamma \rightarrow F_{\mathfrak{p}}^\times$, such that $a_n(\mathfrak{p}) = \phi_{\mathfrak{p}}(T_n) \in F_{\mathfrak{p}}$ for all $n \geq 1$ (here $T_n \in \mathfrak{h}_\infty^{\text{ord}}$ is the n -th Hecke operator). We remark that if $t_{\mathfrak{p}}$ is the smallest positive integer such that $\psi_{\mathfrak{p}}$ factors through $(1 + p\mathbb{Z}_p)/(1 + p^{t_{\mathfrak{p}}}\mathbb{Z}_p)$ then $m_{\mathfrak{p}} = \max\{1, t_{\mathfrak{p}}\}$. See [20, §5.5] for details.

Let $\mathbb{Q}(\mu_{p^\infty})$ be the p -adic cyclotomic extension of \mathbb{Q} and factor the cyclotomic character

$$\epsilon_{\text{cyc}} : \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_p^\times$$

as a product $\epsilon_{\text{cyc}} = \epsilon_{\text{tame}} \epsilon_{\text{wild}}$ with ϵ_{tame} (respectively, ϵ_{wild}) taking values in μ_{p-1} (respectively, Γ). Write $\gamma \mapsto [\gamma]$ for the inclusion $\Gamma \hookrightarrow \Lambda^\times$ of group-like elements and define

$$\Theta := \epsilon_{\text{tame}}^{(k+j-2)/2} [\epsilon_{\text{wild}}^{1/2}] : \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \longrightarrow \Lambda^\times$$

where $\epsilon_{\text{wild}}^{1/2}$ is the unique square root of ϵ_{wild} with values in Γ . In the obvious way, we shall also view Θ as \mathcal{R}^\times -valued. We associate with Θ the character θ and, for every arithmetic prime \mathfrak{p} , the characters $\Theta_{\mathfrak{p}}$ and $\theta_{\mathfrak{p}}$ given by

$$\begin{aligned} \theta : \mathbb{Z}_p^\times &\xrightarrow{\epsilon_{\text{cyc}}^{-1}} \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \xrightarrow{\Theta} \mathcal{R}^\times, \\ \Theta_{\mathfrak{p}} : \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) &\xrightarrow{\Theta} \mathcal{R}^\times \xrightarrow{\pi_{\mathfrak{p}}} F_{\mathfrak{p}}^\times, \\ \theta_{\mathfrak{p}} : \mathbb{Z}_p^\times &\xrightarrow{\epsilon_{\text{cyc}}^{-1}} \text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \xrightarrow{\Theta_{\mathfrak{p}}} F_{\mathfrak{p}}^\times. \end{aligned}$$

According to convenience, the characters Θ and $\Theta_{\mathfrak{p}}$ will also be viewed as defined on $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ via the canonical projection. If we write $[\cdot]_{\mathfrak{p}}$ for the composition of $[\cdot]$ with $\mathcal{R}^\times \rightarrow F_{\mathfrak{p}}^\times$ then $(\theta_{\mathfrak{p}}|_{\Gamma})^2 = [\cdot]_{\mathfrak{p}}$, and for every arithmetic prime \mathfrak{p} of weight 2 the modular form $f_{\mathfrak{p}}^\dagger := f_{\mathfrak{p}} \otimes \theta_{\mathfrak{p}}^{-1}$ has trivial nebentype (see [14, p. 806]).

As in [14, Definition 2], we give

Definition 2.1. An arithmetic prime \mathfrak{p} of weight 2 is *generic* for θ if one of the following (mutually exclusive) conditions is satisfied:

- (1) $f_{\mathfrak{p}}$ is the p -stabilization (in the sense of [14, Definition 1]) of a newform in $S_2(\Gamma_0(N), F_{\mathfrak{p}})$ and $\theta_{\mathfrak{p}}$ is trivial;
- (2) $f_{\mathfrak{p}}$ is a newform in $S_2(\Gamma_0(Np), F_{\mathfrak{p}})$ and $\theta_{\mathfrak{p}} = \omega^{(p-1)/2}$;

(3) f_p has non-trivial nebentype.

Let $\text{Gen}_2(\theta)$ denote the set of all weight 2 arithmetic primes of \mathcal{R} which are generic for θ . Observe that, since only a finite number of weight 2 arithmetic primes do not satisfy condition (3) in Definition 2.1, the set $\text{Gen}_2(\theta)$ contains all but finitely many weight 2 arithmetic primes.

Before concluding this subsection, let us fix some more notations. If E is a number field denote \mathbb{A}_E its ring of adeles and write $\text{art}_E : \mathbb{A}_E^\times \rightarrow \text{Gal}(E^{\text{ab}}/E)$ for the Artin reciprocity map. Furthermore, let \mathfrak{h}_m be the \mathcal{O}_F -Hecke algebra acting on the \mathbb{C} -vector space $S_2(\Phi_m, \mathbb{C})$ and write $\mathfrak{h}_m^{\text{ord}}$ for the ordinary \mathcal{O}_F -subalgebra of \mathfrak{h}_m , which is defined as the product of the local summands of \mathfrak{h}_m where U_p is invertible (therefore $\mathfrak{h}_\infty^{\text{ord}} = \varprojlim \mathfrak{h}_m^{\text{ord}}$, cf. [20, §5.1]). If M is an \mathfrak{h}_m -module set $M^{\text{ord}} := M \otimes_{\mathfrak{h}_m} \mathfrak{h}_m^{\text{ord}}$. Finally, define the Θ^{-1} -twist of a Λ -module M endowed with an action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ as follows: let Λ^\dagger denote Λ viewed as a module over itself but with $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -action given by Θ^{-1} and set $M^\dagger := M \otimes_\Lambda \Lambda^\dagger$.

3. REVIEW OF BIG HEEGNER POINTS

Let B denote the quaternion algebra over \mathbb{Q} of discriminant N^- . Fix an isomorphism of \mathbb{Q}_p -algebras $i_p : B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$ and choose for every integer $t \geq 0$ an Eichler order R_t of level $p^t N^+$ such that $i_p(R_t \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ is equal to the order of $M_2(\mathbb{Q}_p)$ consisting of the matrices in $M_2(\mathbb{Z}_p)$ which are upper triangular matrices modulo p^t and $R_{t+1} \subset R_t$ for all t . Let \hat{B} and \hat{R}_t denote the finite adelizations of B and R_t and define U_t to be the compact open subgroup of \hat{R}_t^\times obtained by replacing the p -component of \hat{R}_t^\times with the subgroup of $(R_t \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ consisting of those elements γ such that $i_p(\gamma) \equiv \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \pmod{p^t}$.

Fix an integer $t \geq 0$ and define the set

$$(2) \quad \tilde{X}_t^{(K)} = U_t \backslash (\hat{B}^\times \times \text{Hom}_{\mathbb{Q}}(K, B)) / B^\times,$$

where $\text{Hom}_{\mathbb{Q}}$ denotes homomorphisms of \mathbb{Q} -algebras. See [20, §2.1] for details. An element $[(b, \phi)] \in \tilde{X}_t^{(K)}$ is said to be a *Heegner point of conductor p^t* if the following two conditions are satisfied.

- (1) The map ϕ is an optimal embedding of the order \mathcal{O}_{p^t} of K of conductor p^t into the Eichler order $b^{-1} \hat{R}_t b \cap B$ of B .
- (2) Let x_p denote the p -component of an adele x , define $U_{t,p}$ to be the p -component of U_t , set $\mathcal{O}_{p^t,p} := \mathcal{O}_{p^t} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $\mathcal{O}_{K,p} := \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and let ϕ_p denote the morphism obtained from ϕ by extending scalars to \mathbb{Z}_p . Then

$$\phi_p^{-1}(\phi_p(\mathcal{O}_{p^t,p}^\times) \cap b_p^{-1} U_{t,p}^\times b_p) = \mathcal{O}_{p^t,p}^\times \cap (1 + p^t \mathcal{O}_{K,p})^\times.$$

For every $t \geq 0$ let H_t be the ring class field of K of conductor p^t and set $L_t := H_t(\mu_{p^t})$.

3.1. Indefinite Shimura curves. For every $t \geq 0$ we consider the indefinite Shimura curve \tilde{X}_t over \mathbb{Q} whose complex points are described as the compact Riemann surface

$$\tilde{X}_t(\mathbb{C}) := U_t \backslash (\hat{B}^\times \times (\mathbb{C} - \mathbb{R})) / B^\times.$$

The set $\tilde{X}_t^{(K)}$ can be identified with a subset of $\tilde{X}_t(K^{\text{ab}})$ where K^{ab} is the maximal abelian extension of K . For every number field E let

$$\mathcal{G}_E := \text{Gal}(\bar{\mathbb{Q}}/E).$$

Then \mathcal{G}_E acts naturally on the geometric points $\tilde{X}_t(\bar{\mathbb{Q}})$ of \tilde{X}_t .

3.2. Definite Shimura curves. For every $t \geq 0$ we consider the definite Shimura curve

$$\tilde{X}_t := U_t \backslash (\hat{B}^\times \times \mathbb{P}) / B^\times$$

where \mathbb{P} is the genus 0 curve over \mathbb{Q} defined by setting

$$\mathbb{P}(A) := \{x \in B \otimes_{\mathbb{Q}} A \mid x \neq 0, \text{Norm}(x) = \text{Trace}(x) = 0\} / A^\times$$

for every \mathbb{Q} -algebra A (here Norm and Trace are the norm and trace maps of B , respectively). There is a canonical identification between the set $\tilde{X}_t^{(K)}$ in (2) and the K -rational points $\tilde{X}_t(K)$ of \tilde{X}_t .

If E/K is an abelian extension then we define

$$\mathcal{G}_E := \text{Gal}(K^{\text{ab}}/E);$$

in particular, \mathcal{G}_K is the abelianization of the absolute Galois group of K . We explicitly remark that this group is different from the group denoted by the same symbol in the indefinite case. The set $\tilde{X}_t^{(K)}$ is equipped with a canonical action of $\mathcal{G}_K \simeq \hat{K}^\times / K^\times$ given by

$$[(b, \phi)]^\sigma := [(b\hat{\phi}(\mathfrak{a}), \phi)],$$

where $\hat{\phi} : \hat{K} \hookrightarrow \hat{B}$ is the adelization of ϕ and $\mathfrak{a} \in \hat{K}^\times$ satisfies $\text{art}_K(\mathfrak{a}) = \sigma$.

3.3. Constructions of points. In this and the next subsection our considerations apply both to the definite case and to the indefinite case. Recall that in the former case the symbol \mathcal{G}_E stands for the Galois group $\text{Gal}(K^{\text{ab}}/E)$ of an abelian extension E of K , while in the latter we use \mathcal{G}_E for the absolute Galois group of a number field E . Given a field extension E/F (with E and F abelian over K in the definite case) and a continuous \mathcal{G}_F -module M , for every integer $i \geq 0$ we let

$$(3) \quad \text{res}_{E/F} : H^i(\mathcal{G}_F, M) \longrightarrow H^i(\mathcal{G}_E, M), \quad \text{cor}_{E/F} : H^i(\mathcal{G}_E, M) \longrightarrow H^i(\mathcal{G}_F, M)$$

be the usual restriction and corestriction maps in Galois cohomology.

Denoting $\alpha_t : \tilde{X}_t \rightarrow \tilde{X}_{t-1}$ the canonical projection, [20, Theorem 1.1] shows the existence of a family of points $\tilde{P}_t \in \tilde{X}_t^{(K)}$ which are fixed by the subgroup \mathcal{G}_{L_t} of \mathcal{G}_K and satisfy

$$U_p(\tilde{P}_{t-1}) = \alpha_t \left(\text{tr}_{L_t/L_{t-1}}(\tilde{P}_t) \right)$$

(here, for a finite Galois extension E/F , the symbol $\text{tr}_{E/F}$ stands for the usual trace map).

Via the Jacquet–Langlands correspondence, the divisor group $\text{Div}(\tilde{X}_t^{(K)}) \otimes_{\mathbb{Z}} \mathcal{O}_F$ is equipped with a standard action of the \mathcal{O}_F -Hecke algebra \mathfrak{h}_t (see [20, §6.3]). Define

$$\mathbb{D}_t := \left(\text{Div}(\tilde{X}_t^{(K)}) \otimes \mathcal{O}_F \right)^{\text{ord}}$$

(notice that here and below we are regarding Θ as a character of \mathcal{G}_K via restriction). Taking the inverse limit, we may define

$$\mathbb{D}_\infty := \varprojlim_t \mathbb{D}_t,$$

which is naturally an $\mathfrak{h}_\infty^{\text{ord}}$ -module. Finally, set

$$\mathbb{D} := \mathbb{D}_\infty \otimes_{\mathfrak{h}_\infty^{\text{ord}}} \mathcal{R}.$$

The canonical structure of Λ -module on these groups makes it possible to define the twisted modules \mathbb{D}_t^\dagger , $\mathbb{D}_\infty^\dagger$ and \mathbb{D}^\dagger .

Let $H = H_0$ be the Hilbert class field of K , write

$$\mathbb{P}_t \in H^0(\mathcal{G}_{H_t}, \mathbb{D}_t^\dagger)$$

for the image of \tilde{P}_t in \mathbb{D}_t (cf. [20, §7.1]) and define

$$\mathcal{P}_t := \text{cor}_{H_t/H}(\mathbb{P}_t) \in H^0(\mathcal{G}_H, \mathbb{D}_t^\dagger).$$

Following [20, Definition 7.3], we give

Definition 3.1. The *big Heegner point of conductor 1* is the element

$$\mathcal{P}_H := \varprojlim_t U_p^{-t}(\mathcal{P}_t) \in H^0(\mathcal{G}_H, \mathbb{D}^\dagger)$$

obtained by taking the inverse limit of the compatible sequence $(U_p^{-t}(\mathcal{P}_t))_{t \geq 1}$ and then taking the image via the map $\mathbb{D}_\infty^\dagger \rightarrow \mathbb{D}^\dagger$.

See [20, Corollary 7.2] for a proof that the inverse limit considered in Definition 3.1 makes sense. Finally, set

$$(4) \quad \mathcal{P} = \mathcal{P}_K := \text{cor}_{H/K}(\mathcal{P}_H) \in H^0(\mathcal{G}_K, \mathbb{D}^\dagger).$$

3.4. Weight 2 arithmetic primes. We introduce some notations in the case of arithmetic primes of weight 2. So fix an arithmetic prime \mathfrak{p} of \mathcal{R} of weight 2, level $\Phi_{m_{\mathfrak{p}}}$ and character $\psi_{\mathfrak{p}}$. In order to simplify notations, in this subsection put $m := m_{\mathfrak{p}}$ and $\psi := \psi_{\mathfrak{p}}$. The action of $\mathfrak{h}_\infty^{\text{ord}}$ on D_m^{ord} factors through $\mathfrak{h}_m^{\text{ord}}$, which, by the Jacquet–Langlands correspondence, can be identified with a quotient of the ordinary \mathcal{O}_F -Hecke algebra of level Φ_m . Define an $F_{\mathfrak{p}}^\times$ -valued character of \mathbb{A}_K^\times by

$$\chi_{\mathfrak{p}}(x) := \Theta_{\mathfrak{p}}(\text{art}_{\mathbb{Q}}(N_{K/\mathbb{Q}}(x)))$$

and denote $\chi_{0,\mathfrak{p}}$ the restriction of $\chi_{\mathfrak{p}}$ to $\mathbb{A}_{\mathbb{Q}}^\times$. Then the nebentype $\psi_{\mathfrak{p}} \omega^{k+j-k_{\mathfrak{p}}}$ of $f_{\mathfrak{p}}$ is equal to $\chi_{0,\mathfrak{p}}^{-1}$ (see [14, §3] for a proof). In particular, since

$$(\mathbb{Z}/p\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p)/(1 + p^{t_{\mathfrak{p}}}\mathbb{Z}_p) \simeq (\mathbb{Z}/p^{t_{\mathfrak{p}}}\mathbb{Z})^\times,$$

the conductor of $\chi_{0,\mathfrak{p}}^{-1}$ is $p^{t_{\mathfrak{p}}}$ unless $\chi_{0,\mathfrak{p}}$ is trivial. For $\sigma \in \text{Gal}(K^{\text{ab}}/K)$ let $x_\sigma \in \mathbb{A}_K^\times$ be such that $\text{art}_K(x_\sigma) = \sigma$ and define $\chi_{\mathfrak{p}}(\sigma) := \chi_{\mathfrak{p}}(x_\sigma)$.

Since $\Theta_{\mathfrak{p}}$ factors through $\mathcal{G}_{\mathbb{Q}(\mu_{p^m})}$, we obtain the equality

$$(5) \quad H^0(\mathcal{G}_{L_m}, \mathbb{D}_m^\dagger) = H^0(\mathcal{G}_{L_m}, \mathbb{D}_m).$$

Using (5), the point $\mathbb{P}_m \in H^0(\mathcal{G}_{H_m}, \mathbb{D}_m^\dagger)$ gives rise, by restriction, to a point $\text{res}_{L_m/H_m}(\mathbb{P}_m)$ in $H^0(\mathcal{G}_{L_m}, \mathbb{D}_m)$. Since $\mathcal{G}_{\mathbb{Q}(\mu_{p^m})}$ is contained in the kernel of $\Theta_{\mathfrak{p}}$, the map $\chi_{\mathfrak{p}}$ can be viewed as a character of $\text{Gal}(L_m/K)$, so we may form the sum

$$(6) \quad \mathbb{P}_m^{\chi_{\mathfrak{p}}} := \sum_{\sigma \in \text{Gal}(L_m/K)} \text{res}_{L_m/H_m}(\mathbb{P}_m)^\sigma \otimes \chi_{\mathfrak{p}}^{-1}(\sigma) \in H^0(\mathcal{G}_{L_m}, \mathbb{D}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}}).$$

Via (5), we may also view $\mathbb{P}_m^{\chi_{\mathfrak{p}}}$ as an element of $H^0(\mathcal{G}_{L_m}, \mathbb{D}_m^\dagger \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$. Moreover, for every $\tau \in \mathcal{G}_K$ we have

$$(\mathbb{P}_m^{\chi_{\mathfrak{p}}})^\tau = \sum_{\sigma \in \text{Gal}(L_m/K)} \mathbb{P}_m^{\sigma\tau|L_m} \otimes \Theta_{\mathfrak{p}}(\text{art}_{\mathbb{Q}}(N_{K/\mathbb{Q}}(x_\tau))) \chi_{\mathfrak{p}}^{-1}(\sigma\tau|L_m) = \Theta(\tau) \mathbb{P}_m^{\chi_{\mathfrak{p}}},$$

so $\mathbb{P}_m^{\chi_{\mathfrak{p}}} \in H^0(\mathcal{G}_K, \mathbb{D}_m^\dagger \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$.

We want to explicitly relate $\mathbb{P}_m^{\chi_{\mathfrak{p}}}$ and \mathcal{P}_m . In order to do this, we note that, by definition, there is an equality

$$\mathbb{P}_m^{\chi_{\mathfrak{p}}} = \text{cor}_{L_m/K} \circ \text{res}_{L_m/H_m}(\mathbb{P}_m),$$

where the restriction and corestriction maps are as in (3) with $E := L_m$, $F := H_m$ and $M := \mathbb{D}_m^\dagger \otimes_{\mathcal{O}_F} F_{\mathfrak{p}}$. Therefore we get

$$(7) \quad \begin{aligned} \mathbb{P}_m^{\chi_{\mathfrak{p}}} &= (\mathrm{cor}_{H/K} \circ \mathrm{cor}_{H_m/H} \circ \mathrm{cor}_{L_m/H_m} \circ \mathrm{res}_{L_m/H_m})(\mathbb{P}_m) \\ &= [L_m : H_m] \mathrm{cor}_{H/K}(\mathcal{P}_m). \end{aligned}$$

In the following we shall also make use of the module

$$\mathbb{D}_{\mathfrak{p}} := \mathbb{D} \otimes_{\mathcal{R}} F_{\mathfrak{p}},$$

its twist $\mathbb{D}_{\mathfrak{p}}^\dagger = \mathbb{D}^\dagger \otimes_{\mathcal{R}} F_{\mathfrak{p}}$ and the image of \mathcal{P} in $H^0(\mathcal{G}_K, \mathbb{D}_{\mathfrak{p}}^\dagger)$ via the canonical map $\mathbb{D} \rightarrow \mathbb{D}_{\mathfrak{p}}$, which we denote $\mathcal{P}_{\mathfrak{p}}$.

Remark 3.2. By extending Hida's arguments in the proof of [12, Theorem 12.1] to the case of divisors, it seems possible to show that the canonical map $\mathbb{D} \rightarrow \mathbb{D}_{\mathfrak{p}}$ factors as

$$(8) \quad \mathbb{D} \longrightarrow \mathbb{D}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}} \longrightarrow \mathbb{D}_{\mathfrak{p}}.$$

Then one would obtain from the above discussion the formula $\mathbb{P}_{\mathfrak{p}} = a_p(\mathfrak{p})^m [L_m : H_m] \mathcal{P}_{\mathfrak{p}}$ in $H^0(\mathcal{G}_K, \mathbb{D}_{\mathfrak{p}}^\dagger)$. Instead of deriving such an equality at this stage, we will obtain two related formulas (see Propositions 4.1 and 5.1) in specific situations when factorizations analogous to (8) are available thanks to known cases of Hida's control theorem.

4. VANISHING OF CENTRAL DERIVATIVES

Let the quaternion algebra B be *indefinite* and write \mathfrak{p} for a weight 2 arithmetic prime of \mathcal{R} of level $\Phi_{m_{\mathfrak{p}}}$ and character $\psi_{\mathfrak{p}}$. Once \mathfrak{p} has been fixed, set $m := m_{\mathfrak{p}}$ and $\psi := \psi_{\mathfrak{p}}$.

4.1. Tate modules. For every integer $t \geq 0$ let $\tilde{\mathcal{J}}_t$ be the Jacobian variety of \tilde{X}_t . Define

$$\mathbb{T}_t := \left(\mathrm{Ta}_p(\tilde{\mathcal{J}}_t) \otimes_{\mathbb{Z}_p} \mathcal{O}_F \right)^{\mathrm{ord}}$$

where, as usual, the upper index *ord* denotes the ordinary submodule, then form the inverse limit

$$\mathbb{T}_{\infty} := \varprojlim_t \mathbb{T}_t.$$

We may also define the twists \mathbb{T}_t^\dagger and $\mathbb{T}_{\infty}^\dagger$. By [20, Corollary 6.5], there is an isomorphism

$$(9) \quad \mathbb{T}_{\infty} \otimes_{\mathfrak{h}^{\mathrm{ord}}} \mathcal{R} \simeq \mathbb{T}$$

of Galois representations. We fix once and for all such an isomorphism, and freely use the notation \mathbb{T} for the Galois module appearing in the left hand side of (9). Now set

$$V_{\mathfrak{p}} := \mathbb{T} \otimes_{\mathcal{R}} F_{\mathfrak{p}}.$$

Then $V_{\mathfrak{p}}^\dagger = \mathbb{T}^\dagger \otimes_{\mathcal{R}} F_{\mathfrak{p}}$ is (a twist of) the self-dual representation attached to $f_{\mathfrak{p}}^\dagger$ by Deligne. Observe that the canonical map $\mathbb{T} \rightarrow V_{\mathfrak{p}}$ can be factored as

$$(10) \quad \mathbb{T} \longrightarrow \mathbb{T}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}} \xrightarrow{\pi_{m,\mathfrak{p}}} V_{\mathfrak{p}}.$$

This follows from Hida's control theorem [12, Theorem 12.1] (or, rather, from its extension to the current setting given in [21, Proposition 2.17]). Following [13, p.101], for every $t \geq 0$ one defines a twisted Kummer map

$$\delta_t : H^0(K, \mathbb{D}_t^\dagger) \longrightarrow H^1(K, \mathbb{T}_t^\dagger).$$

Taking the limit as m varies and using the canonical map $\mathbb{T}_{\infty}^\dagger \rightarrow \mathbb{T}^\dagger$, we get a map

$$\delta_{\infty} : H^0(K, \mathbb{D}^\dagger) \longrightarrow H^1(K, \mathbb{T}^\dagger).$$

Recall the element \mathcal{P} introduced in (4) and define

$$\mathfrak{Z} := \delta_\infty(\mathcal{P}) \in H^1(K, \mathbb{T}^\dagger).$$

Write $\mathfrak{Z}_{\mathfrak{p}}$ for the image of \mathfrak{Z} in $H^1(K, V_{\mathfrak{p}}^\dagger)$. The map δ_∞ gives rise to a map

$$\delta_{\mathfrak{p}} : H^0(K, \mathbb{D}_{\mathfrak{p}}^\dagger) \longrightarrow H^1(K, V_{\mathfrak{p}}^\dagger),$$

and the square

$$\begin{array}{ccc} H^0(K, \mathbb{D}^\dagger) & \xrightarrow{\delta_\infty} & H^1(K, \mathbb{T}^\dagger) \\ \downarrow & & \downarrow \\ H^0(K, \mathbb{D}_{\mathfrak{p}}^\dagger) & \xrightarrow{\delta_{\mathfrak{p}}} & H^1(K, V_{\mathfrak{p}}^\dagger) \end{array}$$

is commutative, so that $\mathfrak{Z}_{\mathfrak{p}}$ is also equal to $\delta_{\mathfrak{p}}(\mathcal{P}_{\mathfrak{p}})$ (in the notations of §3.4). We may also consider the class $\delta_m(\mathbb{P}_m^{\chi_{\mathfrak{p}}}) \in H^1(K, \mathbb{T}_m^\dagger \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$, where $\mathbb{P}_m^{\chi_{\mathfrak{p}}} \in H^0(K, \mathbb{D}_m^\dagger \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$ is the element introduced in (6) and δ_m is extended by F_p -linearity, and define $\mathfrak{X}_{\mathfrak{p}}$ to be the image of $\delta_m(\mathbb{P}_m^{\chi_{\mathfrak{p}}})$ in $H^1(K, V_{\mathfrak{p}}^\dagger)$ via the map $\pi_{m, \mathfrak{p}}$ appearing in (10).

Proposition 4.1. *The formula $\mathfrak{X}_{\mathfrak{p}} = a_{\mathfrak{p}}(\mathfrak{p})^m [L_m : H_m] \mathfrak{Z}_{\mathfrak{p}}$ holds in $H^0(\mathcal{G}_K, \mathbb{T}_{\mathfrak{p}}^\dagger)$.*

Proof. We just need to review the above constructions. Thanks to factorization (10), the class $\delta_{\mathfrak{p}}(\mathcal{P}_{\mathfrak{p}}) = \mathfrak{Z}_{\mathfrak{p}}$ is equal to the corestriction from H to K of the image of $U_p^{-m} \delta_m(\mathcal{P}_m) \in H^0(\mathcal{G}_H, \mathbb{T}_m^\dagger)$ in $H^0(\mathcal{G}_K, V_{\mathfrak{p}}^\dagger)$ under the map induced by the map labeled $\pi_{m, \mathfrak{p}}$ in (10). Since the action of $\mathfrak{h}_\infty^{\text{ord}}$ on $V_{\mathfrak{p}}^\dagger$ is via the character $\phi_{\mathfrak{p}}$ defined in (1), one has

$$\delta_{\mathfrak{p}}(\mathcal{P}_{\mathfrak{p}}) = \pi_{m, \mathfrak{p}} \left(\text{cor}_{H/K} (a_{\mathfrak{p}}(\mathfrak{p})^{-m} \delta_m(\mathcal{P}_m)) \right).$$

Finally, using (7) we find that

$$\delta_{\mathfrak{p}}(\mathcal{P}_{\mathfrak{p}}) = a_{\mathfrak{p}}(\mathfrak{p})^{-m} [L_m : H_m]^{-1} \pi_{m, \mathfrak{p}} (\delta_m(\mathbb{P}_m^{\chi_{\mathfrak{p}}})) = a_{\mathfrak{p}}(\mathfrak{p})^{-m} [L_m : H_m]^{-1} \mathfrak{X}_{\mathfrak{p}},$$

and the result follows. \square

Corollary 4.2. $\mathfrak{Z}_{\mathfrak{p}} \neq 0 \iff \mathfrak{X}_{\mathfrak{p}} \neq 0$.

Proof. Immediate from Proposition 4.1. \square

4.2. Non-vanishing results. Using the fixed embedding $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$, we form the Rankin–Selberg convolution $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, s)$ as in [15, §1]; note that if $\mathbf{1}$ is the trivial character then $L_K(f_{\mathfrak{p}}, \mathbf{1}, s) = L_K(f_{\mathfrak{p}}, s)$. For the definition of the L -function $L_K(f_{\mathfrak{p}}, s)$ of $f_{\mathfrak{p}}$ over K and of its twists the reader is referred also to [4, §2.1]. The results obtained in this subsection are the counterpart in an arbitrary indefinite quaternionic setting of those in [14, Propositions 3 and 4], where the case of the split algebra $M_2(\mathbb{Q})$ is considered. Although many arguments follow those in [14] closely, for the convenience of the reader we give detailed proofs.

Proposition 4.3. *Suppose $\mathfrak{p} \in \text{Gen}_2(\theta)$ satisfies either (2) or (3) in Definition 2.1.*

- (a) $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, s)$ vanishes to odd order at $s = 1$.
- (b) $L'_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, 1) \neq 0 \iff \mathfrak{Z}_{\mathfrak{p}} \neq 0$.

Proof. Let \mathfrak{h} denote the algebra generated over \mathbb{Z} by the Hecke operators T_n with $(n, Np) = 1$ and the diamond operators $\langle d \rangle$ with $(d, Np) = 1$ acting on the \mathbb{C} -vector space of cusp forms in $S_2(\Phi_m, \mathbb{C})$. Write $e_{\mathfrak{p}}$ for the idempotent in $\mathfrak{h} \otimes_{\mathbb{Z}} F_{\mathfrak{p}}$ which projects onto the maximal summand on which every T_n acts as $a_{\mathfrak{p}}(n)$ and $\langle \cdot \rangle$ acts as $\chi_{0, \mathfrak{p}}^{-1}$.

We apply the results in [15] with $\mathfrak{s} = p^m$, $\mathfrak{c} = p^m$ and $\mathfrak{m} = N$. As recalled in [15, p. 829] (cf. also [15, eqs. (2.5), (2.6), (2.8)]), the sign of the functional equation for $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, s)$ is -1 , and this shows part (a).

Let us prove (b). As in [15, §5.1], denote Hg the Hodge embedding defined in [27, §6.2]. We introduce the divisor

$$P := \sum_{\sigma \in \text{Gal}(L_m/K)} \mathbb{P}_m^\sigma \otimes \chi_{\mathfrak{p}}^{-1}(\sigma) \in \text{Div}(\tilde{X}_m^{(K)}) \otimes_{\mathbb{Z}} F_{\mathfrak{p}},$$

whose image in $\mathbb{D}_{\mathfrak{p}}$ under the canonical maps is $\mathbb{P}_{\mathfrak{p}}$. After fixing an appropriate embedding $K \hookrightarrow B$, the image $\text{Hg}(P)$ of our divisor P in $e_{\mathfrak{p}}(\tilde{J}_m(L_m) \otimes_{\mathbb{Z}} F_{\mathfrak{p}})$ is equal to the point $Q_{\chi, \Pi}$ considered in [15, Theorem 4.6.2], hence

$$\text{Hg}(P) \neq 0 \iff L'_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, 1) \neq 0,$$

by [15, Theorem 4.6.2]. It remains to show that $\text{Hg}(P) \neq 0$ if and only if $\mathfrak{Z}_{\mathfrak{p}} \neq 0$, or equivalently, thanks to Corollary 4.2, if and only if $\mathfrak{X}_{\mathfrak{p}} \neq 0$.

To prove this, we start by noticing that our hypotheses ensure that $f_{\mathfrak{p}}$ is a newform of level Np^m , so that Eichler–Shimura theory and multiplicity one show that the summand $e_{\mathfrak{p}}(\mathbb{T}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}}) = e_{\mathfrak{p}}(\text{Ta}_p(\tilde{J}_m) \otimes_{\mathbb{Z}_p} F_{\mathfrak{p}})^{\text{ord}}$ is sent isomorphically onto $V_{\mathfrak{p}}$ under the map in (10). Using [25, Corollary 2.2 and Proposition 2.3], it can be checked that the Kummer map

$$\tilde{J}_m(L_m) \otimes_{\mathbb{Z}} F_{\mathfrak{p}} \longrightarrow H^1(L_m, \text{Ta}_p(\tilde{J}_m) \otimes_{\mathbb{Z}_p} F_{\mathfrak{p}})$$

is injective, hence the above identifications yield an injection

$$e_{\mathfrak{p}}(\tilde{J}_m(L_m) \otimes_{\mathbb{Z}} F_{\mathfrak{p}})^{\text{ord}} \hookrightarrow H^1(L_m, V_{\mathfrak{p}}) \simeq H^1(L_m, V_{\mathfrak{p}}^{\dagger}).$$

By definition, the image of $\text{Hg}(P)$ under this map is equal to the image of $\mathfrak{X}_{\mathfrak{p}}$ under the restriction map $H^1(K, V_{\mathfrak{p}}^{\dagger}) \rightarrow H^1(L_m, V_{\mathfrak{p}}^{\dagger})$, which is injective as its kernel is an $F_{\mathfrak{p}}$ -vector space that is killed by $[L_m : K]$. This concludes the proof. \square

Proposition 4.4. *Suppose $\mathfrak{p} \in \text{Gen}_2(\theta)$ satisfies (1) in Definition 2.1.*

- (a) $L_K(f_{\mathfrak{p}}, s)$ vanishes to odd order at $s = 1$.
- (b) $L'_K(f_{\mathfrak{p}}, 1) \neq 0 \implies \mathfrak{Z}_{\mathfrak{p}} \neq 0$.

Proof. In this case $m = 1$ and there is a normalized newform $f_{\mathfrak{p}}^{\sharp}$ of level $\Gamma_0(N)$, with Fourier coefficients $a_n^{\sharp}(\mathfrak{p})$, whose p -stabilization is $f_{\mathfrak{p}}$. Moreover, $\chi_{\mathfrak{p}}$ is the trivial character $\mathbf{1}$. If ϵ_K denotes the character of K then the L -functions of $f_{\mathfrak{p}}^{\sharp}$ and $f_{\mathfrak{p}}$ are related by the equality

$$(11) \quad \left(1 - \frac{p^{1-s}}{a_p(\mathfrak{p})}\right) \cdot \left(1 - \epsilon_K(p) \frac{p^{1-s}}{a_p(\mathfrak{p})}\right) \cdot L_K(f_{\mathfrak{p}}^{\sharp}, s) = L_K(f_{\mathfrak{p}}, s).$$

The extra Euler factors on the left do not vanish at $s = 1$ because $a_p(\mathfrak{p}) \neq \pm 1$ (since $a_p(\mathfrak{p})$ is a root of the Hecke polynomial $X^2 - a_p^{\sharp}(\mathfrak{p})X + p = 0$, its absolute value is \sqrt{p}), therefore the L -functions of $f_{\mathfrak{p}}^{\sharp}$ and $f_{\mathfrak{p}}$ have the same order of vanishing at $s = 1$, which is odd in light of the functional equation recalled in [15, p. 829] (here we apply again the results of [15] with $\mathfrak{s} = p$, $\mathfrak{c} = p$, $\mathfrak{m} = N$). This proves part (a).

By Corollary 4.2, to prove part (b) one can equivalently show that if $\mathfrak{X}_{\mathfrak{p}} = 0$ then $L'_K(f_{\mathfrak{p}}^{\sharp}, 1) = 0$. Let X_0 and X_1 be the (compact) Shimura curves over \mathbb{Q} associated with the Eichler orders R_0 and R_1 , respectively, so that

$$X_i(\mathbb{C}) = \widehat{R}_i^{\times} \backslash (\widehat{B}^{\times} \times \mathbb{P}) / B^{\times}$$

for $i = 0, 1$. There are degeneracy maps $\alpha, \beta : X_1(\mathbb{C}) \rightarrow X_0(\mathbb{C})$, with α the canonical projection and β corresponding to the action of an element of norm p of R_1 normalising R_1^{\times} (see, e.g., [19, §3.1]). These maps induce homomorphisms by Picard functoriality

$$\alpha^*, \beta^* : J_0 \longrightarrow J_1$$

where J_0 and J_1 are the Jacobian varieties of X_0 and X_1 , respectively. The maps α^* and β^* are compatible with the action of the Hecke operators T_n for $(n, Np) = 1$ and the diamond

operators $\langle d \rangle$ for $(d, Np) = 1$. Write \mathfrak{h} for the Hecke algebra generated over \mathbb{Z} by these operators and denote $e_{\mathfrak{p}}$ the idempotent in $\mathfrak{h} \otimes_{\mathbb{Z}} F_{\mathfrak{p}}$ associated with \mathfrak{p} , as in the proof of Proposition 4.3. Define

$$e_{\mathfrak{p}} \mathrm{Ta}_p(J_i) := e_{\mathfrak{p}}(\mathrm{Ta}_p(J_i) \otimes_{\mathbb{Z}_p} F_{\mathfrak{p}})$$

for $i = 0, 1$. Since the form $f_{\mathfrak{p}}$ is old at p , the idempotent $e_{\mathfrak{p}}$ kills the p -new quotient $J_1/(\alpha^*(J_0) \oplus \beta^*(J_0))$ of J_1 , hence by [2, §1.7] there is a decomposition

$$(12) \quad e_{\mathfrak{p}} \mathrm{Ta}_p(J_1) \simeq \alpha^* e_{\mathfrak{p}} \mathrm{Ta}_p(J_0) \oplus \beta^* e_{\mathfrak{p}} \mathrm{Ta}_p(J_0).$$

The Hecke operator T_n acts on $e_{\mathfrak{p}} \mathrm{Ta}_p(J_0)$ as multiplication by $a_n^{\sharp}(\mathfrak{p}) = a_n(\mathfrak{p})$ and, by strong multiplicity one, T_p acts on it as multiplication by $a_p^{\sharp}(\mathfrak{p})$. On the other hand, we have the relations

$$U_p \circ \alpha^* = p\beta^*, \quad U_p \circ \beta^* = \beta^* \circ T_p - \alpha^*.$$

A proof of the analogous formulas for classical modular curves is given in [3, Lemma 2.1], and the arguments carry over *mutatis mutandis* to the case of Shimura curves attached to division quaternion algebras. For a precise reference, see [9, p. 93].

It follows that U_p acts on $e_{\mathfrak{p}} \mathrm{Ta}_p(J_1)$ via decomposition (12) with characteristic polynomial

$$(X^2 - a_p^{\sharp}(\mathfrak{p})X + p)^2 = (X - a_p(\mathfrak{p}))^2 \cdot (X - p/a_p(\mathfrak{p}))^2$$

and is diagonalizable. Furthermore, the projection to the $a_p(\mathfrak{p})$ -eigenspace corresponds to the ordinary projection, because the other eigenvalue $p/a_p(\mathfrak{p})$ has positive p -adic valuation. By multiplicity one, the $F_{\mathfrak{p}}$ -vector spaces $e_{\mathfrak{p}}(\mathrm{Ta}_p^{\mathrm{ord}}(J_1) \otimes_{\mathbb{Z}_p} F_{\mathfrak{p}})$ and $e_{\mathfrak{p}}(\mathbb{T}_1 \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$ can be identified. Thus, as in the proof of Proposition 4.3, it follows from the Eichler–Shimura relations that the map $\mathbb{T}_1 \otimes_{\mathcal{O}_F} F_{\mathfrak{p}} \rightarrow V_{\mathfrak{p}}$ arising from (10) takes the summand $e_{\mathfrak{p}}(\mathbb{T}_1 \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$ isomorphically onto $V_{\mathfrak{p}}$. Since, as observed in the proof of Proposition 4.3, the Kummer map

$$\tilde{J}_1(L_1) \otimes_{\mathbb{Z}} F_{\mathfrak{p}} \longrightarrow H^1(L_1, \mathrm{Ta}_p(\tilde{J}_1) \otimes_{\mathbb{Z}_p} F_{\mathfrak{p}})$$

is injective, using the above identifications we get an injection

$$(13) \quad e_{\mathfrak{p}}(\tilde{J}_1(L_1) \otimes_{\mathbb{Z}} F_{\mathfrak{p}})^{\mathrm{ord}} \hookrightarrow H^1(L_1, V_{\mathfrak{p}}).$$

If Hg is the Hodge embedding used in the proof of Proposition 4.3, the map in (13) sends $\mathrm{Hg}(\mathbb{P}_{\mathfrak{p}})$ to $\mathfrak{X}_{\mathfrak{p}}$. Taking into account the previous description of the ordinary projection and the injectivity of (13), it follows that

$$(14) \quad \mathfrak{X}_{\mathfrak{p}} \neq 0 \iff \left(U_p - \frac{p}{a_p(\mathfrak{p})} \right) (e_{\mathfrak{p}} \tilde{Q}) \neq 0,$$

where

$$\tilde{Q} := \sum_{\sigma \in \mathrm{Gal}(L_1/K)} \tilde{P}_1^{\sigma} \in \mathrm{Div}(\tilde{X}_1^{(K)})$$

maps to $\mathbb{P}_{\mathfrak{p}}$ under the ordinary projection (recall that the character $\chi_{0,\mathfrak{p}}$ is trivial in this case). To complete the proof we shall relate $e_{\mathfrak{p}} \tilde{Q}$, and thus $\mathfrak{X}_{\mathfrak{p}}$, to classical Heegner points on X_0 .

Let P_1 denote the image of \tilde{P}_1 under the canonical map $\tilde{X}_1 \rightarrow X_1$. This is a Heegner point of conductor p on the indefinite Shimura curve X_1 considered in [2, Section 2], and lives in $X_1(H_1)$. Define also Q to be the image of \tilde{Q} in X_1 , so that

$$Q := \sum_{\sigma \in \mathrm{Gal}(L_1/K)} P_1^{\sigma} = [L_1 : H_1] P \in \mathrm{Div}(X_1(H_1))$$

with

$$P := \sum_{\sigma \in \mathrm{Gal}(H_1/K)} P_1^{\sigma} \in \mathrm{Div}(X_1(H_1)).$$

Since f_p^\sharp has trivial character, by Faltings's isogeny theorem ([7, §5, Korollar 2]) there is an isomorphism

$$e_p \operatorname{Ta}_p(\tilde{J}_1) \simeq e_p \operatorname{Ta}_p(J_1)$$

such that the Kummer maps identify the images of $[H_1 : H] \cdot \operatorname{Hg}(e_p P)$ and $e_p \operatorname{Hg}(Q)$. Using (14), we thus conclude that

$$(15) \quad \mathfrak{X}_p \neq 0 \iff \left(U_p - \frac{p}{a_p(\mathfrak{p})} \right) (e_p P) \neq 0.$$

Let $\alpha_*, \beta_* : J_1 \rightarrow J_0$ be the maps between Jacobians induced from α and β by Albanese (i.e., covariant) functoriality. The relation $\beta_* U_p = p \alpha_*$ combined with (15) then shows that

$$(16) \quad \mathfrak{X}_p = 0 \implies \left(\alpha_* - \frac{\beta_*}{a_p(\mathfrak{p})} \right) (e_p P) = 0.$$

Now we relate the vanishing of $e_p P$ with special values of L -series. If p splits in K then let σ and σ^* denote the Frobenius elements in $\operatorname{Gal}(H_0/K)$ corresponding to the two primes of K above p . Write u for half the order of \mathcal{O}_K^\times and consider the points $\alpha(P_1), \beta(P_1) \in X_0(H_1)$. By [20, Proposition 4.6], the point $\alpha(P_1)$ is a Heegner point of conductor p . Moreover, as already pointed out, the map β is given by the composition of α with the Atkin–Lehner involution at p (see, e.g., [19, §3.1]). Comparing with the constructions of [20, §§4.1–4.2] (see, in particular, [20, eq. (12)]), it follows that $\beta(P_1)$ is a Heegner point of conductor 1 living in $X_0(H_0)$. The Euler system relations in [2, §2.4] can then be applied, and we get

$$(17) \quad \operatorname{utr}_{H_1/H} \alpha(P_1) = \begin{cases} T_p(\beta(P_1)) & \text{if } p \text{ is inert in } K \\ (T_p - \sigma - \sigma^*)(\beta(P_1)) & \text{if } p \text{ is split in } K \end{cases}$$

as divisors on X_0 . Thanks to (17) and the relation $u[H_1 : H] = p - \epsilon_K(p)$, one has

$$(18) \quad \alpha_*(P) = \frac{T_p - 1 - \epsilon_K(p)}{p - \epsilon_K(p)} (\beta_*(P)).$$

Combining (16) and (18) we see that if $\mathfrak{X}_p = 0$ then

$$\left(\frac{1}{a_p(\mathfrak{p})} - \frac{a_p^\sharp(\mathfrak{p}) - 1 - \epsilon_K(p)}{p - \epsilon_K(p)} \right) \cdot \beta_*(P) = 0.$$

Since $a_p(\mathfrak{p})$ is a root of the polynomial $X^2 - a_p^\sharp(\mathfrak{p})X + p$ and $a_p(\mathfrak{p}) \neq \pm 1$, the coefficient multiplying $\beta_*(P)$ is nonzero, hence we conclude that if \mathfrak{X}_p is zero then $\beta_*(P)$ is zero too. However, thanks to [26, Theorem C], this possibility is ruled out by our assumption that $L'_K(f_p, 1) \neq 0$. \square

If \mathfrak{p} is an arithmetic prime of \mathcal{R} of weight 2 then we say that the pair (f_p, χ_p) has *analytic rank one* if $L_K(f_p, \chi_p, 1) = 0$ and $L'_K(f_p, \chi_p, 1) \neq 0$.

Corollary 4.5. *The following are equivalent:*

- (1) *there is a $\mathfrak{p} \in \operatorname{Gen}_2(\theta)$ such that (f_p, χ_p) has analytic rank one;*
- (2) *there is an arithmetic prime \mathfrak{p} such that $\mathfrak{Z}_p \neq 0$;*
- (3) *\mathfrak{Z} is not \mathcal{R} -torsion;*
- (4) *$\mathfrak{Z}_p \neq 0$ for all but finitely many arithmetic primes \mathfrak{p} ;*
- (5) *(f_p, χ_p) has analytic rank one for all but finitely many $\mathfrak{p} \in \operatorname{Gen}_2(\theta)$.*

Proof. The implication (1) \Rightarrow (2) is immediate from Propositions 4.3 and 4.4. Now write $\mathbb{T}_{\mathfrak{p}}^{\dagger}$ for the localization of \mathbb{T}^{\dagger} at an arithmetic prime \mathfrak{p} of \mathcal{R} , so that the triangle

$$(19) \quad \begin{array}{ccc} H^1(K, \mathbb{T}^{\dagger}) & \longrightarrow & H^1(K, V_{\mathfrak{p}}^{\dagger}) \\ \downarrow & \nearrow & \\ H^1(K, \mathbb{T}_{\mathfrak{p}}^{\dagger}) & & \end{array}$$

is commutative. Observe that, localization being an exact functor, there is also a canonical identification $H^1(K, \mathbb{T}_{\mathfrak{p}}^{\dagger}) = H^1(K, \mathbb{T}^{\dagger})_{\mathfrak{p}}$. As in [20, §10.1], define $\mathfrak{a}_{\mathcal{R}}$ to be the annihilator in \mathcal{R} of the finitely generated torsion \mathcal{R} -module $\prod_{\ell|N-} H^1(K_{\ell}, \mathbb{T}^{\dagger})_{\text{tors}}$ (cf. [23, Proposition 4.2.3]), then choose a nonzero $\lambda \in \mathfrak{a}_{\mathcal{R}}$. Let $\text{Sel}_{\text{Gr}}(K, \mathbb{T}^{\dagger}) \subset H^1(K, \mathbb{T}^{\dagger})$ denote the Greenberg Selmer group as defined, e.g., in [20, §5.6]. By [20, Proposition 10.1], $\lambda \cdot \mathfrak{Z} \in \text{Sel}_{\text{Gr}}(K, \mathbb{T}^{\dagger})$. According to [23, Proposition 12.7.13.4 (iii)] plus [13, eq. (21)], the localization $\text{Sel}_{\text{Gr}}(K, \mathbb{T}^{\dagger})_{\mathfrak{p}}$ at an arithmetic prime \mathfrak{p} is free of finite rank over $\mathcal{R}_{\mathfrak{p}}$, so if (2) holds then diagram (19) implies that $\lambda \cdot \mathfrak{Z}$ has nonzero, hence non-torsion, image in $\text{Sel}_{\text{Gr}}(K, \mathbb{T}^{\dagger})_{\mathfrak{p}}$ for some arithmetic prime \mathfrak{p} . It follows that $\lambda \cdot \mathfrak{Z}$ is not \mathcal{R} -torsion, hence \mathfrak{Z} is not \mathcal{R} -torsion as well. This shows that (2) \Rightarrow (3). To prove that (3) \Rightarrow (4) one can proceed exactly as in the proof of the implication (c) \Rightarrow (d) in [14, Corollary 5]. Finally, the fact that (4) \Rightarrow (5) is a straightforward consequence of Proposition 4.3, while (5) \Rightarrow (1) is obvious. \square

Now let $\tilde{H}_f^1(K, \mathbb{T}^{\dagger})$ be Nekovář's extended Selmer group (see [23, Ch. 6]).

Theorem 4.6. *Suppose there is a $\mathfrak{p} \in \text{Gen}_2(\theta)$ such that $(f_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank one. Then $\tilde{H}_f^1(K, \mathbb{T}^{\dagger})$ is an \mathcal{R} -module of rank one.*

Proof. As in the proof of Corollary 4.5, choose a nonzero $\lambda \in \mathfrak{a}_{\mathcal{R}}$. In the course of proving Corollary 4.5 we showed that the existence of an arithmetic prime $\mathfrak{p} \in \text{Gen}_2(\theta)$ such that $(g_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank one implies that the class $\lambda \cdot \mathfrak{Z}$ is not \mathcal{R} -torsion. In other words, under this analytic condition [20, Conjecture 10.3] is true, and then the \mathcal{R} -module $\tilde{H}_f^1(K, \mathbb{T}^{\dagger})$ has rank one by [20, Theorem 10.6]. \square

5. VANISHING OF SPECIAL VALUES

Let the quaternion algebra B be *definite* and write \mathfrak{p} for a weight 2 arithmetic prime of \mathcal{R} of level $\Phi_{m_{\mathfrak{p}}}$ and character $\psi_{\mathfrak{p}}$. Once \mathfrak{p} has been fixed, set $m := m_{\mathfrak{p}}$ and $\psi := \psi_{\mathfrak{p}}$.

5.1. Picard groups. For every integer $t \geq 0$ define the $\mathfrak{h}_t^{\text{ord}}$ -module

$$\mathbb{J}_t := \mathcal{O}_F[U_t \backslash \widehat{B}^{\times} / B^{\times}]^{\text{ord}} \simeq \left(\text{Pic}(\tilde{X}_t) \otimes_{\mathbb{Z}} \mathcal{O}_F \right)^{\text{ord}}$$

and form the inverse limit

$$\mathbb{J}_{\infty} := \varprojlim_t \mathbb{J}_t.$$

Consider the \mathcal{R} -component $\mathbb{J} := \mathbb{J}_{\infty} \otimes_{\mathfrak{h}_{\infty}^{\text{ord}}} \mathcal{R}$ of \mathbb{J}_{∞} and define $\mathbb{J}_{\mathfrak{p}} := \mathbb{J} \otimes_{\mathcal{R}} F_{\mathfrak{p}}$. By [20, Proposition 9.1], $\mathbb{J}_{\mathfrak{p}}$ is a one-dimensional $F_{\mathfrak{p}}$ -vector space. Note that the canonical projection $\mathbb{J} \rightarrow \mathbb{J}_{\mathfrak{p}}$ can be factored as

$$(20) \quad \mathbb{J} \longrightarrow \mathbb{J}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}} \xrightarrow{\pi_{m, \mathfrak{p}}} \mathbb{J}_{\mathfrak{p}}.$$

This follows from Hida's control theorem [12, Theorem 12.1] (or, rather, from its extension [21, Proposition 3.6] to our present context). Using the natural maps

$$\eta_t : \mathbb{D}_t \longrightarrow \mathbb{J}_t$$

arising from Picard (i.e., contravariant) functoriality, we get a map

$$\eta_K : H^0(\mathcal{G}_K, \mathbb{D}^\dagger) \longrightarrow \mathbb{J}.$$

Recall the point \mathcal{P} introduced in (4) and define

$$\mathfrak{Z} := \eta_K(\mathcal{P}) \in \mathbb{J}.$$

Let $\mathfrak{Z}_{\mathfrak{p}}$ denote the image of \mathfrak{Z} in $\mathbb{J}_{\mathfrak{p}}$. The map η_K induces a map

$$\eta_{\mathfrak{p}} : H^0(\mathcal{G}_K, \mathbb{D}_{\mathfrak{p}}^\dagger) \longrightarrow \mathbb{J}_{\mathfrak{p}},$$

and the square

$$(21) \quad \begin{array}{ccc} H^0(\mathcal{G}_K, \mathbb{D}^\dagger) & \xrightarrow{\eta_K} & \mathbb{J} \\ \downarrow & & \downarrow \\ H^0(\mathcal{G}_K, \mathbb{D}_{\mathfrak{p}}^\dagger) & \xrightarrow{\eta_{\mathfrak{p}}} & \mathbb{J}_{\mathfrak{p}} \end{array}$$

is commutative, so that $\mathfrak{Z}_{\mathfrak{p}}$ is also equal to $\eta_{\mathfrak{p}}(\mathcal{P}_{\mathfrak{p}})$ (in the notations of §3.4). We may also consider the class $\eta_m(\mathbb{P}_m^{\chi_{\mathfrak{p}}}) \in H^0(K, \mathbb{J}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$, where $\mathbb{P}_m^{\chi_{\mathfrak{p}}} \in H^0(K, \mathbb{D}_m^\dagger \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$ is the element introduced in (6) and η_m is extended by $F_{\mathfrak{p}}$ -linearity, and define $\mathfrak{X}_{\mathfrak{p}}$ to be the image of $\eta_m(\mathbb{P}_m^{\chi_{\mathfrak{p}}})$ in $H^1(K, \mathbb{J}_{\mathfrak{p}})$ via the map $\pi_{m,\mathfrak{p}}$ appearing in (20).

The following result is the analogue in the definite case of Proposition 4.1.

Proposition 5.1. *The formula $\mathfrak{X}_{\mathfrak{p}} = a_{\mathfrak{p}}(\mathfrak{p})^m [L_m : H_m] \mathfrak{Z}_{\mathfrak{p}}$ holds in $H^0(\mathcal{G}_K, \mathbb{J}_{\mathfrak{p}})$.*

Proof. We argue as in the proof of Proposition 4.1. Thanks to factorization (20), the class $\eta_{\mathfrak{p}}(\mathcal{P}_{\mathfrak{p}}) = \mathfrak{Z}_{\mathfrak{p}}$ is equal to the corestriction from H to K of the image of $U_p^{-m} \eta_m(\mathcal{P}_m) \in H^0(\mathcal{G}_H, \mathbb{J}_m)$ in $H^0(\mathcal{G}_K, \mathbb{J}_{\mathfrak{p}})$ under the map induced by the map labeled $\pi_{m,\mathfrak{p}}$ in (20). Since the action of $\mathfrak{h}_{\infty}^{\text{ord}}$ on $\mathbb{J}_{\mathfrak{p}}$ is via the character $\phi_{\mathfrak{p}}$ defined in (1), one has

$$\eta_{\mathfrak{p}}(\mathcal{P}_{\mathfrak{p}}) = \pi_{m,\mathfrak{p}} \left(\text{cor}_{H/K} (a_{\mathfrak{p}}(\mathfrak{p})^{-m} \eta_m(\mathcal{P}_m)) \right).$$

Finally, using (7) we find that

$$\eta_{\mathfrak{p}}(\mathcal{P}_{\mathfrak{p}}) = a_{\mathfrak{p}}(\mathfrak{p})^{-m} [L_m : H_m]^{-1} \pi_{m,\mathfrak{p}} (\eta_m(\mathbb{P}_m^{\chi_{\mathfrak{p}}})) = a_{\mathfrak{p}}(\mathfrak{p})^{-m} [L_m : H_m]^{-1} \mathfrak{X}_{\mathfrak{p}},$$

and the result follows. \square

We conclude this subsection with the counterpart of Corollary 4.2.

Corollary 5.2. $\mathfrak{Z}_{\mathfrak{p}} \neq 0 \iff \mathfrak{X}_{\mathfrak{p}} \neq 0$.

Proof. Immediate from Proposition 5.1. \square

5.2. Non-vanishing results. Keep the notation of §4.2 for $L_K(f_{\mathfrak{p}}, s)$ and $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, s)$.

Proposition 5.3. *Suppose $\mathfrak{p} \in \text{Gen}_2(\theta)$ satisfies either (2) or (3) in Definition 2.1.*

- (a) $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, s)$ vanishes to even order at $s = 1$.
- (b) $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, 1) \neq 0 \iff \mathfrak{Z}_{\mathfrak{p}} \neq 0$.

Proof. As in the proof of Proposition 4.3, let \mathfrak{h} denote the algebra generated over \mathbb{Z} by the Hecke operators T_n with $(n, Np) = 1$ and the diamond operators $\langle d \rangle$ with $(d, Np) = 1$ acting on the \mathbb{C} -vector space of cusp forms in $S_2(\Phi_m, \mathbb{C})$. Moreover, write $e_{\mathfrak{p}}$ for the idempotent in $\mathfrak{h} \otimes_{\mathbb{Z}} F_{\mathfrak{p}}$ which projects onto the maximal summand on which every T_n acts as $a_{\mathfrak{p}}(n)$ and $\langle \cdot \rangle$ acts as $\chi_{0,\mathfrak{p}}^{-1}$.

We apply the results in [15] with $\mathfrak{s} = p^m$, $\mathfrak{c} = p^m$ and $\mathfrak{m} = N$. As pointed out in [15, p. 829], the sign of the functional equation for $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, s)$ is $+1$, and this implies part (a).

Let us prove (b). Define

$$\tilde{J}_m := \mathcal{O}_F[U_m \backslash \widehat{B}^\times / B^\times] \simeq \text{Pic}(\tilde{X}_m) \otimes_{\mathbb{Z}} \mathcal{O}_F$$

and consider the divisor

$$P := \sum_{\sigma \in \text{Gal}(L_m/K)} \mathbb{P}_m^\sigma \otimes \chi_{\mathfrak{p}}^{-1}(\sigma) \in \text{Div}(\tilde{X}_m^{(K)}) \otimes_{\mathbb{Z}} F_{\mathfrak{p}}.$$

Write Q for the image of P in \tilde{J}_m . After fixing an appropriate embedding $K \hookrightarrow B$, the divisor Q is equal to the divisor Q_χ considered in [15, Theorem 3.3.3]. Let $\phi_{\mathfrak{p}}$ be the (unique up to nonzero multiplicative factors) modular form on B associated with $f_{\mathfrak{p}}$ by the Jacquet–Langlands correspondence. Up to rescaling, one may view $\phi_{\mathfrak{p}}$ as an $F_{\mathfrak{p}}$ -valued function on $U_m \backslash \widehat{B}^\times / B^\times$. By a slight abuse of notation, we adopt the same symbol for the $F_{\mathfrak{p}}$ -linear extension of $\phi_{\mathfrak{p}}$ to $\tilde{J}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}}$. By [15, Theorem 3.3.3], we know that

$$\phi_{\mathfrak{p}}(Q) \neq 0 \iff L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, 1) \neq 0,$$

so it remains to show that $\phi_{\mathfrak{p}}(Q) \neq 0$ if and only if $\mathfrak{Z}_{\mathfrak{p}} \neq 0$, or equivalently, thanks to Corollary 5.2, if and only if $\mathfrak{X}_{\mathfrak{p}} \neq 0$.

To prove this, we compute $\phi_{\mathfrak{p}}(Q)$ as follows. Let $e_{\mathfrak{p}}Q$ be the image of Q in $e_{\mathfrak{p}}(\tilde{J}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$. Since $f_{\mathfrak{p}}$ is a newform of level Np^m , the same is true of $\phi_{\mathfrak{p}}$ and it follows, by multiplicity one, that $e_{\mathfrak{p}}(\tilde{J}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}})$ is one-dimensional. Therefore $\phi_{\mathfrak{p}}(Q) \neq 0$ if and only if $e_{\mathfrak{p}}Q \neq 0$. Now the image of P in $\mathbb{D}_{\mathfrak{p}}$ is $\mathbb{P}_{\mathfrak{p}}$, hence the image of Q in $\mathbb{J}_{\mathfrak{p}}$ is $\mathfrak{X}_{\mathfrak{p}}$, by the commutativity of (21). Finally, notice that, by multiplicity one (cf. [2, §1.9]), there is an isomorphism $e_{\mathfrak{p}}(\tilde{J}_m \otimes_{\mathcal{O}_F} F_{\mathfrak{p}}) \simeq \mathbb{J}_{\mathfrak{p}}$ sending $e_{\mathfrak{p}}Q$ to $\mathfrak{X}_{\mathfrak{p}}$, which completes the proof. \square

Proposition 5.4. *Suppose $\mathfrak{p} \in \text{Gen}_2(\theta)$ satisfies (1) in Definition 2.1.*

- (a) $L_K(f_{\mathfrak{p}}, s)$ vanishes to even order at $s = 1$.
- (b) $L_K(f_{\mathfrak{p}}, 1) \neq 0 \implies \mathfrak{Z}_{\mathfrak{p}} \neq 0$.

Proof. Part (a) follows as in the proof of Proposition 4.4, this time observing that, as explained in [15, p. 829], the order of vanishing of $L_K(f_{\mathfrak{p}}^{\sharp}, s)$ at $s = 1$ is even (here we apply again the results of [15] with $\mathfrak{s} = p$, $\mathfrak{c} = p$, $\mathfrak{m} = N$), equal to that of $L_K(f_{\mathfrak{p}}, s)$ by (11).

By Corollary 5.2, proving (b) is equivalent to proving that if $\mathfrak{X}_{\mathfrak{p}} = 0$ then $L_K(f_{\mathfrak{p}}^{\sharp}, 1) = 0$. Let X_0 and X_1 be the Shimura curves over \mathbb{Q} associated with the Eichler orders R_0 and R_1 , respectively, so that (with notation as in §3.2) one has

$$X_i = \widehat{R}_i^\times \backslash (\widehat{B}^\times \times \mathbb{P}) / B^\times$$

for $i = 0, 1$. There are degeneracy maps $\alpha, \beta : X_1 \rightarrow X_0$ where α is the canonical projection and β is the composition of α with an Atkin–Lehner involution at p (see, e.g., [2, §1.7]). The maps α, β induce by Picard functoriality homomorphisms

$$\alpha^*, \beta^* : J_0 \longrightarrow J_1$$

where J_i is the Picard group of X_i for $i = 0, 1$, so that $J_i = \mathbb{Z}[\widehat{R}_i^\times \backslash \widehat{B}^\times / B^\times]$. The maps α^* and β^* are compatible with the action of the Hecke operators T_n for $(n, Np) = 1$ and the diamond operators $\langle d \rangle$ for $(d, Np) = 1$. Write \mathfrak{h} for the Hecke algebra generated over \mathbb{Z} by these operators and denote $e_{\mathfrak{p}}$ the idempotent in $\mathfrak{h} \otimes_{\mathbb{Z}} F_{\mathfrak{p}}$ associated with \mathfrak{p} , as in the proof of Proposition 5.3. Define

$$e_{\mathfrak{p}}J_i := e_{\mathfrak{p}}(J_i \otimes_{\mathbb{Z}} F_{\mathfrak{p}})$$

for $i = 0, 1$. Since the form $f_{\mathfrak{p}}$ is old at p , the idempotent $e_{\mathfrak{p}}$ annihilates the p -new quotient $J_1/(\alpha^*(J_0) \oplus \beta^*(J_0))$ of J_1 , hence by [2, §1.7] there is a decomposition

$$(22) \quad e_{\mathfrak{p}}J_1 \simeq \alpha^* e_{\mathfrak{p}}J_0 \oplus \beta^* e_{\mathfrak{p}}J_0.$$

The operator T_n acts on $e_p J_0$ as multiplication by $a_n^\sharp(\mathfrak{p}) = a_n(\mathfrak{p})$ and, by strong multiplicity one, T_p acts on it as multiplication by $a_p^\sharp(\mathfrak{p})$. On the other hand, we have the relations

$$U_p \circ \alpha^* = p\beta^*, \quad U_p \circ \beta^* = \beta^* \circ T_p - \alpha^*.$$

For a proof, see [16, Theorem 3.16]. It follows that U_p acts on $e_p J_1$ via decomposition (22) with characteristic polynomial

$$(X^2 - a_p^\sharp(\mathfrak{p})X + p)^2 = (X - a_p(\mathfrak{p}))^2 \cdot (X - p/a_p(\mathfrak{p}))^2$$

and is diagonalizable. Furthermore, the projection to the $a_p(\mathfrak{p})$ -eigenspace corresponds to the ordinary projection, because the other eigenvalue $p/a_p(\mathfrak{p})$ has positive p -adic valuation. By multiplicity one, the F_p -vector spaces $e_p J_1$ and $e_p \tilde{J}_1$ can be identified. Thus the map $\mathbb{J}_1 \otimes_{\mathcal{O}_F} F_p \rightarrow \mathbb{J}_p$ arising from (20) takes the summand $e_p(\mathbb{J}_1 \otimes_{\mathcal{O}_F} F_p)$ isomorphically onto \mathbb{J}_p . Taking the previous description of the ordinary projection into account, it follows that

$$(23) \quad \mathfrak{X}_p \neq 0 \iff \left(U_p - \frac{p}{a_p(\mathfrak{p})} \right) (e_p \tilde{Q}) \neq 0,$$

where

$$\tilde{Q} := \sum_{\sigma \in \text{Gal}(L_1/K)} \tilde{P}_1^\sigma \in \text{Div}(\tilde{X}_1^{(K)})$$

maps to \mathbb{P}_p under the ordinary projection (recall that the character $\chi_{0,p}$ is trivial in this case). To complete the proof we shall relate $e_p \tilde{Q}$, and thus \mathfrak{X}_p , to classical Heegner (or, rather, Gross–Heegner) points on X_0 .

Let P_1 denote the image of \tilde{P}_1 under the canonical map $\tilde{X}_1 \rightarrow X_1$. This is a Heegner point of conductor p on the definite Shimura curve X_1 considered in [2, Section 2]. Define also Q to be the image of \tilde{Q} in X_1 , so that

$$Q := \sum_{\sigma \in \text{Gal}(L_1/K)} P_1^\sigma = [L_1 : H_1]P \in \text{Div}(X_1^{(K)})$$

with

$$P := \sum_{\sigma \in \text{Gal}(H_1/K)} P_1^\sigma \in \text{Div}(X_1^{(K)}).$$

Thus, using (23), we conclude that

$$(24) \quad \mathfrak{X}_p \neq 0 \iff \left(U_p - \frac{p}{a_p(\mathfrak{p})} \right) (e_p P) \neq 0.$$

Let $\alpha_*, \beta_* : J_1 \rightarrow J_0$ be the maps between Jacobians induced from α and β by Albanese functoriality. The relation $\beta_* U_p = p\alpha_*$ combined with (24) then shows that

$$(25) \quad \mathfrak{X}_p = 0 \implies \left(\alpha_* - \frac{\beta_*}{a_p(\mathfrak{p})} \right) (e_p P) = 0.$$

Now we relate $e_p P$ with special values of L -series. If p splits in K then let σ and σ^* denote the Frobenius elements in $\text{Gal}(H_0/K)$ corresponding to the two primes of K above p . Write u for half the order of \mathcal{O}_K^\times and consider the points $\alpha(P_1), \beta(P_1) \in X_0(K)$. By [20, Proposition 4.6], $\alpha(P_1)$ is a Heegner point of conductor p . Moreover, as already pointed out, the map β is given by the composition of α with the Atkin–Lehner involution at p . Comparing with the constructions of [20, §§4.1–4.2] (see, in particular, [19, eq. (12)]), it follows that $\beta(P_1)$ is a Heegner point of conductor 1. The Euler system relations in [2, §2.4] can then be applied, and we get that

$$(26) \quad \text{utr}_{H_1/H} \alpha(P_1) = \begin{cases} T_p(\beta(P_1)) & \text{if } p \text{ is inert in } K \\ (T_p - \sigma - \sigma^*)(\beta(P_1)) & \text{if } p \text{ is split in } K \end{cases}$$

as divisors on X_0 . Thanks to (26) and the relation $u[H_1 : H] = p - \epsilon_K(p)$, one has

$$(27) \quad \alpha_*(P) = \frac{T_p - 1 - \epsilon_K(p)}{p - \epsilon_K(p)} (\beta_*(P)).$$

Combining (25) and (27) we see that if $\mathfrak{X}_{\mathfrak{p}} = 0$ then

$$\left(\frac{1}{a_p(\mathfrak{p})} - \frac{a_p^\sharp(\mathfrak{p}) - 1 - \epsilon_K(p)}{p - \epsilon_K(p)} \right) \cdot \beta_*(e_{\mathfrak{p}}P) = 0.$$

Since $a_p(\mathfrak{p})$ is a root of the polynomial $X^2 - a_p^\sharp(\mathfrak{p})X + p$ and $a_p(\mathfrak{p}) \neq \pm 1$, the coefficient multiplying $\beta_*(e_{\mathfrak{p}}P)$ is nonzero, hence we conclude that if $\mathfrak{X}_{\mathfrak{p}}$ is zero then $\beta_*(e_{\mathfrak{p}}P) = e_{\mathfrak{p}}\beta_*(P)$ is zero too. However, thanks to [27, Theorem 1.3.2], this possibility is ruled out by the non-vanishing condition $L_K(f_{\mathfrak{p}}^\sharp, 1) \neq 0$, which is equivalent to our assumption $L_K(f_{\mathfrak{p}}, 1) \neq 0$ (cf. (11)). Indeed, write $\phi_{\mathfrak{p}}^\sharp$ for the (unique up to nonzero multiplicative factors) modular form on B attached to $f_{\mathfrak{p}}^\sharp$ by the Jacquet–Langlands correspondence. At the cost of rescaling, we may view $\phi_{\mathfrak{p}}^\sharp$ as an $F_{\mathfrak{p}}$ -valued function on $\widehat{R}^\times \backslash \widehat{B}^\times / B^\times$, and we use the symbol $\phi_{\mathfrak{p}}$ also for its $F_{\mathfrak{p}}$ -linear extension to $J_0 \otimes_{\mathbb{Z}} F_{\mathfrak{p}}$. Then [27, Theorem 1.3.2] shows that $\phi_{\mathfrak{p}}^\sharp(\beta_*(P)) \neq 0$ if and only if $L_K(f_{\mathfrak{p}}^\sharp, 1) \neq 0$. But $\phi_{\mathfrak{p}}^\sharp(\beta_*(P)) = e_{\mathfrak{p}}\beta_*(P)$, and the result follows. \square

If \mathfrak{p} is an arithmetic prime of \mathcal{R} of weight 2 then we say that the pair $(f_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has *analytic rank zero* if $L_K(f_{\mathfrak{p}}, \chi_{\mathfrak{p}}, 1) \neq 0$.

Corollary 5.5. *The following are equivalent:*

- (1) *there is a $\mathfrak{p} \in \text{Gen}_2(\theta)$ such that $(f_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank zero;*
- (2) *there is an arithmetic prime \mathfrak{p} such that $\mathfrak{Z}_{\mathfrak{p}} \neq 0$;*
- (3) *\mathfrak{Z} is not \mathcal{R} -torsion;*
- (4) *$\mathfrak{Z}_{\mathfrak{p}} \neq 0$ for all but finitely many arithmetic primes \mathfrak{p} ;*
- (5) *$(f_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank zero for all but finitely many primes $\mathfrak{p} \in \text{Gen}_2(\theta)$.*

Proof. The implication (1) \Rightarrow (2) is immediate from Propositions 5.3 and 5.4. On the other hand, the localization of \mathbb{J} at \mathfrak{p} is free of rank one over $\mathcal{R}_{\mathfrak{p}}$ by [20, Proposition 9.1], and the implication (2) \Rightarrow (3) follows from this. If (3) holds then, since \mathbb{J} is finitely generated over \mathcal{R} , [13, Lemma 2.1.7] shows that $\mathfrak{Z} \notin \mathfrak{p}(\mathbb{J} \otimes_{\mathcal{R}} \mathcal{R}_{\mathfrak{p}})$ for all but finitely many \mathfrak{p} , which proves (4). Finally, (4) \Rightarrow (5) is a direct consequence of Proposition 5.3, while (5) \Rightarrow (1) is obvious. \square

To conclude this subsection, we prove

Theorem 5.6. *Suppose there is a $\mathfrak{p} \in \text{Gen}_2(\theta)$ such that $(f_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank zero. Then $\widetilde{H}_f^1(K, \mathbb{T}^\dagger)$ is \mathcal{R} -torsion.*

Proof. Combining our assumption and Corollary 5.5, we see that $(f_{\mathfrak{p}}, \chi_{\mathfrak{p}})$ has analytic rank zero for all but finitely many primes $\mathfrak{p} \in \text{Gen}_2(\theta)$. For every arithmetic prime \mathfrak{p} of \mathcal{R} let $H_f^1(K, V_{\mathfrak{p}}^\dagger)$ be the Bloch–Kato Selmer group, whose definition (in terms of Fontaine’s ring B_{cris}) can be found in [5, Sections 3 and 5]. A result of Nekovář ([24, Theorem A]) then shows that $H_f^1(K, V_{\mathfrak{p}}^\dagger) = 0$ for infinitely many $\mathfrak{p} \in \text{Gen}_2(\theta)$. (Observe that, in the notation of [24], $\chi = {}^c\chi$ is our character $\chi_{\mathfrak{p}}$, ω is our character $\chi_{0,\mathfrak{p}}^{-1}$ and $V \otimes \chi$ is our Galois representation $V_{\mathfrak{p}}^\dagger$.) Now one argues as in the proof of [20, Theorem 9.8]. Briefly, as explained in the proof of [13, Corollary 3.4.3], for every arithmetic prime \mathfrak{p} of \mathcal{R} there is an injection

$$(28) \quad \widetilde{H}_f^1(K, \mathbb{T}^\dagger)_{\mathfrak{p}} / \mathfrak{p} \widetilde{H}_f^1(K, \mathbb{T}^\dagger)_{\mathfrak{p}} \hookrightarrow \widetilde{H}_f^1(K, V_{\mathfrak{p}}^\dagger).$$

By [20, Proposition 5.5], for every $\mathfrak{p} \in \text{Gen}_2(\theta)$ there is an isomorphism

$$\widetilde{H}_f^1(K, V_{\mathfrak{p}}^\dagger) \simeq H_f^1(K, V_{\mathfrak{p}}^\dagger),$$

hence the left hand side in (28) is trivial for infinitely many \mathfrak{p} . By [13, Lemma 2.1.7], this shows that $\tilde{H}_f^1(K, \mathbb{T}^\dagger)$ is \mathcal{R} -torsion. \square

6. LOW RANK RESULTS OVER ARBITRARY QUADRATIC FIELDS

In this final section we give some complementary results in the spirit of Theorems 4.6 and 5.6 but whose proofs do not use the elements \mathfrak{Z} introduced before and work for any quadratic number field K (not necessarily imaginary). In fact, these results are essentially corollaries of those obtained by Howard in [14, Section 4], and we add them here for completeness.

6.1. Splitting the Selmer groups. Fix a quadratic field K of discriminant D prime to Np and let ϵ_K denote the character of K/\mathbb{Q} . For each arithmetic prime \mathfrak{p} of \mathcal{R} consider the twisted Galois representation $V_{\mathfrak{p}} \otimes \epsilon_K$ and the twisted modular form

$$f_{\mathfrak{p}} \otimes \epsilon_K = \sum_{n \geq 1} \epsilon_K(n) a_n(\mathfrak{p}) q^n.$$

Since D is prime to Np , the form $f_{\mathfrak{p}} \otimes \epsilon_K$ is again a p -stabilized newform, of weight $k_{\mathfrak{p}}$, level ND^2p and character $\vartheta_{\mathfrak{p}} \epsilon_K^2 := \psi_{\mathfrak{p}} \omega^{k+j-k_{\mathfrak{p}}} \epsilon_K^2$, where $k_{\mathfrak{p}}$ is the weight of $f_{\mathfrak{p}}$ and we denote $\vartheta_{\mathfrak{p}} := \psi_{\mathfrak{p}} \omega^{k+j-k_{\mathfrak{p}}}$ the character of $f_{\mathfrak{p}}$ (so $\vartheta_{\mathfrak{p}} = \theta_{\mathfrak{p}}^2 = \chi_{0,\mathfrak{p}}^{-1}$ if \mathfrak{p} has weight 2). See [18, Theorem 9] for a proof of this fact (see also the discussion in the introduction to [1]). Since ϵ_K is quadratic, we shall simply write $\vartheta_{\mathfrak{p}}$ for $\vartheta_{\mathfrak{p}} \epsilon_K^2$, with the convention that $\vartheta_{\mathfrak{p}}$ is viewed as a character of conductor $Dp^{m_{\mathfrak{p}}}$ via the canonical projection $(\mathbb{Z}/Dp^{m_{\mathfrak{p}}}\mathbb{Z})^\times \rightarrow (\mathbb{Z}/p^{m_{\mathfrak{p}}}\mathbb{Z})^\times$. It follows that $V_{\mathfrak{p}} \otimes \epsilon_K$ is the p -adic Galois representation attached to $f_{\mathfrak{p}} \otimes \epsilon_K$ and $V_{\mathfrak{p}}^\dagger \otimes \epsilon_K$ is the representation attached to $f_{\mathfrak{p}}^\dagger \otimes \epsilon_K$.

Now let g be a weight 2, p -ordinary normalized eigenform on $\Gamma_0(Mp)$ for some integer $M \geq 1$ prime to p with (non-necessarily primitive) Dirichlet character ψ , and let $\tilde{\psi}$ denote the primitive character associated with ψ . We say that g has an *exceptional zero* if $\tilde{\psi}(p) \neq 0$ and the p -th Fourier coefficient of g (which is equal to the eigenvalue of U_p acting on g) is ± 1 . See [22, Ch. I, §15] for more general definitions and details.

Since we will need it in the following, we recall the definition of the (strict) Greenberg Selmer group. Suppose that L is a number field, v is a finite place of L and $D_v \subset \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is a decomposition group at v . Let M be a $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module, which we require, for simplicity, to be a two-dimensional vector space over a finite field extension E of \mathbb{Q}_p . We also assume that M is ordinary at all places $v|p$, which means that for all $v|p$ there is a short exact sequence

$$(29) \quad 0 \longrightarrow M_v^+ \longrightarrow M \xrightarrow{\pi} M_v^- \longrightarrow 0$$

of D_v -modules such that M_v^+ and M_v^- are one-dimensional E -vector spaces and the inertia subgroup $I_v \subset D_v$ acts trivially on M_v^- . For $v \nmid p$ the Greenberg local conditions at v are given by the group

$$H_{\text{Gr}}^1(L_v, M) := \ker \left(H^1(L_v, M) \longrightarrow H^1(L_v^{\text{unr}}, M) \right)$$

where L_v^{unr} is the maximal unramified extension of the completion L_v of L at v , while for $v \mid p$ the Greenberg local conditions at v are defined as

$$H_{\text{Gr}}^1(L_v, M) := \ker \left(H^1(L_v, M) \xrightarrow{\pi} H^1(L_v, M_v^-) \right);$$

here, by a slight abuse of notation, we write π for the map in cohomology induced by the corresponding map in (29).

The (strict) Greenberg Selmer group $\text{Sel}_{\text{Gr}}(L, M)$ is defined as

$$\text{Sel}_{\text{Gr}}(L, M) := \ker \left(H^1(L, M) \xrightarrow{\prod_v \text{res}_v} \prod_v H^1(L_v, M) / H_{\text{Gr}}^1(L_v, M) \right),$$

where the product is over all finite places of L and the maps res_v are the usual restriction maps. The groups $\text{Sel}_{\text{Gr}}(L, M)$ and $\tilde{H}_f^1(L, M)$ sit in an exact sequence

$$0 \rightarrow \tilde{H}_f^0(L, M) \rightarrow H^0(L, M) \rightarrow \bigoplus_{v|p} H^0(L_v, M_v^-) \rightarrow \tilde{H}_f^1(L, M) \rightarrow \text{Sel}_{\text{Gr}}(L, M) \rightarrow 0$$

(see [23, Lemma 9.6.3]).

For $L = \mathbb{Q}$ or $L = K$ and $v|p$, we know that there is an exact sequence

$$(30) \quad 0 \rightarrow (V_{\mathfrak{p}}^{\dagger})_v^+ \rightarrow V_{\mathfrak{p}}^{\dagger} \rightarrow (V_{\mathfrak{p}}^{\dagger})_v^- \rightarrow 0$$

as (29) above for $M = V_{\mathfrak{p}}^{\dagger}$. When $L = \mathbb{Q}$ we can twist (30) by ϵ_K to get a D_p -equivariant short exact sequence

$$(31) \quad 0 \rightarrow (V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K)_p^+ \rightarrow V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K \rightarrow (V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K)_p^- \rightarrow 0$$

where $(V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K)_p^{\pm} := (V_{\mathfrak{p}}^{\dagger})_p^{\pm} \otimes \epsilon_K$. Furthermore, since p is unramified in K , we have $\epsilon_K(\sigma) = 1$ for all $\sigma \in I_v$. Therefore (31) is an exact sequence of type (29) for $V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K$, hence we may define $\text{Sel}_{\text{Gr}}(\mathbb{Q}, V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K)$ by the previous recipe.

If $f_{\mathfrak{p}}$ does not have an exceptional zero then [13, eq. (21)] ensures that

$$\tilde{H}_f^1(L, V_{\mathfrak{p}}^{\dagger}) \simeq \text{Sel}_{\text{Gr}}(L, V_{\mathfrak{p}}^{\dagger})$$

for $L = \mathbb{Q}$ or $L = K$. Following [13, Lemma 2.4.4], we extend this isomorphism to the case of the twisted representation $V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K$.

Lemma 6.1. *If $f_{\mathfrak{p}} \otimes \epsilon_K$ does not have an exceptional zero then*

$$\tilde{H}_f^1(\mathbb{Q}, V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K) \simeq \text{Sel}_{\text{Gr}}(\mathbb{Q}, V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K).$$

Proof. For simplicity, put $N := (V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K)_p^-$. We only need to show that $H^0(\mathbb{Q}_p, N) = 0$. Recall that D_v acts on $V_{\mathfrak{p}}^{\dagger}$ by $\eta\Theta_{\mathfrak{p}}^{-1}$ where $\eta : D_p/I_p \rightarrow F_{\mathfrak{p}}$ sends the arithmetic Frobenius to the eigenvalue $\alpha_{\mathfrak{p}}$ of U_p acting on $f_{\mathfrak{p}}$. Hence D_p acts on the twist N by $\eta\epsilon_K\Theta_{\mathfrak{p}}^{-1}$. If $H^0(\mathbb{Q}_p, N) \neq 0$ then $\eta\epsilon_K\Theta_{\mathfrak{p}}^{-1}$ is trivial on D_p . Since $\Theta_{\mathfrak{p}}$ factors through $\text{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$, it follows that $\eta\epsilon_K$ is trivial on $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{p^\infty}))$. On the other hand, $\eta\epsilon_K$ is unramified, so, since it factors through the maximal abelian extension of \mathbb{Q}_p , it must be trivial. Thus $\alpha_{\mathfrak{p}} = \epsilon_K(p)$ and $\Theta_{\mathfrak{p}}$ is trivial too. Now we have

$$\Theta_{\mathfrak{p}} = \epsilon_{\text{tame}}^{(k+j-2)/2} \cdot \epsilon_{\text{wild}}^{(k_{\mathfrak{p}}-2)/2} \cdot (\psi_{\mathfrak{p}} \circ \epsilon_{\text{wild}}^{1/2}),$$

so if $\Theta_{\mathfrak{p}}$ is trivial then $k_{\mathfrak{p}} = 2$ necessarily. Finally, the fact that $\Theta_{\mathfrak{p}}$ is trivial implies that $\theta_{\mathfrak{p}}$ is trivial too, hence $\vartheta_{\mathfrak{p}}$ is trivial and the character of $f_{\mathfrak{p}} \otimes \epsilon_K$ is ϵ_K . Since p is unramified in K , $\epsilon_K(p) \neq 0$. Therefore, by definition, $f_{\mathfrak{p}} \otimes \epsilon_K$ has an exceptional zero. \square

For lack of a convenient reference, we also add a proof of the following result, which is certainly known.

Proposition 6.2. *If \mathfrak{p} is an arithmetic prime of \mathcal{R} such that neither $f_{\mathfrak{p}}$ nor $f_{\mathfrak{p}} \otimes \epsilon_K$ has an exceptional zero then there is a decomposition*

$$\tilde{H}_f^1(K, V_{\mathfrak{p}}^{\dagger}) = \tilde{H}_f^1(\mathbb{Q}, V_{\mathfrak{p}}^{\dagger}) \oplus \tilde{H}_f^1(\mathbb{Q}, V_{\mathfrak{p}}^{\dagger} \otimes \epsilon_K).$$

Proof. Our fixed embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_{\ell}$ allow us to define for every finite place v of $L = \mathbb{Q}$ or $L = K$ the decomposition group $D_v \subset \text{Gal}(\bar{\mathbb{Q}}/L)$ and its inertia subgroup $I_v \subset D_v$. Now let v be a finite place of L and let ℓ denote the rational prime over which v lies. Thus $D_v \subset D_{\ell}$ with index 2 if ℓ does not split in K and $D_v = D_{\ell}$ otherwise. If ℓ is split in K and $\ell = v\bar{v}$ is a factorization of ℓ into prime ideals v and \bar{v} of the ring of integers of K then $D_v = \tau^{-1}D_{\bar{v}}\tau$ where τ is the generator of $\text{Gal}(K/\mathbb{Q})$. Moreover, $I_v \subset I_{\ell}$ with index 2 if ℓ ramifies K and

$I_v = I_\ell$ otherwise. Since, by assumption, neither f_p nor $f_p \otimes \epsilon_K$ has an exceptional zero, [13, eq. (21)] and Lemma 6.1 ensure that

$$\tilde{H}_f^1(L, V_p^\dagger) \simeq \text{Sel}_{\text{Gr}}(L, V_p^\dagger), \quad \tilde{H}_f^1(\mathbb{Q}, V_p^\dagger \otimes \epsilon_K) \simeq \text{Sel}_{\text{Gr}}(\mathbb{Q}, V_p^\dagger \otimes \epsilon_K)$$

for $L = \mathbb{Q}$ or $L = K$. A direct computation using the explicit description given above shows that $\text{Sel}_{\text{Gr}}(K, V_p^\dagger)$ is stable under the action of $\text{Gal}(K/\mathbb{Q})$. Since F_p is a field of characteristic zero and τ is an involution on $\text{Sel}_{\text{Gr}}(K, V_p^\dagger)$, there is a decomposition

$$\text{Sel}_{\text{Gr}}(K, V_p^\dagger) = \text{Sel}_{\text{Gr}}(K, V_p^\dagger)^+ \oplus \text{Sel}_{\text{Gr}}(K, V_p^\dagger)^-$$

where $\text{Sel}_{\text{Gr}}(K, V_p^\dagger)^\pm$ is the \pm -eigenspaces for τ . One has

$$\text{Sel}_{\text{Gr}}(K, V_p^\dagger)^+ = \text{Sel}_{\text{Gr}}(K, V_p^\dagger)^{\text{Gal}(K/\mathbb{Q})}, \quad \text{Sel}_{\text{Gr}}(K, V_p^\dagger)^- = \text{Sel}_{\text{Gr}}(K, V_p^\dagger \otimes \epsilon_K)^{\text{Gal}(K/\mathbb{Q})}$$

(for the last equality use the fact that $\epsilon_K(\tau) = -1$). Restriction maps give isomorphisms

$$\text{res}_K : H^1(\mathbb{Q}, V_p^\dagger) \xrightarrow{\simeq} H^1(K, V_p^\dagger)^{\text{Gal}(K/\mathbb{Q})}$$

and

$$\text{res}_K : H^1(\mathbb{Q}, V_p^\dagger \otimes \epsilon_K) \xrightarrow{\simeq} H^1(K, V_p^\dagger \otimes \epsilon_K)^{\text{Gal}(K/\mathbb{Q})},$$

a consequence of the fact that the relevant kernels and cokernels are 2-torsion groups as well as F_p -vector spaces. To conclude, we have to show that the last two isomorphisms respect the local conditions defining Selmer groups. We deal with both cases simultaneously, letting V denote either V_p^\dagger or $V_p^\dagger \otimes \epsilon_K$. Thus we need to check that the isomorphism

$$\text{res}_K : H^1(\mathbb{Q}, V) \xrightarrow{\simeq} H^1(K, V)^{\text{Gal}(K/\mathbb{Q})}$$

takes $\text{Sel}_{\text{Gr}}(\mathbb{Q}, V)$ isomorphically onto $\text{Sel}_{\text{Gr}}(K, V)^{\text{Gal}(K/\mathbb{Q})}$. Since, with notation as before, $D_v \subset D_\ell$ and $I_v \subset I_\ell$, the image of $\text{Sel}_{\text{Gr}}(\mathbb{Q}, V)$ under res_K is contained in $\text{Sel}_{\text{Gr}}(K, V)$, hence in $\text{Sel}_{\text{Gr}}(K, V)^{\text{Gal}(K/\mathbb{Q})}$. To prove surjectivity, fix $c \in \text{Sel}_{\text{Gr}}(K, V)^{\text{Gal}(K/\mathbb{Q})}$ and let $\tilde{c} \in H^1(K, V)$ be such that $\text{res}_K(\tilde{c}) = c$: we want to show that $\tilde{c} \in \text{Sel}_{\text{Gr}}(\mathbb{Q}, V)$. In other words, our goal is to prove that $\text{res}_\ell(\tilde{c}) \in H_{\text{Gr}}^1(\mathbb{Q}_\ell, V)$ for every prime number ℓ . First suppose that $\ell \neq p$. Since $I_\ell = I_v$ whenever ℓ is unramified in K , and since we already know that $\text{res}_v(c) = 0$ in $H^1(I_v, V)$, we can assume that ℓ is ramified in K , in which case $[I_\ell : I_v] = 2$. But restriction from I_ℓ to I_v yields an isomorphism

$$H^1(I_\ell, V) \xrightarrow{\simeq} H^1(I_v, V)^{I_\ell/I_v}$$

(kernel and cokernel are both 2-torsion groups and F_p -vector spaces) and $\text{res}_v(c)$ actually belongs to $H^1(I_v, V)^{I_\ell/I_v}$, which shows that $\text{res}_\ell(\tilde{c}) = 0$. Suppose now that v lies above p . Since $[D_p : D_v] \leq 2$, there is an isomorphism

$$H^1(D_p, V^-) \xrightarrow{\simeq} H^1(D_v, V^-)^{D_p/D_v}$$

(again, kernel and cokernel are 2-torsion F_p -vector spaces) and $\pi(c)$ actually belongs to $H^1(D_v, V^-)^{D_p/D_v}$, which gives $\pi(\tilde{c}) = 0$. Therefore $\tilde{c} \in \text{Sel}_{\text{Gr}}(\mathbb{Q}, V)$, as was to be shown. \square

6.2. Twists of Hida families and low rank results. The representations $V_p \otimes \epsilon_K$, or equivalently the cusp forms $f_p \otimes \epsilon_K$, can be p -adically interpolated by a Hida family γ_∞ of tame level ND^2 , in the same way as the V_p and the f_p are interpolated by ϕ_∞ (cf. [6, Section 7]); such a Hida family will play an auxiliary role in our subsequent considerations. More precisely, if $\mathfrak{H}_\infty^{\text{ord}}$ denotes Hida's ordinary Hecke algebra over \mathcal{O}_F of tame level $\Gamma_0(ND^2)$ then

$$\gamma_\infty : \mathfrak{H}_\infty^{\text{ord}} \longrightarrow \mathcal{R}'$$

where \mathcal{R}' is the branch of the Hida family on which $f \otimes \epsilon_K$ lives. In particular, the ring \mathcal{R}' is a complete local noetherian domain, finite and flat over Λ , which is defined in terms of projective limits of Hecke algebras of tame level ND^2 . As before, the Hida family γ_∞ can be

viewed as a formal power series $g_\infty \in \mathcal{R}'[[q]]$, whose specializations at arithmetic primes Ω of \mathcal{R}' will be denoted g_Ω . It follows that if \mathfrak{p} is an arithmetic prime of \mathcal{R} then $f_{\mathfrak{p}} \otimes \epsilon_K = g_{\mathfrak{p}}$ for a suitable arithmetic prime \mathfrak{P} of \mathcal{R}' of weight $k_{\mathfrak{p}}$. In the following, whenever we deal with an arithmetic prime \mathfrak{p} of \mathcal{R} we shall use the symbol \mathfrak{P} to indicate the arithmetic prime of \mathcal{R}' such that $g_{\mathfrak{P}} = f_{\mathfrak{p}} \otimes \epsilon_K$ (and analogously for \mathfrak{p}' and \mathfrak{P}'); more generally, we adopt lowercase gothic letters to denote primes of \mathcal{R} , while we reserve uppercase letters for primes of \mathcal{R}' .

One can attach to γ_∞ a big Galois representation \mathbb{T}' in exactly the same manner as \mathbb{T} is associated with ϕ_∞ (see [20, §5.5]), and there are specializations $W_\Omega := \mathbb{T}' \otimes_{\mathcal{R}'} F'_\Omega$ of \mathbb{T}' at arithmetic primes Ω of \mathcal{R}' , where F'_Ω is the residue field of \mathcal{R}' at Ω . In particular, with the convention introduced before, $W_{\mathfrak{P}} \simeq V_{\mathfrak{p}} \otimes \epsilon_K$ for every arithmetic prime \mathfrak{p} of \mathcal{R} . Finally, write \mathbb{T}'^\dagger for the Θ^{-1} -twist of \mathbb{T}' , so that $W_{\mathfrak{P}}^\dagger \simeq V_{\mathfrak{p}}^\dagger \otimes \epsilon_K$ for every arithmetic prime \mathfrak{p} of \mathcal{R} , and introduce the notion of *generic* weight 2 arithmetic prime of \mathcal{R}' in terms of an \mathcal{R}'^\times -valued character θ' defined as in Section 2.

Theorem 6.3. *Assume that there exists an arithmetic prime \mathfrak{p} of even weight and trivial character such that*

- (a) $(f_{\mathfrak{p}}, \mathbf{1})$ has analytic rank zero;
- (b) neither $f_{\mathfrak{p}}$ nor $f_{\mathfrak{p}} \otimes \epsilon_K$ has an exceptional zero.

Then

- (1) $\tilde{H}_f^1(K, \mathbb{T}^\dagger)$ is \mathcal{R} -torsion;
- (2) $(f_{\mathfrak{p}'}, \mathbf{1})$ has analytic rank zero for all but finitely many arithmetic primes \mathfrak{p}' of even weight and trivial character.

Proof. First of all, for every arithmetic prime \mathfrak{q} with trivial character there is a factorization of L -functions

$$(32) \quad L_K(f_{\mathfrak{q}}, s) = L_K(f_{\mathfrak{q}}, \mathbf{1}, s) = L(f_{\mathfrak{q}}, s) \cdot L(f_{\mathfrak{q}} \otimes \epsilon_K, s).$$

Now let \mathfrak{p} be as in the statement of the theorem. Since $(f_{\mathfrak{p}}, \mathbf{1})$ has analytic rank zero, it follows from (32) that $L(f_{\mathfrak{p}}, k_{\mathfrak{p}}/2) \neq 0$ and $L(f_{\mathfrak{p}} \otimes \epsilon_K, k_{\mathfrak{p}}/2) \neq 0$. Applying the results of Kato ([17]) to the Hida families ϕ_∞ and γ_∞ as in the proof of [14, Theorem 7], and recalling that $W_{\mathfrak{P}}^\dagger \simeq V_{\mathfrak{p}}^\dagger \otimes \epsilon_K$, we obtain that

$$\tilde{H}_f^1(\mathbb{Q}, V_{\mathfrak{p}'}^\dagger) = 0, \quad \tilde{H}_f^1(\mathbb{Q}, V_{\mathfrak{p}'}^\dagger \otimes \epsilon_K) \simeq \tilde{H}_f^1(\mathbb{Q}, W_{\mathfrak{P}'}^\dagger) = 0$$

for infinitely many arithmetic primes \mathfrak{p}' of \mathcal{R} . Now Proposition 6.2 gives $\tilde{H}_f^1(K, V_{\mathfrak{p}'}^\dagger) = 0$ for infinitely many \mathfrak{p}' , and for part (1) we conclude as in the proof of Theorem 5.6. To prove part (2) we apply [14, Theorem 7] to the families ϕ_∞ and γ_∞ and get that $L(f_{\mathfrak{p}'}, k_{\mathfrak{p}'}/2) \neq 0$ and $L(f_{\mathfrak{p}'} \otimes \epsilon_K, k_{\mathfrak{p}'}/2) \neq 0$ for all but finitely many \mathfrak{p}' of \mathcal{R} of even weight and trivial character. On the other hand, equality (32) with $\mathfrak{q} = \mathfrak{p}'$ implies that

$$L_K(f_{\mathfrak{p}'}, k_{\mathfrak{p}'}/2) = L(f_{\mathfrak{p}'}, k_{\mathfrak{p}'}/2) \cdot L(f_{\mathfrak{p}'} \otimes \epsilon_K, k_{\mathfrak{p}'}/2),$$

and we are done. \square

Under narrower assumptions, we can also offer a result in rank one analogous to Theorem 6.3. Here $w = \pm 1$ is the sign appearing in the functional equation for the Mazur–Tate–Teitelbaum p -adic L -function (see [13, Proposition 2.3.6]).

Theorem 6.4. *Assume that $w = 1$ and there exists an arithmetic prime \mathfrak{p} of weight 2 and trivial character such that*

- (a) $(f_{\mathfrak{p}}, \mathbf{1})$ has analytic rank one;
- (b) $\mathfrak{p} \in \text{Gen}_2(\theta)$ and $\mathfrak{P} \in \text{Gen}_2(\theta')$.

Then $\tilde{H}_f^1(K, \mathbb{T}^\dagger)$ is an \mathcal{R} -module of rank one.

Proof. Let \mathfrak{p} be as in the statement. In light of factorization (32), two possibilities can occur:

- (i) $L(f_{\mathfrak{p}}, 1) \neq 0$, $L(f_{\mathfrak{p}} \otimes \epsilon_K, 1) = 0$ and $L'(f_{\mathfrak{p}} \otimes \epsilon_K, 1) \neq 0$.
- (ii) $L(f_{\mathfrak{p}}, 1) = 0$, $L(f_{\mathfrak{p}} \otimes \epsilon_K, 1) \neq 0$ and $L'(f_{\mathfrak{p}}, 1) \neq 0$.

Suppose that we are in case (i). By [14, Theorem 7] applied to the Hida family ϕ_{∞} , the form $f_{\mathfrak{p}'}$ has analytic rank zero and $\tilde{H}_f^1(\mathbb{Q}, V_{\mathfrak{p}'}^{\dagger}) = 0$ for all but finitely many $\mathfrak{p}' \in \text{Gen}_2(\theta)$. On the other hand, by [14, Theorem 8] applied to the Hida family γ_{∞} , the form $f_{\mathfrak{p}'} \otimes \epsilon_K = g_{\mathfrak{P}'}$ has analytic rank one and $\tilde{H}_f^1(\mathbb{Q}, V_{\mathfrak{p}'}^{\dagger} \otimes \epsilon_K) \simeq \tilde{H}_f^1(\mathbb{Q}, W_{\mathfrak{P}'}^{\dagger})$ is one-dimensional over $F_{\mathfrak{p}'} = F_{\mathfrak{P}'}$ for all but finitely many $\mathfrak{P}' \in \text{Gen}_2(\theta')$ (see the proof of [13, Corollary 3.4.3] with \mathbb{Q} in place of K). Finally, Proposition 6.2 implies that $\tilde{H}_f^1(K, V_{\mathfrak{p}'}^{\dagger})$ is one-dimensional for almost all \mathfrak{p}' , and the theorem follows by arguing as in the final part of the proof of [13, Corollary 3.4.3]. Case (ii) is symmetric to (i) and is left to the reader. \square

REFERENCES

- [1] A. O. L. Atkin, W.-C. W. Li, Twists of newforms and pseudo-eigenvalues of W -operators, *Invent. Math.* **48** (1978), no. 3, 221–243.
- [2] M. Bertolini, H. Darmon, Heegner points on Mumford–Tate curves, *Invent. Math.* **126** (1996), no. 3, 413–456.
- [3] M. Bertolini, H. Darmon, Euler systems and Jochnowitz congruences, *Amer. J. Math.* **121** (1999), no. 2, 259–281.
- [4] M. Bertolini, H. Darmon, A. Iovita, M. Spiess, Teitelbaum’s exceptional zero conjecture in the anticyclotomic setting, *Amer. J. Math.* **124** (2002), no. 2, 411–449.
- [5] S. Bloch, K. Kato, L -functions and Tamagawa numbers of motives, in *The Grothendieck Festschrift, vol. I*, P. Cartier, L. Illusie, N. M. Katz, G. Laumon, Y. Manin and K. A. Ribet (eds.), Progress in Mathematics **86**, Birkhäuser, Boston, MA, 1990, 333–400.
- [6] M. Dimitrov, E. Ghate, On classical weight one forms in Hida families, available at <http://www.math.jussieu.fr/~dimitrov/>, submitted (2010).
- [7] G. Faltings, Endlichkeitssätze für abelsche Varietäten über Zahlkörpern, *Invent. Math.* **73** (1983), no. 3, 349–366.
- [8] R. Greenberg, Elliptic curves and p -adic deformations, in *Elliptic curves and related topics*, H. Kisilevsky and M. Ram Murty (eds.), CRM Proceedings and Lecture Notes **4**, American Mathematical Society, Providence, RI, 1994, 101–110.
- [9] D. Helm, On maps between modular Jacobians and Jacobians of Shimura curves, *Israel J. Math.* **160** (2007), no. 1, 61–117.
- [10] H. Hida, Iwasawa modules attached to congruences of cusp forms, *Ann. Sci. École Norm. Sup. (4)* **19** (1986), no. 2, 231–273.
- [11] H. Hida, Galois representations into $\text{GL}_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms, *Invent. Math.* **85** (1986), no. 3, 545–613.
- [12] H. Hida, On p -adic Hecke algebras for GL_2 over totally real fields, *Ann. of Math. (2)* **128** (1988), no. 2, 295–384.
- [13] B. Howard, Variation of Heegner points in Hida families, *Invent. Math.* **167** (2007), no. 1, 91–128.
- [14] B. Howard, Central derivatives of L -functions in Hida families, *Math. Ann.* **339** (2007), no. 4, 803–818.
- [15] B. Howard, Twisted Gross–Zagier theorems, *Canad. J. Math.* **61** (2009), no. 4, 828–887.
- [16] B. W. Jordan, R. Livné, Integral Hodge theory and congruences between modular forms, *Duke Math. J.* **80** (1995), no. 2, 419–484.
- [17] K. Kato, p -adic Hodge theory and values of zeta functions of modular forms, *Astérisque* **295** (2004), 117–290.
- [18] W.-C. W. Li, Newforms and functional equations, *Math. Ann.* **212** (1975), no. 4, 285–315.
- [19] M. Longo, V. Rotger, S. Vigni, On rigid analytic uniformizations of Jacobians of Shimura curves, *Amer. J. Math.*, to appear.
- [20] M. Longo, S. Vigni, Quaternion algebras, Heegner points and the arithmetic of Hida families, *Manuscripta Math.* **135** (2011), no. 3–4, 273–328.
- [21] M. Longo, S. Vigni, A note on control theorems for quaternionic Hida families of modular forms, arXiv:1110.1977, submitted.
- [22] B. Mazur, J. Tate, J. Teitelbaum, On p -adic analogues of the conjectures of Birch and Swinnerton-Dyer, *Invent. Math.* **84** (1986), no. 1, 1–48.
- [23] J. Nekovář, Selmer complexes, *Astérisque* **310** (2006).

- [24] J. Nekovář, Level raising and Selmer groups for Hilbert modular forms of weight two, *Canad. J. Math.*, to appear.
- [25] J. Tate, Relations between K_2 and Galois cohomology, *Invent. Math.* **36** (1976), no. 1, 257–274.
- [26] S. Zhang, Heights of Heegner points on Shimura curves, *Ann. of Math. (2)* **153** (2001), no. 1, 27–147.
- [27] S. Zhang, Gross–Zagier formula for GL_2 , *Asian J. Math.* **5** (2001), no. 2, 183–290.

DIPARTIMENTO DI MATEMATICA PURA E APPLICATA, UNIVERSITÀ DI PADOVA, VIA TRIESTE 63, 35121 PADOVA, ITALY

E-mail address: `mlongo@math.unipd.it`

DEPARTMENT OF MATHEMATICS, KING'S COLLEGE LONDON, STRAND, LONDON WC2R 2LS, UNITED KINGDOM

E-mail address: `stefano.vigni@kcl.ac.uk`