REAL ANALYSIS EXAMS
A.A 2011–12

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Exercise 1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) For $f \in L^+(X)$ define the integral of f, and making use of this definition prove that $\int_X f = 0$ only if $\mu(\{f \neq 0\}) = 0$.
(ii) Prove that if $f, g \in L^1_\mu(X, \mathbb{R})$ then $f = g$ a.e. if and only if $\int_E f = \int_E g$ for every $E \in \mathcal{M}$ (hint: consider $E = \{f < g\} \dots$).
(iii) Prove that if $f, g \in L^1_\mu(X, \mathbb{C})$ then $f = g$ a.e. if and only if $\int_E f = \int_E g$ for every $E \in \mathcal{M}$.
(iv) Define Dynkin classes and state Dynkin's theorem.
(v) Let $\mathcal{E} \subseteq \mathcal{M}$ be closed under intersection, assume that $\mathcal{M}(\mathcal{E}) = \mathcal{M}$, and that X is covered by a countable subset of $\mathcal{E}$. Assume that $f, g \in L^1(\mu)$ are such that $\int_E f = \int_E g$ for every $E \in \mathcal{E}$. Prove that $f = g$ a.e.

Solution. (i) Definition: $\int_X f = \sup\{\int_X \varphi : 0 \le \varphi \le f, \varphi \text{ simple}\}$. If $\int_X f = 0$ then $\int_X \varphi = 0$ for every positive simple φ under f; in particular, if $E(n) = \{f > 1/n\}$ then $(1/n)\chi_{E(n)}$ is a simple function under f, so that $\int_X (1/n)\chi_{E(n)} = 0$, that is $(1/n)\mu(E(n)) = 0$, which implies $\mu(E(n)) = 0$; since

$$\{f \neq 0\} = \{f > 0\} = \bigcup_{n=1}^{\infty} E(n), \quad \text{we have } \mu(\{f > 0\}) = 0.$$

(ii) From $\int_E f = \int_E g$ we get $\int_E (g - f) = 0$; but we have $g(x) - f(x) > 0$ for every $x \in E$, so that $\int_E (g - f) = 0$ implies $\mu(E) = 0$. In the same way, if $F = \{f > g\}$ we get $\mu(F) = 0$, so that $f = g$ a.e. in X.

(iii) $\int_E f = \int_E g$ is equivalent to $\int_E \operatorname{Re} f = \int_E \operatorname{Re} g$ and $\int_E \operatorname{Im} f = \int_E \operatorname{Im} g$. By (ii) this happens for every measurable E iff $\operatorname{Re} f = \operatorname{Re} g$ and $\operatorname{Im} f = \operatorname{Im} g$ a.e., that is $f = g$ a.e.

(iv) See Lecture Notes, 3.4.1

(v) The proof mimics the proof of LN, 3.4.3. Given a set $E \in \mathcal{E}$ consider the set $\mathcal{E}_E = \{F \cap E : F \in \mathcal{E}\}(= \{G \subseteq E : G \in \mathcal{E}\})$, and the set $\mathcal{C}_E = \{A \in \mathcal{M} : A \subseteq E, \int_A f = \int_A g\}$. This set is a Dynkin class of parts of E, as is easy to check: closure under countable disjoint union is countable additivity of the integral: if $f \in L^1(\mu)$ and $(A(n))_{n \in \mathbb{N}}$ is a countable disjoint sequence of elements of $\mathcal{M}$, then $\sum_{n=0}^{\infty} f \chi_{E(n)}$ is a normally convergent series in $L^1(\mu)$, so $\int_{\bigcup_n A(n)} f \chi_{A(n)} = \sum_{n=0}^{\infty} \int_X f \chi_{A(n)} = \sum_{n=0}^{\infty} \int_{A(n)} f$; same for g. And since $\int_{E \smallsetminus A} f = \int_E f - \int_A f$ for every measurable subset A of E, and the same for g, we also have closure under complementation. Since this set $\mathcal{C}_E$ contains $\mathcal{E}_E$, it contains the Dynkin class generated by it, and since $\mathcal{E}_E$ is closed under intersection, by Dynkin' s theorem $\mathcal{C}_E$ contains the $\sigma-$algebra generated by $\mathcal{E}_E$, which is $\mathcal{M}_E = \{A \in \mathcal{M} : A \subseteq E\}$. Now X can be written as a countable union of members of $\mathcal{E}$, say $X = \bigcup_{k \in \mathbb{N}} E_k$; by the usual technique ($F_k = E_k \smallsetminus \left(\bigcup_{j=0}^{k-1} E_j\right)$) we can write X as a countable disjoint union of members F_k of $\mathcal{M}(\mathcal{E})$ with $F_k \subseteq E_k$; given $A \in \mathcal{M}(\mathcal{E})$ we have $A = \bigcup_{k=0}^{\infty} A \cap F_k$, a countable disjoint union, and $\int_{A \cap F_k} f = \int_{A \cap F_k} g$ for every k, since $A \cap F_k \in \mathcal{M}_{E_k}$. □

Remark. Of course, considering $h = f - g$, (i),(iii),(iv), (v) may be stated as $\int_E h = 0$ for every $E \in \dots$ implies $h = 0$ a.e.. The statement:

- *If $f, g \in L^+(X)$ are such that $\int_E f = \int_E g$ for every $E \in \mathcal{M}$, then $f = g$ a.e.*

is FALSE unless some additional hypothesis is made on μ: take an uncountable set X with the $\sigma-$algebra of countable or co–countable subsets, and the measure μ that is ∞ for co–countable, and 0 for countable sets: the constants 1 and 2 have integral 0 on countable and ∞ on co–countable sets, but are never equal. We can prove (but the proof is much more complicated than (ii) above, owing to possibly infinite integrals):

- *If μ is semifinite, and $f, g \in L^+(X)$ are such that $\int_E f = \int_E g$ for every $E \in \mathcal{M}$, then $f = g$ a.e.*

Proof. Let $A = \{f < g\}$; it is enough to prove that $\mu(A) = 0$ (an analogous proof will work for $B = \{g < f\}$). Given $n \in \mathbb{N}$, let $E(n) = \{g \le n\} \cap A$. Then $\mu(E(n)) = 0$; in fact, if not, by semifiniteness we get $E \subseteq E(n)$ with $0 < \mu(E) < \infty$. Then $\int_E f = \int_E g \le \int_E n = n\,\mu(E) < \infty$; it follows that $\int_E (g - f) = 0$, but $g(x) - f(x) > 0$ for every $x \in E$, impossible if $\mu(E) > 0$. Then $\mu(E(n)) = 0$ for every n, so that $\mu(\{g < \infty\} \cap A) = 0$ (since $\{g < \infty\} = \bigcup_{n=1}^{\infty} \{g \le n\}$). If $\mu(\{g = \infty\} \cap A) > 0$ we still get a contradiction: notice that since $f(x) < \infty$ for every $x \in A$ we still have $\{g = \infty\} \cap A = \bigcup_{n=1}^{\infty} \{g = \infty\} \cap A \cap \{f \le n\}$; unless these sets have all measure zero we can get $E \subseteq \{g = \infty\} \cap A \cap \{f \le n\}$ with $0 < \mu(E) < \infty$; then $\int_E f \le n\,\mu(E) < \infty$, but $\int_E g = \infty$. Then $\mu(A) = \mu(A \cap \{g < \infty\}) + \mu(A \cap \{g = \infty\}) = 0$. □

EXERCISE 2. (i) Let U be an open subset of $\mathbb{R}^n$. Prove that U is a countable union of compact intervals (or even compact cubes).

(ii) Prove that if X is an open subset of $\mathbb{R}^n$ then the $\sigma-$algebra of Borel subsets of X is generated by the compact intervals contained in X.

From now on U and V are open subintervals of $\mathbb{R}$ and $\phi : U \to V$ is a C^1 diffeomorphism (a C^1 bijective map with C^1 inverse).

(iii) We define on the $\sigma-$algebra $\mathcal{B}$ of Borel subsets of U the set functions:

$$\mu(E) = \lambda(\phi(E)); \qquad \nu(E) = \int_E |\phi'(x)|\, d\lambda(x),$$

where of course $\lambda = \lambda_1$ is the one dimensional Lebesgue measure. Prove that μ and ν are measures, and that $\mu = \nu$ on $\mathcal{B}$.

(iv) The measure μ can be considered as an image measure, in which way? Using this fact prove that, for every $f \in L^1_\lambda(V, \mathbb{K})$ we have the *change of variable formula*:

$$\int_V f(y)\, dy = \int_U f(\phi(x))\, |\phi'(x)|\, dx.$$

Solution. (i) Consider the set of all closed cubes $Q(c, r] = \{x \in \mathbb{R}^n : \|x - c\|_\infty \le r\}$ with centers $c \in \mathbb{Q}^n$ and half–sides $r \in \mathbb{Q}^>$ which are contained in U: this is a countable set of compact cubes, whose union is U. In fact, given $a \in U$, pick $c \in \mathbb{Q}^n$ such that $\|a - c\|_\infty < d = \operatorname{dist}(a, \mathbb{R}^n \setminus U)/3$, where $\operatorname{dist}(a, \mathbb{R}^n \setminus U) = \inf\{\|a - y\|_\infty : y \in \mathbb{R}^n \setminus U\}$. Picking a rational number r such that $d < r < 2d$ we get $a \in Q(c, r] \subseteq U$.

(ii) The compact intervals are Borel sets, so they generate a $\sigma-$algebra contained in the Borel sets of X. But as shown in (i), every open set is a countable union of compact intervals, so the generators of the Borel $\sigma-$algebra are all contained in the $\sigma-$algebra generated by compact intervals, and so these $\sigma-$algebras coincide.

(iii) Answering now to part of (iii) we can observe that $\mu = \lambda\phi^{-1\leftarrow}$ is the image measure of the Lebesgue measure on V, by means of the map $\phi^{-1} : V \to U$(see LN, 3.3.7.2). Anyway the direct verification that μ is a measure (thanks to the fact that ϕ is a homeomorphism) is trivial. We know that ν is a measure (the one with density $|\phi'|$ with respect to Lebesgue measure), owing to countable additivity of integrals of positive functions (LN, 3.3.5.2). Remember now that a diffeomorphism between intervals of $\mathbb{R}$ has necessarily a derivative always strictly positive or strictly negative: it cannot vanish, and intervals are connected. If $[a, b]$ is a compact subinterval of U we have $\phi([a, b]) = [\phi(a), \phi(b)]$ if ϕ is increasing ($\phi'(x) > 0$), and $\phi([a, b]) = [\phi(b), \phi(a)]$ if ϕ is decreasing ($\phi'(x) < 0$). And we have

$$\int_{[a,b]} |\phi'(x)|\, dx = \pm \int_a^b \phi'(x)\, dx = \pm(\phi(b) - \phi(a)),$$

where $+$ holds if ϕ is increasing, $-$ in the other case. Then μ and ν coincide and are finite on compact intervals, a class of sets closed under finite intersection which generates the $\sigma-$algebra $\mathcal{B}$, with U also a countable union of compact intervals; so the measures coincide on $\mathcal{B}$ (we are using the uniqueness result in LN, 3.4.3).

(iv) We have seen that $\mu = \lambda\phi^{-1\leftarrow}$; then for $g \in L^+(\lambda\phi^{-1\leftarrow})$ we have

$$\int_U g(x)\, d\lambda\phi^{-1\leftarrow}(x) = \int_V g \circ \phi^{-1}(y)\, d\lambda(y);$$

on the other hand, since $d\lambda\phi^{-1\leftarrow} = |\phi'|\, d\lambda$ we have also (see LN, 3.3.5.2):

$$\int_U g(x)\, d\lambda\phi^{-1\leftarrow}(x) = \int_U g(x)\, |\phi'(x)|\, d\lambda(x),$$

so that, equating the right–hand sides of the two preceding equations:

$$\int_V g\circ\phi^{-1}(y)\,d\lambda(y) = \int_U g(x)\,|\phi'(x)|\,d\lambda(x),$$

for every $g \in L^+(U)$; and setting $f(y) = g\circ\phi^{-1}(y)$ we get $g(x) = f\circ\phi(x)$, and we conclude. □

Exercise 3. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) State the dominated convergence theorem.
(ii) Prove that if $f, g \in L^+$, then $(g - f)^+ \le g$.
(iii) Let f_n in $L^+(X)$ converge a.e. to $f \in L^+(X)$, and assume that all integrals are finite and $\int_X f_n \to \int_X f < \infty$. Prove that then f_n converges to f in $L^1(\mu)$, i.e. $\|f - f_n\|_1 \to 0$ (by (ii) we have $(f - f_n)^+ \le \dots$, then apply (i) ...).
(iv) We now assume that f_n in $L^+(X)$ converge a.e. to $f \in L^+(X)$, that $f_n \le f$ for every n, and that all integrals are finite. Is it true that f_n converges to f in $L^1(\mu)$?

Solution. (i) See the Lecture Notes, 3.3.2. (ii) If $(g - f)^+(x) = 0$ the assertion is trivial, since $g(x) \ge 0$ for every $x \in X$. If $(g - f)^+(x) > 0$, then $(g - f)^+(x) = g(x) - f(x) > 0$; and since $f(x) \ge 0$ by the hypothesis $f \in L^+(X)$, we conclude that $(g - f)^+(x) = g(x) - f(x) \le g(x)$.

(iii) If all integrals are finite then all functions are in $L^1(\mu)$, being all positive. Then $(f - f_n)^+ \le f$ is a sequence which converges to 0 a.e and is dominated by $f \in L^1(\mu)$. By dominated convergence we have $\lim_n \int_X (f - f_n)^+ = 0$. But then, since $(f - f_n)^- = (f - f_n)^+ - (f - f_n)$ we get

$$\lim_n \int_X (f - f_n)^- = \lim_n \left(\int_X (f - f_n)^+ - \int_X (f - f_n)\right) = \lim_n \int_X (f - f_n)^+ - \int_X f + \lim_n \int_X f_n = 0;$$

hence also

$$\lim_n \int_X |f - f_n| = \lim_n \left(\int_X (f - f_n)^+ + \int_X (f - f_n)^-\right) = 0.$$

(iv) Since all integrals, including that of f, are finite, we have that $f \in L^1(\mu)$; since $0 \le f_n \le f$, dominated convergence is applicable (one–sided limits, 3.3.17.6), hence we have convergence in $L^1(\mu)$.

Remark. Of course (iii) can also be obtained from the generalized dominated convergence theorem; in fact, the proof suggested here follows essentially the same route as the proof of that result (LN, 3.3.17.7).

□

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Exercise 4. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) State Fatou's lemma.
(ii) Using Fatou's lemma prove the monotone convergence theorem for functions in $L^+(X)$.
(iii) Let $g : [0, \infty[\to [0, \infty[$ be continuous. Given $a > 0$ let

$$M_a = \{f \in L^1(\mu) : \|g(|f|)\|_1 \le a\}.$$

Prove that M_a is closed in $L^1(\mu)$ (if $f_n \in M_a$ converges to f in $L^1(\mu)$, then some subsequence converges to f also ...).

Solution. (i) LN, 3.3.6.

(ii) If $f_n \in L^+(X)$ and $f_n \uparrow f$, Fatou's lemma says that $\int_X f \le \liminf_n \int_X f_n$. But the sequence f_n is increasing, hence also the sequence $\int_X f_n$ is increasing, so that $\lim_n \int_X f_n = \sup_n \int_X f_n$ exists. Then the preceding assertion implies $\int_X f \le \lim_n \int_X f_n$; and since $\int_X f_n \le \int_X f$ for every n, we have also $\lim_n \int_X f_n \le \int_X f$, and hence equality, $\lim_n \int_X f_n = \int_X f$.

(iii) If $f_n \in M_a$ converges to f in $L^1(\mu)$, then some subsequence converges to f also a.e.; let's assume that the entire sequence converges a.e. to f. Then $|f_n|$ converges a.e. to $|f|$, and by continuity of g on $[0, \infty[$ we have that $g(|f_n(x)|)$ converges to $g(|f(x)|)$ if $|f_n(x)|$ converges to $|f(x)|$. Then Fatou's lemma says that

$$\int_X g\circ|f| \le \liminf_n \int_X g\circ|f_n| \le a,$$

so that $f \in M_a$, and M_a is closed in $L^1(\mu)$.

Remark. (ii) Many have proved monotone convergence by applying Fatou's lemma to the sequence $f - f_n$ to get the inequality $\limsup_n \int_X f_n \le \int_X f$; this not only makes the proof uselessly longer, it is strictly speaking an incomplete proof, because it excludes the case $\int_X f_n = \infty$.

□

Exercise 5. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Prove that if $f \in L^1(\mu)$ then for every $\alpha > 0$ we have $\|f\|_1 \ge \int_{\{|f|\ge\alpha\}} |f|$, and prove that $\lim_{\alpha\to\infty} \int_{\{|f|\ge\alpha\}} |f| = 0$ (i.e. prove that for every sequence $\alpha_n \to \infty$ we have $\lim_n \int_{\{|f|\ge\alpha_n\}} |f| = 0$).

(ii) If $f \in L^\infty(\mu)$, given $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(E) \le \delta$ implies $\left|\int_E f\right| \le \varepsilon$ (trivial, 1 point). Prove that the same is true if $f \in L^1(\mu)$: given $\varepsilon > 0$ and $\alpha > 0$ write

$$\left|\int_E f\right| \le \int_E |f| = \int_{E\cap\{|f|\ge\alpha\}} |f| + \int_{E\cap\{|f|<\alpha\}} |f|,$$

and estimate separately the two terms.

(iii) State and prove Čebičeff's inequality: $\mu(\{|f| \ge \alpha\}) \le \dots$, and use it to prove that if f_n is a sequence in $L^1(\mu)$ converging to f in $L^1(\mu)$ then, for every $\alpha > 0$:

$$\lim_{n\to\infty} \mu(\{|f - f_n| \ge \alpha\}) = 0.$$

Solution. (i) Clearly $|f| \ge |f|\,\chi_\alpha$, if χ_α is the characteristic function of the set $\{|f| \ge \alpha\}$. Then

$$\int_{\{|f|\ge\alpha\}} |f| = \int_X |f|\,\chi_\alpha = \int_{\{|f|\ge\alpha\}} |f| \le \int_X |f| = \|f\|_1.$$

If $\alpha(n)$ tends to ∞, then $|f|\,\chi_{\alpha(n)} \to 0$ everywhere, and $|f|\,\chi_{\alpha(n)} \le |f|$ for every n, so that dominated convergence implies $\lim_n \int_{\{|f|\ge\alpha(n)\}} |f| = 0$.

(ii) Clearly we have

$$\left|\int_E f\right| \le \int_E |f| \le \int_E \|f\|_\infty = \|f\|_\infty\,\mu(E) \quad \text{for every } E \in \mathcal{M} \text{ of finite measure,}$$

so that given ε we simply take $\delta = \varepsilon/\|f\|_\infty$. Following the hint, we write

$$\left|\int_E f\right| \le \int_{E\cap\{|f|\ge\alpha\}} |f| + \int_{E\cap\{|f|<\alpha\}} |f|;$$

given $\varepsilon > 0$ we first pick $\alpha > 0$ so that $\int_{\{|f|\ge\alpha\}} |f| \le \varepsilon/2$. Then we have also

$$\int_{E\cap\{|f|\ge\alpha\}} |f| \le \int_{\{|f|\ge\alpha\}} |f| \le \varepsilon/2 \quad \text{for every } E \in \mathcal{M}.$$

Keeping now α fixed we have, if $\mu(E) \le \delta$

$$\int_{E\cap\{|f|<\alpha\}} |f| \le \alpha\,\mu(E \cap \{|f| < \alpha\}) \le \alpha\mu(E) \le \alpha\delta,$$

so that we need only to pick $\delta = \varepsilon/(2\alpha)$ to conclude.

(iii) The inequality is $\mu(\{|f| \ge \alpha\} \le (1/\alpha)\|f\|_1$, and the proof is immediate, the first part already done in (i):

$$\|f\|_1 = \int_X |f| \ge \int_{\{|f|\ge\alpha\}} |f| \ge \int_{\{|f|\ge\alpha\}} \alpha = \alpha\,\mu(\{|f| \ge \alpha\}).$$

Then we have

$$\mu(\{|f - f_n| \ge \alpha\}) \le \frac{1}{\alpha}\,\|f - f_n\|_1 \to 0 \text{ for } n \to \infty.$$

Remark. (i) Many wanted to use monotone convergence, or the fact that $E \mapsto \int_E |f|$ is a measure; this is possible if $\alpha_n \uparrow \infty$. Now this can be assumed without loss of generality. In fact we have:

. *Let $\varphi : D \to \mathbb{R}$ be a function, and assume that $c \in \tilde{\mathbb{R}}$ is an accumulation point for $D\cap]-\infty, c[$. Then $\lim_{x\to c^-} \varphi(x)$ exists and is ℓ if and only if $\lim_{n\to\infty} \varphi(x_n) = \ell$ for every increasing sequence $x_n \in D$ with $x_n \uparrow c$.*

In fact we know that $\lim_{x\to c^-}\varphi(x)$ exists and is ℓ if and only if $\lim_{n\to\infty}\varphi(x_n)=\ell$ for every sequence $x_n\in D$ with $x_n\to c$; now every real valued sequence has a monotone subsequence, and if $x_n<c$ and $x_n\to c$ this subsequence must be increasing, since it has c as limit.

This fact, applied to $\varphi(\alpha)=\int_{\{|f|\ge\alpha\}}|f|$ shows that we can assume $\alpha_n\uparrow\infty$. But a proof ought to be given.

It is however impossible to prove (i) using Čebičeff's inequality, or vague arguments such as

$$\lim_n\int_{\{|f|\ge\alpha_n\}}|f|=\int_{\{|f|=\infty\}}|f|$$

stated without proof. In this respect also notice that, by definition, functions in $L^1(\mu)$ are finite valued, so that $\{|f|=\infty\}$ is empty for $f\in L^1(\mu)$, and not only of measure zero: this is a minor point, but is worth noticing.

□

EXERCISE 6. Let F, $G:\mathbb{R}\to\mathbb{R}$ be increasing and right continuous; recall that we have the formula of integration by parts, if $a,b\in\mathbb{R}$ and $a<b$ then

$$(*)\qquad \int_{]a,b]}F(x^-)\,dG(x)+\int_{]a,b]}G(x)\,dF(x)=F(b)G(b)-F(a)G(a).$$

In the sequel we assume also $F(-\infty)=G(-\infty)=0$.

(i) Prove that

$$\int_{\mathbb{R}}F(x^-)\,dG(x)+\int_{\mathbb{R}}G(x)\,dF(x)=F(\infty)G(\infty)$$

(infinite values are possible),

(a): directly using Tonelli's theorem

(b): using formula (*) and passing to the limit with $a\downarrow\dots$ and $b\uparrow\dots$.

(ii) Prove that if F and G do not have a common point of discontinuity then we may replace $F(x^-)$ with $F(x)$ in the preceding formula.

(iii) Assuming F bounded and continuous prove that

$$\int_{\mathbb{R}}F(x)\,dF(x)=\frac{1}{2}\,(F(\infty))^2.$$

Now we take $F(x)=\chi_{[0,\infty[}$ the Heaviside step, and

$$G(x)=\begin{cases}e^x & x<0\\ 3-e^{-x} & x\ge 0\end{cases},$$

(iv) Plot the graph of G and compute

$$\int_{\mathbb{R}}F(x^-)\,dG(x);\qquad \int_{\mathbb{R}}F(x)\,dG(x).$$

(v) Compute $(dG-dF)(]a,b])$ for every $a,b\in\mathbb{R}$ with $a<b$. Prove that there is a function $\rho\in L^+_\lambda(\mathbb{R})$ such that $(dG-dF)(E)=\int_E\rho\,d\lambda$ for every Borel $E\subseteq\mathbb{R}$, and find it.

Solution. (i) (a) We compute $dF\otimes dG(T)$, where $T=\{(x,y)\in\mathbb{R}^2:\ x\le y\}$. Since all measures are $\sigma-$finite, and T is a Borel subset of $\mathbb{R}^2$, hence measurable, Tonelli's theorem is applicable and gives ($T_x=\{y\in\mathbb{R}:\ (x,y)\in T\}=]-\infty,x]$)

$$dF\otimes dG(T)=\int_{\mathbb{R}}\left(\int_{T_x}dG\right)\,dF(x)=\int_{\mathbb{R}}\left(\int_{]-\infty,x]}dG\right)dF(x)=\int_{\mathbb{R}}G(x)\,dF(x);$$

reversing the order of integration ($T^y=[x,+\infty[$):

$$dF\otimes dG(T)=\int_{\mathbb{R}}\left(\int_{T^y}dF\right)dG(y)=\int_{\mathbb{R}}(F(\infty)-F(x^-))dG(x),$$

so that

$$\int_{\mathbb{R}}G(x)\,dF(x)=\int_{\mathbb{R}}(F(\infty)-F(x^-))dG(x).$$

Caution: we cannot say that $\int_{\mathbb{R}}(F(\infty) - F(x^-))dG(x) = F(\infty)\, G(\infty) - \int_{\mathbb{R}} F(x^-)\, dG(x)$ because of possible infinities. Adding to both sides $\int_{\mathbb{R}} F(x^-)\, dG(x)$, which certainly exists since $x \mapsto F(x^-)$ is positive measurable, we get that

$$\int_{\mathbb{R}} F(x^-)\, dG(x) + \int_{\mathbb{R}} G(x)\, dF(x) = \int_{\mathbb{R}} F(x^-)\, dG(x) + \int_{\mathbb{R}} (F(\infty) - F(x^-))dG(x) = \int_{\mathbb{R}} F(\infty)\, dG(x) = F(\infty)G(\infty).$$

(b) Let $a_n \downarrow -\infty$ and $b_n \uparrow \infty$. Then $f_n(x) = F(x^-)\, \chi_{]a_n a, b_n]}$ and $g_n = G(x)\, \chi_{]a_n a, b_n]}$ are increasing sequences of positive functions such that $f_n(x) \uparrow F(x^-)$ and $g_n(x) \uparrow G(x)$, for every $x \in \mathbb{R}$. Then monotone convergence implies that

$$\int_{\mathbb{R}} f_n\, dG + \int_{\mathbb{R}} g_n\, dF \uparrow \int_{\mathbb{R}} F(x^-)\, dG(x) + \int_{\mathbb{R}} G(x)\, dF(x),$$

and since

$$\int_{\mathbb{R}} f_n\, dG + \int_{\mathbb{R}} g_n\, dF = \int_{]a_n, b_n]} F(x^-)\, dG(x) + \int_{]a_n, b_n]} G(x)\, dF(x) = F(b_n)\, G(b_n) \uparrow F(\infty)\, G(\infty)$$

we conclude.

(ii) Clear: discontinuities of F are a countable set of dG measure 0, so that $F(x)$ and $F(x^-)$ are $dG-$almost equal.

(iii) Is a trivial application of the second formula, given continuity of F.

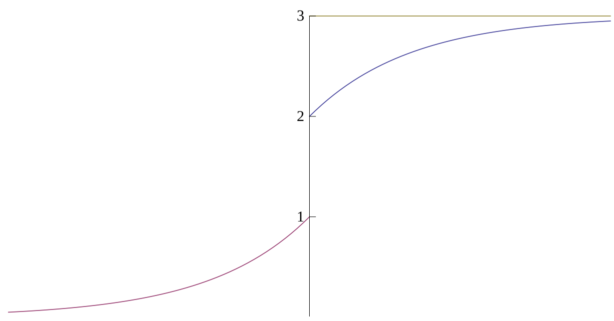

Figure 1. Plot of G

(iv) Notice that $x \mapsto F(x^-) = \chi_{]0,\infty[}$. Then

$$\int_{\mathbb{R}} F(x^-)\, dG(x) = \int_{]0,\infty[} dG(x) = dG(]0,\infty[) = G(\infty) - G(0) = 3 - 2 = 1.$$

And

$$\int_{\mathbb{R}} F(x)\, dG(x) = \int_{[0,\infty[} dG = G(\infty) - G(0^-) = 3 - 1 = 2.$$

(v) We have

$$(dG - dF)(]a, b]) = dG(]a, b]) - dF(]a, b]) = G(b) - G(a) - (F(b) - F(a) = (G - F)(b) - (G - F)(a)$$

for every pair $a, b \in \mathbb{R}$ with $a < b$. Observe that $H = G - F$ is still an increasing function: we have

$$H(x) = \begin{cases} e^x & \text{for} \quad x < 0 \\ 2 - e^{-x} & \text{for} \quad x \geq 0 \end{cases};$$

then $dG - dF = dH$ is the Radon–Stieltjes measure associated to H. And we have

$$(dG - dF)(]a, b]) = H(b) - H(a) = \begin{cases} e^{-a} - e^{-b} & \text{for} \quad 0 \leq a < b \\ 2 - e^{-b} - e^{-a} & \text{for} \quad a < 0 < b \\ e^b - e^a & \text{for} \quad 0 < b \leq 0 \end{cases}.$$

It is clear that H is a C^1 function: it is continuous, and its derivative is e^x for $x < 0$, while $H'(x) = e^{-x}$ for $x > 0$, so that $H'(0) = 1$ also exists; we have that $H'(x) = e^{-|x|}$ for every $x \in \mathbb{R}$, so the density function is $\rho(x) = e^{-|x|}$.

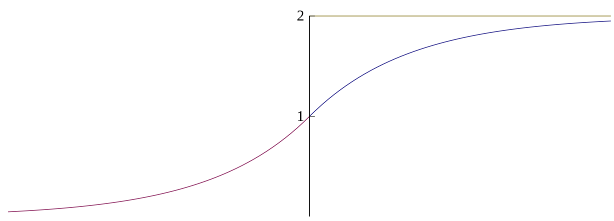

Figure 2. Plot of $H = G - F$, a function of class C^1.

Remark. (i), part (a): A more elegant solution has been found by a student: write the first term of the formula to be proved as $\int_{\mathbb{R}} F(y^-)\,dG(y)$; then, since $F(y^-) = dF(]-\infty, y[) = \int_{]-\infty,y[} dF(x)$ we have

$$\int_{\mathbb{R}} F(y^-)\,dG(y) = \int_{\mathbb{R}} \left(\int_{]-\infty,y[} dF(x) \right) dG(y) = \int_S dF \otimes dG(x,y) \quad \text{where } S = \{(x,y) \in \mathbb{R}^2 : x < y\}.$$

For the second term we get

$$\int_{\mathbb{R}} G(x)\,dF(x) = \int_{\mathbb{R}} \left(\int_{]-\infty,x]} dG(y) \right) dF(x) = \int_T dF \otimes dG(x,y) \quad \text{where } T = \{(x,y) \in \mathbb{R}^2 : y \le x\},$$

so that, observing that $\mathbb{R}^2 = S \cup T$, disjoint union of the two half–planes S, T

$$\int_{\mathbb{R}} F(x^-)\,dG(x) + \int_{\mathbb{R}} G(x)\,dF(x) = \int_S dF \otimes dG(x,y) + \int_T dF \otimes dG(x,y) = \int_{\mathbb{R}^2} dF \otimes dG(x,y) =$$
$$(F(\infty) - F(-\infty))\,(G(\infty) - G(-\infty)) = F(\infty)\,G(\infty).$$

□

1. Analisi Reale per Matematica–Secondo Precompitino 18 gennaio 2012

Exercise 7. (i) Define a signed measure $\nu : \mathcal{M} \to \tilde{\mathbb{R}}$. If $\nu(A)$ is not finite, and $B \supseteq A$, is $\nu(B)$ also not finite? and if $B \subseteq A$ is $\nu(B)$ also not finite? or what else can be said ?(of course $A, B \in \mathcal{M}$)

(ii) Prove that a signed measure can assume only one of the values $\pm\infty$.

(iii) Prove that if $A_0 \subseteq A_1 \subseteq \dots$ is an increasing sequence in $\mathcal{M}$, and $A = \bigcup_{n=0}^{\infty} A_n$, then

$$\lim_{n\to\infty} \nu(A_n) = \nu(A).$$

Is there an analogous proposition for decreasing sequences? if so, state and prove it.

(iv) Assume that $\nu(X) \in \mathbb{R}$. Is it true that $\nu(\mathcal{M})$ has a maximum? and a minimum?

Solution. (i) Let $(X, \mathcal{M})$ be a measurable space. A signed measure is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that $\nu(\emptyset) = 0$, and which is countably additive, that is, for every disjoint sequence $(A_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$ we have

$$\nu\left(\bigcup_{n=0}^{\infty} A_n \right) = \sum_{n=0}^{\infty} \nu(A_n).$$

If $B \supseteq A$ we have $\nu(B) = \nu(A) + \nu(B \setminus A)$; if $\nu(A) = \pm\infty$, any meaningful addition $\nu(A) + c$, with $c \in \tilde{\mathbb{R}}$ has $\nu(A)$ as the resulting sum, so $\nu(B) = \nu(A)$. Similarly, if $B \subseteq A$ we have $\nu(A) = \nu(B) + \nu(A \setminus B)$; $\nu(B)$ may be finite, but then we have $\nu(A \setminus B) = \nu(A) = \pm\infty$.

(ii) Since $X \supseteq A$ and $X \in \mathcal{M}$, as seen above we have $\nu(X) = \nu(A)$ when $\nu(A) = \pm\infty$.

(iii) We can write $A = \bigcup_{n=1}^{\infty}(A_n - A_{n-1})$, disjoint union, so that by countable additivity we get, setting $A_{-1} = \emptyset$:

$$\nu(A) = \sum_{n=0}^{\infty} \nu(A_n \setminus A_{n-1}) := \lim_{m\to\infty} \sum_{n=0}^{m} \nu(A_n \setminus A_{n-1}) =$$

(by finite additivity, since $\bigcup_{n=0}^{m}(A_n \setminus A_{n-1}) = A_m$)

$$\lim_{m\to\infty} \nu(A_m).$$

The statement for decreasing sequences requires the additional hypothesis that $\nu(A_m)$ be finite for some m (hence, by (i), also for all $n > m$):

. *Let $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ be a decreasing sequence in $\mathcal{M}$, with intersection A. If for some $m \in \mathbb{N}$ the measure $\nu(A_m)$ is finite, we have $\nu(A) = \lim_{n\to\infty} \nu(A_n)$.*

Proof. It is not restrictive to assume $\nu(A_0)$ finite, re–indexing if necessary. Then, by (i), every A_n and A have finite $\nu-$measure; the sequence $A_0 \setminus A_n$ is increasing and has $B = A_0 \setminus A$ as its union, so that, by the result on increasing sequences we get $\nu(A_0 \setminus A) = \lim_{n\to\infty} \nu(A_0 \setminus A_n)$. Since every set involved has finite $\nu-$measure we get $\nu(A_0 \setminus A) = \nu(A_0) - \nu(A)$ and $\nu(A_0 \setminus A_n) = \nu(A_0) - \nu(A_n)$; then we have:

$$\nu(A_0) - \nu(A) = \lim_{n\to\infty} (\nu(A_0) - \nu(A_n)) = \nu(A_0) - \lim_{n\to\infty} \nu(A_n) \Longrightarrow \nu(A) = \lim_{n\to\infty} \nu(A_n).$$

□

(iv) If $\nu(X)$ is finite, by (i) every $A \in \mathcal{M}$ has finite $\nu-$measure. If we consider a Hahn decomposition for ν, let's say $X = P \cup Q$, with P positive and Q negative, $\nu(P)$ and $\nu(Q)$ are both finite and they are respectively $\max \nu(\mathcal{M})$ and $\min \nu(\mathcal{M})$: $\nu(A) = \nu(A \cap P) + \nu(A \cap Q) \le \nu(A \cap P) \le \nu(P)$ (because $\nu(A \cap Q) \le 0$ and $\nu(P \setminus A) \ge 0$); and also $\nu(A) = \nu(A \cap P) + \nu(A \cap Q) \ge \nu(A \cap Q) \ge \nu(Q)$ (because $\nu(A \cap P) \ge 0$, and $\nu(Q \setminus A) \le 0$).

□

Exercise 8. (12) Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) [1] Assume that $g : X \to \mathbb{C}$ is measurable and such that $\|g\|_q < \infty$ for some $q > 0$. Then $\lim_{p\to\infty} \|g\|_p = \dots$ (no proof required, simply state the result).

(ii) [7] Let $f \in L^+(X)$ be such that $\int_X f^n$ is finite for $n \in \mathbb{N}$ large, and

$$\lim_{n\to\infty} \int_X f^n = a \in \mathbb{R}.$$

Prove that then $f \in L^\infty(\mu)$, find the possible values of $\|f\|_\infty$, and prove that $f^n(x)$ converges a.e. in X to a function g to be described. Is this convergence also in $L^1(\mu)$?

(iii) [1] In $\mathbb{R}$ with Lebesgue measure give an example of an f for which the preceding limit is a given $a > 0$.

(iv) [3] In (ii) we remove the assumption that $f \ge 0$, we assume f real–valued but of arbitrary sign, leaving the other hypotheses intact. What can you say about f and the sequence f^n?

Solution. (i) $\lim_{p\to\infty} \|g\|_p = \|g\|_\infty$.

(ii) If $c_n = \|f\|_n$, we gave that $c_n < \infty$ for large n, so that $c_n \to \|f\|_\infty$. But by hypothesis c_n^n has a finite limit $a \in \mathbb{R}$. This implies that either $\|f\|_\infty = 0$ or $\|f\|_\infty = 1$. In fact, if $\|f\|_\infty > 1$, and $1 < \alpha < \|f\|_\infty$, then $\alpha < c_n$ for n large, and then $\alpha^n < c_n^n$ for n large, implying that $c_n^n \to \infty$, against the hypothesis. Then $\|f\|_\infty \le 1$. Then we have $0 \le f(x) \le 1$ for a.e. $x \in X$, implying that for a.e. $x \in X$ we have either $f^n(x) \to 0$ (if $f(x) < 1$) or $f^n(x) = f(x) = 1$ for all n. In other words

$$f^n(x) \text{ converges pointwise a.e. in } X \text{ to } \chi_C, \text{ where } C = \{f = 1\}.$$

Morever the sequence is decreasing, $f^0 \ge f^1 \ge f^2 \ge f^3 \ge \dots$; if $m \in \mathbb{N}$ is such that $f^m \in L^1(\mu)$ then dominated convergence (or decreasing monotone convergence) says that f^n converges to its pointwise limit χ_C also in $L^1(\mu)$. In particular we have

$$a = \lim_n \int_X f^n = \int_X \chi_C = \mu(C) = \mu(f^{\leftarrow}\{1\});$$

Notice that if $\|f\|_\infty < 1$ then $\mu(C) = 0$ and hence $a = 0$.

(iii) Simply take for f the characteristic function of any set of measure a, e.g, $\chi_{[0,a]}$. The sequence f^n is constantly f, then also $\int_X f^n = a$ is constant.

(iv) We have that $f^{2n} = (f^2)^n$ verifies the hypotheses of (i), then $\|f^2\|_\infty \le 1$, hence also $\|f\|_\infty \le 1$, and f^{2n} converges decreasing and in $L^1(\mu)$ to the characteristic function of $\{f^2 = 1\} = \{f = 1\} \cup \{f = -1\}$. If this set has measure 0 then $\|f\|_\infty < 1$, and the entire sequence f^n converges to 0, pointwise and in $L^1(\mu)$. Otherwise this set has a positive measure $a = \lim_{k\to\infty} \int_X f^{2k}$. We claim that the limit $\lim_n \int_X f^n$ exists and is a iff $\mu(\{f = -1\}) = 0$. In fact, if $f = f^+ - f^-$ we have, for $k \ge 1$:

$$f^{2k} = (f^+)^{2k} + (f^-)^{2k}; \quad f^{2k-1} = (f^+)^{2k-1} - (f^-)^{2k-1};$$

now the sequences $(f^+)^n$ and $(f^-)^n$ are exactly in the situation of f in the hypotheses in (i): that is, they are in $L^1(\mu)$ for n large enough and converge decreasing to $\chi_{\{f=1\}}$ and $\chi_{\{f=-1\}}$ respectively; then

$$\lim_k \int_X f^{2k-1} = \lim_k \left(\int_X (f^+)^{2k-1} - \int_X (f^-)^{2k-1} \right) = \lim_k \int_X (f^+)^{2k-1} - \lim_k \int_X (f^-)^{2k-1} = \mu(\{f=1\}) - \mu(\{f=-1\});$$

and analogously

$$\lim_k \int_X f^{2k} = \lim_k \int_X (f^+)^{2k} + \lim_k \int_X (f^-)^{2k} = \mu(\{f=1\}) + \mu(\{f=-1\}),$$

and the two limits coincide if and only if $\mu(\{f=-1\}) = 0$.

Summing up: the limit $\lim_n \int_X f^n$ exists finite for f real measurable of arbitrary sign if and only if $|f(x)| \le 1$ for a.e $x \in X$, $f^n \in L^1(\mu)$ for n large, and moreover $\mu(\{f=-1\}) = 0$; the limit a is $\mu(\{f=1\})$, the limit function is a.e. $\chi_{\{f=1\}}$, and convergence to this function is also in $L^1(\mu)$. □

Exercise 9. Let $F : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$F(x) = \begin{cases} -e^x & \text{if } x < 0 \\ \sqrt{1-x^2} & \text{if } 0 \le x < 1 \\ 1 - e^{-x} & \text{if } 1 \le x \end{cases}$$

(i) Find $T(x) = VF(]-\infty, x])$ and plot it.
(ii) Plot $T^{\pm}(x) = (T(x) \pm F(x))/2$.
(iii) Find a Hahn decomposition for the measure $\mu = dF$.
(iv) Find the absolutely continuous and the singular parts of $\mu = dF$.
(iv) Let $G(x) = x$ be the identity of $\mathbb{R}$. For every integer $k > 0$ compute the integral

$$\int_{]-k,k]} G(x)\, dF(x),$$

both directly and with the partial integration formula:

$$\int_{]a,b]} G(x^-)\, dF(x) = G(b)F(b) - G(a)F(a) - \int_{]a,b]} F(x)\, dG(x).$$

(v) Find

$$\int_{\mathbb{R}} G(x)\, dF(x).$$

Solution. We plot also a graph of F:

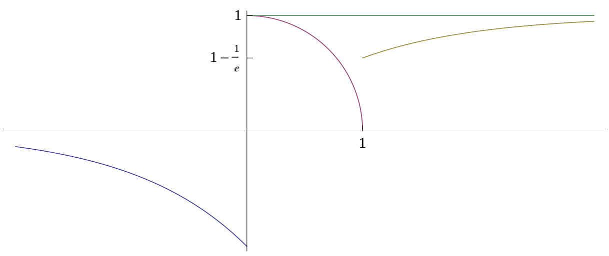

Figure 3. Graph of F.

(i) Since F is decreasing in $]-\infty, 0[$ we have $T(x) = e^x$ in this interval. The jump of F at 0 is 2, so $T(0) = T(0^-) + 2 = 3$. Again F is decreasing in $[0, 1[$ so that $VF([0, x] = F(0) - F(x) = 1 - \sqrt{1-x^2}$ in this interval, hence $T(x) = T(0) + 1 - \sqrt{1-x^2} = 4 - \sqrt{1-x^2}$ for $x \in [0, 1[$. Next we get $T(1) = T(1^-) + 1 - 1/e = 5 - 1/e$ (the jump at 1 is $1 - 1/e$). Finally $VF([1, x]) = 1 - e^{-x} - (1 - 1/e) = 1/e - e^{-x}$, so that $T(x) = 5 - e^{-x}$ for $x \ge 1$.

(ii) We get

$$T^+(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2 & \text{if } 0 \le x < 1 \\ 3 - e^{-x} & \text{if } 1 \le x \end{cases}; \qquad T^-(x) = \begin{cases} e^x & \text{if } x < 0 \\ 2 - \sqrt{1-x^2} & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x \end{cases}.$$

(iii) A positive set for μ is $P = \{0\} \cup \{1\} \cup]0, \infty[$, its complement is a negative set.

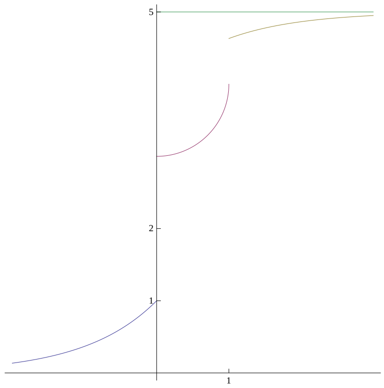

FIGURE 4. Graph of T.

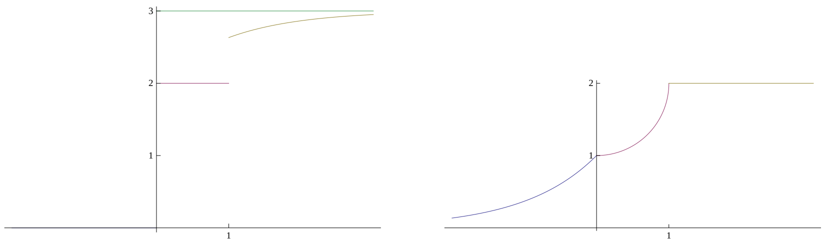

FIGURE 5. From left to right: graphs of T^+, T^-.

(iv) The absolutely continuous part is of course $F'(x)\,dx$, where F' is the derivative of F, which clearly exists in $\mathbb{R} \setminus \{0, 1\}$ and is

$$F'(x) = \begin{cases} -e^x & \text{if} \quad x < 0 \\ -x/\sqrt{1-x^2} & \text{if} \quad 0 \le x < 1 \\ e^{-x} & \text{if} \quad 1 \le x \end{cases} .$$

The singular part is $2\,\delta_0 + (1 - 1/e)\,\delta_1$.

(v) We have directly, using the Radon–Nikodym decomposition

$$\int_{]-k,k]} G(x)\,dF(x) = \int_{-k}^{k} G(x)\,F'(x)\,dx + \int_{]-k,k]} G(x)\,d(2\,\delta_0 + (1 - 1/e)\,\delta_1) =$$
$$\int_{-k}^{0} x(-e^x)\,dx + \int_0^1 x\,\frac{-x}{\sqrt{1-x^2}}\,dx + \int_1^k x\,e^{-x}\,dx + 1 - e^{-1} =$$

(in the first integral we put $t = -x$, in the last $t = x$)

$$\int_0^k t\,e^{-t}\,dt + \int_0^1 \frac{1 - x^2 - 1}{\sqrt{1-x^2}}\,dx + \int_1^k te^{-t}\,dt + 1 - 1/e =$$
$$1 - 1/e + \int_0^1 t\,e^{-t}\,dt + 2\int_1^k t\,e^{-t}\,dt + \int_0^1 \sqrt{1-x^2}\,dx - \int_0^1 \frac{dx}{\sqrt{1-x^2}} =$$

(una primitiva di $t\,e^{-t}$ è $-(1+t)\,e^{-t}$)

$$1 - 1/e + \left[-(1+t)\,e^{-t}\right]_0^1 + 2\left[-(1+t)\,e^{-t}\right]_1^k + \frac{\pi}{4} - \frac{\pi}{2} =$$
$$1 - e^{-1} - 2(1+k)e^{-k} + 2\,e^{-1} + 1 + \frac{\pi}{4} = 2 + \frac{1}{e} - 2(1+k)e^{-k} - \frac{\pi}{4}.$$

With the partial integration formula we get, calling for simplicity $I(k)$ the required integral

$$I(k) = k\,F(k) - (-k)\,F(-k) - \int_{-k}^{k} F(x)\,dx = k\left(1 - e^{-k} - e^{-k}\right) -$$

$$-\int_{-k}^{0}(-e^x)\,dx-\int_0^1\sqrt{1-x^2}\,dx-\int_1^k(1-e^{-x})\,dx=$$
$$k\left(1-2e^{-k}\right)+[e^x]_{-k}^0-\frac{\pi}{4}-\left[x+e^{-x}\right]_1^k=k\left(1-2e^{-k}\right)+1-e^{-k}-\frac{\pi}{4}-k-e^{-k}+1+\frac{1}{e}=$$
$$2+\frac{1}{e}-2(1+k)e^{-k}-\frac{\pi}{4}.$$

(vi) We have that $G_k = G\,\chi_{]-k,k]}$ converges to G on $\mathbb{R}$. And $G\in L^1(\mu)$, because $x\,e^{-|x|}\in L^1(m)$ (more on this below). Then by dominated convergence we can we just take the limit:

$$\int_{\mathbb{R}} G(x)\,dF(x)=\lim_{k\to\infty}\int_{]-k,k]}G(x)\,dF(x)=2+\frac{1}{e}-\frac{\pi}{4}.$$

The function G is a continuous function, hence Borel measurable and bounded on compact subsets of $\mathbb{R}$; the measure μ is finite on compacta, hence for every compact subset K of $\mathbb{R}$ we have that $G\in L^1_\mu(K)$. We need to prove that $G\in L^1_\mu(\mathbb{R}\setminus[-a,a])$, where $a>0$, say $a=2$. On the open set $]-\infty,-a[\cup]a,\infty[$ the measure $|\mu|=dT$ is absolutely continuous, with $d|\mu|(x)=e^{-|x|}\,dx$, as is easy to see. Then $G\in L^1(|\mu|)$ if and only if $|x|\,e^{-|x|}\in L^1_m(]-\infty,-a[\cup]a,\infty[)$, where m is Lebesgue measure. And this is immediate. □

Analisi Reale per Matematica– Secondo compitino–28 gennaio 2012

Exercise 10. Let $(X,\mathcal{M})$ be a measurable space, and let $\nu:\mathcal{M}\to\tilde{\mathbb{R}}$ be a signed measure; as usual $\nu^\pm$ and $|\nu|$ are the positive/negative parts and the total variation of ν.

(i) Define the notion of positive/negative set for ν, and prove that positive sets form a $\sigma-$ideal of $\mathcal{M}$ (closed under countable union and formation of subsets).
(ii) Assume that $A\in\mathcal{M}$ contains no negative subset of strictly negative measure. Is it true that then A is a positive subset?
(iii) For $A\in\mathcal{M}$ we have $\nu(A)\in\mathbb{R}\iff|\nu|(A)<\infty$. True or false? Is the fact that X is covered by a sequence of sets in $\mathcal{M}$ of finite $\nu-$measure equivalent to $\sigma-$finiteness of $|\nu|$?
(iv) Assume that $\mu:\mathcal{M}\to\tilde{\mathbb{R}}$ is another signed measure. Define mutual singularity of μ and ν. Is it equivalent to mutual singularity of $|\mu|$ and $|\nu|$?
(v) Let $\lambda:\mathcal{M}\to\tilde{\mathbb{R}}$ be a third signed measure; assume that $\lambda\ll|\mu|$ and $\lambda\ll|\nu|$, and that $\mu\perp\nu$. Is it true that $\lambda=0$?

Solution. (i) $A\in\mathcal{M}$ is said to be positive/negative for ν if for every $B\in\mathcal{M}$ contained in A we have $\nu(B)\ge 0/\nu(B)\le 0$. Given this definition, trivially the set $\mathcal{P}$ of positive sets is closed under the formation of measurable subsets. And if $(A_n)_{n\in\mathbb{N}}$ is a sequence of positive sets, making the union A of these sets a disjoint union of sets $(B_n)_{n\in\mathbb{N}}$ with the usual trick, $B_n=A_n\setminus\bigcup_{k=0}^{n-1}A_k$, each B_n is positive, being a subset of the positive set A_n, and if $B\subseteq A$ then $B=\bigcup_{n=0}^{\infty}(B\cap B_n)$, a disjoint union, so that

$$\nu(B)=\sum_{n=0}^{\infty}\nu(B\cap B_n)\ge 0\quad\text{because }\nu(B\cap B_n)\ge 0\text{ for every }n\in\mathbb{N}.$$

(ii) Let $P\cup Q$ be a Hahn decomposition for ν; consider $A\cap Q$; then we have $\nu(A\cap Q)(=-\nu^-(A\cap Q))=0$, since otherwise $A\cap Q$ would be a negative set of strictly negative measure contained in A. Then $A=(A\cap P)\cup(A\cap Q)$, the union of the positive set $A\cap P$ and the null set $A\cap Q$, is a positive set.

Remark. We have proved a lemma, preparatory to the Hahn decomposition theorem, which says that if $\infty\notin\nu(\mathcal{M})$ then a set which does not contain positive sets of strictly positive measure is a negative set. One can apply this result in the opposite direction, but we need to know that $-\infty$ is not a value assumed by ν. It is simpler to apply the Hahn decomposition: strictly speaking there is a circularity of arguments in the case $-\infty\notin\nu(\mathcal{M})$, which needs however not concern us.

(iii) We know that if $\nu(A)\in\mathbb{R}$ then every measurable subset of A has finite measure, in particular $\nu(A\cap P)$ and $\nu(A\cap Q)$ are finite, so that $\nu^\pm(A)$ are both finite, hence $|\nu|(A)=\nu^+(A)+\nu^-(A)<\infty$; since $|\nu(A)|\le|\nu|(A)$ the converse is trivial. This of course immediately implies that the answer to the second question is yes.

(iv) We say that μ and ν are mutually singular if there is a partition $X=M\cup N$, $M,\,N\in\mathcal{M}$, with N null for μ and M null for ν. Since a set null for a signed measure is clearly null also for its total variation as we show immediately after, the two conditions are clearly equivalent.

If M is null for ν then it is also null for $|\nu|$; in fact (always assuming that $P \cup Q$ is a Hahn decomposition for ν) we have $\nu^+(M) = \nu(M \cap P) = 0$ and $-\nu^-(M) = -\nu(M \cap Q) = 0$, so that $\nu^{\pm}(M) = 0$, hence also $|\nu|(M) = 0$.

(v) Clearly true: $|\mu|(N) = 0$ implies that N is null for λ, and $|\nu|(M) = 0$ implies that M is null for λ. Then $X = M \cup N$ is null for λ. □

Exercise 11. Let $f,\, g : \mathbb{R}^n \to \mathbb{K}$ be Borel measurable functions, with $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$ (we consider Lebesgue measure on all spaces $\mathbb{R}^n$).

(i) Prove that the formula

$$(*) \qquad f * g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

defines a function $f * g : \mathbb{R}^n \to \mathbb{K}$, and prove that $\|f * g\|_\infty \le \|f\|_1 \|g\|_\infty$. Prove also that

$$f * g(x) = g * f(x) = \int_{\mathbb{R}^n} f(t)\, g(x-t)\, dt.$$

(ii) Assume that $g \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and that also all derivatives $\partial_k g$ belong to $L^\infty(\mathbb{R}^n)$, for $k = 1, \dots, n$. Prove that then $f * g \in C^1(\mathbb{R}^n)$ and that $\partial_k(f * g) = f * (\partial_k g)$.

We now assume $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, with $p, q > 1$ conjugate exponents, i.e. $1/p + 1/q = 1$.

(iii) Prove that formula (*) defines a function $f * g : \mathbb{R}^n \to \mathbb{C}$, and prove that $\|f * g\|_\infty \le \|f\|_p \|g\|_q$. (use Hölder's inequality ...).

Finally assume that $f,\, g \in L^1(\mathbb{R}^n)$.

(iv) Prove that the formula (*) now defines a.e. on $\mathbb{R}^n$ a function $f*g$ that is Borel measurable, belongs to $L^1(\mathbb{R}^n)$, and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$ (consider $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{K}$ given by $F(x, y) = f(x-y)\, g(y)$ and apply Fubini–Tonelli's theorem ...).

Solution. (i) Trivially we have

$$|f * g(x)| = \left| \int_{\mathbb{R}^n} (x-y)\, g(y)\, dy \right| \le \int_{\mathbb{R}^n} |f(x-y)\, g(y)|\, dy \le \int_{\mathbb{R}^n} |f(x-y)|\, \|g\|_\infty\, dy;$$

Now the change of variables $t = x - y$ says that

$$\int_{\mathbb{R}^n} |f(x-y)|\, dy = \int_{\mathbb{R}^n} |f(t)|\, dt = \|f\|_1$$

(remember that we are in $\mathbb{R}^n$, so the coordinate change is $t_k = x_k - y_k \iff y_k = x_k - t_k$ $1 \le k \le n$, an affine self diffeomorphism of $\mathbb{R}^n$, with jacobian matrix -1_n, opposite of the identity matrix, hence determinant $(-1)^n$, with absolute value 1). Then

$$|f * g(x)| \le \int_{\mathbb{R}^n} |f(x-y)|\, \|g\|_\infty\, dy = \|f\|_1 \|g\|_\infty \Longrightarrow \|f * g\|_\infty \le \|f\|_1 \|g\|_\infty.$$

The change of variables $t = x - y$ above considered says also that

$$\int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy = \int_{\mathbb{R}^n} f(t)\, g(x-t)\, dt.$$

(ii) We use the second expression for $f * g$:

$$f * g(x) = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy;$$

we have

$$\frac{\partial}{\partial x_k}(f(y)\, g(x-y)) = f(y)\, \partial_k g(x-y);$$

moreover $|f(y)\, \partial_k g(x-y)| \le \|\partial_k g\|_\infty |f(y)|$; since $y \mapsto \|\partial_k g\|_\infty |f(y)|$ is in $L^1(\mathbb{R}^n)$ the theorem of differentiation under the integral sign applies to say that

$$\partial_k(f * g) = \int_{\mathbb{R}^n} f(y)\, \partial_k g(x-y)\, dy = (f * (\partial_k g))(x);$$

and the theorem on continuity of parameter depending integrals says that these derivatives are continuous.

(iii) We have. for every $x \in \mathbb{R}^n$:

$$|f * g(x)| = \left| \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \right| \le \int_{\mathbb{R}^n} |f(x-y)\, g(y)|\, dy \le$$

$$\le \left(\int_{\mathbb{R}^n} |f(x-y)|^p\,dy\right)^{1/p} \left(\int_{\mathbb{R}^n} |g(y)|^q\,dy\right)^{1/q} = \|f\|_p\,\|g\|_q,$$

(the usual change of variables $t = x - y$ says that $\int_{\mathbb{R}^n} |f(x-y)|^p\,dy = \|f\|_p^p$) which immediately implies the thesis.

(iv) Let us prove that F belongs to $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Clearly F is Borel measurable, since so are f and g. And the iterated integral:

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\,dx\right) = \int_{\mathbb{R}^n} |g(y)|\left(\int_{\mathbb{R}^n} |f(x-y)|\,dx\right)dy = \int_{\mathbb{R}^n} |g(y)|\,\|f\|_1\,dy = \|f\|_1\,\|g\|_1,$$

is finite. By Tonelli's theorem $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$. Then Fubini's theorem says that for a.e $x \in \mathbb{R}^n$ the integral

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy$$

is finite, the resulting a.e. defined function is Borel measurable, and moreover, since

$$|f * g(x)| = \left|\int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy\right| \le \int_{\mathbb{R}^n} |f(x-y)\,g(y)|\,dy$$

we have

$$\|f * g\|_1 = \int_{\mathbb{R}^n} |f * g(x)|\,dx \le \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(x-y)\,g(y)|\,dy\right)dx;$$

and since $|F| : (x,y) \mapsto |f(x-y)\,g(y)|$ belongs to $L^1(\mathbb{R}^n \times \mathbb{R}^n)$ this iterated integral is the double integral over $\mathbb{R}^n \times \mathbb{R}^n$ of $|F|$, just computed above, with value $\|f\|_1\,\|g\|_1$. □

Exercise 12. Let $F : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$F(x) = \begin{cases} e^{x+1} & \text{if } x < -1 \\ -x & \text{if } -1 \le x < 1 \\ e^{-(x-1)} & \text{if } 1 \le x \end{cases}$$

(i) Plot the graph of F; find $T(x) = VF(]-\infty, x])$ and plot it.
(ii) Plot $T^{\pm}(x) = (T(x) \pm F(x))/2$. What are $\mu^+(\mathbb{R})$ and $\mu^-(\mathbb{R})$?
(iii) Find a Hahn decomposition for the measure $\mu = dF$.
(iv) Find the absolutely continuous and the singular parts of $\mu = dF$.
(iv) Let $G(x) = \cos(\alpha x)$, where $\alpha > 0$ is a constant. For every $a > 1$ compute the integral

$$\int_{]-a,a]} G(x)\,dF(x) = \int_{]-a,a]} G(x)\,d\mu^+ - \int_{]-a,a]} G(x)\,d\mu^-,$$

(compute both integrals) , and also by the partial integration formula

$$\int_{]a,b]} G(x^-)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_{]a,b]} F(x)\,dG(x).$$

(v) Prove that $G \in L^1(\mu)$ and find

$$\int_{\mathbb{R}} G(x)\,dF(x).$$

Solution. (i) Graph of F is easy; note that there is only one jump at 1, $F(1^+) - F(1^-) = 2$.

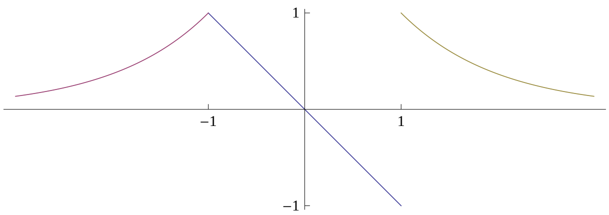

Figure 6. Plot of F.

We have (notice that F is increasing in $]-\infty,-1]$, and that $T(-\infty)=0$ so that $T(x)=F(x)$ in this interval; T is decreasing in $[-1,1[$ so that $VF([-1,x]) = F(-1)-F(x) = 1+x$ for $x\in[-1,1[$, and $T(1)=T(1^-)+(F(1^+)-F(1^-))$, etc):

$$T(x)=\begin{cases} e^{x+1} & \text{for} \quad x<-1 \\ 2+x & \text{for} \quad -1\le x<1 \\ 6-e^{-(x-1)} & \text{for} \quad 1\le x \end{cases},$$

consequently

$$T^+(x)=\begin{cases} e^{x+1} & \text{for} \quad x<-1 \\ 1 & \text{for} \quad -1\le x<1 \\ 3 & \text{for} \quad 1\le x \end{cases} \qquad T^-(x)=\begin{cases} 0 & \text{for} \quad x<-1 \\ 1+x & \text{for} \quad -1\le x<1 \\ 3-e^{-(x-1)} & \text{for} \quad 1\le x \end{cases}.$$

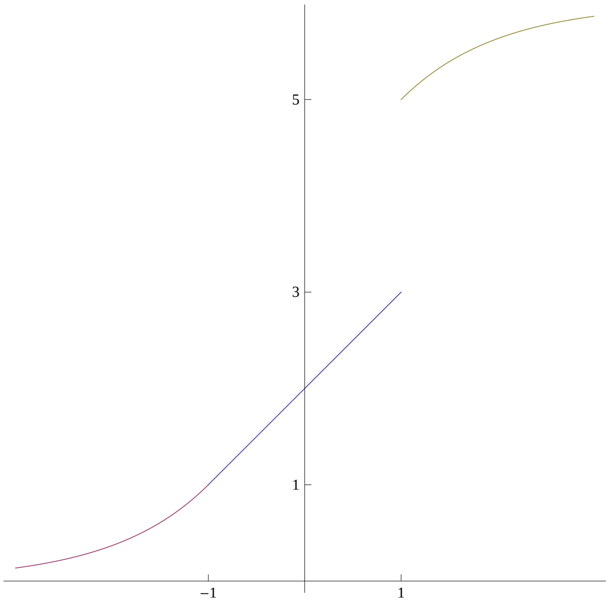

FIGURE 7. Plot of T.

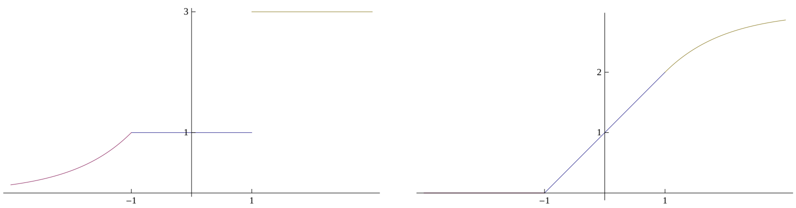

FIGURE 8. Plot of $T^\pm$

□

Since $T^\pm(\infty)-T^\pm(-\infty)=3-0$ we have $\mu^\pm(\mathbb{R})=3$, hence $|\mu|(\mathbb{R})=6$.
(iii) A Hahn decomposition is $P=]-\infty,-1]\cup\{1\}$, $Q=[-1,1[\cup]1,\infty[$.
(iv) The derivative $F'(x)$ exists for every $x\in\mathbb{R}\setminus\{-1,1\}$ and we have

$$F'(x)=\begin{cases} e^{x+1} & \text{for} \quad x<-1 \\ -1 & \text{for} \quad -1\le x<1 \\ -e^{-(x-1)} & \text{for} \quad 1\le x \end{cases};$$

The singular part is clearly $2\,\delta_1$ so that $dF = F'\,dm + 2\,\delta_1$.

(v) Clearly $d\mu^+ = \chi_{-]\infty,-1]}\, e^{x+1}\, dx + 2\,\delta_1$ so that

$$\int_{]-a,a]} G(x)\, d\mu^+(x) = \int_{-a}^{-1} \cos(\alpha x)\, e^{x+1}\, dx + 2\, G(1) = 2\cos\alpha + e\int_1^a \cos(\alpha t)\, e^{-t}\, dt;$$

A primitive of $e^{-t}\cos(\alpha t)$ is $e^{-t}(\alpha\,\sin(\alpha t) - \cos(\alpha t))/(1+\alpha^2)$ so that

(*)
$$\int_1^a \cos(\alpha t)\, e^{-t}\, dt = \left[\frac{e^{-t}}{1+\alpha^2}(\alpha\,\sin(\alpha t) - \cos(\alpha t))\right]_{t=1}^{t=a} =$$
$$\frac{e^{-a}}{1+\alpha^2}(\alpha\,\sin(\alpha a) - \cos(\alpha a)) - \frac{e^{-1}}{1+\alpha^2}(\alpha\,\sin(\alpha) - \cos(\alpha)),$$

and

$$\int_{]-a,a]} G(x)\, d\mu^+(x) = \frac{e^{1-a}}{1+\alpha^2}(\alpha\,\sin(\alpha a) - \cos(\alpha a)) - \frac{1}{1+\alpha^2}(\alpha\,\sin(\alpha) - \cos(\alpha)) + 2\cos\alpha.$$

We have next, since $d\mu^- = (\chi_{]-1,1[} + e^{-(x-1)}\,\chi_{]1,\infty[})\, dx$:

$$\int_{]-a,a]} G(x)\, d\mu^- = \int_{-1}^1 \cos(\alpha x)\, dx + \int_1^a \cos(\alpha x)\, e^{-(x-1)}\, dx = 2\frac{\sin\alpha}{\alpha} + \int_1^a \cos(\alpha x)\, e^{-(x-1)}\, dx;$$

the last integral has already been computed (see (*)). Taking the difference:

$$\int_{]-a,a]} G(x)\, dF(x) = \int_{]-a,a]} G(x)\, d\mu^+ - \int_{]-a,a]} G(x)\, d\mu^- = 2\cos\alpha - 2\frac{\sin\alpha}{\alpha}.$$

By partial integration:

$$\int_{]-a,a]} G(x)\, dF = G(a)F(a) - G(-a)F(-a) - \int_{]-a,a]} F(x)\,\alpha\,\sin(\alpha x)\, dx =$$
$$\int_{-a}^{-1} e^{x+1}\,\alpha\,\sin(\alpha x)\, dx + \int_{-1}^1 (-x)\alpha\,\sin(\alpha x)\, dx + \int_1^a e^{1-x}\,\alpha\,\sin(\alpha x)\, dx;$$

the first integral and the third cancel; we are left with:

$$\int_{-1}^1 (-x)\,\alpha\sin(\alpha x)\, dx = 2\int_0^1 x(-\alpha\,\sin(\alpha x))\, dx = 2\,[x\,\cos(\alpha x)]_0^1 - 2\int_0^1 \cos(\alpha x)\, dx = 2\cos\alpha - 2\,\frac{\sin\alpha}{\alpha}.$$

(vi) The entire space has finite measure, $|\mu|(\mathbb{R}) = 6$. Every bounded measurable function is then in $L^1(\mu) = L^1(|\mu|)$, in particular $G \in L^1(\mu)$. Clearly we have

$$\int_{\mathbb{R}} G\, dF = \int_{[-1,1]} G\, dF = 2\cos\alpha - 2\frac{\sin\alpha}{\alpha}$$

(the integrals over $]-\infty,-1[$ and $]1,\infty[$ are finite, and cancel with each other).

Analisi Reale–Primo appello–7 febbraio 2012

Exercise 13. Let $(X,\mathcal{M},\mu)$ be a measure space, and let $L^+ = L^+(X,\mathcal{M})$ denote the set of all $\mathcal{M}-$measurable functions from X to $[0,\infty]$ (as usual).

(i) Prove that if $f \in L^+$ and $\int_X f < \infty$, then $\mu(\{f=\infty\}) = 0$. If $\int_X f = 0$, what can we say about $\{f>0\}$?
(ii) State Fatou's lemma.

From now on f_n is a sequence in L^+ that converges pointwise everywhere to $f \in L^+$.

(iii) Assume that $\{f=\infty\}$ has strictly positive measure. Then $\lim_{n\to\infty}\int_X f_n = \infty$: true or false?
(iv) Suppose that there exists $g \in L^+$, with finite integral, such that $f_n(x) \le g(x)$ for every $x \in X$. Then $\int_X f = \lim_{n\to\infty}\int_X f_n$.
(iv) Assume now that there is a constant $a \in [0,\infty[$ such that, for every $n \in \mathbb{N}$

$$\int_X f_0 \vee \cdots \vee f_n \le a; \quad \text{prove that then } \lim_{n\to\infty}\int_X f_n = \int_X f.$$

Solution. (i) if $E = \{f = \infty\}$, then for every $n > 0$ we have $n\,\chi_E \le f$ so that $n\,\mu(E) = \int_X n\chi_E \le \int_X f$, which clearly implies $\mu(E) = 0$ (otherwise we may choose $n > \int_X f/\mu(E)$). If the integral of a positive f is zero, then $\mu(\{f > 0\}) = 0$: in fact nf is an increasing sequence of functions in L^+, all with zero integral, whose pointwise limit is the function constantly ∞ on $\{f > 0\}$; by monotone convergence this pointwise limit has integral 0, hence finite, and so its infinity set has zero measure. Otherwise, every measurable positive simple function dominated by f has integral 0, hence $\{f > 1/n\}$ has measure 0 for all $n \ge 1$, hence $\{f > 0\} = \bigcup_{n\ge1}\{f > 1/n\}$ has measure 0.

(ii) See the Lecture Notes.

(iii) By Fatou's lemma we get (recalling that $\liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x) = f(x)$):

$$\int_X f \le \liminf_{n\to\infty} \int_X f_n;$$

since $\{f = \infty\}$ has strictly positive measure we have $\int_X f = \infty$; then $\infty \le \liminf_{n\to\infty} \int_X f_n$, clearly equivalent to $\lim_{n\to\infty} \int_X f_n = \infty$.

(iv) This is essentially the dominated convergence theorem; the only difference is that f_n and g might be infinite valued, so we simply set all the f_n, f and g to be 0 on the set $\{g = \infty\}$, which has measure 0 by (i): no integral has been modified, and all functions are now in $L^1(\mu)$.

(v) Setting $g_n = f_0 \vee \cdots \vee f_n$, g_n is an increasing sequence of functions in L^+, with integrals all dominated by a; then $g_n \uparrow g$, and $\int_X g \le a < \infty$, by the monotone convergence theorem. We are now in the hypotheses of (iv), since clearly $f_n \le g_n \le g$ for every n. □

Exercise 14. Let $(X, \mathcal{M}, \mu)$ be a measure space. Given q, with $1 < q < \infty$ and $a > 0$ consider $a\,\bar{B} = \{f \in L^q(\mu) : \|f\|_q \le a\}$ (the closed ball of center 0 and radius $a > 0$ in $L^q(\mu)$).

(i) Prove that if the sequence $f_n \in a\,\bar{B}$ converges pointwise a.e. to f, then $f \in a\,\bar{B}$ (Fatou's lemma ...)

(ii) Let $E \in \mathcal{M}$ be a subset of X of finite measure. Prove that for every $p \in [1, q[$ and every $f \in a\,\bar{B}$ we have:

$$\left(\int_E |f|^p\,d\mu\right)^{1/p} \le \mu(E)^{\alpha(p,q)}\,a,$$

where the exponent $\alpha(p, q)$ is to be found (hint: consider $|f|^p$ and 1, with convenient conjugate exponents ...).

(iii) Deduce from (ii) that for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for every $E \in \mathcal{M}$ with $\mu(E) \le \delta$, every $f \in a\,\bar{B}$ and every $p \in [1, q[$ we have

$$\left(\int_E |f|^p\,d\mu\right)^{1/p} \le \varepsilon.$$

From now on X is assumed of finite measure, $\mu(X) < \infty$.

(iv) State the Severini–Egoroff's theorem on almost uniform convergence. Assume that the sequence $f_n \in a\,\bar{B}$ converges pointwise a.e. to f. Using this theorem and (iii) prove that f_n converges to f in $L^p(\mu)$, for every $p \in [1, q[$.

Solution. (i) If f_n converges a.e. to f, then $|f_n|^q$ converges a.e. to $|f|^q$, and Fatou's lemma says that:

$$\int_X |f|^q \left(= \int_X \liminf_{n\to\infty} |f_n|^q\right) \le \liminf_{n\to\infty} \int_X |f_n|^q \le a^q.$$

(ii) We use q/p and $(q/p)/(q/p-1) = q/(q-p)$ as conjugate exponents, and consider E as the ambient space, obtaining

$$\int_E |f|^p \le \left(\int_E |f|^q\right)^{p/q} \left(\int_E 1^{q/(q-p)}\right)^{(q-p)/q} = \mu(E)^{1-p/q} \left(\int_E |f|^q\right)^{p/q} \le \mu(E)^{1-p/q} \left(\int_X |f|^q\right)^{p/q};$$

taking p^{th}−roots of both sides we get

$$\left(\int_E |f|^p\,d\mu\right)^{1/p} \le \mu(E)^{1/p-1/q}\,\|f\|_q \le \mu(E)^{1/p-1/q}\,a.$$

(iii) Immediate: since $\delta^{1/p-1/q}\,a$ has to be smaller than ε we get $\delta \le (\varepsilon/a)^{pq/(q-p)}$; any such δ will do.

(iv) For the statement we refer to the Lecture Notes. Next, by (i) we have $f \in a\,\bar{B}$; considering $f - f_n$ in place of f we can assume that $f = 0$, and we have to prove that $\|f_n\|_p$ has limit 0. Given $\varepsilon > 0$ we find δ such that $\mu(E) \le \delta$ implies $\left(\int_E |f_n|^p\right)^{1/p} \le \varepsilon$ for every $n \in \mathbb{N}$; since the convergence to 0 is almost uniform we can find a set E such that $\mu(E) \le \delta$ and on $X \setminus E$ the sequence converges uniformly to 0. Then, if $\|f_n\|_{X\setminus E} = \sup\{|f_n(x)| : x \in X \setminus E\}$:

$$\int_X |f_n|^p = \int_E |f_n|^p + \int_{X\setminus E} |f_n|^p \le \varepsilon^p + \|f_n\|^p_{X\setminus E}\, \mu(X \setminus E) \le \varepsilon^p + \|f_n\|^p_{X\setminus E}\, \mu(X);$$

since $\lim_{n\to\infty} \|f_n\|^p_{X\setminus E} = 0$, we conclude. □

Exercise 15. Assume that $f \in L^p(\mathbb{R}^n)$, with $1 \le p < \infty$.

(i) Prove that

$$\lim_{r\to\infty} \int_{|x|>r} |f|^p\, dm = 0.$$

We now define $F : \mathbb{R}^n \to \mathbb{K}$ by

$$F(x) = \int_{B(x,1[} f(y)\, dy \quad \text{where, as usual, } B(x, 1[= \{y \in \mathbb{R}^n : |y - x| < 1\}.$$

(ii) Prove that the preceding formula effectively defines a function $F : \mathbb{R}^n \to \mathbb{K}$; prove that F is continuous and bounded, and find an estimate for $\|F\|_\infty$ involving $\|f\|_p$.

(iii) Prove that $\lim_{x\to\infty} F(x) = 0$ (use (i)).

Solution. (i) By definition of $L^p(\mathbb{R}^n)$ we have $|f|^p \in L^1(\mathbb{R}^n)$; clearly $|f|^p\, \chi_{\mathbb{R}^n \setminus rB}$ tends to 0 as $r \to \infty$, and is dominated by $|f|^p$, so that the limit of integrals $\lim_{r\to\infty} \int_{|x|>r} |f|^p\, dm = 0$ by dominated convergence.

(ii) If $p = 1$ there is nothing to prove. If $p > 1$ the usual estimates for L^p spaces on sets of finite measure give (we apply Hölder's inequality to $|f|$ and 1 of $B(x, 1[$, with conjugate exponents p and $q = p/(p-1)$):

$$(*) \qquad |F(x)| \le \int_{B(x,1[} |f(y)| \le \left(\int_{B(x,1[} |f(y)|^p\right)^{1/p} (m(B(x, 1[)^{1/q} \le v_n^{1/q}\, \|f\|_p,$$

which immediately implies

$$\|F\|_\infty \le v_n^{1/q}\, \|f\|_p.$$

In other words, we have proved the well known fact that if $f \in L^p(\mathbb{R}^n)$ then $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We know that if x_j tends to x in $\mathbb{R}^n$ then $\chi_{B(x_j,1[}$ tends a.e. to $\chi_{B(x,1[}$, and the sequence is dominated by $\chi_{B(x,1+R]}$ with $R = \max_j\{|x - x_j|\}$. Then F is continuous, by the dominated convergence theorem.

(iii) By (i), given $\varepsilon > 0$ there is $r(\varepsilon)$ such that $\int_{\{|x|\ge r(\varepsilon)\}} |f|^p \le \varepsilon^p$. If $|x| \ge r(\varepsilon) + 1$ we have that $B(x, 1[\subseteq \{|x| \ge r(\varepsilon)\}$ so that, for these x:

$$|F(x)| \le \left(\int_{B(x,1[} |f(y)|^p\right)^{1/p} (m(B(x, 1[)^{1/q} \le v_n^{1/q} \left(\int_{\{|x|\ge r(\varepsilon)\}} |f|^p\right)^{1/p} \le v_n^{1/p}\, \varepsilon.$$

□

Exercise 16. For every $n = 1, 2, 3, \ldots$ and every $x \in \mathbb{R}$ define $F_n(x) = \int_0^x nt^{n-1}\, \chi_{[0,1]}(t)\, dt$.

(i) Plot some F_n and the limit function $F(x) = \lim_{n\to\infty} F_n(x)$. What is the measure $\mu = \mu_F$?

(ii) Setting $\mu_n = \mu_{F_n}$, compute

$$\lim_{n\to\infty} \mu_n(]-\infty, a])\ (0 < a < 1); \quad \lim_{n\to\infty} \mu_n([0, 1[); \quad \lim_{n\to\infty} \mu_n([0, 1]).$$

(iii) Assume that $f : \mathbb{R} \to \mathbb{R}$ is bounded and Borel measurable. Prove that $f \in L^1(\mu_n)$ for every n, and moreover, if f is also left–continuous at 1 then:

$$\lim_{n\to\infty} \int_{\mathbb{R}} f\, d\mu_n = f(1) \left(= \int_{\mathbb{R}} f\, d\mu\right)$$

(prove first that if $f(1) = 0$ then the limit is 0; split the integral in $\int_{]-\infty,a]} + \int_{]a,1]}$ and use (ii)).

Solution. (i) We have $F_n(x) = 0$ for $x < 0$; $F_n(x) = x^n$ for $0 \le x < 1$ and $F(x) = 1$ for $x \ge 1$. Then $F(x) = 0$ for $x < 1$, and $F(x) = 1$ for $1 \le x$; F is the characteristic function of $[1, \infty[$, and hence $\mu = \delta_1$, unit mass at 1. Notice that all these measures are supported by $[0, 1]$.

(ii) Clearly, if $0 < a < 1$

$$\mu_n(]-\infty, a]) = a^n - 0 = a^n \quad \text{so that} \quad \lim_{n\to\infty} \mu_n(]-\infty, a]) = 0;$$

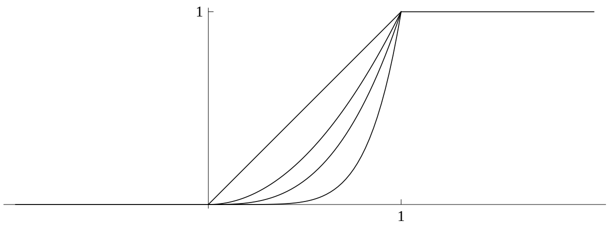

FIGURE 9. Plots of some F_n.

and we have $\mu_n([0,1[) = \mu_n([0,1]) = 1$, so that the limit is 1.

(iii) Since $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_n)$ is a finite measure space for every n, all bounded measurable functions are in $L^1(\mu_n)$ for every n. Next, if f is left continuous and 0 at 1, given $\varepsilon > 0$ find $a \in]0,1[$ such that $|f(x)| \le \varepsilon$ if $x \in [a,1]$ so that

$$\left|\int_{\mathbb{R}} f\,d\mu_n\right| = \left|\int_{[0,1]} f\,d\mu_n\right| = \left|\int_{[0,a[} f\,d\mu_n + \int_{[a,1]} f\,d\mu_n\right| \le \int_{[0,a[} |f|\,d\mu_n + \int_{[a,1]} |f|\,d\mu_n \le$$
$$\le \|f\|_\infty \int_{[0,a]} d\mu_n + \int_{[a,1]} \varepsilon\,d\mu_n = \|f\|_\infty\, a^n + \varepsilon\,(1-a^n),$$

since this expression has limit ε as $n \to \infty$, we conclude that

$$\lim_{n\to\infty} \left|\int_{\mathbb{R}} f\,d\mu_n\right| = 0,$$

for f bounded left continuous and zero at 1. For f bounded left continuous at 1 we simply write $f = f - f(1) + f(1)$ and note that $\int_{\mathbb{R}} f(1)\,d\mu_n = f(1)$ for every n, while by what just proved we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} (f - f(1))\,d\mu_n = 0.$$

□

ANALISI REALE– SECONDO APPELLO–28–02–2012

EXERCISE 17. (10) Let $(X, \mathcal{M}, \mu)$ be a measure space

(i) [2] Assume that $f, g : X \to \mathbb{R}$ are measurable, that $E \in \mathcal{M}$, that $f(x) < g(x)$ for every $x \in E$, and that $f, g \in L^1_\mu(E)$. Prove that if $\int_E f < \int_E g$ iff $\mu(E) > 0$.

(ii) [2] Let $E \in \mathcal{M}$ be such that $0 < \mu(E) < \infty$, and let $f \in L^1(\mu)$ be real valued. Prove that there exists $x \in E$ such that

$$f(x) \le \fint_E f\,d\mu := \frac{1}{\mu(E)} \int_E f\,d\mu;$$

more precisely, prove that the set $\{x \in E : f(x) \le \fint_E f\,d\mu\}$ has strictly positive measure.

This expresses the intuitively obvious fact that not all values of f on E can be larger than its average on E: not everybody can be above the mean!

(iii) [2] Let now $f,\, g \in L^1(\mu)$ be real functions. Prove that $f(x) \le g(x)$ for a.e. $x \in X$ if and only if $\int_E f \le \int_E g$ for every $E \in \mathcal{M}$ (consider $E = \{f > g\}\ \dots$).

(iv) [4] For $f, g, h \in L^+(X, \mathcal{M})$ assume that $f^2(x) \le g(x)\,h(x)$ for a.e. $x \in X$. Prove that then, for every $E \in \mathcal{M}$ we have

$$\left(\int_E f\right)^2 \le \left(\int_E g\right)\left(\int_E h\right) \qquad (*)$$

($f^2 \le g\,h$ is equivalent to $f \le g^{1/2}\,h^{1/2}$; apply a convenient inequality ...).

Solution. (i) We have

$$\int_E f < \int_E g \iff \int_E (g - f) > 0 \iff \int_X (g-f)\,\chi_E > 0;$$

by hypothesis $(g-f)(x) = g(x) - f(x) > 0$ for every $x \in E$, so that $\mathrm{Coz}((g-f)\,\chi_E = E$; we know that a positive measurable function has integral 0 if and only if its cozero set has measure 0, so we conclude.

(ii) Setting for simplicity $c = \fint_E f$, if there is no $x \in E$ such that $f(x) \le c$, then $c < f(x)$ for every $x \in E$. Since $\mu(E) < \infty$, the constant c is in $L^1_\mu(E)$, so that (i) is applicable and gives that

$$\int_E c < \int_E f \iff c\,\mu(E) < \int_E f \iff c < \fint_E f = c,$$

a contradiction. Since we can alter f on any subset of E of zero measure without altering the average c, the set $\{x \in E : f(x) \le c\}$ must be of strictly positive measure.

(iii) If $f \le g$ a.e. then $\int_E f \le \int_E g$, by isotony of the integral, as is well–known. And if it is not true that $f(x) \le g(x)$ for a.e. $x \in X$, then if $E = \{f - g > 0\}$ has strictly positive measure; by (i)

$$\int_E f > \int_E g,$$

contradicting the hypothesis.

(iv) Integrating over E the inequality $f \le g^{1/2}\,h^{1/2}$ we get

$$\int_E f \le \int_E g^{1/2}\,h^{1/2};$$

By Cauchy–Schwarz inequality for integrals we have

$$\int_E g^{1/2}\,h^{1/2} \le \left(\int_E g\right)^{1/2} \left(\int_E h\right)^{1/2},$$

so that

$$\int_E f \le \left(\int_E g\right)^{1/2} \left(\int_E h\right)^{1/2},$$

and squaring both sides we conclude.

□

Exercise 18. (12)

(i) [2] State the Radon–Nikodym theorem.

(ii) [4] Let $(X, \mathcal{M})$ be a measurable space, and let $\mu, \nu : \mathcal{M} \to [0, \infty]$ be positive measures, both $\sigma-$finite. Prove that the following are equivalent:
  (a) We have $\nu \ll \mu$ and $\mu \ll \nu$.
  (b) μ and ν have the same null sets.
  (c) There is $\rho \in L^+(X, \mathcal{M})$ such that $\rho(x) > 0$ for every $x \in X$ and

$$\nu(E) = \int_E \rho\,d\mu \qquad \text{for every} \quad E \in \mathcal{M}.$$

Let now $(X, \mathcal{M}, \mu)$ be a measure space.

(iii) [4] Assume that there exists $f \in L^1(\mu)$ such that $f(x) \ne 0$ for every $x \in X$. Prove that then X has $\sigma-$finite measure. Conversely, if X has $\sigma-$finite measure then there is $f \in L^1(\mu)$ such that $f(x) > 0$ for every $x \in X$, and $\int_X f\,d\mu = 1$.

(iv) [2] Prove that if $(X, \mathcal{M}, \mu)$ is $\sigma-$finite there exists a measure $\nu : \mathcal{M} \to [0, \infty[$ such that $\nu(X) = 1$, $\nu \ll \mu$ and $\mu \ll \nu$.

Solution. (i) OK

(ii):(a) $\iff$ (b) is by definition of absolute continuity. And by Radon–Nikodym theorem, since all measures are $\sigma-$finite we have that (a), more precisely the hypothesis $\nu \ll \mu$, implies the existence of $\rho \in L^+(X, \mathcal{M})$ such that

$$\nu(E) = \int_E \rho\,d\mu \qquad \text{for every} \quad E \in \mathcal{M}.$$

But since $\nu(E) = 0$ implies also $\mu(E) = 0$, the set $Z = \{\rho = 0\}$, having $\nu-$measure 0, has also $\mu-$measure 0; we can the alter ρ on this set, e.g. set $\rho(x) = 1$ for $x \in Z$, and make $\rho(x) > 0$ everywhere.

(iii) Any $f \in L^1(\mu)$ has the cozero set of $\sigma-$finite measure ($\mathrm{Coz}(f) = \bigcup_{n>1}\{|f| > 1/n\}$, and $\mu(\{|f| > 1/n\}) \le n\int_X |f|$). And if a measurable set $A \in \mathcal{M}$ has $\sigma-$finite measure then it is the cozero set of a positive measurable function with integral 1; simply write A as a disjoint union of a sequence of sets of finite nonzero measure, $A = \bigcup_{n=0}^\infty A_n$, and consider $f : X \to \mathbb{R}$ defined by

$$f = \sum_{n=0}^\infty \frac{1}{2^n\,\mu(A_n)}\,\chi_{A_n}.$$

(iv) is now obvious: take $d\nu = \rho\, d\mu$ where $\rho \in L^1(\mu)$ is everywhere positive with integral 1; ρ exists by (iii), by (i) we have $\mu \ll \nu$. □

Exercise 19. (11) Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) [2] Compute
$$\lim_{n\to\infty} n \log(1 + (t/n)^\alpha),$$
for $t > 0$ and $\alpha > 0$. Hint:
$$n \log(1 + (t/n)^\alpha) = n(t/n)^\alpha \frac{\log(1 + (t/n)^\alpha)}{(t/n)^\alpha}; \quad \text{remember that} \quad \lim_{u\to 0} \frac{\log(1 + u)}{u} = \dots$$
What is the limit for $t = 0$?

Let now f be a positive function in $L^1(\mu)$, and assume that $c = \int_X f > 0$. We want to compute
$$(*) \qquad \lim_{n\to\infty} \int_X f_n\, d\mu$$
for various values of $\alpha > 0$, here $f_n(x)(= f_{\alpha,n}(x)) = n \log(1 + (f(x)/n)^\alpha)$ for $n = 1, 2, 3, \dots$ and $x \in X$.

(ii) [1] Compute $g(x) = (g_\alpha(x) =) \lim_{n\to\infty} f_n(x)$ (distinguish the cases $0 < \alpha < 1$, $\alpha = 1$, $\alpha > 1$).

(iii) [3] Suppose that $0 < \alpha < 1$. Prove that in this case Fatou's lemma is applicable and gives (*).

(iv) [2] Prove that $\log(1 + t^\alpha) < \alpha\, t$ for every $\alpha \ge 1$, $t > 0$ (consider $\alpha\, t - \log(1 + t^\alpha)$ and differentiate ...).

(v) [3] Compute the limit (*) for $\alpha = 1$ and for $\alpha > 1$.

Solution. (i) Recall that $\lim_{u\to 0} \log(1 + u)/u = 1$; then:
$$\lim_{n\to\infty} n \log(1 + (t/n)^\alpha) = \lim_{n\to\infty} n\, (t/n)^\alpha \frac{\log(1 + (t/n)^\alpha)}{(t/n)^\alpha} = \lim_{n\to\infty} n^{1-\alpha}\, t^\alpha \frac{\log(1 + (t/n)^\alpha)}{(t/n)^\alpha};$$
since $\lim_{n\to\infty} \log(1 + (t/n)^\alpha)/(t/n)^\alpha = 1$ we get
$$\lim_{n\to\infty} n \log(1 + (t/n)^\alpha) = \begin{cases} \infty & \text{for} \quad 0 < \alpha < 1 \\ t & \text{for} \quad \alpha = 1 \\ 0 & \text{for} \quad \alpha > 1 \end{cases}.$$
For $t = 0$ all the terms are 0, so the limit is 0.

(ii) By (i) we have, for $0 < \alpha < 1$ that $g(x) = \liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x) = \infty$ if $f(x) > 0$, and 0 if $f(x) = 0$. Then $g_\alpha = \infty\, \chi_{\mathrm{Coz}(f)}$, for $0 < \alpha < 1$. For $\alpha = 1$ we have $g_1 = f$. For $\alpha > 1$ we have $g_\alpha = 0$.

(iii) By Fatou's lemma (notice that all functions f_n are positive, since $1 + (f(x)/n)^\alpha \ge 1$)
$$\int_X g \le \liminf_{n\to\infty} \int_X f_n.$$
Since $\int_X f > 0$ by hypothesis, we have that $\mu(\mathrm{Coz}(f)) > 0$, and hence $\int_X g = \infty$, so that $\liminf_{n\to\infty} \int_X f_n = \infty$, which implies $\lim_{n\to\infty} \int_X f_n = \infty$.

(iv) Differentiating we get
$$\alpha - \frac{\alpha\, t^{\alpha-1}}{1 + t^\alpha} > 0 \iff 1 > \frac{t^{\alpha-1}}{1 + t^\alpha},$$
clearly true if $t > 0$ and $\alpha \ge 1$ because $0 < t^{\alpha-1} \le t^\alpha < 1 + t^\alpha$ if $t \ge 1$, while if $0 < t < 1$ then $t^{\alpha-1} \le 1 < 1 + t^\alpha$. Then the function $\alpha\, t - \log(1 + t^\alpha)$ is zero at 0, continuous in $[0, \infty[$, and strictly increasing on $]0, \infty[$, so that $\alpha\, t - \log(1 + t^\alpha) > 0$ for $t > 0$(if $\alpha \ge 1$).

(v) By (iii) we have $0 \le f_n(x) \le n(\alpha(f(x)/n)) = \alpha\, f(x)$, so that dominated convergence may be applied. For $\alpha = 1$ we have, by (ii):
$$f(x) = \lim_{n\to\infty} f_n(x),$$
so that $\lim_{n\to\infty} \int_X f_n = \int_X f = c$. For $\alpha > 1$ we have $\lim_{n\to\infty} f_n(x) = 0$, so that the required limit is 0. □

Exercise 20. (12) Let $\mu : \mathcal{B}_1 \to [0, \infty]$ be defined by $\mu = (e - 1) \sum_{n=1}^\infty e^{-n}\, \delta_n$, where δ_n is the unit mass at n, and $\mathcal{B}_1$ is the $\sigma-$algebra of Borel subsets of $\mathbb{R}$.

(i) [2] Find $\mu(\mathbb{R})$ and the smallest closed set that supports μ. Is μ singular with respect to Lebesgue measure $m = \lambda_1$?

(ii) [3] Find a formula for the distribution function $F(x) = \mu(]-\infty, x])$, and plot F (it is convenient to express F with the integer part function $[x]$).

Let now ν be the Radon measure defined on $\mathcal{B}_1$ by $d\nu = \chi_{]-\infty,0]}(x)\, dx/(x-1)^3$, and consider the measure $\lambda = \nu + \mu$ on $\mathcal{B}_1$.

(iii) [2] Find the absolutely continuous and the singular part of λ (with respect to Lebesgue measure m), find $\lambda^\pm$, and also a Hahn decomposition for λ.

(iv) [2] Find a formula for the total variation function $T(x) = |\lambda|(]-\infty, x])$, and plot T.

(v) [3] Given $f(x) = x$, determine the set of $p > 0$ such that $f \in L^p(|\lambda|)$. Compute the integral

$$\int_{\mathbb{R}} x\, d\lambda(x)$$

if this integral exists (it may be useful to know that $\sum_{n=1}^{\infty} n\, z^{n-1} = 1/(1-z)^2$ for $|z| < 1$).

Solution. (i) We have

$$\mu(\mathbb{R}) = (e-1)\sum_{n=1}^{\infty} e^{-n} = (e-1)\frac{1/e}{1-1/e} = 1.$$

Plainly $\mu(\mathbb{R} \setminus \mathbb{N}^>) = 0$, and every larger set has strictly positive measure. Since $\mathbb{N}^>$ is closed, it is the required set. Since $m(\mathbb{N}^>) = 0$, we have $\mu \perp m$.

(ii) We clearly have $F(x) = 0$ for $x < 1$. If $x \geq 1$, we have $F(x) = F([x])$, and

$$F([x]) = (e-1)\sum_{n=1}^{[x]} e^{-n} = (e-1)\,\frac{1}{e}\,\frac{1-e^{-[x]}}{1-1/e} = 1 - e^{-[x]}.$$

The plot is easily done.

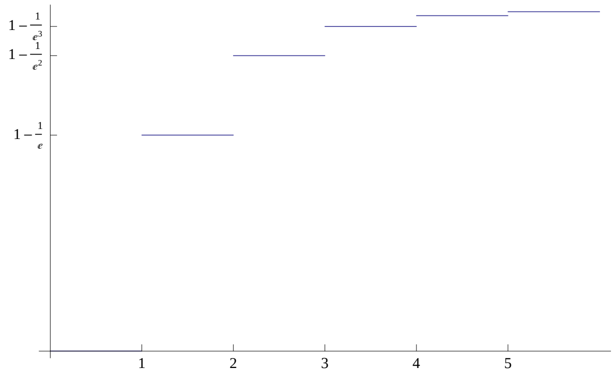

FIGURE 10. Plot of F (not on scale).

(iii) By its very definition ν is absolutely continuous with respect to m, and μ is singular, so that ν is the absolutely continuous part and μ the singular part. Next, ν is negative (notice that $\chi_{]-\infty,0]}(x)/(x-1)^3 \leq 0$ for every $x \in \mathbb{R}$), so that $\lambda^- = -\nu$ and $\lambda^+ = \mu$ (since also $\nu \perp \mu$). A Hahn decomposition for λ is for instance $\mathbb{N}^> \cup (\mathbb{R} \setminus \mathbb{N}^>)$, the first set positive, the second negative.

(iii) For $x < 1$ we have

$$T(x) = |\lambda|(]-\infty, x]) = -\nu(]-\infty, x]) = -\int_{]-\infty,x]} \chi_{]-\infty,0]}(t)\frac{dt}{(t-1)^3};$$

assuming $x \leq 0$ this integral is

$$\int_{-\infty}^{x} \frac{-dt}{(t-1)^3} = \frac{1}{2}\left[\frac{1}{(t-1)^2}\right]_{-\infty}^{x} = \frac{1}{2(x-1)^2};$$

then $T(0) = 1/2$ and $T(x) = 1/2$ for $x \in [0, 1[$. For $x \geq 1$ we have $T(x) = 1/2 + F(x)$.

(iv) We have that $L^p(|\lambda|) = L^p(\lambda^+) \cap L^p(\lambda^-) = L^p(\mu) \cap L^p(-\nu)$. Thus $f \in L^p(|\lambda|)$ iff

$$\int_{-\infty}^{0} \frac{|x|^p}{(1-x^3)}\, dx; \qquad \sum_{n=1}^{\infty} n^p\, e^{-n}$$

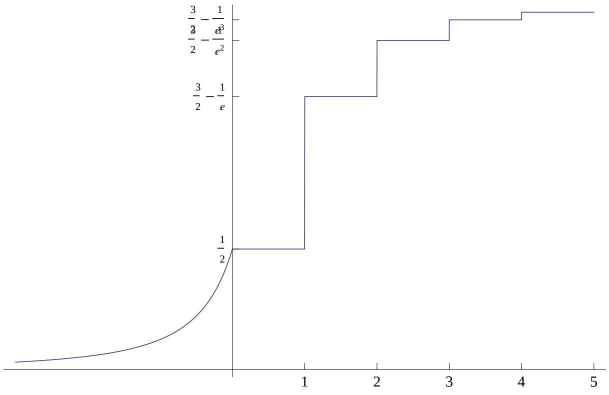

FIGURE 11. Plot of T (not on scale).

are both finite. The second series converges for every $p \in \mathbb{R}$ (e.g. by the root test), while the integral is finite if and only if $3 - p > 1 \iff p < 2$: in fact the function is continuous and hence locally summable on $]-\infty, 0]$, and at $-\infty$ it is asymptotic to $1/|x|^{3-p}$. So the answer is: for $0 < p < 2$. The integral is

$$\int_{-\infty}^{0} \frac{x}{(x-1)^3}\,dx + \sum_{n=1}^{\infty} n e^{-n} = \left[\frac{-x}{2(x-1)^2}\right]_{x=-\infty}^{x=0} + \frac{1}{2}\int_{-\infty}^{0} \frac{dx}{(x-1)^2} + \frac{1}{e}\sum_{n=1}^{\infty} n\,\frac{1}{e^{n-1}}$$

$$\frac{1}{2}\left[\frac{-1}{x-1}\right]_{-\infty}^{0} + \frac{1}{e}\,\frac{1}{(1-1/e)^2} = \frac{1}{2} + \frac{e}{(e-1)^2}.$$

□

ANALISI REALE PER MATEMATICA – APPELLO DI RICUPERO – 18 LUGLIO 2012

EXERCISE 21. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Let $g_n \in L^+$ be a sequence of measurable positive functions; assume that $\int_X g_n < \infty$ for every $n \in \mathbb{N}$. Consider the following statements:
  (a) The series of functions $\sum_{n=0}^{\infty} g_n(x)$ converges to a finite sum for a.e. $x \in X$.
  (b) The series $\sum_{n\in\mathbb{N}} \int_X g_n$ of the integrals is convergent, that is $\sum_{n\in\mathbb{N}} \int_X g_n < \infty$.
  Are these statements equivalent? or does (b) imply (a)? or conversely does (a) imply (b)? Give proofs, or counterexamples.

(ii) Given any function $g \in L^+(\mathbb{R})$, with $\int_{\mathbb{R}} g = a > 0$ (the measure is Lebesgue measure), and a sequence $c_n \in \mathbb{R}$, prove that the formula

$$f(x) = \sum_{n=0}^{\infty} g(2^n(x - c_n))$$

defines for a.e. $x \in \mathbb{R}$ a function $f \in L^1(\mathbb{R})$. What is the integral of f?

(iii) Let the function g in (ii) be $\log^+(1/|x|) = \max\{-\log|x|, 0\}$, with $g(0) = 0$, and let $n \mapsto c_n$ be a bijection of $\mathbb{N}$ onto the set of rational numbers. Plot g, and prove that for every $\alpha > 0$ and every non–empty open interval I of $\mathbb{R}$ the set $\{x \in I : f(x) > \alpha\}$ has strictly positive measure.

Solution. (i) It is true that (b) implies (a), but not the converse. If $h_m = \sum_{n=0}^{m} g_n$, then $h_m \in L^+$, and the sequence h_m is increasing to a limit h with $h(x) = \sum_{n=0}^{\infty} g_n(x)$; by the monotone convergence theorem we have

$$\int_X h = \lim_{m\to\infty} \int_X h_m = \lim_{m\to\infty} \sum_{n=0}^{m} \int_X g_n = \sum_{n=0}^{\infty} \int_X g_n < \infty \quad \text{(by (b))};$$

then $\int_X h < \infty$ implies that $E = \{h = \infty\}$ has measure 0; and E is exactly the set of all $x \in X$ such that $\sum_{n=0}^{\infty} g_n(x) = \infty$. Pointwise convergence everywhere of the series does not ensure convergence of the series of integrals: take e.g. $g_n(x) = g(x - n)$, where $g = \chi_{[0,1[}$.

(ii) The change of variable $t = 2^n(x - c_n) \iff x = t/2^n + c_n$ reduces the integral to

$$\int_{\mathbb{R}} g(2^n(x - c_n))\,dx = \int_{\mathbb{R}} g(t)\,\frac{dt}{2^n} = \frac{a}{2^n},$$

so that the series of integrals of the given series is

$$\sum_{n=0}^{\infty}\int_{\mathbb{R}} g(2^n(x-c_n))\,dx = \sum_{n=0}^{\infty}\frac{a}{2^n} = 2a,$$

and by (i) the series the converges pointwise a.e. to a measurable positive function f with $\int_{\mathbb{R}} f(x)\,dx = 2a$.

(iii) The plot is easy.

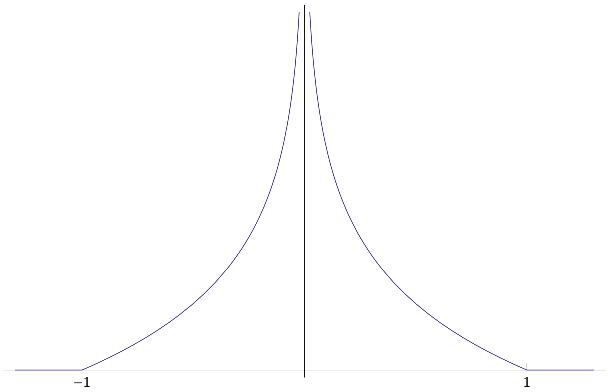

FIGURE 12. Plot of g

If I is non–empty open interval, by density of $\mathbb{Q}$ there are infinitely many $n \in \mathbb{N}$ such that $c_n \in I$, and if n is such that $2^{-n} < m(I)$ (m Lebesgue measure) then either the right or the left half of the interval $]c_n - 1/2^n, c_n + 1/2^n[$ are contained in I. Since the series has positive terms, $f(x) > \alpha$ is ensured if $g(2^n(x-c_n)) > \alpha$ for at least one n; and

$$g(2^n(x-c_n)) > \alpha \iff 2^n|x-c_n| < e^{-\alpha} \iff c_n - e^{-\alpha}/2^n < x < c_n + e^{-\alpha}/2^n,$$

so that the set $\{f > \alpha\} \cap I$ has measure not less than $e^{-\alpha}/2^n$.

□

EXERCISE 22. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Let $\mathcal{S} \subseteq \mathcal{M}$ be closed under union (that is, $A, B \in \mathcal{S}$ imply $A \cup B \in \mathcal{S}$). Let $s = \sup\{\mu(A) : A \in \mathcal{S}\}$. Prove that there exists an increasing sequence $A_0 \subseteq A_1 \subseteq \dots$ of elements of $\mathcal{S}$ such that $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = s$. Prove that if $\mathcal{S}$ is closed under countable union then $s = \max\{\mu(A) : A \in \mathcal{S}\}$.

Given $E \in \mathcal{M}$ let $\mathcal{S}(E) = \{A \in \mathcal{M} : A \subseteq E,\ \mu(A) < \infty\}$, and set $\mu_0(E) = \sup\{\mu(A) : A \in \mathcal{S}(E)\}$.

(ii) Prove that $\mathcal{S}(E)$ is closed under union, and that the following are equivalent:
  (a) $\mu_0(E) = \max\{\mu(A) : A \in \mathcal{S}(E)\}$.
  (b) $\mu_0(E) < \infty$.
  (c) $\mathcal{S}(E)$ is closed under countable union.

Let's call *atom* in a measure space $(X, \mathcal{M}, \mu)$ any $A \in \mathcal{M}$ such that $0 < \mu(A) \le \infty$, and for every $B \subseteq A$, $B \in \mathcal{M}$, we have either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Prove that if for some $E \in \mathcal{M}$ we have $\mu_0(E) < \mu(E)$ then E contains an atom of infinite measure.

Solution. (i) There is of course a sequence $S_n \in \mathcal{S}$ such that $\sup_n \mu(S_n) = s$. Set $A_n = S_0 \cup \cdots \cup S_n$. Then $A_n \in \mathcal{S}$ because $\mathcal{S}$ is closed under union, and clearly A_n is increasing. We have $\mu(S_n) \le \mu(A_n)$, and $\mu(A_n) \le s$ because $A_n \in \mathcal{S}$. Then

$$s = \sup_n \mu(S_n) \le \lim_{n\to\infty} \mu(A_n) \le s \quad \text{so that} \quad s = \lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right).$$

Trivially we then have $s = \max\{\mu(A) : A \in \mathcal{S}\}$ if $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{S}$.

(ii) Subadditivity implies immediately that $\mathcal{S}(E)$ is closed under $\cup$: $\mu(A \cup B) \le \mu(A) + \mu(B) < \infty$ if both $\mu(A)$ and $\mu(B)$ are finite. Let us next show that (a) implies (b) implies (c) implies (a):

(a) implies (b) Since $\mathcal{S}(E)$ is closed under union, there is an increasing sequence $A_0 \subseteq A_1 \subseteq \dots$ of elements of $\mathcal{S}$ such that $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = s$; then $s = \mu_0(E) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right)$; if $\mu_0(E) < \infty$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{S}(E)$, and $s = \max\{\mu(A) : A \in \mathcal{S}(E)\}$.

(b) implies (c) Since $\mathcal{S}(E)$ is closed under finite union, we only have to prove that the union of an increasing sequence in $\mathcal{S}(E)$ belongs to $\mathcal{S}(E)$. If $A_0 \subseteq A_1 \subseteq \dots$ is such a sequence we have $\mu(A_n) \le \mu_0(E)$ for every n; if $A = \bigcup_{n\in\mathbb{N}} A_n$ we then get $\mu(A) = \lim_{n\to\infty} \mu(A_n) \le \mu_0(E)$, so that $\mu(A) \le \mu_0(E) < \infty$; thus $A \in \mathcal{S}(E)$.

(c) implies (a) is immediate by (i).

Last question: if $\mu_0(E) < \mu(E)$ then certainly $\mu_0(E) < \infty$; if $\mu(E)$ is finite, then trivially $\mu_0(E) = \mu(E)$ so that the hypothesis implies $\mu_0(E)$ finite and $\mu(E) = \infty$; by (ii) there is $A \subseteq E$ such that $\mu(A) = \mu_0(E) = \max\{\mu(B) : B \in \mathcal{S}(E)\}$ Then $E \setminus A$ is the required atom; it clearly has infinite measure, and if $B \subseteq E \setminus A$ has finite measure then $A \cup B \in \mathcal{S}(E)$ and $\mu(A) = \mu(A) + \mu(B)$ implies $\mu(B) = 0$ (otherwise $\mu(A \cup B) > \mu(A) = \mu_0(E)$, a contradiction)

□

Exercise 23. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) If $0 < p < q$, for every $f \in L(X)$ we have

$$\|f\|_q \le \|f\|_p^{p/q}\, \|f\|_\infty^{1-p/q};$$

Prove it, and say when equality holds, assuming the right–hand side finite and nonzero.

On every set X the spaces $\ell^p = \ell^p(X, \mathbb{K})$ are defined, and also $\|f\|_p$ is defined for every $f : X \to \mathbb{K}$.

(ii) Explain how these spaces can be defined within the general theory of L^p spaces (that is, they are $L^p(X, \mathcal{M}, \mu)$ for some $\sigma-$ algebra $\mathcal{M}$ on X and some measure μ). Prove that $\|f\|_\infty \le \|f\|_p$ for every $p > 0$, and determine the functions f for which equality holds. Prove that if $0 < p < q < \infty$ then $\|f\|_q \le \|f\|_p$.

(iii) Prove that if $\ell^p(X) = \ell^q(X)$ for $p, q > 0$ and $p < q$, then X is finite (remember that $\sum_{n=1}^\infty 1/(n+1)^\alpha$ is in $\ell^p(\mathbb{N})$ iff $p\alpha \dots$).

Solution. (i) For a.e. $x \in X$ we have

(*) $$|f(x)|^q = |f(x)|^p\, |f(x)|^{q-p} \le |f(x)|^p\, \|f\|_\infty^{q-p} \quad \text{integrating}$$

$$\int_X |f|^q \le \int_X |f|^p\, \|f\|_\infty^{q-p} = \left(\int_X |f|^p\right) \|f\|_\infty^{q-p};$$

taking $q-$th roots of both sides:

$$\|f\|_q \le \|f\|_p^{p/q}\, \|f\|_\infty^{1-p/q}.$$

To avoid trivialities we consider the case in which the right–hand side is finite and nonzero. When integrating in (*), the inequality becomes an equality if and only if the set

$$\{x \in X : |f(x)|^q < |f(x)|^p\, \|f\|_\infty^{q-p}\} \quad \text{has measure } 0;$$

this set is clearly contained in the cozero set $\{|f| > 0\}$ of f, and coincides with

$$\{x \in X : |f(x)| > 0,\, |f(x)|^{q-p} < \|f\|_\infty^{q-p}\},$$

and clearly it has measure 0 if and only if $|f(x)|$ is constantly a.e. equal to its esssupnorm on $\{|f| > 0\}$, in other words $|f| = \|f\|_\infty\, \chi_{\mathrm{Coz}(f)}$; and for the right–hand side to be finite we need $\|f\|_\infty < \infty$ and $\mu(\mathrm{Coz}(f)) < \infty$. To sum up: the inequality is an equality with finite nonzero sides if and only if $|f|$ is of the form $r\,\chi_E$, with $r > 0$ and $0 < \mu(E) < \infty$.

(ii) We know that $\ell^p(X) = L^p(X, \mathcal{M}, \mu)$ if $\mathcal{M} = \mathcal{P}(X)$, the power set of X, and μ the counting measure on $\mathcal{P}(X)$. It is trivial to see that $\|f\|_\infty \le \|f\|_p$ for every p with $0 < p < \infty$: for every $c \in X$ one has $|f(c)|^p \le \sum_{x\in X} |f(x)|^p = \|f\|_p^p$, so that $|f(c)| \le \|f\|_p$ for every $c \in X$, and then $\|f\|_\infty = \sup\{|f(c)| : c \in X\} \le \|f\|_p$. Equality holds when $\|f\|_\infty = \infty$ or when $f = 0$; excluding these cases $\|f\|_p$ has to be finite; then $\|f\|_\infty = \max\{|f(x)| : x \in X\}$; if $\|f\|_\infty = |f(c)| > 0$, then we must have $f(x) = 0$ for all $x \in X \setminus \{c\}$; if not we have $\|f\|_\infty^p = |f(c)|^p < |f(c)|^p + |f(x)|^p \le \|f\|_p^p$ when $f(x) \ne 0$. Then equality holds in non–trivial cases iff the cozero set of f is a singleton. Finally, from (i) we get $\|f\|_q \le \|f\|_p^{p/q}\, \|f\|_\infty^{1-p/q}$ if $0 < p < q < \infty$; and since $\|f\|_\infty \le \|f\|_p$ we conclude that $\|f\|_q \le \|f\|_p^{p/q}\, \|f\|_p^{1-p/q} = \|f\|_p$.

(iii) If X is infinite, then X contains a countably infinite subset $N = \{x_0, x_1, x_2, \dots\}$. Given $\alpha > 0$ we consider the function $f = f_\alpha : X \to \mathbb{R}$ given by $f(x) = 0$ if $x \in X \setminus N$, and $f(x_n) = 1/(n+1)^\alpha$. Clearly $f \in \ell^p(X)$ iff $p\alpha > 1$; the conclusion is immediate. □

Exercise 24. Define $\alpha : \mathbb{R} \to \mathbb{R}$ by $\alpha(x) = 1/(1-x)$ for $x < 0$, $\alpha(0) = 0$, $\alpha(x) = e^{-[1/x]}$ for $x > 0$ (as usual, $[t]$ is the integer part of t, for every $t \in \mathbb{R}$).

(i) Find all points of discontinuity of α, the jump of α at these points, and determine left or right continuity of α at these points.
(ii) Plot α and the total variation function $T(x) = V\alpha(]-\infty, x])$; compute $\mu(\mathbb{R})$, where μ is the total variation measure $|\lambda_\alpha|$ of the measure λ_α determined by α. Find the largest open set null for λ_α.
(iii) For λ_α find a Hahn decomposition, and describe the absolutely continuous and singular part with respect to Lebesgue measure m on $\mathcal{B}_1$.
(v) Given $f(x) = x^+ = \max\{x, 0\}$, determine the set of all $p > 0$ such that $f \in L^p(\mu)$. Is it true that $f \in L^\infty(\mu)$?

Solution. (i) Clearly α is continuous on $]-\infty, 0[$. For $x > 1$ we have $[1/x] = 0$ so that $\alpha(x) = 1$ for $x > 1$. We have $[1/x] = n \in \mathbb{N}$ iff $n \le 1/x < n+1$, that is iff $1/(n+1) < x \le 1/n$. Then on the left–open interval $]1/(n+1), 1/n]$ the function α has the constant value e^{-n}; α is discontinuous at all points $1, 1/2, 1/3, \dots$, and at these points it is left continuous, with $\alpha(1/n) = e^{-n} = \lim_{x\to(1/n)^-}\alpha(x)$, while $\lim_{x\to(1/n)^+}\alpha(x) = e^{-(n-1)}$; the jump at $1/n$ is then $\sigma_\alpha(1/n) = e^{-(n-1)} - e^{-n} = e^{-n}(e-1)$. Another point of discontinuity is 0, with $\lim_{x\to 0^-}\alpha(x) = 1$, and $\lim_{x\to 0^+}\alpha(x) = 0 = \alpha(0)$; at 0 we have right continuity, and $\sigma_\alpha(0) = -1$.

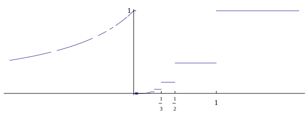

FIGURE 13. Plot of α

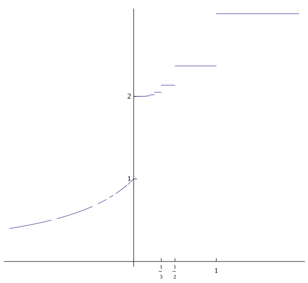

FIGURE 14. Plot of T

(ii) With the previous information the plot of α is easy. For T: since α is increasing on $]-\infty, 0[$ and 0 at $-\infty$, for $x < 0$ we get $T(x) = \alpha(x) = 1/(1-x)$, while $T(0) = 2$; next we get $T(x) = 2 + \beta(x)$. where $\beta :]0, \infty[\to \mathbb{R}$ is the right–continuous modification of α; $T(+\infty) = 3 = \mu(\mathbb{R})$. Sets null for λ_α are those of $|\lambda_\alpha|$–measure 0; it is quite clear that the largest open set of μ–measure 0 is $]0, \infty[\setminus \{1/n : n \ge 1\}$ (any larger open set will either contain a point $1/n$, with measure $\mu(\{1/n\}) = e^{-n}(e-1)$, or 0, with measure $\mu(\{0\}) = 1$, or an open interval I of $]-\infty, 0[$, with measure $\alpha(\sup I) - \alpha(\inf I) > 0$).

(iii) The function α is increasing on $]-\infty, 0]$ and on $]0, \infty[$, so that both these are positive sets; and $\{0\}$ is a negative set. Thus a Hahn decomposition is $P = \mathbb{R}^*$, $N = \{0\}$. The absolutely continuous part is $(\chi_{]-\infty,0[}/(1-x)^2)\, dm$ (or the measure associated to the monotone function $x \mapsto 1/(1-x)$ for $x < 0$, $x \mapsto 1$ for $x \ge 0$); the singular part is the measure $-\delta_0 + \sum_{n=1}^\infty e^{-n}(e-1)\,\delta_{1/n}$.

(iv) Clearly the integral of f^p is

$$\int_{\mathbb{R}} f^p(x)\, d\mu(x) = \sum_{n=1}^\infty \frac{1}{n^p} e^{-n}(e-1);$$

this sum is clearly finite for every $p > 0$. Then $f \in L^p(\mu)$ for every $p > 0$. And it is easy to see that $f \in L^\infty(\mu)$: the set $\{f > 1\} =]1, \infty[$ has clearly μ–measure 0 (it is also easy to see that $\|f\|_\infty = 1$). □

ANALISI REALE PER MATEMATICA – APPELLO DI RICUPERO – 4 SETTEMBRE 2012

EXERCISE 25. Let $(X, \mathcal{M}, \mu)$ be a measure space; as usual we denote by $L^+(X) = L^+_{\mathcal{M}}(X)$ the set of all measurable functions with values in $[0, \infty]$.

(i) Given $f \in L^+(X)$ which of the following two statements is correct?
  (a) If $\int_X f$ is finite, then $f \in L^1(\mu)$.
  (b) If $\int_X f$ is finite, then f coincides a.e. with a function $g \in L^1(\mu)$.

(ii) In $X = \mathbb{R}$ with Lebesgue measure consider the sequence $f_n = \chi_{[n,\infty[}$. Notice that f_n is a decreasing sequence in $L^+(\mathbb{R})$, and find the limit f; is it true that $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$?

(iii) State Fatou's lemma. Next, state and prove the analogous of Fatou's lemma for lim sup (with the necessary modifications).

Solution. (i) The correct statement is (b). Functions in $L^+(X)$ may assume the value $+\infty$; we know (LN, 3.3.5, corollary) that if the integral is finite then $\{f = \infty\}$ is measurable with zero measure.

(ii) The limit function f is identically 0, with zero integral, whereas $\int_{\mathbb{R}} f_n = \infty$ for every n, so that also $\lim_n \int_{\mathbb{R}} f_n = \infty$

(iii) For Fatou's lemma see LN, 3.3.6. We were reminded, from (ii) that for a decreasing sequence of functions to have passage to the limit under the integral sign an hypothesis of finiteness of the integral has to be added. Then we can state:

. *Let f_n be a sequence of functions in $L^+(X)$. Assume that for some $m \in \mathbb{N}$ the integral of $f^*_m = \bigvee_{n \ge m} f_n$ is finite. Then*

$$\int_X \limsup_n f_n \ge \limsup_n \int_X f_n.$$

Proof. The sequence f^*_k is decreasing, converges pointwise to $f^* = \limsup_n f_n$ and $\int_X f^*_k$ is finite as soon as $k \ge m$; then $\int_X f^* = \lim_{k\to\infty} \int_X f^*_k$ (LN, 3.3.6.2; alternatively, dominated convergence); since $f^*_k \ge f_l$ for $l \ge k$ we have $\int_X f^*_k \ge \int_X f_l$ for every $l \ge k$, hence also $\int_X f^*_k \ge \sup_{l \ge k} \int_X f_l$; passing to the limit in this inequality as $k \to \infty$ we get

$$\int_X f^* \left(= \int_X \limsup_n f_n\right) \ge \limsup_k \int_X f_k,$$

as required. □

Remark. The hypothesis that f^*_m has finite integral for some m is equivalent to the hypothesis that some function in $L^1(\mu)$ dominates a.e. all functions f_k for $k \ge m$: combined with Fatou's lemma for lim inf the above in fact gives the dominated convergence theorem (another proof of)

□

Exercise 26. Let $(X, \mathcal{M})$ be a measurable space and let $\mu, \nu : \mathcal{M} \to [0, \infty]$ be positive measures on it.

(i) Define absolute continuity of ν with respect to μ, $\nu \ll \mu$.

The $\varepsilon - \delta$ notion of absolute continuity is the following:

Definition. The measure ν is said to be $\varepsilon - \delta$ absolutely continuous with respect to μ if for every $\varepsilon > 0$ there is $\delta = \delta_\varepsilon > 0$ such that $(|\nu(E)| =)\nu(E) \le \varepsilon$ for every $E \in \mathcal{M}$ with $\mu(E) \le \delta$.

(ii) Prove that $\varepsilon - \delta$ absolute continuity implies absolute continuity.

(iii) With $X = \mathbb{R}$ and $\mathcal{M} = \mathcal{B}(\mathbb{R})$, Borel $\sigma-$algebra of $\mathbb{R}$, let $\mu = m$=Lebesgue measure, and $d\nu = x^2\, dm$. Prove that $\nu \ll m$, but that ν is not $\varepsilon - \delta$ absolutely continuous with respect to m.

(iv) On a measure space $(X, \mathcal{M}, \mu)$ let ρ be a positive function in $L^\infty(\mu)$, and let $d\nu = \rho\, d\mu$. Prove that ν is $\varepsilon - \delta$ absolutely continuous with respect to μ.

(v) Prove that if ν is a finite measure, and $\nu \ll \mu$ then ν is also $\varepsilon - \delta$ absolutely continuous with respect to μ.

Solution. (i) For every $E \in \mathcal{M}$, if $\mu(E) = 0$ then also $\nu(E) = 0$. (ii) If $\mu(E) = 0$, then $\mu(E) < \delta$ for every $\delta > 0$ so that $\nu(E) \le \varepsilon$ for every $\varepsilon > 0$, hence $\nu(E) = 0$.

(iii) We compute $\nu([a, a + \delta])$:

$$\nu([a, a+\delta]) = \int_a^{a+\delta} x^2\, dm = \left[\frac{x^3}{3}\right]_{x=a}^{x=a+\delta} = \frac{(a+\delta)^3 - a^3}{3} = \frac{a^3}{3}((1 + \delta/a)^3 - 1).$$

For $a \to +\infty$ we have $\nu([a, a + \delta]) \to \infty$, and the $\varepsilon - \delta$ condition cannot hold.

(iv) Trivial: for every $E \in \mathcal{M}$ of finite $\mu-$measure we have

$$\nu(E) = \int_E \rho\, d\mu = \int_E g\, d\mu \le \int_E \|\rho\|_\infty\, d\mu = \|\rho\|_\infty\, \mu(E),$$

so that, given $\varepsilon > 0$ we take $\delta = \varepsilon/\|\rho\|_\infty$ and we get $\nu(E) \le \varepsilon$ if $\mu(E) \le \delta$.

(v) See LN,6.2.5.3: when ν is finite then $\nu \ll \mu$ implies that ν verifies also the $\varepsilon - \delta$ condition.

□

EXERCISE 27. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\mu(X) < \infty$. Assume that $0 < p < q \le \infty$

(i) Prove that there exists a constant $C(p, q) > 0$ such that for every measurable $f : X \to \mathbb{C}$ we have

$$\|f\|_p \le C(p, q)\,\|f\|_q;$$

and find such a constant.

(ii) We have $L^p(\mu) \supseteq L^q(\mu)$, and convergence of a sequence in $L^q(\mu)$ implies convergence of the sequence in $L^p(\mu)$, to the same limit; prove these statements.

(iii) Prove that $L^p([0,1]) \supsetneqq L^q([0,1])$, and that $L^\infty([0,1]) \subsetneqq \bigcap_{0<p<\infty} L^p([0,1])$; the measure is Lebesgue measure.

Solution. (i) Remember (LN, 5.1.8) that we have

$$\|f\|_p \le \mu(X)^{1/p-1/q}\,\|f\|_q,$$

for every measurable $f \in L(X)$. In fact, assuming first $q < \infty$, and applying Hölder's inequality to the pair of functions $|f|^p, 1$ with conjugate exponents q/p, $q/(q-p)$ we get:

$$\int_X |f|^p = \int_X |f|^p\,1 \le \left(\int_X |f|^q\right)^{p/q} \left(\int_X 1^{q/(q-p)}\right)^{(q-p)/q} = \mu(X)^{1-p/q}\,\|f\|_q^p,$$

so we need only to take $p-$th roots of both sides. For $q = \infty$ the inequality is immediate.

(ii) Is now trivial: $f \in L^q(\mu)$ means that f is measurable and that $\|f\|_q < \infty$; the preceding inequality says that then also $\|f\|_p < \infty$, so that $f \in L^p(\mu)$. Similarly, $f_n \to f$ in $L^q(\mu)$ means that $\|f - f_n\|_q \to 0$; since for $p < q$

$$\|f - f_n\|_p \le \mu(X)^{1/p-1/q}\,\|f - f_n\|_q$$

this implies $\|f - f_n\|_p \to 0$ and hence $f_n \to f$ also in $L^p(\mu)$.

(iii) The function $f_\alpha(x) = 1/x^\alpha$ is in $L^p([0,1])$ iff $p\alpha < 1$; if $p\alpha < 1$ but $q\alpha > 1$, that is for $\alpha \in]1/q, 1/p[$ then $f_\alpha \in L^p \setminus L^q$. And $\log x$ is in $\bigcap_{0<p<\infty} L^p([0,1]) \setminus L^\infty([0,1])$.

□

EXERCISE 28. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = -e^{-|x|}$ for $x < 0$, $F(x) = \sqrt{(2x - x^2)^+}$ for $x \ge 0$ (as usual, $(2x - x^2)^+ = \max\{2x - x^2, 0\}$ is the positive part of $2x - x^2$, for every $x \in \mathbb{R}$).

(i) Plot F.

(ii) Find the total variation function $T(x) = VF(]-\infty, x])$, the positive and negative variation $F_\pm$ of F, and plot all these functions.

(iii) For the signed measure $\nu = \mu_F$ associated to F describe a Hahn decomposition, and describe the Lebesgue–Radon–Nikodym decomposition of $\nu^\pm$ with respect to Lebesgue measure m on $\mathcal{B}_1$.

Let now $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^+ = \max\{x, 0\}$.

(iv) Compute

$$\int_{\mathbb{R}} f\,d|\nu|.$$

(v) Prove that $f \in L^\infty(|\nu|)$ and compute $\|f\|_\infty$ in this space.

Solution. (i) Easy:

(ii) We have $T(x) = e^x$ for $x < 0$; $T(0) = 2$; $T(x) = 2 + \sqrt{2x - x^2}$ for $0 \le x < 1$; $T(x) = 4 - \sqrt{2x - x^2}$ for $1 \le x < 2$; $T(x) = 4$ for $x \ge 2$. And we have $F_\pm(x) = (T(x) \pm F(x))/2$ so that

$$F_+(x) = 0 \quad (x < 0);\ F_+(x) = 1 + \sqrt{2x - x^2} \quad (0 \le x < 1);\ F_+(x) = 2 \quad (1 \le x).$$

$$F_-(x) = e^x \quad (x < 0);\ F_-(x) = 1 \quad 0 \le x < 1;\ F_-(x) = 2 - \sqrt{(x - 2x^2)^+} \quad 1 \le x$$

(in particular, $F_-(x) = 2$ if $2 \le x$).

(iii) We can take $P = [0, 1[$ and $Q = \mathbb{R} \setminus P$. Also

$$d\nu^+ = \delta_0 + \frac{1-x}{\sqrt{2x - x^2}}\chi_{]0,1[}(x)\,dx; \quad d\nu^- = e^x\,\chi_{[-\infty,0[}(x)\,dx + \frac{x-1}{\sqrt{2x - x^2}}\chi_{]1,2[}(x)\,dx.$$

(iv) We have

$$\int_{\mathbb{R}} f\,d|\nu| = \int_{\mathbb{R}} f\,d\nu^+ + \int_{\mathbb{R}} f\,d\nu^- = \int_0^1 x\,\frac{1-x}{\sqrt{2x - x^2}}\,dx + \int_1^2 x\,\frac{x-1}{\sqrt{2x - x^2}}\,dx =$$

$$\left[x\sqrt{2x-x^2}\right]_{x=0}^{x=1} - \int_0^1 \sqrt{2x-x^2}\,dx + \left[-x\sqrt{2x-x^2}\right]_{x=1}^{x=2} + \int_1^2 \sqrt{2x-x^2}\,dx =$$
$$1 - \int_0^1 \sqrt{2x-x^2}\,dx + 1 + \int_1^2 \sqrt{2x-x^2}\,dx = 2.$$

(v) Since $|\nu|([2,+\infty[) = 0$ we have $\|f\|_\infty = 2$.

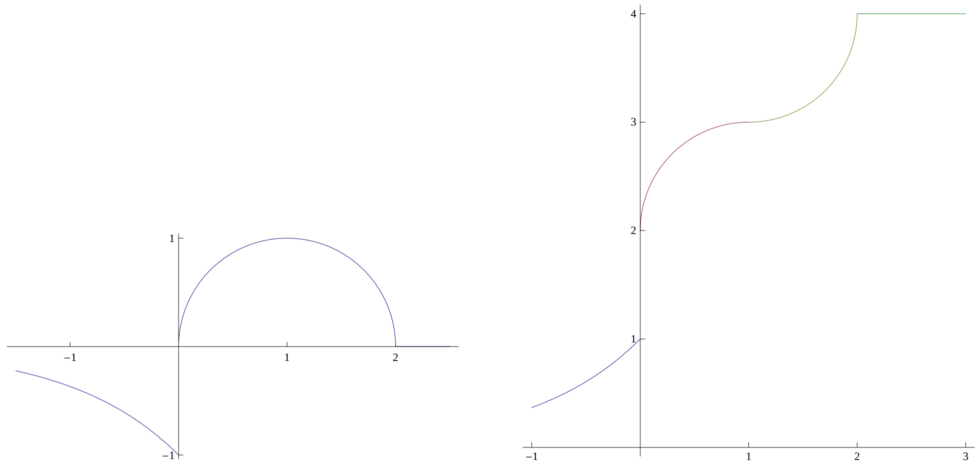

FIGURE 15. Plots of F (left) and T (right).

(the plots of $F_\pm$ are omitted). □

ANALISI REALE PER MATEMATICA – APPELLO DI RICUPERO – 19 SETTEMBRE 2012

EXERCISE 29. Let $(X, \mathcal{M}, \mu)$ be a measure space; for simplicity we consider only real valued functions on X, in particular here $L^1(\mu)$ consists of real–valued functions only.

(i) Define (real valued) measurable simple functions, and prove that such a function f is in $L^1(\mu)$ if and only if its cozero–set $\{f \neq 0\}$ has finite measure.

We call $S(\mu)$ the set of all simple functions which belong to $L^1(\mu)$.

(ii) Prove that a positive measurable function $f : X \to [0,\infty[$ is in $L^1(\mu)$ if and only if it is the limit in $L^1(\mu)$ of an increasing sequence of positive simple functions in $L^1(\mu)$, and deduce from this that $S(\mu)$ is dense in $L^1(\mu)$.

Assume now that $(X, \mathcal{M}, \mu)$ is the Carathèodory extension of a premeasure (still called μ) defined on an algebra $\mathcal{A}$ of parts of X.

(iii) Prove that if $E \in \mathcal{M}$ and $\mu(E) < \infty$ then for every $\varepsilon > 0$ there is $A \in \mathcal{A}$ such that $\mu(A \triangle E) \le \varepsilon$; deduce from this fact that the subspace of $\mathcal{A}$–simple functions in $S(\mu)$ is still dense in $L^1(\mu)$.

Solution. (i) A simple function is a function with finite range; real–valued measurable simple functions are then functions f of the form $f = \sum_{k=1}^m \alpha_k\,\chi_{E(k)}$, where $\{E(k) : k = 1,\dots,m\}$ is a finite partition of X into members of $\mathcal{M}$, and $\{\alpha_1,\dots,\alpha_m\}$ is set of m different real numbers (we are here talking of the standard representation). The absolute value of such a function is then $|f| = \sum_{k=1}^m |\alpha_k|\,\chi_{E(k)}$, and by definition the integral of such a function is $\int_X |f| = \sum_{k=1}^m |\alpha_k|\,\mu(E(k))$; this is a finite value if and only if $\mu(E(k)) = \infty$ implies $|\alpha_k| = 0$, that is, on a set of infinite measure the simple function must be identically zero. The cozero–set of f is $\{|f| > 0\}$ and is $\bigcup\{E(k) : |\alpha_k| > 0\}$; so $f \in L^1(\mu) \iff |f| \in L^1(\mu) \iff \mu(\{|f| > 0\}) < \infty$ has been proved.

(ii) Recall that every positive measurable function f is the pointwise limit of an increasing sequence of positive measurable simple functions φ_n (LN, 3.2.3). If $f \in L^1(\mu)$ then clearly $\varphi_n \in L^1(\mu)$, and by monotone convergence $\int_X f = \lim \int_x \varphi_n$, which implies

$$\|f - \varphi_n\|_1 = \int_X (f - \varphi_n) = \int_X f - \int_X \varphi_n \to 0 \quad \text{for} \quad n \to \infty.$$

Clearly any L^1 limit of a sequence of functions in L^1 is in L^1. Given a real $f \in L^1(\mu)$ we simply split $f$as $f = f^+ - f^-$; if φ_n, ψ_n are sequences of simple functions converging in $L^1(\mu)$ to $f^\pm$ respectively, then $\varphi_n - \psi_n$ converges to f in $L^1(\mu)$.

(iii) The first part is LN, 2.3.4. Given a simple function in $L^1(\mu)$, $f=\sum_{k=1}^m \alpha_k\,\chi_{E(k)}$ (where now the value 0 of f, if present, is omitted so that $\alpha_k\neq 0$ and $\mu(E(k))<\infty$ for every $k\in\{1,\dots,m\}$) and $\varepsilon>0$ we can pick for every $k\in\{1,\dots,m\}$ a set $A(k)\in\mathcal{A}$ such that $\mu(E(k)\vartriangle A(k))\le\varepsilon/\alpha$, where $\alpha=\sum_{k=1}^m|\alpha_k|$. If $g=\sum_{k=1}^m\alpha_k\,\chi_{A(k)}$ then g is $\mathcal{A}-$simple, belongs to $L^1(\mu)$, and

$$\|f-g\|_1=\int_X|f-g|=\int_X\left|\sum_{k=1}^m\alpha_k\,\chi_{E(k)}-\sum_{k=1}^m\alpha_k\,\chi_{A(k)}\right|\le\int_X\sum_{k=1}^m|\alpha_k|\,|\chi_{E(k)}-\chi_{A(k)}|=$$
$$\sum_{k=1}^m|\alpha_k|\int_X|\chi_{E(k)}-\chi_{A(k)}|=\sum_{k=1}^m|\alpha_k|\,\mu(E(k)\vartriangle A(k))\le\varepsilon,$$

Then the closure of the set of $\mathcal{A}-$simple functions in $L^1(\mu)$ contains $S(\mu)$, which is dense in $L^1(\mu)$; then this closure is all of $L^1(\mu)$.

Remark. A more direct proof of the above is in LN, 3.3.15.

□

Exercise 30. Let $(X,\mathcal{M},\mu)$ be a measure space.

(i) What does it mean that $E\in\mathcal{M}$ is of $\sigma-$finite measure? when is the measure space called $\sigma-$finite?

An *atom* in the measure space $(X,\mathcal{M},\mu)$ is a set $A\in\mathcal{M}$ with $\mu(A)>0$ such that for every $E\in\mathcal{M}$ contained in A we either have $\mu(E)=0$ or $\mu(A\smallsetminus E)=0$.

(ii) If $A,B\in\mathcal{M}$ are atoms, then either $\mu(A\cap B)=0$, or $\mu(A\cap B)=\mu(A)=\mu(B)$.

(iii) Prove that in a $\sigma-$finite measure space an atom has finite measure.

Two sets $A,B\in\mathcal{M}$ are said to be *almost disjoint* if $\mu(A\cap B)=0$.

(iv) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise almost disjoint sets in $\mathcal{M}$, and let $A=\bigcup_{n\in\mathbb{N}}A_n$. Prove that
$$\mu(A)=\sum_{n=0}^\infty\mu(A_n).$$

(v) Prove that in a $\sigma-$finite measure space a family of pairwise almost disjoint atoms is at most countable.

Solution. (i) Lecture notes, 2.2.8. (ii) If $\mu(A\cap B)>0$, then $\mu(A\smallsetminus(A\cap B))=\mu(B\smallsetminus(A\cap B))=0$ because A and B are atoms. Then $\mu(A)=\mu(A\cap B)+\mu(A\smallsetminus(A\cap B))=\mu(A\cap B)$ and $\mu(B)=\mu(A\cap B)+\mu(B\smallsetminus(A\cap B))=\mu(A\cap B)$ by finite additivity; by transitivity $\mu(A)=\mu(B)(=\mu(A\cap B))$. (iii) We can reproduce the argument given above that $\sigma-$finiteness implies semifiniteness; at any rate, if $A_n\in\mathcal{M}$ is an increasing sequence of sets of finite measure with union X, and A is an atom, we also have $A\cap A_n\uparrow A$, so that if $\mu(A\cap A_n)=0$ for every n we get $\mu(A)=0$, a contradiction; then $\mu(A\cap A_n)>0$ for some n, which implies $\mu(A\smallsetminus A_n)=0$, and $\mu(A)=\mu(A\cap A_n)+\mu(A\smallsetminus A_n)=\mu(A\cap A_n)<\infty$.

(iv)Let's apply the usual trick for making a disjoint union, $B_k=A_k\smallsetminus\left(\bigcup_{j=0}^{k-1}A_j\right)$. We have $B_k\subseteq A_k$, and if the A_k's are pairwise almost disjoint then $\mu(B_k)=\mu(A_k)$: in fact $A_k\smallsetminus B_k=A_k\cap\left(\bigcup_{j=0}^{k-1}A_j\right)=\bigcup_{j=0}^{k-1}A_k\cap A_j$ is a finite union of sets of measure zero, and has then measure zero.

(v) Assume that $E\subseteq X$ has finite measure, and let $(A_\lambda)_{\lambda\in\Lambda}$ be a family of almost disjoint atoms contained in E; we prove that $\sum_{\lambda\in\Lambda}\mu(A_\lambda)\ \big(:=\sup\{\sum_{\lambda\in F}\mu(A_\lambda):\ F\text{ a finite subset of }\Lambda\}\big)\le\mu(E)$; this implies that Λ is countable (Lecture Notes, lemma 1.2.4). In fact, for every finite subset $F\subseteq\Lambda$ we have, by (i), $\sum_{\lambda\in F}\mu(A_\lambda)=\mu\left(\bigcup_{\lambda\in F}A_\lambda\right)\le\mu(E)$. We have proved that any subset of X of finite measure contains an at most countable set $\mathcal{A}(E)$ of pairwise almost disjoint atoms; since $X=\bigcup_{n\in\mathbb{N}}E_n$, where $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of sets of finite measure, we have that $\mathcal{A}(X)=\bigcup_{n\in\mathbb{N}}\mathcal{A}(E_n)$ is a countable union of countable sets, hence countable.

□

Exercise 31. Let $(X,\mathcal{M},\mu)$ be a measure space, and let $f\in L(X)$ be a measurable function.

(i) Prove that
$$\liminf_{p\to\infty}\|f\|_p\ge\|f\|_\infty$$
(given $0<\alpha<\|f\|_\infty$ use Čebičeff's inequality for L^p to prove that $\liminf_{p\to\infty}\|f\|_p\ge\alpha$).

(ii) Assuming $f\in L^p(\mu)$ for some $p>0$ prove that $\limsup_{p\to\infty}\|f\|_p\le\|f\|_\infty$.

(iii) Find a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{p\to\infty} \|f\|_p$ exists, but is not equal to $\|f\|_\infty$.

(iv) Compute the limit
$$\lim_{n\to\infty} \left(\int_0^\infty \frac{dx}{(1+x^2)^n}\right)^{1/n}$$
and deduce from it the value of $\lim_{n\to\infty}((2n)!/(n!)^2)^{1/n}$ (use the Beta and Gamma functions to evaluate the preceding integral; do this last part only if you spare some time).

Solution. (i) and (ii): LN, 5.1.1. (iii) Take the constant 1: its $p-$norms are all infinite, but $\|1\|_\infty = 1$.

(iv) Clearly all $p-$norms are finite, so that the limit is the $L^\infty-$norm of $f(x) = 1/(1+x^2)$ in $[0, \infty[$, which is 1. To compute the integrals: first use the change of variables $x^2 = t$, which gives
$$\int_0^\infty \frac{dx}{(1+x^2)^n} = \frac{1}{2}\int_0^\infty \frac{t^{-1/2}}{(1+t)^n}\,dt = \frac{1}{2}\,B(1/2, n-1/2) = \frac{1}{2}\,\frac{\Gamma(1/2)\,\Gamma(n-1/2)}{\Gamma(n)}.$$
We have $\Gamma(1/2) = \pi^{1/2}$ and
$$\Gamma(n-1/2) = \frac{(n-1/2)\,\Gamma(n-1/2)}{n-1/2} = \frac{\Gamma(n+1/2)}{n-1/2} = \frac{(2n)!}{2^{2n}n!}\,\pi^{1/2},$$
so that
$$\left(\int_0^\infty \frac{dx}{(1+x^2)^n}\right)^{1/n} = \left(\frac{\pi}{2}\,\frac{(2n)!}{2^{2n}n!\Gamma(n)}\right)^{1/n} = \frac{1}{4}\left(\frac{n\pi}{2}\right)^{1/n}\left(\frac{(2n)!}{(n!)^2}\right)^{1/n};$$
as $n \to \infty$ the left–hand side tends to 1, and also $(n\,\pi/2)^{1/n}$ tends to 1; then the required limit is 4. □

Exercise 32. Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = \operatorname{sgn} x\, e^{-|x|}$.

(i) Plot F.

(ii) Find the total variation function $T(x) = VF(]-\infty, x])$, the positive and negative variation $F_\pm$ of F, and plot all these functions.

(iii) For the signed measure $\nu = \mu_F$ associated to F describe a Hahn decomposition, and describe the Lebesgue–Radon–Nikodym decomposition of $\nu^\pm$ with respect to Lebesgue measure m on $\mathcal{B}_1$.

Let now $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$.

(iv) Compute
$$\int_{\mathbb{R}} f^p\, d|\nu|,$$
for every $p > 0$

(v) Is it true that $f \in L^\infty(|\nu|)$?

Solution. (schematic) The plots are easy and we omit them. The total variation is $T(x) = e^x$ for $x < 0$, $T(0) = 2$, $T(x) = 4 - e^{-x}$ for $x > 0$. The positive variation is $F_+(x) = 0$ for $x < 0$, $F_+(0) = 1$, $F_+(x) = 2$ for $x > 0$; the negative is $F_-(x) = e^x$ for $x < 0$, $F_-(x) = 2 - e^{-x}$ for $x \geq 0$. A Hahn decomposition is $P = \{0\}$ and $Q = \mathbb{R} \setminus \{0\}$. The singular part is $2\,\delta_0$, the absolutely continuous part is $-e^{-|x|}\,dm$. We have
$$\int_{\mathbb{R}} |f|^p\, d|\nu| = 2|f(0)|^p\,\delta_0 + \int_{\mathbb{R}\setminus\{0\}} |x|^p\, e^{-|x|}\, dm(x) = 2\int_0^\infty x^p\, e^{-x}\, dx = 2\Gamma(p+1).$$
Then $f \in L^p(|\nu|)$ for every $p > 0$. Clearly f is not in $L^\infty(|\nu|)$): for every $\alpha > 0$ the set $\{x \in \mathbb{R} : |x| > \alpha\}$ is the union of the two half lines $]-\infty, -\alpha[\cup]\alpha, \infty[$, of $|\nu|-$measure $2\,\exp(-\alpha) > 0$.

□