Lezione 21

mercoledì 27 novembre 2013
14:07

ES. $$\lim_{x\to 0^+} \frac{e^{\alpha x^2} - \cos x + [\log(1+x)]^2}{x^3} \qquad \alpha \in \mathbb{R}$$

- $$[\log(1+x)]^2 = x^2 - x^3 + o(x^3) \qquad x \to 0.$$

Sviluppo del numeratore

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$e^{\alpha x^2} = 1 + \alpha x^2 + \frac{1}{2}(\alpha x^2)^2 + o\left((\alpha x^2)^2\right)$$

$$= 1 + \alpha x^2 + \frac{\alpha^2}{2}x^4 + o(x^4) = 1 + \alpha x^2 + o(x^3)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{1}{4!}x^4 + o(x^4) = 1 - \frac{x^2}{2} + o(x^3)$$

$$N(x) = e^{\alpha x^2} - \cos x + [\log(1+x)]^2$$

$$= \cancel{1} + \alpha x^2 + o(x^3) - \cancel{1} + \frac{x^2}{2} + o(x^3) + x^2 - x^3 + o(x^3)$$

$$= \left(\frac{3}{2} + \alpha\right)x^2 - x^3 + o(x^3)$$

Limite:

$$L = \lim_{x\to 0^+} \frac{N(x)}{x^3} = \lim_{x\to 0^+} \frac{\left(\frac{3}{2}+\alpha\right)x^2 - x^3 + o(x^3)}{x^3}$$

1° caso $\frac{3}{2} + \alpha \neq 0$

$$\square \quad L = \lim_{x\to 0^+} \boxed{\frac{1}{x}} \underbrace{\left(\frac{3}{2} + \alpha - x + o(x)\right)}_{\downarrow\ x\to 0} = \begin{cases} \infty & \frac{3}{2}+\alpha>0 \\ -\infty & \frac{3}{2}+\alpha<0 \end{cases}$$

$$\xrightarrow{x \to 0} \frac{3}{2} + \alpha$$

2° Caso $\frac{3}{2} + \alpha = 0$

$$\Delta \quad L = \lim_{x \to 0^+} \frac{-x^3 + x^3 \cdot o(1)}{x^3}$$

$$= \lim_{x \to 0^+} \left(-1 + o(1) \right) = -1 \quad \text{per } \alpha = -\frac{3}{2}.$$

Esercizio Calcolare il seguente limite

$$L = \lim_{x \to 0^+} \frac{(\cosh x - 1) \log x + x^{x+1} - x}{\sin^2(x) \log x} \cdot \left[\frac{0}{0}\right]$$

Attenzione! $\log x$ NON si sviluppa per $x \to 0^+$

Sviluppi:

$$\sin x = x + o(x) = x\,(1+o(1))$$

$$\sin^2 x = x^2 (1+o(1))^2 = x^2 (1+o(1))$$

$$\cosh x = 1 + \frac{x^2}{2} + o(x^2)$$

$$\cosh x - 1 = \frac{x^2}{2} + o(x^2) = x^2 \left(\frac{1}{2} + o(1) \right)$$

Poi c'è $x^{x+1} = x \cdot x^x = x \cdot e^{\log x^x} = x \cdot e^{x \log x}$

Parto da:

$$e^x = 1 + x + \frac{1}{2} x^2 + o(x^2) \quad x \to 0$$

Siccome $x \cdot \log x \to 0$ per $x \to 0$ posso usare lo regola di sostituzione ed ottenere:

$$e^{x\log x} = 1 + x\log x + \frac{1}{2}x^2\log^2 x + o\left(x^2\log^2 x\right)$$

Dunque

$$x^{x+1} = x + x^2\log x + \frac{1}{2}x^3\log^2 x + x^3\log^2 x \cdot o(1)$$

Numeratore:

$$\begin{aligned}
N(x) &= (\cosh x - 1)\log x + x^{x+1} - x \\
&= x^2\left(\frac{1}{2} + o(1)\right)\log x + \cancel{x} + x^2\log x + \frac{1}{2}x^3\log^2 x \\
&\quad + x^3\log^2 x \cdot o(1) - \cancel{x} \\
&= x^2\log x\left(\frac{1}{2} + 1\right) + \frac{1}{2}x^3\log^2 x \\
&\quad + x^2\log x \cdot o(1) + x^3\log^2 x \cdot o(1) \\
&= \frac{3}{2}x^2\log x + o\left(x^2\log x\right) \\
&= x^2\log x\left(\frac{3}{2} + o(1)\right)
\end{aligned}$$

$$\begin{aligned}
D(x) &= \sin^2 x \log x = x^2(1 + o(1)) \cdot \log x \\
&= x^2\log x\,(1 + o(1))
\end{aligned}$$

$$L = \lim_{x\to 0^+}\frac{N(x)}{D(x)} = \lim_{x\to 0^+}\frac{\cancel{x^2\log x}\left(\frac{3}{2} + o(1)\right)}{\cancel{x^2\log x}\,(1 + o(1))}$$

$$= \lim_{x\to 0^+}\frac{\frac{3}{2} + o(1)}{1 + o(1)} = \frac{3}{2}. \qquad \square$$

ES Calcolare il seguente limite:

$$L = \lim_{x\to 0} \frac{\log\left(\frac{1+\sin x}{1+x}\right)}{\sin x + \log(1-x) + 1 - \cos x} \qquad \left[\frac{0}{0}\right]$$

Soluzione:

$$N(x) = \log\left(\frac{1+\sin x}{1+x}\right)$$

$$= \log(1+\sin x) - \log(1+x)$$

Sviluppi:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

Siccome $\sin x \to 0$ per $x \to 0$ posso usare la regola di sost.:

$$\log(1+\sin x) = \sin x - \frac{1}{2}\sin^2 x + \frac{1}{3}\sin^3 x + o(\sin^3 x)$$

$$\sin x = x - \frac{1}{6}x^3 + o(x^3)$$

$$\sin^2 x = \left(x - \frac{1}{6}x^3 + o(x^3)\right)^2 = x^2 + o(x^3)$$

$$\sin^3 x = \left(x + o(x)\right)^3 = \left(x(1+o(1))\right)^3 = x^3(1+o(1))^3$$

$$= x^3(1+o(1))$$

Sostituisco sopra:

$$\log(1+\sin x) = x - \frac{1}{6}x^3 + o(x^3) - \frac{1}{2}\left(x^2 + o(x^3)\right) +$$

$$+ \frac{1}{3}x^3(1+o(1)) + o(\sin^3 x) \qquad x \to 0$$

Affermo che $o(\sin^3 x) = o(x^3)$:

$$o(\sin^3 x) = \sin^3 x \cdot o(1) = x^3\big(1+o(1)\big)\cdot o(1)$$
$$= x^3 \cdot o(1) = o(x^3)$$

Numeratore:

$$N(x) = x - \frac{1}{2}x^2 + \left(-\frac{1}{6} + \frac{1}{3}\right)x^3 + o(x^3)$$
$$- \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)\right)$$
$$= \left(-\frac{1}{6} + \frac{1}{3} - \frac{1}{3}\right)x^3 + o(x^3)$$
$$= -\frac{1}{6}x^3 + o(x^3) = x^3\left(-\frac{1}{6} + o(1)\right)$$

Denominatore

$$\sin x = x - \frac{1}{6}x^3 + o(x^3)$$
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$
$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + o(x^3)$$
$$\cos x = 1 - \frac{x^2}{2} + o(x^3)$$

$$D(x) = \sin x + \log(1-x) + 1 - \cos x$$
$$= \cancel{x} - \frac{1}{6}x^3 + o(x^3) - \cancel{x} \boxed{\cancel{-\frac{x^2}{2}}} - \frac{x^3}{3} + o(x^3)$$
$$+ 1 - 1 + \boxed{\cancel{\frac{x^2}{2}}} + o(x^3)$$

$$= \left(-\frac{1}{6} - \frac{1}{3}\right)x^3 + o(x^3) = -\frac{1}{2}x^3 + o(x^3)$$
$$= x^3\left(-\frac{1}{2} + o(1)\right)$$

$$L = \lim_{x\to 0} \frac{N(x)}{D(x)} = \lim_{x\to 0} \frac{\cancel{x^3}\left(-1/6 + o(1)\right)}{\cancel{x^3}\left(-1/2 + o(1)\right)}$$

$= 1/3 .$ $\square$

Criterio del Confronto Asintotico per Serie Numeriche

TEOREMA Siano $(a_n)_{n\in\mathbb{N}}$ e $(b_n)_{n\in\mathbb{N}}$ due successioni numeriche positive, $a_n > 0$ e $b_n > 0$. Supponiamo che esista finito e diverso da 0 il limite

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} \neq 0 \qquad L \in \mathbb{R} .$$

Allora:

la serie $\sum_{n=0}^{\infty} a_n < \infty$ converge $\Longleftrightarrow$ la serie $\sum_{n=0}^{\infty} b_n < \infty$ converge.

Dim. Semplice. Esiste $\bar{n} \in \mathbb{N}$ tale che $\forall\, n \geq \bar{n}$:

$$0 < \frac{L}{2} < \frac{a_n}{b_n} < 2L < \infty$$

ovvero

$$\frac{L}{2}\, b_n < a_n < 2L \cdot b_n \qquad \forall\, n \geq \bar{n}$$

La tesi segue dal teor. del Confronto $\square$

Esempio Supponiamo di dover studiare la serie

$$\sum_{n=0}^{\infty} a_n \qquad \text{con} \quad a_n > 0$$

Confronto questa serie con $\sum_{n=0}^{\infty} b_n$ con $b_n = \frac{1}{n^\alpha}$ con α parametro da discutere. Precisamente si cerca α tale che esista si cerca α tale che esista

$$L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n^\alpha a_n \quad \neq 0, \quad L \in \mathbb{R}$$

Per il Crit. Confr. Asintotico

$$\sum a_n \text{ converge} \iff \sum \frac{1}{n^\alpha} < \infty$$

$$\iff \alpha > 1 .$$

<u>Esercizio</u> Stabilire se converge la seguente serie:

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)\left(\log(n+1) - \log n\right)}{\sqrt[4]{n+1} - \sqrt[4]{n}} .$$

Soluzione. Serie a termini positivi. Idea: Criterio del Confronto Asintotico

$$\sin(x) = x + o(x) \quad \text{per } x \to 0$$

Quindi

$$\sin\left(\frac{1}{n}\right) = \frac{1}{n} + o\left(\frac{1}{n}\right) \quad \text{per } n \to \infty$$

$$= \frac{1}{n}\left(1 + o(1)\right) \quad \text{dove } o(1) \to 0 \text{ per } n \to \infty$$

Poi

$$\log(n+1) - \log n = \log\frac{n+1}{n} = \log\left(1 + \frac{1}{n}\right)$$

$$\log(1+x) = x + o(x) \quad \text{per } x \to 0$$

$$\log\left(1 + \frac{1}{n}\right) = \frac{1}{n} + o\left(\frac{1}{n}\right) = \frac{1}{n}\left(1 + o(1)\right) \quad n \to \infty$$

Infine:

$$\sqrt[4]{1+n} - \sqrt[4]{n\left(1+\frac{1}{n}\right)} \quad \sqrt[4]{\ } \quad \sqrt[4]{1+\frac{1}{n}}$$

Infine:

$$\sqrt[4]{1+n} = \sqrt[4]{n\left(1+\frac{1}{n}\right)} = \sqrt[4]{n}\,\sqrt[4]{1+\frac{1}{n}}$$

Uso lo sviluppo

$$(1+x)^{\alpha} = 1 + \alpha x + o(x) \qquad x \to 0$$

Quindi

$$\sqrt[4]{1+\frac{1}{n}} = \left(1+\frac{1}{n}\right)^{1/4} = 1 + \frac{1}{4}\,\frac{1}{n} + o\left(\frac{1}{n}\right) \quad n \to \infty$$

Quindi

$$\sqrt[4]{n+1} - \sqrt[4]{n} = \sqrt[4]{n}\,\sqrt[4]{1+\frac{1}{n}} - \sqrt[4]{n}$$

$$= \sqrt[4]{n}\left(\sqrt[4]{1+1/n} - 1\right)$$

$$= \sqrt[4]{n}\left(\cancel{1} + \frac{1}{4n} + o\left(\frac{1}{n}\right) - \cancel{1}\right)$$

$$= n^{1/4}\,\frac{1}{n}\left(\frac{1}{4} + o(1)\right)$$

$$= n^{1/4-1}\left(\frac{1}{4} + o(1)\right)$$