<u>Esercizio</u> $f : \mathbb{R} \to \mathbb{R}$ sia tale che $e^{tx} f(t) \in L^1(\mathbb{R})$ $\forall x \in (-1,1)$

Sia $\varphi : (-1,1) \to \mathbb{R}$ tale che

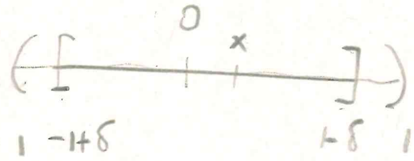
$$\varphi(x) = \int_{\mathbb{R}} e^{tx} f(t)\, dt$$

<u>Studiare derivabilità</u>

Derivata integrando

$$\frac{\partial}{\partial x}\left(e^{tx} f(t)\right) = t\, e^{tx} f(t)$$

Sia $\delta > 0$ piccolo. Avremo



$$-1+\delta \leq x \leq 1-\delta$$

$$|t\, e^{tx} f(t)| \leq |t|\, e^{t(1-\delta)}\, |f(t)| \qquad t>0$$

$$\leq |t|\, e^{t(\delta-1)}\, |f(t)| \qquad t<0$$

$$\begin{cases} \leq |t|\, e^{-t\delta/2}\, e^{t(1-\delta/2)}\, |f(t)| & t>0 \\ \leq |t|\, e^{-|t|\delta/2}\, e^{t(-1+\delta/2)}\, |f(t)| & t<0 \end{cases}$$

$$\leq \underbrace{|t|\, e^{-|t|\delta/2}}_{\substack{\wedge \\ C_\delta \ \forall t \in \mathbb{R} \\ \wedge \\ +\infty}} \underbrace{\left(e^{t(1-\delta/2)} + e^{t(\delta/2-1)}\right)|f|}_{\in L^1(\mathbb{R})}$$

$\Longrightarrow$ Posso derivare sotto integrale.

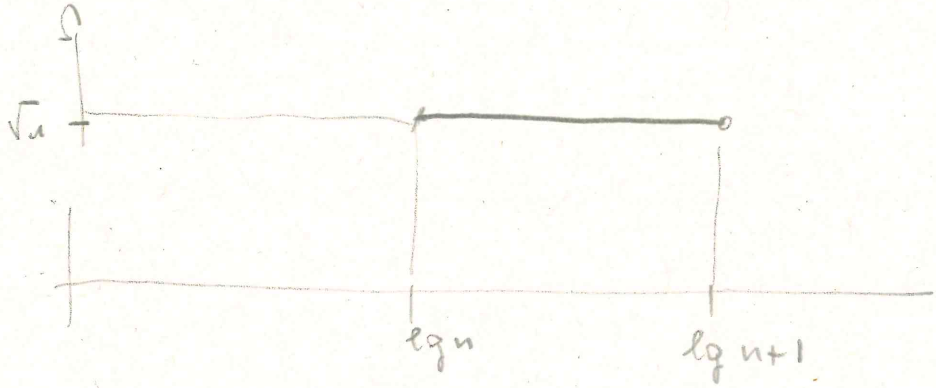
ESERCIZIO χ_n = caratteristica di $[\lg n, \lg n+1]$

$\varphi_n = \sqrt{n}\, \chi_n$

(1) Per quali $1 \leq p < \infty$ ho conv. forte $L^p(0,\infty)$?

(2) " " " " " debole $L^p(0,\infty)$?

Soluzione



(1)

$$\lim_{n\to\infty} \varphi_n(x) = 0 \qquad \forall x > 0$$

Dunque

$$\varphi_n \xrightarrow{L^p \text{ forte}} \varphi \implies \varphi = 0 \quad \text{q.o.}$$

Esaminiamo

$$\int_0^\infty |\varphi_n|^p\, dx = \int_{\log n}^{\lg n+1} n^{p/2}\, dx = n^{p/2} (\lg(n+1) - \lg n$$

$$= n^{p/2} \lg \frac{n+1}{n} = n^{p/2} \lg\left(1 + \frac{1}{n}\right)$$

$$= n^{p/2}\left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right) = n^{p/2-1}\underset{n\to\infty}{(1 + o(1))}$$

Dunque

$$\varphi_n \xrightarrow[n\to\infty]{L^p} 0 \iff \frac{p}{2} - 1 < 0$$

$$\iff \frac{p}{2} < 1$$

$$\iff p < 2.$$

(2) Sappiamo:

conv. forte $\Rightarrow$ conv. debole

Dunque per $1 \le p < 2$; $\varphi_n \xrightarrow[n\to\infty]{L^p\text{-debole}} 0$

Potrebbero esserci altri p.

Consideriamo $p \geq 2$ e sia $1 < q \leq 2$ coniugato

Prendiamo $g \in L^q(0,\infty)$

$$\left| \int_0^\infty \varphi_n(x) g(x)\, dx \right| \le \int_{\log n}^{\log n+1} \sqrt{n}\, |g(x)|\, dx$$

$$\le \left(\int_{\log n}^{\log(n+1)} n^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \left(\int_{\log n}^{\log n+1} |g|^q dx \right)^{\frac{1}{q}}$$

$$= \left(n^{\frac{p}{2}-1} (1+o(1)) \right)^{\frac{1}{p}} \left(\int_{\log n}^{\log n+1} |g|^q dx \right)^{\frac{1}{q}}$$

$$\overset{p=2}{=} \sqrt{1+o(1)} \left(\int_{\log n}^{\log n+1} |g|^2\, dx \right)^{1/2}$$

dove $g \in L^2(0,\infty)$. Ma allora

$$\lim_{n\to\infty} \int_{\log n}^{\log n+1} |g|^2\, dx \overset{CD}{=} 0$$

ovvero

$$\lim_{n\to\infty} \int_0^\infty \varphi_n(x) g(x)\, dx = 0 \quad \forall g \in L^2(0,\infty)$$

ovvero

$$\varphi_n \xrightarrow[n\to\infty]{L^2\text{-debole}} 0$$

Rimane da considerare il caso $p > 2$ e quindi $q < 2$.

In questo caso

$$\int_0^\infty |\varphi_n|^p \, dx \underset{n \to \infty}{\longrightarrow} +\infty$$

Questo lascia supporre che non ci sia conv. debole. Cerco $g \in L^q(0,\infty)$ tale che

$$\lim_{n \to \infty} \int_0^\infty \varphi_n(x)\, g(x)\, dx \quad \text{NON esiste finito}$$

$$\left(\text{Ad es: } \limsup_{n \to \infty} \quad = +\infty \right)$$

Per "ottimizzare" i conti vorrei avere

$$g = n^\alpha \quad \text{su } [\ell_{n}, \ell_{n+1}]$$

con $\alpha \in \mathbb{R}$ da discutere

$$\int_{\ell_n}^{\ell_{n+1}} \sqrt{n}\, n^\alpha \, dx = n^{\frac{1}{2}+\alpha-1} (1+o(1))$$

Diverge se $\alpha - \frac{1}{2} > 0 \iff \boxed{\alpha > \frac{1}{2}}$ (*)

Non posso farlo su tutti gli intervalli ($\Rightarrow g \in L^q$).

Però

$$\int_{\ell_n}^{\ell_{n+1}} |n^\alpha|^q = \boxed{\underset{\substack{\downarrow \\ 0}}{n^{\alpha q - 1}} (1+o(1)) \quad \text{per } \alpha q - 1 < 0}$$

$$\alpha q - 1 < 0 \iff \alpha q < 1 \iff \boxed{\alpha < 1/q} \qquad (**)$$

Sono compatibili (*) e (**)?

$$\frac{1}{2} < \alpha < \frac{1}{q} \Rightarrow q < 2$$ È il nostro caso

Quindi posso scegliere questo α.

ORA: Esiste successione $n_k \in \mathbb{N}$, $k \in \mathbb{N}$, t.c.

$$\int_{\lg n_k}^{\lg(n_k+1)} |n_k^{\alpha}|^q \, dx \leq \frac{1}{2^k}$$

Definisco $g : (0,\infty) \to \mathbb{R}$ così

$$g(x) = \begin{cases} n_k^{\alpha} & \text{se } x \in [\lg n_k, \lg(n_k+1)] \text{ per qualche } k \\ 0 & \text{altrimenti} \end{cases}$$

ORA:

$$\int_0^{\infty} |g|^q \, dx \leq 1$$

E

$$\limsup_{n\to\infty} \int_0^{\infty} \varphi_n(x)\, g(x)\, dx = +\infty.$$