Lezione 17

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Concetto di limite quando "$x \longmapsto \infty$"
$x \in \mathbb{R}^2$

Data $f : \mathbb{R}^2 \to \mathbb{R}$
Posso definire

$$\lim_{|x| \to \infty} f(x) = L \in \mathbb{R}$$

$\Updownarrow$ def

$\forall \varepsilon > 0 \ \exists R > 0$ tale che

$$|x| > R \Rightarrow |f(x) - L| < \varepsilon$$

Funzione esponenziale in campo complesso

Definiamo $\exp : \mathbb{C} \to \mathbb{C}$

$$e^z := \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad z \in \mathbb{C}$$

Raggio di conv. $R = \infty$.

Possiamo anche definire

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

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$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$R = \infty$

$z = x \in \mathbb{R}$ $\longrightarrow$ troviamo la formula nota

TEOR La funzione $\exp : \mathbb{C} \to \mathbb{C}$ ha le seg. proprietà:

1) $\exp(z+w) = \exp(z) \cdot \exp(w)$ $z, w \in \mathbb{C}$

Identità Fond. per exp.

2) $\overline{\exp(z)} = \exp(\bar{z})$

3) $|\exp(ix)| = 1$ $\forall x \in \mathbb{R}$

4) $\exp(ix) = \cos(x) + i \sin(x)$ $\forall x \in \mathbb{R}$

Formula di Eulero.

Dim (1)

$= e^z \cdot e^w \overset{?}{=} e^{z+w}$

$$= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \cdot \left(\sum_{n=0}^{\infty} \frac{w^n}{n!} \right)$$

$$n=0 \qquad n=0$$

$$= \left(1+\frac{z}{1!}+\frac{z^2}{2!}+\dots\right)\cdot\left(1+\frac{w}{1!}+\frac{w^2}{2!}+\dots\right)$$

$$= 1+\frac{z}{1!}+\frac{w}{1!}+\frac{z^2}{2!}+\frac{zw}{1!\,1!}+\frac{w^2}{2!}+$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{z^k}{k!}\,\frac{w^{n-k}}{(n-k)!}\cdot n!$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} z^k\cdot w^{n-k}$$

Newton

$$= \sum_{n=0}^{\infty}\frac{1}{n!}(z+w)^n = e^{z+w}.$$

2) $$\overline{e^z} = \overline{\sum_{k=0}^{\infty}\frac{z^k}{k!}} = \sum_{k=0}^{\infty}\frac{\overline{z^k}}{k!}$$

$$\overline{\exp(z)}$$

$$= \sum_{k=0}^{\infty}\frac{(\bar{z})^k}{k!} = \exp(\bar{z})$$

3) $$|\exp(ix)|^2 = \exp(ix)\cdot\overline{\exp(ix)} =$$

$$|z|^2 = z\cdot\bar{z} \qquad (x\in\mathbb{R})$$

$$|z|^2 = z \cdot \bar{z}$$

$x \in \mathbb{R}$

$$= \exp(ix) \cdot \exp(-ix)$$

$$= \exp(ix - ix) = \exp(0)$$

$$= 1$$

4) $$\exp(ix) = \sum_{k=0}^{\infty} \frac{i^k \cdot x^k}{k!} =$$

$$= \sum_{n=0}^{\infty} \frac{i^{2n} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\boxed{i^{2n+1}} x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \boxed{\cos x + i \sin x} .$$

□

Commento

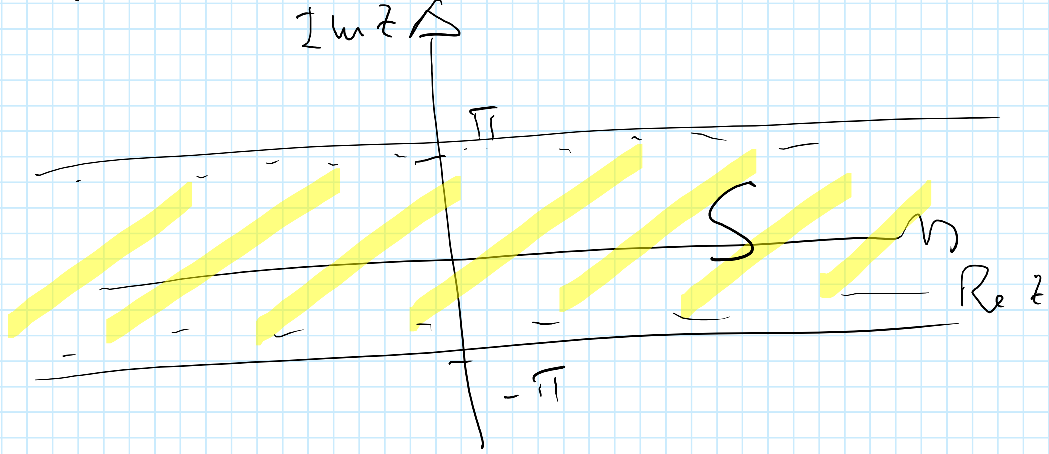
(3) (4)

$$1 \overset{(3)}{=} |\exp(ix)|^2 \overset{(4)}{=} \cos^2 x + \sin^2 x$$

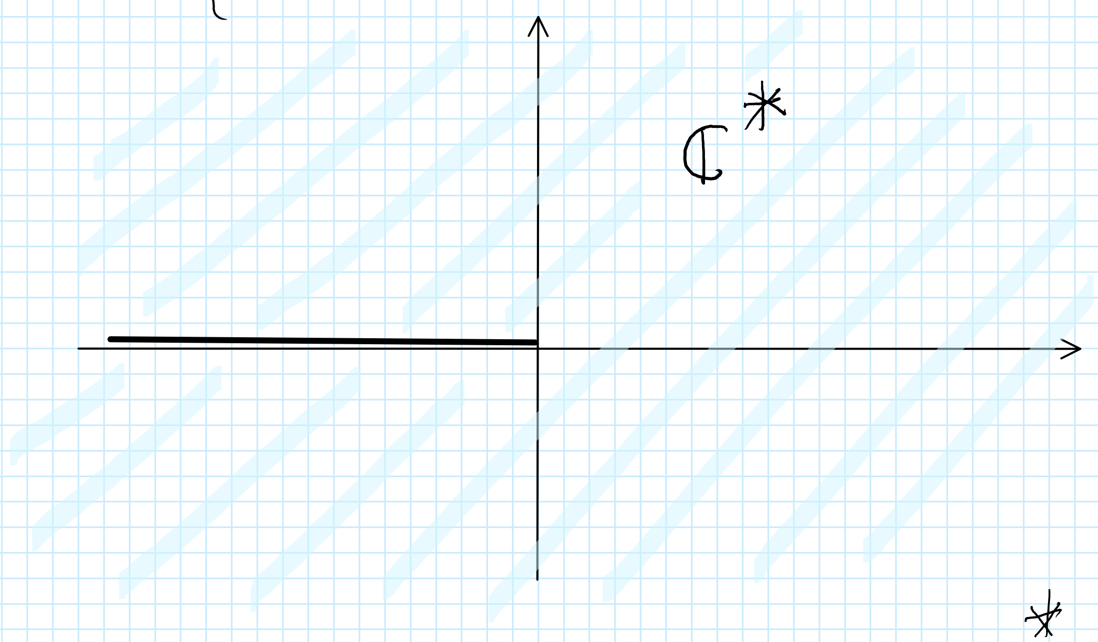
Logaritmo complesso

Introduciamo gli insiemi:

$$S = \{ z \in \mathbb{C} : -\pi < Im(z) < \pi \}$$



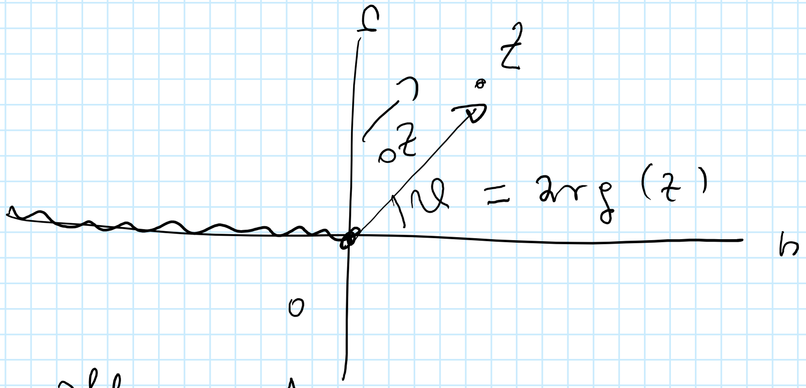
$$\mathbb{C}^* = \mathbb{C} \setminus \{ z \in \mathbb{C} : Im(z) = 0 \text{ e } Re(z) \leq 0 \}$$



Possiamo definire la funzione $\arg : \mathbb{C}^* \longrightarrow (-\pi, \pi)$

$\arg(z)$ = angolo con segno formato

dal segmento $\vec{Oz}$ con l'asse positivo delle $x = \mathrm{Re}\, z$



Sappiamo allora che

$$z = |z|(\cos\vartheta + i\sin\vartheta) \qquad \vartheta = \arg(z)$$

$$= |z|\, e^{i\vartheta} = |z|\, e^{i\arg(z)}.$$

$$z \in \mathbb{C}^*$$

Commenti:

$\exp : S \longrightarrow \mathbb{C}^*$ è iniettiva

Prendo $z = x + iy \in S$

$w = s + it \in S$ $\qquad x\, y\, s\, t \in \mathbb{R}$

$s, t \in (-\pi, \pi)$

$$e^z = e^w \iff e^{x+iy} = e^{s+it}$$

$$\iff e^x \cdot e^{iy} = e^s e^{it}$$

$\neq 0$ $\qquad \neq 0$

$$\iff e^{x-s} = e^{i(t-y)}$$

$\parallel$

$$\parallel$$

$$\cos(t-y) + i\sin(t-y)$$

$$\parallel$$

$$\Downarrow$$

$$t - y = 0$$

$$(\Rightarrow) \quad e^{x-s} = 1$$

$$\Rightarrow \quad x = s$$

$$\Rightarrow \quad z = w$$

Quindi se $\exp : S \to \mathbb{C}^*$

è iniettiva.

Dato $z \in \mathbb{C}^*$ cerco $w \in S$

tale che $\exp(w) = z = |z| e^{i \arg z}$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$e^w \qquad\qquad = e^{\log|z|} e^{i \arg z}$$

Dunque $w = \log|z| + i \arg(z)$ $= e^{\log|z| + i \arg z}$

Risolve $\exp(w) = z$

Quindi esiste una la funzione

inversa

$$S \xrightarrow{\exp} \mathbb{C}^*$$

$\longleftarrow$

log complesso.

Dunque il logaritmo complesso è la funzione

$$\log : \mathbb{C}^* \longrightarrow S \subset \mathbb{C}$$

$$\log(z) := \log|z| + i \arg(z),$$

ESERCIZI

ES 1 Sia exp : $\mathbb{C} \to \mathbb{C}$

$$f(z) = e^z = \exp(z).$$

Verificare che

$$\frac{d f(z)}{dz} = e^z \quad \forall z \in \mathbb{C}.$$

Soluzione: Devo verificare che $\forall z_0 \in \mathbb{C}$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = e^{z_0}$$

$$f(z) = e^z$$

1º caso: $z_0 = 0$

Tesi:

$$= \lim_{z \to 0} \frac{e^z - 1}{z} = 1$$

$$= \lim_{z \to 0} \frac{\quad}{z} = 1$$

$$= \lim_{z \to 0} \frac{1}{z} \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} - 1 \right)$$

$$= \lim_{z \to 0} \frac{1}{z} \sum_{k=1}^{\infty} \frac{z^k}{k!}$$

$$= \lim_{z \to 0} \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!}$$

$k-1 = n$

$k = n+1$

$$= \lim_{z \to 0} \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = f(z)$$

$f(0) = 1$

$= 1$ perché f è cont.

Ora consideriamo $z_0 \in \mathbb{C}$ generico

$$\lim_{z \to z_0} \frac{e^z - e^{z_0}}{z - z_0} =$$

$$\lim_{z \to z_0} \frac{\quad}{z - z_0} = \lim_{h \to 0} \frac{e^{z_0 + h} - e^{z_0}}{h}$$

$$= \lim_{h \to 0} \frac{e^{z_0}(e^h - 1)}{h}$$

$$= e^{z_0} \cdot 1 .$$

ES 2 Al variare di $\alpha \geq 0$ si consideri la serie di pot.

$$\sum_{n=1}^{\infty} \frac{n z^n}{n^\alpha + 1} \qquad z \in \mathbb{C} .$$

i) Calcolare il Raggio di conv.

ii) studiare la conv. nei punti $|z| = R$

iii) Studiare la conv. uniforme.

i) R è definito dalla formula

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{\frac{n}{n^\alpha + 1}}$$

? ∃ il limite

$$\lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{n^\alpha+1}}$$

Noto che $\lim_{n\to\infty} \sqrt[n]{n} = 1$

$$\lim_{n\to\infty} \sqrt[n]{n^\alpha+1} = 1 \quad \underline{\underline{OK}}$$

$$\sqrt[n]{n^\alpha} < \sqrt[n]{n^\alpha+1} \leq \sqrt[n]{n^\alpha+n^\alpha}$$

$$\sqrt[n]{n^\alpha} = \left(\sqrt[n]{n}\right)^\alpha \to 1$$

$$\sqrt[n]{n^\alpha+n^\alpha} = \sqrt[n]{2n^\alpha} = \sqrt[n]{2}\,\sqrt[n]{n^\alpha}, \quad \sqrt[n]{2} \to 1, \quad \sqrt[n]{n^\alpha} \to 1$$

$$\Downarrow$$

$$\frac{1}{R} = 1$$

$$R = 1$$

Per il crit. di C-H:

- Conv. Assoluta in tutti i punti $|z| < 1$
- No conv. nei punti $|z| > 1$
- $\forall \delta < 1$ la serie conv. unif. su $A_\delta = \{ |z| \leq \delta \}$

ii) Devo studiare il caso $|z| = 1$

$(\Rightarrow)$ $z = e^{i\vartheta}$ $\vartheta \in [0, 2\pi]$

$$\downarrow$$

$$\sum_{n=1}^{\infty} \frac{n}{n^\alpha + 1} e^{in\vartheta} \qquad \vartheta \in [0, 2\pi]$$

1° caso $\theta = 0$

2° caso $\vartheta = \pi$

1° C

$$\sum_{n=1}^{\infty} \frac{n}{n^\alpha + 1}$$

∞ n $(\ldots)^n$

$\underline{2^{\circ}\ C}$

$$\sum_{n=1}^{\infty} \frac{n}{n^{\alpha}+1} (-1)^n$$