REAL ANALYSIS EXAMS
A.A 2012–13

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1. Analisi Reale per Matematica – Precompitino – 7 novembre 2012

Exercise 1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Recall that $E \in \mathcal{M}$ is said to be of $\sigma-$finite measure if it can be covered by a sequence of sets in $\mathcal{M}$ of finite measure. Prove that for $E \in \mathcal{M}$ the following are equivalent:
  (a) E has $\sigma-$finite measure.
  (b) E can be written as a countable disjoint union of sets in $\mathcal{M}$ of finite measure.
  (c) E can be written as the union of an increasing sequence of sets in $\mathcal{M}$ of finite measure.

(ii) Prove that if $\mathcal{S} = \{E \in \mathcal{M} : E \text{ has } \sigma-\text{finite measure}\}$ then $\mathcal{S}$ is a $\sigma-$ideal of $\mathcal{M}$, that is, $\mathcal{S}$ is closed under countable union and the formation of subsets (i.e., if $E \in \mathcal{S}$, $F \in \mathcal{M}$ and $F \subseteq E$, then $F \in \mathcal{S}$).

(iii) Recall that an atom of infinite measure is a set $A \in \mathcal{M}$ such that $\mu(A) = \infty$, and for every $E \in \mathcal{M}$ with $E \subseteq A$ we have either $\mu(E) = 0$ or $\mu(A \setminus E) = 0$. Prove that if A is an atom of infinite measure and E has $\sigma-$finite measure then $\mu(E \cap A) = 0$.

The questions that follow are not related to the preceding ones

(iv) Let $(c_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers, with $c_0 = 0$, and let $f_0 \in L^1_m(\mathbb{R})$. If we define, for $n \in \mathbb{N}$, $f_n(x) = 2^n\, f_0(4^n(x - c_n))$, then the formula

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

defines for $m-$a.e. $x \in \mathbb{R}$ a function $f \in L^1_m(\mathbb{R})$: give a careful explanation, quoting the relevant theorems (m is Lebesgue measure).

(v) Prove that if $g \in L^1_m(\mathbb{R})$ then $\liminf_{x\to\infty} |g(x)| = 0$; find a continuous $g \in L^1_m(\mathbb{R})$ such that $\limsup_{x\to\infty} |g(x)| = \infty$ (and this limsup remains infinite even after modification of g on a set of measure 0).

Solution. (i) (a) implies (b): if $E = \bigcup_{k=0}^{\infty} A_k$, with$A_k \in \mathcal{M}$, with the usual trick we make the union disjoint, setting $B_0 = A_0$ and $B_k = A_k \setminus \bigcup_{j=0}^{k-1} A_j$; clearly $\mu(B_k) \le \mu(A_k) < \infty$, for every $k \in \mathbb{N}$. (b) implies (c): if $E = \bigcup_{k\in\mathbb{N}} B_k$, with $B_k \in \mathcal{M}$ (disjoint or not) and $\mu(B_k) < \infty$, setting $A_m = \bigcup_{k=1}^{m} B_k$ we have $A_m \uparrow E$ and by subadditivity $\mu(A_m) \le \sum_{k=0}^{m} \mu(B_k) < \infty$. (c) implies (a): trivial.

(ii) If $(E_m)_{m\in\mathbb{N}}$ is a sequence of sets of $\sigma-$finite measure, and $E_m = \bigcup_{n\in\mathbb{N}} A_{m\,n}$, with each $A_{m\,n} \in \mathcal{M}$ of finite measure, we have

$$\bigcup_{m\in\mathbb{N}} E_m = \bigcup_{m\in\mathbb{N}} \left(\bigcup_{n\in\mathbb{N}} A_{m\,n} \right) = \bigcup_{(m,\,n)\in\mathbb{N}\times\mathbb{N}} A_{m\,n},$$

a countable union of sets of finite measure, since $\mathbb{N}\times\mathbb{N}$ is countable. Any measurable subset F of a set E of $\sigma-$finite measure is of course of $\sigma-$finite measure: if $E = \bigcup_{k\in\mathbb{N}} A_k$ we have $F = \bigcup_{k\in\mathbb{N}} F \cap A_k$, and $\mu(F \cap A_k) \le \mu(A_k) < \infty$.

(iii) $E \cap A$ has $\sigma-$finite measure, being a subset of E, as just proved; since every subset of finite measure of an atom of infinite measure has measure 0, $E \cap A$ is countable union of sets of measure 0, and has then measure 0.

(iv) We have:

$$\|f_n\|_1 = \int_{\mathbb{R}} 2^n\, |f_0(2^n(x - c_n))|\, dx = 2^n \int_{\mathbb{R}} |f_0(t)|\, \frac{dt}{4^n} = \frac{1}{2^n} \int_{\mathbb{R}} |f_0(t)|\, dt = \frac{\|f_0\|_1}{2^n},$$

so that the series $\sum_{n=0}^{\infty} \|f_n\|_1 = 2\|f_0\|_1$ is convergent. The theorem on normally convergent series says that then the series of functions $\sum_{n=0}^{\infty} f_n(x)$ converges a.e. an in $L^1_m(\mathbb{R})$ to an $f \in L^1_m(\mathbb{R})$. We also have $\int_{\mathbb{R}} f(x)\,dx = 2\int_{\mathbb{R}} f_0(x)\,dx$.

(v) If $\liminf_{x\to\infty} |g(x)| = \alpha > 0$, given $\beta \in \mathbb{R}$ with $0 < \beta < \alpha$ there is $a \in \mathbb{R}$ such that $|g(x)| > \beta$ for every $x \ge a$. Then $|g|$ cannot have a finite integral: $\int_{\mathbb{R}} |g| \ge \int_{\mathbb{R}} \beta\,\chi_{[a,\infty[} = \infty$. To construct g as required we may take f_0 continuous with support in $[0,1]$, e.g, $f_0(x) = (1 - |2x-1|) \vee 0$, and $c_n = n$; since $f_n(x) = 2^n f_0(4^n(x-n))$ has $[n, n+1/4^n]$ as support, the sum $f = \sum_{n=0}^{\infty} f_n$ is continuous (on the interval $]-m, m[$ the function f coincides with $\sum_{n=0}^{m} f_n$, a finite sum of continuous functions, hence a continuous function, see the figure). It is clear that $\limsup_{x\to\infty} f(x) = \infty$, and that changing f on a set of measure 0 cannot destroy this fact (for every $a > 0$ the essential supremum of f on $[a, \infty[$ is ∞). We set $g = f$.

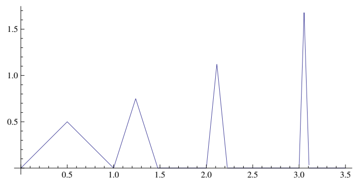

FIGURE 1. Plot of $g = f$ (not on scale).

□

EXERCISE 2. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be increasing; denote by $\mu = d\alpha$ the Radon–Stieltjes measure associated to α. A Borel measurable function $f : \mathbb{R} \to \mathbb{C}$ is said to be locally in $L^1(\mu)$ if for every compact subset K of $\mathbb{R}$ we have $f\,\chi_K \in L^1(\mu)$. For such an f, assuming for simplicity that $c \in \mathbb{R}$ is such that α is continuous at c, we define $F : \mathbb{R} \to \mathbb{C}$ by

$$F(x) = \int_{]c,x]} f\,d\mu \left(= \int_{[c,x]} f\,d\mu \right) \quad \text{for } x \ge c, \quad F(x) = -\int_{]x,c]} f\,d\mu \quad \text{for } x < c.$$

(i) Prove that F is right–continuous and has finite left limits at every point (use dominated convergence: if $x, x_n \in I$ and $x_n \downarrow x$, resp $x_n \uparrow x$ increasing strictly, then the sequence χ_n of the characteristic functions of the intervals of extremes c, x_n tends to ...). Compute the jump $F(x) - F(x^-)$ and prove that if α is continuous at x then also F is continuous at x.

(ii) Prove that if $f \ge 0$, then F is increasing, and as such defines a Radon–Stieltjes measure dF on the Borel subsets of $\mathbb{R}$. Prove that $dF = f\,d\mu$.

From now on we assume that α is continuous. Recall that if F, G are right–continuous increasing functions $F, G : I \to \mathbb{R}$ we have the formula of integration by parts:

$$\int_{]a,b]} F(x^-)\,dG(x) + \int_{]a,b]} G(x)\,dF(x) = F(b)\,G(b) - F(a)\,G(a),$$

for every $a, b \in I$, with $a < b$, so that in particular, if F is as above, with $f \ge 0$ locally in $L^1(\mu)$ we have

$$(*) \qquad \int_a^b G(x)\,f(x)\,d\mu(x) = F(b)G(b) - F(a)\,G(a) - \int_{]a,b]} F(x)\,dG(x),$$

(iii) Prove that formula (*) holds for every f locally in $L^1(\mu)$, of any sign and also complex–valued, and not only for $f \ge 0$.

Solution. (i) Assuming first $x > c$, let χ_n be the characteristic function of the interval $]c, x_n]$; if $x_n \downarrow x$ then $\chi_n \le \chi_{[c,x_0]}$, and χ_n converges pointwise everywhere to $\chi_{]c,x]}$.

If $x_n \uparrow x$ (with x_n strictly increasing) then $\chi_n \le \chi_{[c,x]}$, for every n, and χ_n converges pointwise everywhere to $\chi_{]c,x[}$. In any case $|f\,\chi_n| \le |f|\,\chi_K$, with K compact ($K = [c, x_0]$ in the first case, $K = [c, x]$ in the second case); since this function is in $L^1(\mu)$ we can apply dominate convergence to show that

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_{\mathbb{R}} f\,\chi_n\,d\alpha = \int_{\mathbb{R}} (\lim_{n\to\infty} f\,\chi_n)\,d\alpha = \int_{\mathbb{R}} f\,(\lim_{n\to\infty} \chi_n)\,d\alpha.$$

If $x_n \downarrow x$, then as observed above we have $\lim_{n\to\infty}\chi_n = \chi_{]c,x]}$ so that

$$\lim_{n\to\infty} F(x_n) = \int_{\mathbb{R}} f\,(\lim_{n\to\infty}\chi_n)\,d\alpha = \int_{]c,x]} f\,d\alpha = F(x),$$

so that F is right–continuous at x. And if $x_n \uparrow x$ with $x_n < x$ for every n, then

$$\lim_{n\to\infty} F(x_n) = \int_{\mathbb{R}} f\,(\lim_{n\to\infty}\chi_n)\,d\alpha = \int_{]c,x[} f\,d\alpha;$$

this proves that $F(x^-) = \int_{]c,x[} f\,d\alpha$; then

$$F(x) - F(x^-) = \int_{]c,x]} f\,d\alpha - \int_{]c,x[} f\,d\alpha = \int_{\{x\}} f\,d\alpha = f(x)\,(\alpha(x^+) - \alpha(x^-));$$

in particular F is continuous wherever α is continuous. The proofs for $x \le c$ are similar, we only have to change some signs.

(ii) Clearly F is increasing : one easily sees that if $x_1 < x_2$, with $x_1,\, x_2 \in \mathbb{R}$ then

$$F(x_2) - F(x_1) = \int_{]x_1,x_2]} f\,d\alpha \ge 0 \quad \text{(by positivity of } f).$$

Moreover, for every compact interval $[a,b]$ we have

$$F(b) - F(a^-) = \int_{[a,b]} f\,d\alpha,$$

so that the measure dF and $f\,d\alpha$ coincide and are finite on compact intervals, and hence on every Borel set, since the set of compact intervals is closed under intersection and generates the Borel σ−algebra.

(iii) For real f we write $f = f^+ - f^-$, and we have the formulae:

$$\int_a^b G(x)\,f^+(x)\,d\mu(x) = F_+(b)G(b) - F_+(a)\,G(a) - \int_{]a,b]} F_+(x)\,dG(x)$$

$$\int_a^b G(x)\,f^-(x)\,d\mu(x) = F_-(b)G(b) - F_-(a)\,G(a) - \int_{]a,b]} F(x)\,dG(x),$$

where of course $F_\pm(x) = \operatorname{sgn}(x-c)\int_{]c,x]} f^\pm\,d\alpha$. Subtracting the second formula from the first we get the result. Similarly, for complex f we use real and imaginary parts: the general formula, for a non necessarily positive f, is due to its linearity in f and F. $\square$

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be defined by $f(x,y) = e^{-xy^2}\,e^{ix}$. For $a > 0$ let $E(a) = [0,a] \times [0,\infty[$, $E = [0,\infty[^2$ the first quadrant.

(i) Prove that $f \in L^1(E(a))$ and that $f \notin L^1(E)$.

(ii) Reduce the integral of f on $E(a)$ to one dimensional integrals; compute then the limit

$$\lim_{a\to\infty} \int_{E(a)} f(x,y)\,dxdy$$

in terms of these integrals, and deduce from it the value of the generalized integrals:

$$\int_0^\infty \frac{\cos t}{\sqrt{t}}\,dt; \qquad \int_0^\infty \frac{\sin t}{\sqrt{t}}\,dt$$

(*Fresnel's integrals*; they are not Lebesgue integrals, being non–absolutely convergent).

A careful application of the theorems of Tonelli and Fubini is required. It is useful to know that

$$\int_0^\infty e^{-\alpha\,y^2}\,dy = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \quad (\alpha > 0); \qquad \int_0^\infty \frac{y^2}{1+y^4}\,dy = \int_0^\infty \frac{dy}{1+y^4} = \frac{\pi}{2\sqrt{2}}.$$

Solution. (i) The function f is continuous and hence Borel measurable. We have $|f(x,y)| = e^{-xy^2}$ so that

$$\int_{E(a)} |f(x,y)|\,dxdy = \int_{x=0}^{x=a}\left(\int_0^\infty e^{-xy^2}\,dy\right)dx = \int_0^a \frac{\sqrt{\pi}}{2\sqrt{x}}\,dx < \infty \quad \text{for every } a > 0,$$

while

$$\int_E |f(x,y)|\,dxdy = \int_0^\infty \frac{\sqrt{\pi}}{2\sqrt{x}}\,dx = \infty;$$

(i) is proved: by Tonelli's theorem we have $f \in L^1(E(a))$ and $f \notin L^1(E)$.

(ii) Since $f \in L^1(E(a))$ Fubini's theorem applies and we get

$$\int_{E(a)} f(x,y)\,dxdy = \int_{x=0}^{x=a}\left(\int_{y=0}^{y=\infty} e^{-xy^2}\,dy\right) e^{ix}\,dx = \frac{\sqrt{\pi}}{2}\int_0^a \frac{e^{ix}}{\sqrt{x}}\,dx;$$

$$= \int_{y=0}^{y=\infty}\left(\int_{x=0}^{x=a} e^{-(y^2-i)x}\,dx\right) dy = \int_0^\infty \left[-\frac{e^{-(y^2-i)x}}{y^2-i}\right]_{x=0}^{x=a} dy =$$

$$\int_0^\infty \frac{1-e^{-(y^2-i)a}}{y^2-i}\,dy = \int_0^\infty \frac{dy}{y^2-i} - \int_0^\infty \frac{e^{-(y^2-i)a}}{y^2-i}\,dy.$$

Notice now that $|e^{-(y^2-i)a}| = e^{-y^2a}\,|e^{ia}| = e^{-y^2a}$ and that $1/|y^2-i| \le 1$; as $a \to +\infty$ the function $y \mapsto e^{-(y^2-i)a}$ converges to zero for every $y > 0$ (its module is e^{-y^2a}), and for $a \ge 1$ all integrands are dominated by e^{-y^2}, which is in $L^1([0,\infty[)$. Then

$$\lim_{a\to\infty}\int_0^\infty \frac{e^{-(y^2-i)a}}{y^2-i}\,dy = 0.$$

Since

$$\int_{E(a)} f(x,y)\,dxdy = \frac{\sqrt{\pi}}{2}\int_0^a \frac{e^{ix}}{\sqrt{x}}\,dx = \int_0^\infty \frac{dy}{y^2-i} - \int_0^\infty \frac{e^{-(y^2+i)a}}{y^2-i}\,dy,$$

taking limits as $a \to +\infty$ we get

$$\frac{\sqrt{\pi}}{2}\int_0^{\uparrow\infty} \frac{e^{ix}}{\sqrt{x}}\,dx = \int_0^\infty \frac{dy}{y^2-i} = \left(\int_0^\infty \frac{y^2}{1+y^4}\,dy + i\int_0^\infty \frac{dy}{1+y^4}\right) = \frac{\pi}{2\sqrt{2}}(1+i),$$

and equating real and imaginary parts:

$$\int_0^{\uparrow\infty} \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}; \qquad \int_0^{\uparrow\infty} \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}.$$

□

2. Analisi Reale per Matematica – Primo Compitino – 17 novembre 2012

Exercise 4. Let $(X,\mathcal{M},\mu)$ be a measure space, and let u_n, f_n, v_n be sequences in $L^1_\mu(X,\mathbb{R})$, pointwise converging a.e. to u, f, v, respectively; assume that $u,v \in L^1(\mu)$ and

$$u_n(x) \le f_n(x) \le v_n(x) \quad \text{for every } n \in \mathbb{N} \text{ and a.e. } x \in X; \quad \lim_{n\to\infty}\int_X u_n = \int_X u; \quad \lim_{n\to\infty}\int_X v_n = \int_X v.$$

(i) Prove that in these hypotheses also $f \in L^1_\mu(X,\mathbb{R})$ and $\lim_{n\to\infty}\int_X f_n = \int_X f$; use only Fatou's lemma for the proof, and not the dominated convergence theorem.
(ii) State the dominated convergence theorem.
(iii) The generalized dominated convergence theorem says that if f_n, g_n are sequences in $L^1(\mu)$, pointwise converging to f and g respectively, $|f_n| \le g_n$ and $\int_X g_n \to \int_X g < \infty$, then $f \in L^1(\mu)$ and $\int_X f_n \to \int_X f$. Prove this theorem using the previous result (i) on the three sequences.

Solution. (i) Clearly $u(x) \le f(x) \le v(x)$ for a.e. $x \in X$. Since $u,v \in L^1(\mu)$ by hypothesis we have also $f \in L^1(\mu)$ (e.g. because $-v \le -f \le -u$, so that $|f| = f \vee (-f) \le v \vee (-u)$, and the $\vee$ of two functions in $L^1(\mu)$ is in $L^1(\mu)$; at any rate, it is clear that $|f| \le |u| + |v|$, and this function is in $L^1(\mu)$ because u and v are in $L^1(\mu)$ by hypothesis).

Apply Fatou's lemma to $f_n - u_n \ge 0$, obtaining

$$\int_X \liminf_{n\to\infty}(f_n - u_n) \le \liminf_{n\to\infty}\int_X (f_n - u_n) = \liminf_{n\to\infty}\left(\int_X f_n - \int_X u_n\right) =$$

$$\liminf_{n\to\infty}\int_X f_n - \lim_{n\to\infty}\int_X u_n = \liminf_{n\to\infty}\int_X f_n - \int_X u;$$

the left hand side is $\int_X (f-u) = \int_X f - \int_X u$, so that we get

$$\int_X f - \int_X u \le \liminf_{n\to\infty}\int_X f_n - \int_X u \iff \int_X f \le \liminf_{n\to\infty}\int_X f_n.$$

Now apply Fatou's lemma to the sequence $v_n - f_n \geq 0$, obtaining

$$\int_X \liminf_{n\to\infty}(v_n - f_n) \leq \liminf_{n\to\infty} \int_X (v_n - f_n) = \liminf_{n\to\infty} \left(\int_X v_n - \int_X f_n\right) = \lim_{n\to\infty} \int_X v_n + \liminf_{n\to\infty}\left(-\int_X f_n\right) = \int_X v - \limsup_{n\to\infty} \int_X f_n;$$

the left hand side is $\int_X (v - f) = \int_X v - \int_X f$, so that we get

$$\int_X v - \int_X f \leq \int_X v - \limsup_{n\to\infty} \int_X f_n \iff \limsup_{n\to\infty} \int_X f_n \leq \int_X f,$$

which combined with the previous result yields $\int_X f = \lim_{n\to\infty} \int_X f_n$; (i) has been proved.

(ii) See the Lecture Notes.

(iii) $|f_n| \leq g_n$ is equivalent to $-g_n \leq f_n \leq g_n$ if f_n is a real valued function; we simply set $u_n = -g_n$ and $v_n = g_n$, and the hypotheses are all verified: clearly $\int_X u_n = -\int_X g_n \to -\int_X g$, etc. For f_n complex valued, use real and imaginary parts. □

Remark. Fatou's lemma applies only to sequences of positive functions! Many applied the lemma directly to the sequences $u_n \leq f_n \leq v_n$, a very serious blunder.

Moreover, some incorrectly presumed the following: if $g_n \in L^1_\mu(X, \mathbb{R})$ is a sequence converging a.e. to $g \in L^1_\mu(X, \mathbb{R})$, and $\int_X g_n \to \int_X g$, then $\int_X |g_n| \to \int_X |g|$, or $\int_X g_n^\pm \to \int_X g^\pm$, or even more, g_n converges to g in $L^1_\mu(X, \mathbb{R})$. This is in general not true when g_n may change sign. Let $X = [0, 1]$ with Lebesgue measure m, and let $g_n = n^2 \left(\chi_{]0,1/(2n)]} - \chi_{]1/(2n),1/n]}\right)$. Then $g_n \in L^1(m)$, $\lim_{n\to\infty} g_n(x) = 0$ for every $x \in [0, 1]$, so that the limit function 0 is in $L^1(m)$; moreover $\int_{[0,1]} g_n = 0$ for every n, so that $\lim_{n\to\infty} \int_{[0,1]} g_n = \int_{[0,1]} g = 0$; but $g_n^+ = n^2 \chi_{]0,1/(2n)]}$ and $g_n^- = n^2 \chi_{]1/(2n),1/n]}$ are such that

$$\int_{[0,1]} g_n^+ = \int_{[0,1]} g_n^- = \frac{n}{2} \to \infty,$$

whereas, of course, $\lim_{n\to\infty} g_n^+(x) = \lim_{n\to\infty} g_n^-(x) = 0$ for every $x \in [0, 1]$.

Exercise 5. Let μ be a positive finite measure on the Borel subsets of $\mathbb{R}$, $0 < \mu(\mathbb{R}) = a < \infty$; we also suppose that $\mu(]-\infty, 0[) = 0$. Let $F(x) = \mu(]-\infty, x])$ be the right continuous distribution function of μ, with initial point $-\infty$.

(i) Under what condition on μ is $F(0) = 0$?

(ii) Denoting by m the one–dimensional Lebesgue measure, compute $\mu \otimes m(T)$, where

$$T = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x\},$$

and deduce from it the formula

$$\int_{[0,\infty[} x \, d\mu(x) = \int_0^\infty (F(\infty) - F(x)) \, dx \quad (dx = dm(x)).$$

Is it true that the identity function $f(x) = x$ belongs to $L^1(\mu)$ if and only if $x \mapsto (F(\infty) - F(x))$ belongs to $L^1_m([0, \infty[)$?

(iii) Prove that the formula

$$\varphi(x) = \int_{[0,\infty[} \cos(xt) \, d\mu(t) \quad \text{defines a continuous function } \varphi : \mathbb{R} \to \mathbb{R}.$$

(iv) Assume that $F(t) = F(\infty) + O(1/t^2)$ for $t \to \infty$. Prove that then the function φ defined in (iii) belongs to $C^1(\mathbb{R})$.

Solution. (i) Clearly $F(x) = 0$ for every $x < 0$, so that $F(0^-) = 0$; then the jump of F at 0, namely $F(0) - F(0^-) = \mu(\{0\})$ coincides with $F(0)$: $F(0) = 0$ iff $\mu(\{0\}) = 0$.

(ii) T is closed in $\mathbb{R}^2$, hence Lebesgue measurable; both measures are σ−finite, μ even finite; then, if $T^y = \{x \in \mathbb{R} : (x, y) \in T\} = [y, \infty[$ if $y \geq 0$, and otherwise $T^y = \emptyset$:

$$\mu \otimes m(T) = \int_{[0,\infty]} \mu(T^y) \, dm(y) = \int_{[0,\infty[} (F(\infty) - F(y^-)) \, dm(y);$$

The set of discontinuities of the monotone function F is at most countable, hence of Lebesgue measure 0, so that $F(y^-) = F(y)$ for m a.e. $y \in \mathbb{R}$, and

$$\int_{[0,\infty[} (F(\infty) - F(y^-))\, dm(y) = \int_{[0,\infty[} (F(\infty) - F(y))\, dm(y).$$

Now we integrate exchanging the variables; for every $x \in \mathbb{R}$ we consider the $x-$section of T, $T_x = \{y \in \mathbb{R} : (x, y) \in T\} = [0, x]$ if $x \geq 0$, otherwise $T_x = \emptyset$. We get

$$\mu \otimes m(T) = \int_{[0,\infty]} m(T_x)\, d\mu(x) = \int_{[0,\infty[} x\, d\mu(x).$$

We have proved, as requested, that

$$\mu \otimes m(T) = \int_{[0,\infty[} (F(\infty) - F(y))\, dm(y) = \int_{[0,\infty[} x\, d\mu(x).$$

Since $\mu(]-\infty, 0[) = 0$, $f(x) = x$ coincides $\mu-$a.e. with $f^+(x) = |f(x)| = |x|$ on $\mathbb{R}$, so that

$$\int_{[0,\infty]} x\, d\mu(x) = \int_{\mathbb{R}} |x|\, d\mu(x),$$

and the formula just proved implies that $\|f\|_1 = \int_0^\infty (F(\infty) - F(y))\, dm(y)$; the answer is yes (notice also that $F(\infty) - F(y) \geq 0$, because F is increasing).

(iii) The function $x \mapsto \cos(xt)$ is continuous for every t, and $|\cos(xt)| \leq 1$, with the constant $1 \in L^1(\mu)$ since $\mu(\mathbb{R}) < \infty$. The theorem on continuity of parameter depending integrals then applies, and proves continuity of φ.

(iv) We have

$$\frac{\partial}{\partial x}(\cos(xt)) = -t \sin(xt), \quad \text{so that} \quad \left|\frac{\partial}{\partial x}(\cos(xt))\right| = |t|\,|\sin(xt)| \leq |t|.$$

If $t \mapsto |t|$ is in $L^1(\mu)$, the theorem on differentiation of parameter depending integrals says that $\varphi'(x)$ exists for every $x \in \mathbb{R}$, and

$$\varphi'(x) = \int_{\mathbb{R}} (-t\, \sin(xt))\, d\mu(t);$$

and then the continuity part of the theorem implies that this function φ' is continuous. In (ii) we have seen that the identity function of $\mathbb{R}$ is in $L^1(\mu)$ if and only if $t \mapsto F(\infty) - F(t)$ belongs to $L^1_m([0, \infty[)$. The hypothesis says that there is a constant $k > 0$ and $b > 0$ such that $0 \leq F(\infty) - F(t) \leq k/t^2$ for $t \geq b$; on $[0, b]$ the function is of course bounded. Then $t \mapsto F(\infty) - F(t)$ belongs to $L^1_m([0, \infty[)$. Thus $\varphi \in C^1(\mathbb{R})$, and the derivative is obtained by differentiating under the integral sign. □

Exercise 6. Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that a sequence f_n of measurable functions *converges to* 0 *in measure* if for every $t > 0$ we have $\lim_{n\to\infty} \mu(\{|f_n| > t\}) = 0$.

(i) Using Čebičeff inequality prove that if $\|f_n\|_1 \to 0$, then f_n converges to 0 in measure.

(ii) With $X = [0, 1]$ and μ Lebesgue measure, let $f_n = n\, \chi_{]0,1/n]}$. Is it true that f_n converges to 0 in measure? and in $L^1(\mu)$ also?

(iii) Assume now that f_n is a uniformly bounded sequence of measurable functions on X (that is, there is a constant $M > 0$ such that $\|f_n\|_\infty \leq M$ for every $n \in \mathbb{N}$), and that $\mu(X) < \infty$. Prove that if f_n converges to 0 in measure then it converges to 0 in $L^1(\mu)$ (given $\varepsilon > 0$ write

$$\int_X |f_n| = \int_{\{|f_n|>\varepsilon\}} |f_n| + \int_{\{|f_n|\leq\varepsilon\}} |f_n|$$

and estimate separately the two terms).

(iv) A sequence f_n of real–valued measurable functions converges to 0 in measure if and only if the sequence $\arctan f_n$ converges to 0 in measure.

(v) On a finite measure space a sequence f_n of real–valued measurable functions converges to 0 in measure if and only if the sequence $\arctan f_n$ converges to 0 in $L^1(\mu)$.

Solution. (i) For every $t > 0$ and every $n \in \mathbb{N}$ we have $\mu(\{f_n > t\}) \leq (1/t) \int_X |f_n| = (1/t)\, \|f_n\|_1$; letting $n \to \infty$ in this inequality we get $\lim_{n\to\infty} \mu(\{f_n > t\}) = 0$.

(ii) Given $t > 0$ we have that $\{|f_n| > t\} = \{f_n > t\} =]0, 1/n]$ for $n > t$ (and $\{f_n > t\} = \emptyset$ for $n \leq t$) so that $\mu(\{|f_n| > t\} = 1/n$ tends to 0 as $n \to \infty$, and f_n converges to 0 in measure. On the other hand

$\|f_n\|_1 = \int_{[0,1]} f_n\, dm = n\, m(]0, 1/n]) = 1$ for every n, so that f_n does not converge in $L^1_m([0,1])$ (to 0, or to any other function).

(iii) Accepting the hint we write

$$(*) \qquad \int_X |f_n| = \int_{|f_n|>\varepsilon} |f_n| + \int_{\{|f_n|\le\varepsilon\}} |f_n| \le \int_{|f_n|>\varepsilon} M + \int_{\{|f_n|\le\varepsilon\}} \varepsilon \le$$
$$\le M\,\mu(\{|f_n|>\varepsilon\}) + \varepsilon\,\mu(\{|f_n|\le\varepsilon\}) \le M\,\mu(\{|f_n|>\varepsilon\}) + \varepsilon\,\mu(X);$$

by hypothesis $\lim_{n\to\infty} \mu(\{|f_n| > \varepsilon\}) = 0$, so that we may pick $n_\varepsilon \in \mathbb{N}$ such that if $n \ge n_\varepsilon$ then $\mu(\{|f_n| > \varepsilon\}) \le \varepsilon/M$; then

$$\|f_n\|_1 = \int_X |f_n| \le (1+\mu(X))\,\varepsilon \quad \text{for} \quad n \ge n_\varepsilon;$$

the proof of (iii) is completed.

(iv) Since arctan is odd we have $|\arctan f_n| = \arctan(|f_n|)$ for every real valued function f_n. Then, if $0 < t < \pi/2$ we have $\{|\arctan f_n| > t\} = \{\arctan|f_n| > t\} = \{|f_n| > \tan t\}$ (if $t \ge \pi/2$ we have $\{|\arctan f_n| > t\} = \emptyset$). From this the result is immediate: if $|f_n|$ tends to 0 in measure then $\lim_{n\to\infty} \mu(\{|f_n| > \tan t\}) = 0$ for every $t \in]0, \pi/2[$, implying that $\arctan f_n$ tends to 0 in measure. And if $\arctan f_n$ tends to 0 in measure then for every $t > 0$ we have $\lim_{n\to\infty} \mu(\{\arctan|f_n| > \arctan t\}) = 0$, proving that $|f_n|$ tends to 0 in measure, since $\{|f_n| > t\} = \{|\arctan(f_n)| > \arctan t\}$

(v) Simply combine (i), (iii) and (iv): if f_n tends to 0 in measure then $\arctan f_n$ also tends to 0 in measure, by (iv); since $\mu(X) < \infty$, and $|\arctan f_n(x)| \le \pi/2$ for every $n \in \mathbb{N}$ and every $x \in X$ (iii) implies that $\arctan f_n$ tends to 0 in $L^1(\mu)$. And if this happens, then $\arctan f_n$ tends to 0 in measure, by (i), and by (iv) then also f_n tends to 0 in measure.

Remark. In (ii) many write $\int_{[0,1]} f_n = n\,\mu(]0, 1/n])$ (correctly); then instead of saying that $\mu(]0, 1/n]) = 1/n$ an hence that the integral is always 1, for every n, make complicated computations ending with the conclusion that $\lim_{n\to\infty} \int_{[0,1]} f_n = 0$!

In the proof of (iii) many argue in the following way: passing to the limit as n tends to ∞ in (*) one gets $\lim_{n\to\infty} \int_X |f_n| \le \varepsilon\,\mu(X)$, hence the limit is 0 because "one can take ε tending to 0". This way of arguing is of course incorrect, we cannot write $\lim_{n\to\infty} \int_X |f_n|$ if we do not yet know that the limit exists. A correct way of reasoning along these lines is: in the inequality

$$\int_X |f_n| \le M\,\mu(\{|f_n|>\varepsilon\}) + \varepsilon\,\mu(X)$$

take the lim sup on both sides as $n \to \infty$, obtaining (since $\lim_{n\to\infty} \mu(\{|f_n| > \varepsilon\}) = 0$)

$$\limsup_{n\to\infty} \int_X |f_n| \le \limsup_{n\to\infty} (M\,\mu(\{|f_n|>\varepsilon\}) + \varepsilon\,\mu(X)) = \varepsilon\,\mu(X).$$

Since $\varepsilon > 0$ is arbitrary, this implies $\limsup_{n\to\infty} \int_X |f_n| = 0$, hence also $\lim_{n\to\infty} \int_X |f_n| = 0$, since $\int_X |f_n| \ge 0$.

□

Analisi Reale per Matematica – Secondo precompitino – 21 gennaio 2013

Exercise 7. (10) Let $\mathcal{B}_n$ be the $\sigma-$algebra of Borel subsets of $\mathbb{R}^n$, and let $\mu : \mathcal{B}_n \to [0,\infty]$ be a Radon measure. We consider the set $\mathcal{V} = \{V \subseteq \mathbb{R}^n : V \text{ open},\ \mu(V) = 0\}$ and set $A = \bigcup_{V\in\mathcal{V}} V$.

(i) [3] Prove that $\mu(A) = 0$ (caution: in general $\mathcal{V}$ is not countable. However, $\mathbb{R}^n$ has a countable base for its topology . . .).

The closed set $S = \mathrm{Supp}(\mu) = \mathbb{R}^n \setminus A$ is the *support* (topological support if emphasis is needed) of the measure μ: S is the smallest *closed* set that supports μ, in the sense that $\mathbb{R}^n \setminus S$ is null for μ.

(ii) [1] What is the support of Lebesgue measure on $\mathbb{R}^n$?

(iii) [2] Let $D \subseteq \mathbb{R}^n$ be a countable set, let $\rho : D \to]0,\infty[$ be summable (i.e. $\sum_{x\in D} \rho(x) < \infty$) and let $\nu : \mathcal{B}_n \to [0,\infty[$ be defined by $\nu(A) = \sum_{x\in A\cap D} \rho(x)$. What is $\mathrm{Supp}(\nu)$? (remember that it has to be a closed set, with complement of null measure . . .).

(iv) [1] Give an example of two mutually singular measures, both having as topological support all the space $\mathbb{R}^n$.

(v) [3] Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$ be positive, and define $\mu(E) := \int_E f\,dm$ for every Borel set E. As usual we set

$$A_r f(x) = \fint_{B(x,r]} f\,dm \qquad \text{for every } x \in \mathbb{R}^n \text{ and } r > 0.$$

Prove that if $\limsup_{r\to 0+} A_r f(x) > 0$ then $x \in \mathrm{Supp}(\mu)$. Conversely, assuming $x \in \mathrm{Supp}(\mu)$ does it follow that $\limsup_{r\to 0+} A_r f(x) > 0$?

Solution. (i) Given a countable base $\mathcal{C}$ for the topology of $\mathbb{R}^n$ (e.g. all open cubes with center in $\mathbb{Q}^n$ and rational side length) we have $V = \bigcup\{C \in \mathcal{C} : C \subseteq V\}$ for every open set V, so that

$$A = \bigcup_{V\in\mathcal{V}} V = \bigcup_{V\in\mathcal{V}} \bigcup\{C \in \mathcal{C} : C \subseteq V\};$$

now $C \subseteq V$ and $\mu(V) = 0$ implies $\mu(C) = 0$, and $C \in \mathcal{C}$, $\mu(C) = 0$ implies $C \in \mathcal{V}$. It follows that the set

$$\{C \in \mathcal{C} : C \subseteq V, \text{ for some } V \in \mathcal{V}\},$$

coincides with the set $\{C \in \mathcal{C} : \mu(C) = 0\} = \mathcal{C} \cap \mathcal{V}$. Then

$$A = \bigcup\{C \in \mathcal{C} : \mu(C) = 0\},$$

and hence $\mu(A) = 0$ by countable subadditivity, because this last is a countable union.

(ii) Every non–empty open subset of $\mathbb{R}^n$ has strictly positive Lebesgue measure, as often remarked. Then the support of Lebesgue measure is all of $\mathbb{R}^n$.

(iii) An open subset of $\mathbb{R}^n$ has $\nu-$measure 0 if and only if it is disjoint from D. Then A is the union of all open subsets of $\mathbb{R}^n$ disjoint from D, and its complement, support of ν, is $\bar{D}$, the closure of D in $\mathbb{R}^n$.

(iv) We may take Lebesgue measure m and ν as above, ν, with $D = \mathbb{Q}^n$ dense in $\mathbb{R}^n$, so that $\mathrm{Supp}(\nu) = \mathbb{R}^n$, too. Since $m(\mathbb{Q}^n) = 0$ and $\nu(\mathbb{R}^n \setminus \mathbb{Q}^n) = 0$ the measures are mutually singular.

(v) If the limsup is strictly positive then we have

$$A_r f(x) = \frac{\mu(B(x,r])}{m(B(x,r])} > 0 \qquad \text{for every } r > 0,$$

in particular $\mu(B(x,r]) > 0$ for every $r > 0$. Then x belongs to the support of μ, since $\mu(B(x,r]) = \mu(B(x,r[) > 0$ for every $r > 0$. But $x \in \mathrm{Supp}(\mu)$ is exactly equivalent to $\mu(B(x,r[) > 0$ for every $r > 0$ and does not imply $\limsup A_r f(x) > 0$: take e.g $f(x) = |x|$ on $\mathbb{R}^1$ and $x = 0$; we have $\mathrm{Supp}(\mu) = \mathbb{R}$, since every open non empty interval has clearly strictly positive measure; and since f is continuous

$$\lim_{r\to 0^+} A_r f(0) = f(0) = |0| = 0.$$

□

Exercise 8. (10) For $1 \le p \le \infty$ and Ω an open subset of $\mathbb{R}^n$ we denote by $L^p_{\rm loc}(\Omega)$ the set of all measurable functions $f : \Omega \to \mathbb{K}$ such that $f\,\chi_K \in L^p(\Omega)$ for every compact subset K of Ω.

(i) [4] Prove that $L^p_{\rm loc}(\Omega)$ is a vector subspace of the space of all measurable functions from Ω to $\mathbb{K}$, containing all bounded measurable functions and in particular all constants, and that if $p < q$ then $L^p_{\rm loc}(\Omega) \supsetneqq L^q_{\rm loc}(\Omega)$ (for this last, you may assume $n = 1$).

(ii) [6] Given $1 \le p < \infty$ and $f \in L^p_{\rm loc}(\Omega)$ we say that $x \in \Omega$ is a Lebesgue point for f, as a function of $L^p_{\rm loc}(\Omega)$ if

$$\lim_{r\to 0^+} \fint_{B(x,r]} |f(y) - f(x)|^p\,dy = 0.$$

By imitating, mutatis mutandis, the proof given for $L^1_{\rm loc}$ prove that almost all points of Ω are Lebesgue points for f as a function of $L^p_{\rm loc}$.

Solution. (i) Since $m(K) < \infty$, spaces $L^p_m(K)$ decrease as p increases, and $L^p(K) \supseteq L^\infty(K)$ for every p. If $c \in \Omega$ we know that the function $f_\alpha(x) = 1/|x - c|^\alpha$ is summable in a nbhd of c iff $\alpha < n$; then f_α is in $L^p_{\rm loc}(\Omega)$ iff $\alpha\,p < n \iff \alpha < n/p$; if $n/q < \alpha < n/p$ then $f_\alpha \in L^p_{\rm loc}(\Omega) \setminus L^q_{\rm loc}(\Omega)$.

(ii) For every $c \in \mathbb{K}$ and $f \in L^p_{\rm loc}(\Omega)$ the function $x \mapsto |f(x) - c|^p$ is in $L^1_{\rm loc}(\Omega)$, so that, by the differentiation theorem:

$$\lim_{r\to 0^+} A_r|f - c|^p = |f(x) - c|^p \quad \text{for every } x \in \mathbb{R}^n \setminus E(c), \text{ where } m(E(c)) = 0.$$

Let D be a countable dense subset of $\mathbb{K}$, and let $E = \bigcup_{c\in D} E(c)$. Then $m(E) = 0$. Let's prove that $\lim_{r\to 0^+} \fint_{B(x,r]} |f(y) - f(x)|^p\, dy = 0$ for every $x \in \Omega \setminus E$. Given $x \in \Omega \setminus E$ and $\varepsilon > 0$ pick $c \in D$ such that $|f(x) - c|^p \le \varepsilon$; then

$$\limsup_{r\to 0^+} \fint_{B(x,r]} |f(y) - f(x)|^p\, dy \le \limsup_{r\to 0^+} \left(\fint_{B(x,r]} (|f(y) - c| + |c - f(x)|)^p\, dy \right) =$$

$$\limsup_{r\to 0^+} \left(2^{p-1} \fint_{B(x,r]} |f(y) - c|^p\, dy + 2^{p-1}\, |c - f(x)|^p \right) =$$

$$2^{p-1} \limsup_{r\to 0^+} \fint_{B(x,r]} |f(y) - c|^p\, dy + 2^{p-1}\, |f(x) - c|^p \le$$

$$\le 2^{p-1}\, 2|f(x) - c|^p \le 2^p\, \varepsilon.$$

□

Exercise 9. (16) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{-x^2}$ if $x < 0$; $f(x) = (1 - \cos(\pi x))/2$ if $0 \le x < 1$; $f(x) = e^{-(x-1)}$ if $x \ge 1$.

(i) [3] Plot f. Describe the function $T(x) = V_{]-\infty,x]} f$, plot it, and write f as the difference of two increasing functions.
(ii) [1] Find a Hahn decomposition of the signed measure μ.
(iii) [4] State the Lebesgue–Radon–Nikodym theorem, and find the decomposition for μ into absolutely continuous and singular part with respect to Lebesgue measure m.
(iv) [4] Given $u(x) = x$, compute all four integrals

$$\int_{\mathbb{R}} u^{\pm}\, d\mu^{\pm} \quad \text{and also} \quad \int_{\mathbb{R}} u\, d\mu.$$

(v) [4] Define now $g : \mathbb{R} \to \mathbb{R}$ as $f(x)$ above if $x \notin [0, 1[$, and for $0 \le x < 1$ set $g(x) = \psi(x)$, where $\psi : [0, 1] \to \mathbb{R}$ is the Cantor function with $\delta_n = (2/3)^n$. How does the answer to (iii) change, with $\nu = dg$? can you still compute (with $u(x) = x$, as in (iv))

$$\int_{\mathbb{R}} u\, d\nu?$$

(you may use the fact that $\int_0^1 \psi(x)\, dx = 1/2$).

Solution. (i) The plot of f is easy:

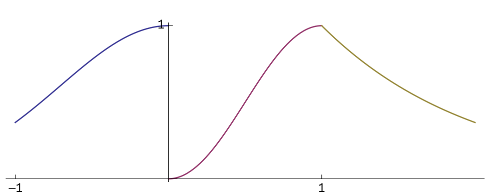

Figure 2. Plot of the function f.

Since $x \mapsto f(x)$ has limit 0 at $-\infty$ and is increasing in $-\infty, 0[$ we have $T(x) = f(x) = e^{-x^2}$ for $x < 0$. Since f is right–continuous T is also right–continuous; moreover $T(0) = T(0^+) = T(0^-) + 1 = 2$, where 1 is the absolute value of the jump of f at 0. For $x \in [0, 1[$ we have $T(x) = T(0) + f(x) - f(0) = 2 + (1 - \cos(\pi x))/2$. Finally on $[1, \infty[$ f is decreasing so that $T(x) = T(2) + f(2) - f(x) = 4 - e^{-(x-1)}$. Here is the plot of T:

Write $A(x) = (T(x) + f(x))/2$; we have $A(x) = e^{-x^2}$ for $x < 0$, $A(x) = 1 + (1 - \cos(\pi x))/2$ for $0 \le x \le 1$ and $A(x) = 2$ for $x \ge 1$, while $B(x) = (T(x) - f(x))/2$ is 0 for $x < 0$, is 1 for $0 \le x \le 1$, and is $3 - e^{-(x-1)}$ for $x \ge 1$.

Of course $\mu^+ = \mu_A$ and $\mu^- = \mu_B$. (ii) A Hahn decomposition is $P =]-\infty, 0[\cup]0, 1[$, positive, with complement $Q = \{0\} \cup [1, \infty[$, negative.

(iii) For the statement of Radon–Nikodym theorem see the Lecture Notes. The singular part of μ is $-\delta_0$, the regular part is $f'\, dm$, where f' is the classical derivative of f where it exists, that is in $\mathbb{R} \setminus \{0, 1\}$:

$$f'(x) = -2x\, e^{-x^2} \quad x < 0; \quad f'(x) = \frac{\pi}{2} \sin(\pi x) \quad 0 < x < 1; \quad f'(x) = -e^{-(x-1)} \quad x > 1.$$

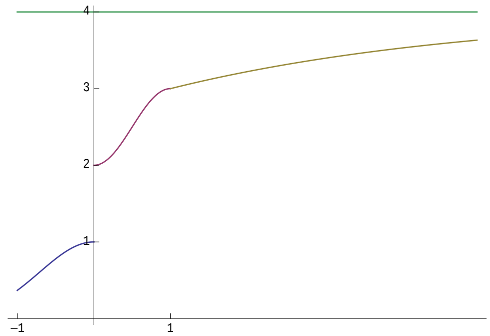

FIGURE 3. Plot of the function T.

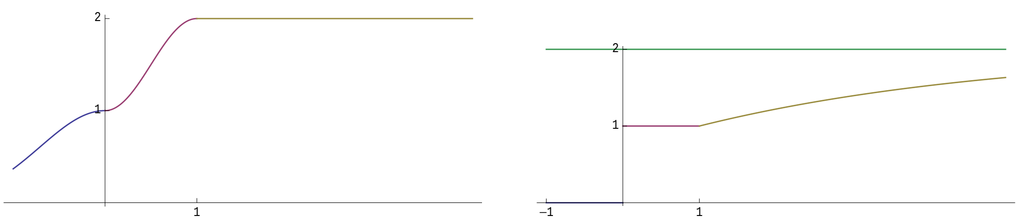

FIGURE 4. Plot of the functions A, B.

(iv) Since $\int_{\mathbb{R}} u\, d\delta_0 = u(0) = 0$ there is no contribution to the integrals from the singular part. Of course $u^+(x) = x$ for $x \geq 0$, and $u^+(x) = 0$ for $x \leq 0$, while $u^-(x) = 0$ for $x > 0$ and $u^-(x) = -x$ for $x \leq 0$. Then

$$\int_{\mathbb{R}} u^+\, d\mu^+ = \int_0^1 x\,(f'(x))\, dx = [x\, f(x)]_{x=0}^{x=1} \int_0^1 f(x)\, dx = f(1) - \int_0^1 \frac{1-\cos(\pi x)}{2}\, dx = 1 - \frac{1}{2} = \frac{1}{2}.$$

$$\int_{\mathbb{R}} u^+\, d\mu^- = -\int_1^\infty x\,(f'(x))\, dx = -\,[x\, f(x)]_{x=0}^{x=\infty} + \int_1^\infty f(x)\, dx = -[0 - f(1)] + 1 = 2.$$

$$\int_{\mathbb{R}} u^-\, d\mu^+ = \int_{-\infty}^0 (-x)\, f'(x)\, dx = \left[(-x)\, e^{-x^2}\right]_{x=-\infty}^{x=0} - \int_{-\infty}^0 (-1)e^{-x^2}\, dx = 0 + \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}.$$

and, finally

$$\int_{\mathbb{R}} u^-\, d\mu^- = 0.$$

Then

$$\int_{\mathbb{R}} u\, d\mu = \int_{\mathbb{R}} u^+\, d\mu^+ - \int_{\mathbb{R}} u^+\, d\mu^- - \int_{\mathbb{R}} u^-\, d\mu^+ + \int_{\mathbb{R}} u^-\, d\mu^- = \frac{1}{2} - 2 - \frac{\sqrt{\pi}}{2} = -\frac{3}{2} - \frac{\sqrt{\pi}}{2}.$$

(v) The singular part is now $-\delta_0 + d\psi$, where $d\psi$ is the Radon measure of the Cantor function. The only integral that may change is $\int_{\mathbb{R}} u^+\, d\mu^+$, which is now $\int_{[0,1]} x\, d\psi$. Using again integration by parts we get

$$\int_{[0,1]} x\, d\psi = [x\, \psi(x)]_{x=0}^{x=1} - \int_{[0,1]} \psi(x)\, dx = 1 - \frac{1}{2} = \frac{1}{2}$$

(unchanged!). □

ANALISI REALE PER MATEMATICA – SECONDO COMPITINO – 26 GENNAIO 2013

EXERCISE 10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/(1-x)^3$ if $x < 0$; $f(x) = (x + \psi(x))/2$ if $0 \leq x < 1$, where ψ is the Cantor function with $\delta_n = (2/3)^n$; $f(x) = 1 - 1/x^3$ if $x \geq 1$.

(i) Plot f. Describe the function $T(x) = V f(]-\infty, x])$, plot it, and write f as the difference of two increasing functions A, B; plot A and B.

(ii) State a theorem which implies that any signed measure can be written as the difference of two positive measures, and find a Hahn decomposition of the signed measure $\mu = df$.

(iii) Find the decomposition for μ into absolutely continuous and singular part with respect to Lebesgue measure m.

(iv) Determine the set of $p > 0$ such that $u(x) = |x|$ belong to $L^p(|\mu|)$. Compute

$$\int_{\mathbb{R}} u\, d|\mu|.$$

Solution. (i) The plot is very easy

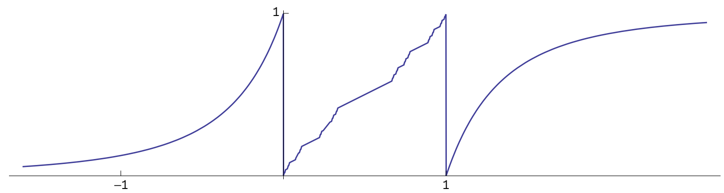

Figure 5. Plot of f.

Notice that f is right–continuous, so that T is also right continuous; we have $T(x) = g(x) = 1/(1-x)^3$ for $x < 0$, $T(0) = 2$, $T(x) = 2 + f(x) = 2 + (x + \psi(x))/2$ for $x \in [0, 1[$, $T(1) = 4$, $T(x) = 5 - 1/x^3$ for $x \geq 1$.

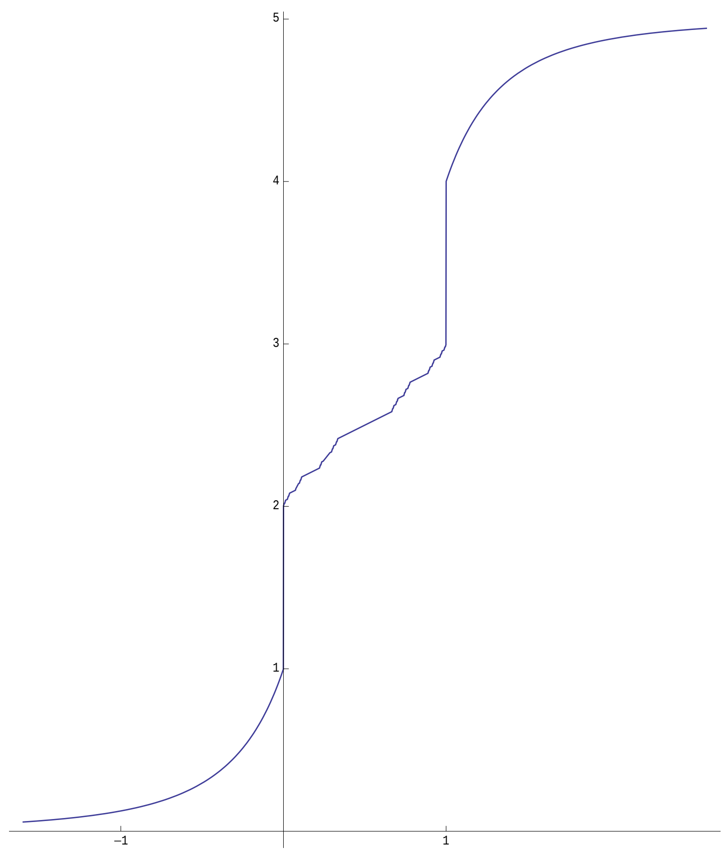

Figure 6. Plot of T.

We have $A(x) = (T(x) + f(x))/2$, and $A(x) = f(x) = 1/(1-x)^3$ for $x < 0$, $A(x) = 1 + f(x)$ for $0 \leq x, 1$, $A(x) = 3 - 1/x^3$ for $x \geq 1$. For $B(x) = (T(x) - f(x))/2$ we have $B(x) = 0$ for $x < 0$; $B(x) = 1$ for $x \in [0, 1[$, $B(x) = 2$ for $x \in [1, \infty[$.

(ii) For the statement see Lecture Notes, 6.1.3, the Hahn decomposition theorem. A Hahn decomposition in our case is $P = \mathbb{R} \smallsetminus \{0, 1\}$, $Q = \{0, 1\}$.

(iii) The absolutely continuous part is $f'(x)\, dm$ where

$$f'(x) = \frac{3}{(1-x)^4} \quad \text{if} \quad x < 0;\; f'(x) = \frac{1}{2} \quad \text{if} \quad 0 < x < 1;\; f'(x) = \frac{3}{x^4} \quad \text{if} \quad x > 1.$$

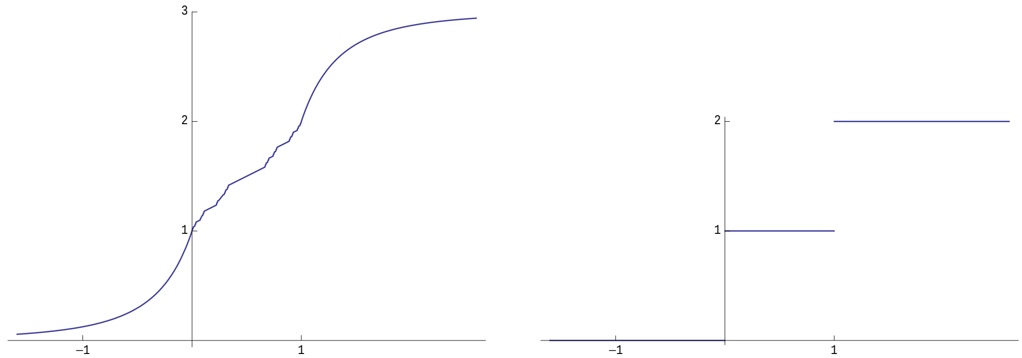

Figure 7. Plot of the functions A, B.

the singular part is

$$\frac{d\psi}{2} - \delta_0 - \delta_1;$$

remember that $\psi'(x) = 0$ a.e. in $\mathbb{R}$. (iv) The measure $|\mu|$ is

$$|\mu| = f'\,dm + \frac{d\psi}{2} + \delta_0 + \delta_1.$$

We have $u \in L^p(|\mu|)$ if and only if $u^p \in L^1$ of each of these four measures. Clearly $u^p \in L^1(\delta_0)$ (with integral 0) and $u^p \in L^1(\delta_1)$ (with integral 1) for every $p > 0$; moreover $u^p \in L^1(d\psi)$ for every p, since u^p is bounded on $[0,1]$, a set of $d\psi$ measure 1 containing the support of $d\psi$. We have to find the set of $p > 0$ such that $u^p \in L^1(f'\,dm)$, the absolutely continuous part of $|\mu|$. This is equivalent to finding the set of all $p > 0$ such that the integrals

$$\int_0^1 x^p\,\frac{dx}{2}; \qquad \int_{-\infty}^0 |x|^p\,\frac{3dx}{(1-x)^4}; \qquad \int_1^\infty x^p\,\frac{3dx}{x^4}$$

are all finite. The first integral is finite for every $p > 0$; the second and third are finite iff $4 - p > 1 \iff p < 3$ (as $x \to \pm\infty$ the integrand is asymptotic to $1/|x|^{4-p}$). So $u \in L^p(|\mu|)$ if $0 < p < 3$. For the integral:

$$\int_{\mathbb{R}} u\,d(\delta_0 + \delta_1) = u(0) + u(1) = 1; \int_0^1 x\,\frac{dx}{2} = \frac{1}{4};$$

$$\int_{-\infty}^0 |x|\,\frac{3dx}{(1-x)^4} = (\text{setting } 1 - x = t) = \int_1^\infty 3(1-t)\,\frac{dt}{t^4};$$

and

$$\int_1^\infty x\,\frac{3dx}{x^4};$$

summing the last three integrals we get the contribution to the integral of the absolutely continuous part, that is

$$\frac{1}{4} + 3\int_1^\infty \frac{dx}{x^4} = \frac{1}{4} + 1 = \frac{5}{4}.$$

It remains to compute the integral $\int_{[0,1]} x\,d\psi/2$; integrating by parts we get

$$\int_0^1 x\,d\psi = [x\,\psi(x)]_0^1 - \int_0^1 \psi(x)\,dx = 1 - \frac{1}{2} = \frac{1}{2}.$$

Then

$$\int_{\mathbb{R}} u\,d|\mu| = 1 + \frac{1}{4} + \frac{5}{4} = \frac{5}{2}.$$

□

Exercise 11. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\mathcal{F} = \mathcal{F}(\mu)$ be the ideal of sets of finite measure, $\mathcal{F} = \{A \in \mathcal{M} :\ \mu(A) < \infty\}$; recall that $\mathcal{F}$ is a metric space under the metric

$$\rho(E, F) = \mu(E \triangle F) = \|\chi_E - \chi_F\|_1,$$

provided that we identify sets E, F with zero distance, i.e. such that $\mu(E \vartriangle F) = 0$. If $f : X \to \mathbb{K}$ is measurable and $f\,\chi_E \in L^1(\mu)$ for every $E \in \mathcal{F}$, we can define a set function $\nu = \nu_f : \mathcal{F} \to \mathbb{K}$ by

$$\nu(E) := \int_E f\, d\mu.$$

(i) Prove that ν is countably additive, and that for E, $F \in \mathcal{F}$ we have

$$|\nu(E) - \nu(F)| = |\nu(E \smallsetminus F) - \nu(F \smallsetminus E)| \le \int_{E \vartriangle F} |f|\, d\mu.$$

(ii) Given $f \in L^\infty(\mu)$, prove that ν can be defined and that $|\nu(E)| \le k\,\mu(E)$ for some $k > 0$ and deduce that ν is Lipschitz continuous from $\mathcal{F}$ to $\mathbb{K}$.

(iii) Assume now that $f \in L^p(\mu)$ for some p, $1 < p < \infty$. Prove that ν can be defined, and that there is $k > 0$ such that

$$|\nu(E)| \le k\,(\mu(E))^{1/q} \quad \text{for every } E \in \mathcal{F};$$

(here $q = p/(p-1)$ is the exponent conjugate to p). Deduce that ν is still a uniformly continuous function from $\mathcal{F}$ to $\mathbb{K}$.

(iv) Finally assume $f \in L^1(\mu)$. In this case the formula $\nu(E) = \int_E f\, d\mu$ defines ν on all of $\mathcal{M}$. Prove that on $\mathcal{F}$ of this function is still uniformly continuous.

A function $f : I \to \mathbb{K}$, where I is an interval of $\mathbb{R}$ is said to satisfy a Hölder condition of exponent α (where $0 < \alpha < 1$) if there is a constant $k > 0$ such that $|f(x_2) - f(x_1)| \le k\,|x_2 - x_1|^\alpha$ for every x_1, $x_2 \in I$.

(v) Prove that if $f : [0,1] \to \mathbb{K}$ is absolutely continuous and $f' \in L^p_m([0,1])$, $p > 1$ then f satisfies a Hölder condition of exponent $1/q = (p-1)/p$.

(vi) Assume that $f : [0,1] \to \mathbb{R}$ is absolutely continuous, $f(0) = 0$ and $f'(x) = 1/(x\,(1 + \log^2 x))$ for $x > 0$. Find f, and prove that f does not satisfy a Hölder condition, for no exponent $\alpha > 0$.

Solution. (i) We have to prove that if $(E_n)_{n\in\mathbb{N}}$ is a disjoint sequence of sets of finite measure, with union $E = \bigcup_{n\in\mathbb{N}} E_n$ still of finite measure, then $\nu(E) = \sum_{n=0}^\infty \nu(E_n)$. Setting $f_n = f\,\chi_{E_n}$ and $g = f\,\chi_E$, this is equivalent to say that

$$\int_X g\, d\mu = \sum_{n=0}^\infty \int_X f_n\, d\mu,$$

and this is an immediate consequence of the theorem on normally convergent series: since the f_n are pairwise disjoint, we have $|g| = \sum_{n=0}^\infty |f_n|$ (pointwise), so that by the theorem on series with positive terms we have

$$\int_X |g|\, d\mu = \sum_{n=0}^\infty \int_X |f_n|\, d\mu, \quad \text{equivalently} \quad \|g\|_1 = \sum_{n=0}^\infty \|f_n\|_1$$

which implies, as well–known, using the dominated convergence theorem, that the integral of the sum is the sum of the series of integrals, exactly what required .

Trivially we have, for every $E \in \mathcal{F}$:

$$|\nu(E)| = \left|\int_E f\, d\mu\right| \le \int_E |f|\, d\mu.$$

By additivity, for $E, F \in \mathcal{F}$ we have:

$$\begin{aligned}|\nu(E) - \nu(F)| =& |(\nu(E \smallsetminus F) + \nu(E \cap F)) - (\nu(F \smallsetminus E) + \nu(E \cap F))| = |\nu(E \smallsetminus F) - \nu(F \smallsetminus E)| \le \\ &\le |\nu(E \smallsetminus F)| + |\nu(F \smallsetminus E)| \le \int_{E\smallsetminus F} |f|\, d\mu + \int_{F \smallsetminus E} |f|\, d\mu = \int_{E \vartriangle F} |f|\, d\mu\end{aligned}$$

(ii) If $f \in L^\infty(\mu)$ clearly $f\,\chi_E \in L^1(\mu)$ for every $E \in \mathcal{F}$, so that ν is defined, and by (i)

$$|\nu(E) - \nu(F)| \le \int_{E \vartriangle F} |f|\, d\mu \le \|f\|_\infty\, \mu(E \vartriangle F) = k\,\rho(E, F),$$

so that ν is Lipschitz continuous,with $k = \|f\|_\infty$.

(iii) Using Hölder inequality applied to $|f_{|E}|$ and the constant 1 on $E \in \mathcal{F}$ we have, for $E \in \mathcal{F}$:

$$|\nu(E)| \le \int_E |f|\, d\mu \le \left(\int_E |f|^p\, d\mu\right)^{1/p} \left(\int_E 1^q\, d\mu\right)^{1/q} \le \left(\int_X |f|^p\, d\mu\right)^{1/p} (\mu(E))^{1/q} = \|f\|_p\, (\mu(E))^{1/q};$$

and arguing as above we get, for $E, F \in \mathcal{F}$:

$$|\nu(E) - \nu(F)| \le \int_{E\smallsetminus F} |f|\, d\mu + \int_{F\smallsetminus E} |f|\, d\mu = \int_{E\Delta F} |f|\, d\mu,$$

and, with $k = \|f\|_p$

$$\int_{E\Delta F} |f|\, d\mu \le \|f\|_p\, (\mu(E \vartriangle F))^{1/q} = k\, (\rho(E, F))^{1/q}.$$

This of course immediately implies uniform continuity of ν: given $\varepsilon > 0$ take $\delta = (\varepsilon/k)^q$.

(iv) In this case $\nu : \mathcal{M} \to \mathbb{K}$ is a finite measure, absolutely continuous with respect to μ, and hence also $(\varepsilon, \delta)-$absolutely continuous; and this is exactly the needed uniform continuity; by the preceding argument in fact we have

$$|\nu(E) - \nu(F)| \le \int_{E\Delta F} |f|\, d\mu = |\nu|(E \vartriangle F);$$

now given $\varepsilon > 0$ we find $\delta > 0$ such that $\mu(G) \le \delta$ implies $|\nu|(G) \le \varepsilon$, we are done (setting $G = E \vartriangle F$). Recall the proof of $(\varepsilon, \delta)-$absolute continuity, by contradiction: if there is $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we find $F_n \in \mathcal{M}$ with $\mu(F_n) \le 2^{-(n+1)}$ and $|\nu|(F_n) > \varepsilon$, then setting $F = \limsup_{n\to\infty} F_n$ we have $\mu(F) = 0$ and $|\nu|(F) \ge \varepsilon$, a contradiction (Lecture Notes, 6.2.5.3,4).

(v) Since f is absolutely continuous we have $f(x_2) - f(x_1) = \int_{[x_1,x_2]} f'(x)\, dx$ so that

$$|f(x_2) - f(x_1)| = \left| \int_{[x_1,x_2]} f'(x)\, dx \right| \le \int_{[x_1,x_2]} |f'(x)|\, dx;$$

and using the proof above given for (ii), with $E = [x_1, x_2]$ we get

$$\int_{[x_1,x_2]} |f'(x)|\, dx \le k\, |x_2 - x_1|^{1/q} \qquad k = \|f'\|_p = \left(\int_{[x_1,x_2]} |f'(x)|^p\, dx \right)^{1/p}.$$

(vi) If f satisfies a Hölder condition then $f(x)/x^\alpha$ ought to be bounded for some $\alpha > 0$; but we have, for every $\alpha > 0$:

$$\lim_{x\to 0^+} \frac{f(x)}{x^\alpha} = (\text{Hôpital's rule}) = \frac{1}{\alpha} \lim_{x\to 0^+} \frac{1}{x^\alpha\, (1 + \log^2 x)} = \infty,$$

recalling that $\lim_{x\to 0^+} x^\alpha \log^2 x = 0$ for every $\alpha > 0$. It is not necessary to evaluate f, however the integral is immediate:

$$f(x) = \int_0^x \frac{dt}{t\,(1 + \log^2 t)} = [\arctan \log t]_{t=0}^{t=x} = \arctan \log x + \frac{\pi}{2}.$$

Remark. Unfortunately the text given at the exam was with $f'(x) = 1/(x \log^2 x)$ instead of the correct version above, so that we get $f(x) = 1/\log(1/x)$; in fact f is not even absolutely continuous on $[0, 1]$, being not continuous at $x = 1$, so the solution is trivial in this case, and the exercise becomes too easy and quite meaningless. I have given full credit to solutions, anyway.

One word on question (i): it is NOT true that $\nu(E) - \nu(F) = \nu(E \vartriangle F)$! We have

$$|\nu(E) - \nu(F)| = \left| \int_X f\, \chi_E - \int_X f\, \chi_F \right| = \left| \int_X f\, (\chi_E - \chi_F) \right|,$$

and now $\chi_E - \chi_F = \chi_{E\smallsetminus F} - \chi_{F\smallsetminus E}$; what is true is that $|\chi_E - \chi_F| = \chi_{E\Delta F}$ so that we may argue as follows

$$\left| \int_X f\, (\chi_E - \chi_F) \right| \le \int_X |f|\, |\chi_E - \chi_F| = \int_X |f|\, \chi_{E\Delta F} = \int_{E\Delta F} |f|,$$

as required. But in general it is NOT true that $|\nu(E) - \nu(F)| \le |\nu(E \vartriangle F)|$.

Many have also the strange delusion that if $f \in L^1(\mu)$ then we have $\left|\int_E f\, d\mu\right| \le \|f\|_1\, \mu(E)$ for every set E of finite measure. This is clearly FALSE: assuming for simplicity $f \ge 0$ this implies that every average of f is less than its integral on X, which in general is not true: consider e.g. $f(x) = \chi_{]0,1]}/(2\sqrt{x})$ in $L^1(\mathbb{R})$ with Lebesgue measure: we have $\|f\|_1 = 1$, and if $E = [0, a]$ with $a < 1$ we have

$$\int_E f(x)\, dx = \int_0^a \frac{dx}{2\sqrt{x}} = \sqrt{a} > a = \|f\|_1\, m(E) \quad \left(\text{and} \quad ⨍_E f\, dm = \frac{1}{\sqrt{a}} > 1 = \|f\|_1 \right).$$

What is true is of course an inequality like ($0 < \mu(E) < \infty$):

$$\left|\fint_E f\,d\mu\right| \le \|f\|_\infty \iff \left|\int_E f\,d\mu\right| \le \|f\|_\infty\,\mu(E)$$

(the average is less than the sup–norm of the function, equivalently the integral is less than the sup–norm times the measure of the set on which we are integrating).

□

Exercise 12. Let $\mathcal{B}_n$ be the $\sigma-$algebra of Borel subsets of $\mathbb{R}^n$, and let $\mu : \mathcal{B}_n \to [0,\infty]$ be a positive measure (not necessarily a Radon measure). We consider the set $\mathcal{V} = \{V \subseteq \mathbb{R}^n : V \text{ open}, \ \mu(V) = 0\}$ and set $A = \bigcup_{V\in\mathcal{V}} V$.

(i) Prove that $\mu(A) = 0$ (caution: $\mathcal{V}$ is in general not countable ...).

The closed set $S = \mathrm{Supp}(\mu) = \mathbb{R}^n \smallsetminus A$ is the *support* (topological support if emphasis is needed) of the measure μ: S is the smallest *closed* set that supports μ, in the sense that $\mathbb{R}^n \smallsetminus S$ is null for μ.

(ii) Let $c \in \mathbb{R}^n$ be given. Prove that the following are equivalent:
  (a) $c \in \mathrm{Supp}(\mu)$.
  (b) For every open set U containing c we have $\mu(U) > 0$.
  (c) For every positive $u \in C_c(\mathbb{R}^n)$ such that $u(c) > 0$ we have $\int_{\mathbb{R}^n} u\,d\mu > 0$.

(iii) Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{K}$ be measurable. Let $\nu : \mathcal{B}(\mathbb{K}) \to [0,\infty]$ be the image measure of μ by means of f, that is $\nu(B) = \mu f^{\leftarrow}(B) := \mu(f^{\leftarrow}(B))$ for every $B \in \mathcal{B}(\mathbb{K})$. The *essential range* of f is, by definition, the support of ν:

$$\mathrm{essrange}(f) = \mathrm{Supp}(\mu f^{\leftarrow}).$$

Prove that $f \in L^\infty(\mu)$ if and only if the essential range of f is a compact subset of $\mathbb{K}$, and that in this case $\|f\|_\infty = \max\{|z| : z \in \mathrm{essrange}(f)\}$.

(4 extra points) If $E \in \mathcal{M}$ with $0 < \mu(E) < \infty$, and $f : X \to \mathbb{K}$ is a measurable function such that $f\,\chi_E \in L^1(\mu)$, then the *average of* f *over* E is defined as

$$A_E f := \fint_E f\,d\mu := \int_E f\,\frac{d\mu}{\mu(E)}.$$

Prove that if $(X, \mathcal{M}, \mu)$ is semifinite and $C \subseteq \mathbb{K}$ is a closed subset of $\mathbb{K}$ that contains all averages of f, then C contains also the essential range of f.

Solution. (i) (i) Given a countable base $\mathcal{C}$ for the topology of $\mathbb{R}^n$ (e.g. all open cubes with center in $\mathbb{Q}^n$ and rational side length) we have $V = \bigcup\{C \in \mathcal{C} : C \subseteq V\}$ for every open set V, so that

$$A = \bigcup_{V\in\mathcal{V}} V = \bigcup_{V\in\mathcal{V}} \bigcup\{C \in \mathcal{C} : C \subseteq V\};$$

now $C \subseteq V$ and $\mu(V) = 0$ implies $\mu(C) = 0$, and $C \in \mathcal{C}$, $\mu(C) = 0$ implies $C \in \mathcal{V}$. It follows that the set

$$\{C \in \mathcal{C} : C \subseteq V, \text{ for some } V \in \mathcal{V}\},$$

coincides with the set $\{C \in \mathcal{C} : \mu(C) = 0\} = \mathcal{C} \cap \mathcal{V}$. Then

$$A = \bigcup\{C \in \mathcal{C} : \mu(C) = 0\},$$

and hence $\mu(A) = 0$ by countable subadditivity, because this last is a countable union (= the union of a countable family of sets).

(ii) (a) is equivalent to (b): immediate by definition, an open set has measure $\mu(U) = 0$ if and only if the support of μ is disjoint from U. (b) implies (c): If $u(c) > 0$ then $U = \{u > u(c)/2\}$ is an open set containing c, so that $\mu(U) > 0$; and by Čebičeff's inequality

$$\mu(U) \le \frac{2}{u(c)} \int_{\mathbb{R}^n} u\,d\mu \quad \text{so that also} \quad \int_{\mathbb{R}^n} u\,d\mu > 0.$$

(c) implies (b): given an open set U containing c, we get a positive function $u \in C_c(\mathbb{R}^n)$ with $\mathrm{Supp}(u) \subseteq U$ and $u(c) > 0$: in fact there is $r > 0$ such that $B(c,r] \subseteq U$ (U is open) and we can take $u(x) = \max\{r - |x-c|\} \vee 0$, which has $B(c,r]$ as support, and is such that $u(c) = r > 0$. Then $\int_{\mathbb{R}^n} u(x)\,d\mu(x) > 0$, and this implies $\mu(\mathrm{Coz}(u)) = \mu(B(c,r[) > 0$, and since $U \supseteq B(c,r[$ we also get $\mu(U) > 0$.

(iii) We have $f \in L^\infty(\mu)$ iff there is $\alpha > 0$ such that $\mu(\{|f| > \alpha\}) = 0$. This is equivalent to say that $\mathrm{Supp}(\nu) \subseteq \{z \in \mathbb{K} : |z| \le \alpha\}$, the closed ball of $\mathbb{K}$ of center 0 and radius α. Then $f \in L^\infty(\mu)$

iff $\operatorname{Supp}(\nu)$ is bounded, and since $\operatorname{Supp}(\nu)$ is closed, then $f \in L^\infty(\mu)$ iff $\operatorname{Supp}(\nu)$ is compact. Moreover $\|f\|_\infty$ is the minimum $\{\alpha \ge 0\}$ such that $\mu(\{|f| > \alpha\})(= \nu(\{z : |z| > \alpha\})) = 0$, the minimum radius of a closed disc centered at the origin of $\mathbb{K}$ that contains $\operatorname{Supp}(\nu)$, and this of course coincides with $\max\{|z| : z \in \operatorname{Supp}(\nu)\}$.

We assume that $c \in \operatorname{essrange}(f) \setminus C$, and get a contradiction. Since C is closed, we find an open disc centered at c disjoint from C, say $B(c, r[= \{z \in \mathbb{K} : |z - c| < r\}$, for some $r > 0$. Since $c \in \operatorname{Supp}(\nu)$, and $B(c, r[$ is an open set containing c we have $0 < \nu(B(c, r[) = \mu(\{|f - c| < r\})$; since μ is semifinite there is $E \in \mathcal{M}$ with $0 < \mu(E) < \infty$ and $E \subseteq \{|f - c| < r\}$. Then we have $A_E f \in B(c, r[$, so that $A_E f \notin C$, contradicting the assumption that C contains all averages of f; in fact

$$|A_E f - c| = \left|\int_E f\,\frac{d\mu}{\mu(E)} - c\right| = \left|\int_E (f - c)\,\frac{d\mu}{\mu(E)}\right| \le \int_E |f - c|\,\frac{d\mu}{\mu(E)} < \int_E r\,\frac{d\mu}{\mu(E)} = r,$$

(the strict inequality is due to the fact that $|f(x) - c| < r$ holds for *every* $x \in E$, and $\mu(E) > 0$; clearly $f\,\chi_E \in L^1(\mu)$ because f is bounded on E and E has finite measure).

□

Analisi Reale per Matematica – Primo appello – 5 febbraio 2013

Exercise 13. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -e^{-x^2}$ if $x < 0$; $f(x) = (x^2 + \psi(x))/2$ if $0 \le x < 1$, where ψ is the Cantor function with $\delta_n = (2/3)^n$; $f(x) = 1 - e^{-(x-1)}$ if $x \ge 1$.

(i) Plot f. Describe the function $T(x) = V f(]-\infty, x])$, plot it, and write f as the difference of two increasing functions A, B; plot A and B.
(ii) Find a Hahn decomposition of the signed measure $\mu = df$.
(iii) Find the decomposition for μ^+ and μ^- into absolutely continuous and singular part with respect to Lebesgue measure m.
(iv) Determine the set of $p > 0$ such that $u(x) = |x - 1| + |x|$ belong to $L^p(|\mu|)$. Compute

$$\int_{\mathbb{R}} u\, d|\mu|.$$

Solution. (i) The plot of f is easy: Since f is right–continuous, so is T. We have $T(x) = e^{-x^2}$ for $x < 0$;

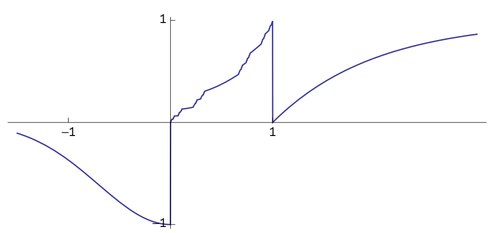

Figure 8. Plot of f.

$T(0^+) = T(0) = T(0^-) + 1$, since the jump of f at 0 is 1, hence $T(x) = T(0) + f(x) - f(0) = 1 + (x^2 + \psi(x))/2$ for $0 < x < 1$, since f is increasing on $[0, 1[$, so $T(1^-) = 3$; and $T(1^+) = T(1) = T(1^-1) + 1 = 4$, since the jump of f at 1 is -1; finally $T(x) = T(1^+) + 1 - e^{-(x-1)} = 5 - e^{-(x-1)}$ for $x > 1$. The plot of T is as follows:

Next we have $A = (T + f)/2$ and $B = (T - f)/2$ as follows

(ii) A positive set for μ is clearly $P = [0, 1[\cup]1, \infty[$, with complement $Q =]-\infty, 0[\cup\{1\}$ a negative set, so P, Q is a Hahn decomposition for μ.

(iii) The absolutely continuous part of $\mu^+ = \mu_A$ is as usual $A'(x)\, dm$, with $A'(x)$ described as

$$A'(x) = 0 \quad \text{if } x < 0;\ A'(x) = x \quad \text{if } 0 < x < 1;\ A'(x) = e^{-(x-1)} \quad \text{if } 1 < x;$$

the singular part is $\delta_0 + d\psi/2$. Similarly, the absolutely continuous part of $\mu^- = \mu_B$ is $B'(x)\, dm$, with $B'(x)$ described as

$$B'(x) = -2x\, e^{-x^2} \quad \text{if } x < 0;\ B'(x) = 0 \quad \text{if } 0 < x < 1;\ B'(x) = 0 \quad \text{if } 1 < x;$$

and the singular part is δ_1.

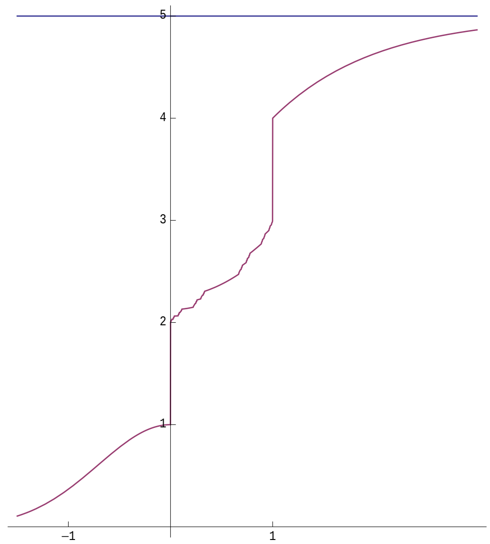

FIGURE 9. Plot of T.

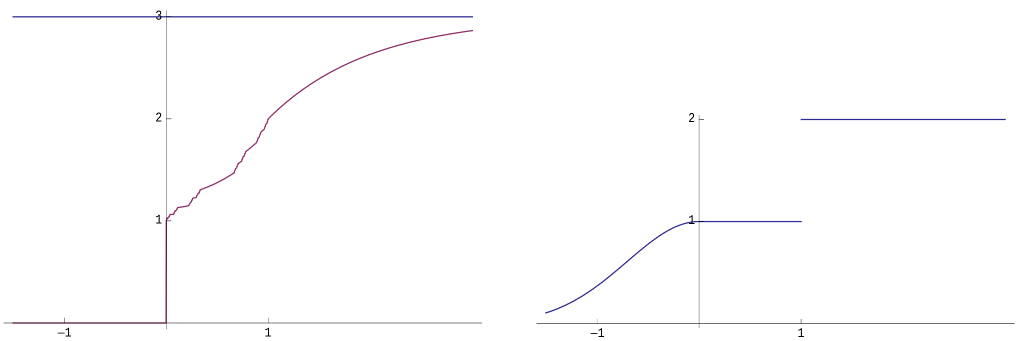

FIGURE 10. Plot of the functions A, B.

(iv) Notice that $u(x) = 1$ if $0 \le x \le 1$; then

$$\int_{[0,1]} u^p\, d|\mu| = \int_{[0,1]} u\, d|\mu| = |\mu|([0,1]) = \delta_0([0,1]) + \delta_1([0,1]) + \int_{[0,1]} x\, dx + \int_{[0,1]} \frac{d\psi}{2} = 1 + 1 + \frac{1}{2} + \frac{1}{2} = 3.$$

For $x < 0$ we have $u(x) = -x + (-(x-1)) = 1 - 2x$; for $x > 1$ we have $u(x) = 2x - 1$ we have to see the values of p for which the integrals:

$$\int_{-\infty}^{0} (1-2x)^p (-2x\, e^{-x^2})\, dx = 2\int_{-\infty}^{0} (1-2x)^p |x|\, e^{-x^2}\, dx; \ \int_1^\infty (2x-1)^p\, e^{-(x-1)}\, dx$$

are both finite; it is immediate that this happens for every $p > 0$ (because of the exponential factors, the integrands are $o(1/|x|^\alpha)$ for every $\alpha > 0$, as $x \to \pm\infty$). So $f \in L^p(|\mu|)$ for every $p > 0$. It remains to compute the last two integrals for $p = 1$. Changing x into $-x$ the first is

$$2\int_0^\infty (2x^2 + x)\, e^{-x^2}\, dx = \int_0^\infty 2x\, e^{-x^2}\, dx + 4\int_0^\infty x^2\, e^{-x^2}\, dx =$$
$$\left[-e^{-x^2}\right]_0^\infty - 2\left[-x\, e^{-x^2}\right]_0^\infty + 2\int_0^\infty e^{-x^2}\, dx = 1 + \sqrt{\pi}.$$

For the second integral we have:

$$\int_1^\infty (2x-1)\, e^{-(x-1)}\, dx = \left[-(2x-1)\, e^{-(x-1)}\right]_1^\infty + 2\int_1^\infty e^{-(x-1)}\, dx = 2 + \left[-e^{-(x-1)}\right]_1^\infty = 3.$$

Collecting the partial results we get

$$\int_{\mathbb{R}} u\, d|\mu| = 7 + \sqrt{\pi}.$$

□

Exercise 14. Let $(X, \mathcal{M})$ be a measurable space, and let $\nu : \mathcal{M} \to \tilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ be a countably additive function, that is, ν is a signed measure.

(i) Prove that if $\nu(E) = -\infty$ (resp. $\nu(E) = +\infty$) for some $E \in \mathcal{M}$ then $\nu(F) = -\infty$ (resp. $\nu(F) = +\infty$) for every $F \in \mathcal{M}$ with $F \supseteq E$; deduce from this that if $\nu(E) = -\infty$ for some $E \in \mathcal{M}$ then $\infty \notin \nu(\mathcal{M})$.

(ii) Is it true that $\sup\{\nu(E) : E \in \mathcal{M}\} = \max\{\nu(E) : E \in \mathcal{M}\}$? and that $\inf\{\nu(E) : E \in \mathcal{M}\} = \min\{\nu(E) : E \in \mathcal{M}\}$?

Assume now that $\mu : \mathcal{M} \to [0, \infty]$ is a positive measure, and that $f : X \to \mathbb{R}$ is a measurable function such that $\nu(E) = \int_E f\, d\mu$, for every $E \in \mathcal{M}$.

(iii) What is a Hahn decomposition for ν, in terms of f? and how can $\nu^{\pm}$ be expressed in terms of f? under which condition on f is ν a finite measure?

(iv) Assume that μ is a $\sigma-$finite measure and that $f \geq 0$, so that ν is also a positive measure, and that $f \in L^1(\mu)$. What condition on f is equivalent to assert that there is a measurable $g \geq 0$ such that $\mu(E) = \int_E g\, d\nu$ for every $E \in \mathcal{M}$?

Solution. (i) We have $\nu(F) = \nu(E) + \nu(F \setminus E)$. If $\nu(E) = -\infty$ (resp: $\nu(E) = \infty$) then the sum $-\infty + \nu(F \setminus E)$, if meaningful, can only have value $-\infty$ (resp: ∞). Then $\nu(F) = \nu(E)$; if some set has measure $-\infty$ (resp: ∞) then $\nu(X) = -\infty$ (resp: ∞) so that ν cannot assume both values.

(ii) There is a lemma that says that if a signed measure does not assume the value $-\infty$, then there is $P \in \mathcal{M}$ such that $\nu(P) = \max\{\nu(E) : E \in \mathcal{M}\}$. Then the answer is affirmative: if ν does not assume the value ∞ the set $\nu(\mathcal{M})$ has a maximum. If $\nu(E) = -\infty$ for some $E \in \mathcal{M}$ then $-\infty = \min \nu(\mathcal{M})$; if $\infty \notin \nu(\mathcal{M})$ the the previously mentioned lemma applied to $-\nu$ implies that for some $Q \in \mathcal{M}$ we have $\nu(Q) = \min \nu(\mathcal{M}) (\in \mathbb{R})$. Clearly this concludes the question with an affirmative answer.

(iii) Trivially $P = \{f > 0\}$ with $Q = X \setminus P = \{f \leq 0\}$, or $P = \{f \geq 0\}$ and $Q = X \setminus P = \{f < 0\}$. Also

$$\nu^+(E) = \nu(P \cap E) = \int_E f^+\, d\mu; \quad \nu^-(E) = -\nu(Q \cap E) = \int_E f^-\, d\mu.$$

Clearly ν is finite if and only if $\nu^{\pm}$ are both finite measures, that is iff

$$\int_X f^+\, d\mu < \infty, \quad \int_X f^-\, d\mu < \infty$$

equivalently $f \in L^1_\mu(X, \mathbb{R})$.

(iv) Since $f \in L^1(\mu)$ the measure ν is finite, and positive since $f \geq 0$. By hypothesis μ is $\sigma-$finite, By the Radon–Nikodym theorem g exists iff $\nu(E) = 0$ implies $\mu(E) = 0$. Now, since $f \geq 0$ we have, calling $Z = Z(f) = \{f = 0\}$ the zero–set of f

$$\nu(E) = \int_E f\, d\mu = 0 \quad \iff \mu(E \setminus Z) = 0$$

Ten, for every $E \in \mathcal{M}$ we have that $\mu(E \setminus Z) = 0$ must imply $\mu(E) = 0$; this is clearly true for every $E \in \mathcal{M}$ iff $\mu(Z) = 0$. In this case of course we have $g(x) = 1/f(x)$ for $x \notin Z$ (and $g(x)$ arbitrary for $x \in Z$, i.e. $g(x) = 0$). □

Exercise 15. Let $(X, \mathcal{M}, \mu)$ be a probability space (that is, a measure space with $\mu(X) = 1$).

(i) For $g : X \to \mathbb{K}$ measurable and $0 < p < q$, how do you compare $\|g\|_p$ and $\|g\|_q$? And what is $\lim_{p \to \infty} \|g\|_p$? (no proof required for this last question; simply state the result).

Assume that $f \in L^1_\mu(X, \mathbb{R})$.

(ii) State Jensen's inequality: if $\omega : \mathbb{R} \to \mathbb{R}$ is convex then $\omega\left(\int_X f\right) \leq \ldots$ (complete the statement)

(iii) Prove that for every $p > 0$ we have

$$\exp\left(\int_X f\right) \leq \left(\int_X e^{p\, f(x)}\, d\mu(x)\right)^{1/p}.$$

(iv) Setting $a(p) = \left(\int_X e^{p\, f(x)}\, d\mu(x)\right)^{1/p}$, prove that $\lim_{p \to \infty} a(p)$ exists in $\tilde{\mathbb{R}}$ and express it by something related to f. Is this limit necessarily finite?

(v) Prove that $\lim_{p \to 0^+} a(p)$ exists and is strictly positive.

(4 extra points) Is it true that $\lim_{p\to 0^+} a(p) = \exp\left(\int_X f\right)$? if not, under which conditions on f does this hold?

Solution. (i) We know that if $p < q$ then $\|g\|_p \le \|g\|_q$: simply use Hölder's inequality applied to functions $|g|^p$ and 1 with conjugate exponents q/p and $1/(1-(p/q)) = q/(q-p)$:

$$\int_X |g|^p \le \left(\int_X (|g|^p)^{q/p}\right)^{p/q} \Longrightarrow \left(\int_X |g|^p\right)^{1/p} \le \left(\int_X |g|^q\right)^{1/q};$$

and $\lim_{p\to\infty} \|g\|_p = \|g\|_\infty$, essential supremum of $|g|$.

(ii) For Jensen's inequality see Weeks, eighth week.

(iii) Apply Jensen's inequality to f, with $\omega(x) = e^{px}$, clearly a strictly convex function, obtaining

$$\exp\left(p\int_X f\,d\mu\right) \le \int_X \exp(p\,f)\,d\mu \iff \exp\left(\int_X f\right) \le \left(\int_X e^{p\,f(x)}\,d\mu(x)\right)^{1/p}.$$

(iv) If we define $g(x) = \exp(f(x))$ we have

$$a(p) = \left(\int_X e^{p\,f(x)}\,d\mu(x)\right)^{1/p} = \left(\int_X g^p(x)\,d\mu(x)\right)^{1/p} = \|g\|_p,$$

so that $p \mapsto a(p)$ is increasing and its limit as $p \to \infty$ is $\|g\|_\infty$, which of course is $e^{\operatorname{esssup} f}$. Taking $X = [0,1]$ with Lebesgue measure, and $f(x) = 1/(2\sqrt{x})$ we have $f \in L^1$ with $\int_X f = 1$, but $e^{p\,f(x)} \notin L^1$, for no $p > 0$ (we have $\lim_{x\to 0^+} e^{p\,f(x)}/x = \lim_{x\to 0^+} \exp(p/(2\sqrt{x}) - \log x) = e^\infty = \infty$), so that for every p we have $a(p) = \infty$).

(v) Since $p \mapsto a(p)$ is increasing, we have that $\lim_{p\to 0^+} a(p)$ exists and coincides with $\inf\{a(p) : p > 0\}$; and since $a(p) \ge \exp\left(\int_X f\right) > 0$ for every p, this limit is strictly positive.

If $a(p) = \infty$ for every $p > 0$ then of course $\lim_{p\to 0^+} a(p) = \infty$; this happens for instance with $f(x) = 1/(2\sqrt{x})$ on $[0,1]$ as above, whereas $\exp\left(\int_X f\right)$ is finite by hypothesis. But if $a(q) < \infty$ for some $q > 0$, then we have $\lim_{p\to 0^+} a(p) = \exp\left(\int_X f\right)$: for a proof see Weekly, Eighth week, Geometric Mean (Exercise 19). □

Exercise 16. Let $I = [a,b]$ be a compact interval of $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a function.

(i) State the $(\varepsilon,\delta)-$condition for the absolute continuity of the function f, and prove that if f is Lipschitz continuous then it is absolutely continuous.

(ii) Assume that f is absolutely continuous, that $f([a,b]) = J$, and that $g : J \to \mathbb{R}$ is Lipschitz continuous. Prove that then the composition $g \circ f$ is absolutely continuous on $[a,b]$.

(iii) For $\alpha > 0$ define $f_\alpha : [0,\infty[\to \mathbb{R}$ by the formula $f_\alpha(x) = |\sin(x^\alpha)|$. Find the values of $\alpha > 0$ for which f_α is absolutely continuous on every compact subinterval of $[0,\infty[$.

Solution. (i) See Lecture Notes, 7.3.2. If there is $k > 0$ such that $|f(x_2) - f(x_1)| \le k\,|x_2 - x_1|$ for every x_1, $x_2 \in [a,b]$ then f verifies the $(\varepsilon,\delta)-$ condition of absolute continuity: given $\varepsilon > 0$ let $\delta = \varepsilon/k$; if $([a_j,b_j])_{1\le j\le m}$ is sequence of non–overlapping subintervals of $[a,b]$ and $\sum_{j=1}^m (b_j - a_j) \le \delta$, then

$$\sum_{j=1}^m |f(b_j) - f(a_j)| \le \sum_{j=1}^m k\,|b_j - a_j| \le k\,\delta \le \varepsilon.$$

(ii) Given $\varepsilon > 0$, let $\rho = \varepsilon/k$; since f is absolutely continuous, we find $\delta > 0$ such that for every sequence $([a_j,b_j])_{1\le j\le m}$ of non overlapping intervals with $\sum_{j=1}^m (b_j - a_j) \le \delta$ we have

$$\sum_{j=1}^m |f(b_j) - f(a_j)| \le \rho = \frac{\varepsilon}{k};$$

then we get

$$\sum_{j=1}^m |g(f(b_j)) - g(f(a_j))| \le \sum_{j=1}^m k\,|f(b_j) - f(a_j)| \le k\,\frac{\varepsilon}{k} = \varepsilon,$$

thus proving absolute continuity of $g \circ f$.

(iii) The function x^α is locally absolutely continuous for every $\alpha > 0$; in fact it is a C^1 function on $]0,\infty[$, and clearly, if $\alpha > 0$ and $x > 0$ then

$$\int_0^x \alpha\,t^{\alpha-1}\,dt = x^\alpha \quad \text{that is, } f(x) = \int_0^x f'(t)\,dt, \text{ for every } x > 0.$$

The function $y \mapsto |\sin y|$ is Lipschitz continuous, so f_α is locally absolutely continuous by (ii), for every $\alpha > 0$.

□

Analisi Reale per Matematica – Secondo appello – 25 febbraio 2013

Exercise 17. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = x\,e^{-(x-1)}\,U(x) + e^x\,U(-x)$, with $U = \chi_{]0,\infty[}$ the characteristic function of the open half–line $]0,\infty[$.

(i) Plot F; is F right–continuous?

Define $\mu = dF(= \mu_F)$ the Radon–Stieltjes signed measure associated to F.

(ii) Find a Hahn decomposition for μ, and find the decomposition for μ^+ and μ^- into absolutely continuous and singular part with respect to Lebesgue measure m.

(iii) Find functions A, B such that $\mu^+ = dA$ and $\mu^- = dB$; plot A and B.

(iv) Given $a > 0$ let $T(a) = \{(x,y) \in \mathbb{R}^2 : 0 < x \le y \le a\}$ Compute

$$m \otimes \mu^+(T(a)) \quad \text{and} \quad m \otimes \mu^-(T(a)),$$

(with m Lebesgue measure).

(v) Using (iii) compute

$$\int_{[0,\infty[} t\,d\mu^+(t) \qquad \int_{[0,\infty[} t\,d\mu^-(t) \qquad \int_{[0,\infty[} t\,d|\mu|(t).$$

Solution. (i) Clearly F is continuous on $\mathbb{R} \setminus \{0\}$ and $\lim_{x\to 0^-} F(x) = 1$, while $\lim_{x\to 0^+} F(x) = 0$; $F(0) = 0$ so that F is right–continuous, but not continuous, at 0. The plot is easy (notice that $F'(x) = (1-x)\,e^{-(x-1)}$ for $x > 0$, so that F is increasing in $[0,1]$ and decreasing in $[1,\infty[$):

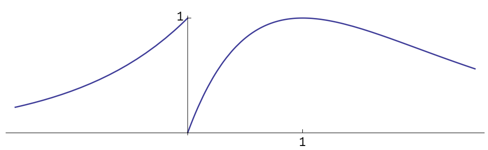

Figure 11. Plot of F.

(ii) A positive set for F is $P =]-\infty, 0[\cup]0,1[$; the complement $Q = \{0\} \cup [1,\infty[$ is negative, so P, Q is a Hahn decomposition for μ. Since $F'(x) = e^{-(x-1)}\,(1-x)\,U(x) + e^x\,U(-x)$, the absolutely continuous part of μ^+ is where $F'(x) \ge 0$, that is $dA(x) = (e^x\,U(-x) + e^{-(x-1)}\,(1-x)\,\chi_{]0,1[}(x))\,dx$, and that of μ^- is $e^{-(x-1)}\,(x-1)\,\chi_{[1,\infty[}(x)\,dx$; we have $\mu^+ \ll m$; the singular part of μ^- is δ_0.

(iii) We have

$$A(x) = e^x\,U(-x) + (1 + x\,e^{-(x-1)})\,\chi_{[0,1[}(x) + 2\,\chi_{[1,\infty[}(x); \qquad B(x) = \chi_{[0,1[}(x) + (2 - x\,e^{-(x-1)})\,\chi_{[1,\infty[}(x).$$

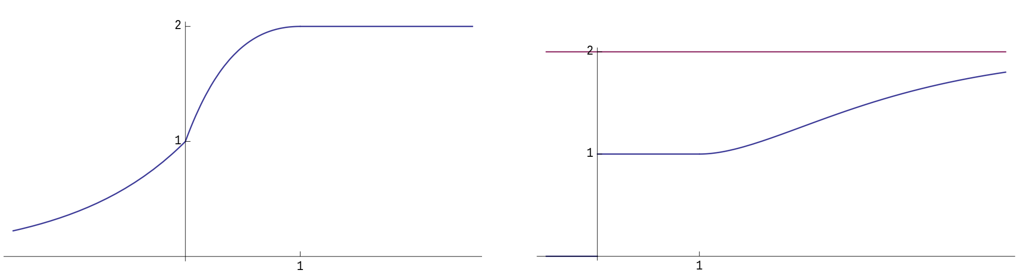

Figure 12. Plot of the functions A, B.

(iv) Clearly the set $T(a)$ is a bounded Borel set, hence of finite $m \otimes \mu^\pm$ measure. Using Fubini's theorem we get (with $T_x(a) = \{y \in \mathbb{R} : (x,y) \in T(a)\} =]x,a]$ the $x-$section of $T(a)$, for $x \in]0,a]$):

$$m \otimes \mu^+(T(a)) = \int_{]0,a]} \left(\mu^+(T_x(a))\right)\,dm(x) = \int_0^a (A(a) - A(x))\,dx = A(a)\,a - \int_0^a A(x)\,dx;$$

now we have, for $0 < a \le 1$:

$$\int_0^a A(x)\,dx = \int_0^a (1 + x\,e^{-(x-1)})\,dx = a + \left[-x\,e^{-(x-1)}\right]_0^a + \int_0^a e^{-(x-1)}\,dx = a - a\,e^{-(a-1)} + e - e^{-(a-1)} =$$
$$= e + a - (a+1)\,e^{-(a-1)},$$

so that $m \otimes \mu^+(T(a)) = (a^2 + a + 1)\,e^{-(a-1)} - e$ for $0 < a \le 1$. Since $\mu^+(]1, \infty[) = 0$ we have that $m \otimes \mu^+(T(a)) = m \otimes \mu^+(T(1))$ for $a > 1$; then

$$m \otimes \mu^+(T(a)) = \begin{cases} (a^2 + a + 1)\,e^{-(a-1)} - e & \text{for} \quad 0 < a \le 1 \\ 3 - e \quad \text{for} \quad a > 1. \end{cases}$$

Next:

$$m \otimes \mu^-(T(a)) = \int_{]0,a]} \left(\mu^-(T_x(a))\right)\,dm(x) = \int_0^a (B(a) - B(x))\,dx = B(a)\,a - \int_0^a B(x)\,dx;$$

Now, for $0 < a < 1$ we have $B(a) = 1$ so that the preceding is $a - a = 0$; if $a \ge 1$

$$\begin{aligned} a\,B(a) - \int_0^a B(x)\,dx =& a\,(2 - a\,e^{-(a-1)}) - \int_0^a (2 - x\,e^{-(x-1)}\,dx = \\ & a\,(2 - a\,e^{-(a-1)}) - \int_0^1 dx - \int_1^a (2 - x\,e^{-(x-1)}\,dx = \\ & a\,(2 - a\,e^{-(a-1)}) - 1 - 2(a-1) + \left[-x\,e^{-(x-1)}\right]_1^a + \int_1^a e^{-(x-1)}\,dx = \\ & 3 - (1 + a + a^2)\,e^{-(a-1)}. \end{aligned}$$

Then

$$m \otimes \mu^-(T(a)) = \begin{cases} 0 \quad \text{for} \quad 0 < a < 1 \\ 3 - (1 + a + a^2)\,e^{-(a-1)} \quad \text{for} \quad a \ge 1. \end{cases}$$

(v) Integrating first in the $x-$coordinate and then in the $y-$coordinate we get (with $T_y(a) = \{x \in \mathbb{R} : (x, y) \in T(a)\} =]0, y]$ if $0 < y \le a$)

$$m \otimes \mu^+(T(a)) = \int_{]0,a]} (m(T_y(a)))\,dA = \int_{]0,a]} y\,dA(y),$$

and similarly

$$m \otimes \mu^-(T(a)) = \int_{]0,a]} (m(T_y(a)))\,dB = \int_{]0,a]} y\,dB(y);$$

by the dominated convergence theorem we have

$$\int_{]0,\infty[} y\,dA(y) = \lim_{a \to +\infty} \int_{]0,a]} y\,dA(y) \quad \text{and} \quad \int_{]0,\infty[} y\,dB(y) = \lim_{a \to +\infty} \int_{]0,a]} y\,dB(y);$$

but we have, for $a > 1$:

$$\int_{]0,a]} y\,dA(y) = m \otimes \mu^+(T(a)) = 3 - e; \qquad \int_{]0,a]} y\,dA(y) = m \otimes \mu^-(T(a)) = 3 - (1 + a + a^2)\,e^{-(a-1)},$$

so that, taking limits as $a \to +\infty$:

$$\int_{]0,\infty[} y\,d\mu^+(y) = 3 - e; \qquad \int_{]0,\infty[} y\,d\mu^-(y) = 3,$$

and of course

$$\int_{]0,\infty[} y\,d|\mu|(y) = \int_{]0,\infty[} y\,d\mu^+(y) + \int_{]0,\infty[} y\,d\mu^-(y) = 6 - e.$$

□

EXERCISE 18. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Assume that f_n and f are measurable functions, that $f_n \to f$ a.e. on X, and that $|f_n| \uparrow |f|$ a.e. on X. Given p with $0 < p < \infty$ prove that $f \in L^p(\mu)$ if and only if $\sup\{\|f_n\|_p : n \in \mathbb{N}\} < \infty$. Does f_n also converge to f in $L^p(\mu)$, under this hypothesis?

We denote by $S(\mu)$ the space of measurable simple functions contained in $L^1(\mu)$.

(ii) Is it true that $S(\mu)$ is dense in every $L^p(\mu)$, for $p < \infty$? sketch a proof, or give a counterexample.

(iii) Let $0 < p, q < \infty$. Assume that there is a constant $C > 0$ such that $\|f\|_q \le C\,\|f\|_p$ for every $f \in S(\mu)$. Prove that then the same inequality holds for every $f \in L^p(\mu)$, and that $L^p(\mu) \subseteq L^q(\mu)$.

Solution. (i) Assume $M = \sup\{\|f_n\|_p : n \in \mathbb{N}\} < \infty$. Then $\int_X |f_n|^p \le M^p$ for every n. By monotone convergence we have

$$\int_X |f|^p = \lim_{n\to\infty} \int_X |f_n|^p \le M^p,$$

so that $f \in L^p(\mu)$. Conversely, if $f \in L^p(\mu)$ then clearly we have, from $|f_n| \le |f|$, that $\int_X |f_n|^p \le \int_X |f|^p$, so that $\sup\{\|f_n\|_p : n \in \mathbb{N}\} \le \|f\|_p < \infty$. Clearly in these hypotheses we also have that $\|f - f_n\|_p \to 0$:

$$|f - f_n|^p \le (|f| + |f_n|)^p \le (|f| + |f|)^p = 2^p|f|^p;$$

and since $2^p|f|^p \in L^1(\mu)$ and $|f - f_n| \to 0$ pointwise a.e., we get that $\|f - f_n\|_p^p \to 0$ by dominated convergence.

(ii) It is well–known that $S(\mu)$ is dense in $L^p(\mu)$ for every $p < \infty$: we know that for every measurable $f : X \to \mathbb{K}$ there exists a sequence s_n of measurable simple functions converging pointwise to f, and such that $|s_n| \uparrow |f|$. If $f \in L^p(\mu)$ then (i) applies to say that $s_n \in L^p(\mu)$ and $\|f - s_n\|_p \to 0$ as $n \to \infty$. Now, a simple function in $L^p(\mu)$ for $p < \infty$ is of course also in $S(\mu)$: simply note that a simple function is always in L^∞, and that $L^1(\mu) \cap L^\infty(\mu) \subseteq L^p(\mu)$ for every $p \ge 1$; trivially, in any case we have $\left|\sum_{k=1}^m \alpha_k\, \chi_{A_k}\right|^p = \sum_{k=1}^m |a_k|^p\, \chi_{A_k}$, if the A_k's are pairwise disjoint, so that if $0 < p < \infty$ we have that a measurable simple function is in $L^1(\mu)$ iff it is in $L^p(\mu)$.

(iii) Given $f \in L^p(\mu)$. pick a sequence s_n of simple functions as in (ii). We have $\|s_n\|_q \le C\,\|s_n\|_p$for every n; since $|s_n| \uparrow |f|$, by monotone convergence the left–hand side tends to $\|f\|_q$, the right–hand side to $\|f\|_p$. Then $\|f\|_q \le C\,\|f\|_p$ for every $f \in L^p(\mu)$, and this of course implies $\|f\|_q < \infty$, that is, $f \in L^q(\mu)$, so that $f \in L^q(\mu)$ when $f \in L^p(\mu)$, in other words $L^p(\mu) \subseteq L^q(\mu)$.

□

Exercise 19. The formula:

(*) $$F(x) = \int_0^\infty \frac{1 - e^{-xt}}{\sinh t}\, dt$$

defines a function $F : [0, \infty[\to \mathbb{R}$ (immediate, accept for the moment this fact).

(i) Using the theorem on differentiation of parameter depending integrals, prove that F is smooth, i.e. $F \in C^\infty([0, \infty[)$.

We have, for $t > 0$:

$$\frac{1}{\sinh t} = \frac{2}{e^t - e^{-t}} = 2\frac{e^{-t}}{1 - e^{-2t}} = 2\sum_{n=0}^\infty e^{-(2n+1)t},$$

so that, for $t > 0$:

(**) $$\frac{1 - e^{-xt}}{\sinh t} = 2\sum_{n=0}^\infty (1 - e^{-xt})\, e^{-(2n+1)t}.$$

(ii) Compute, for $x \ge 0$:

$$\int_0^\infty (1 - e^{-xt})\, e^{-(2n+1)t};$$

is it possible to use the representation of the integrand in the series (**) to express F as the sum of a series of rational functions? in other words, can the series (**) be integrated termwise on $[0, \infty[$, if $x \ge 0$?

(iii) Formula (*) defines F on set D larger than $[0, \infty[$. Find D. Is $F \in C^\infty(D)$?

Solution. (i) The integrand is $f(x, t) = (1 - e^{-xt})/\sinh t$ so that $\partial_x f(x, t) = (t/\sinh t)\, e^{-xt}$, $\partial_x^2 f(x, t) = (-t/\sinh t)\, e^{-xt}$and in general

$$\partial_x^n f(x, t) = (-1)^{n-1}\frac{t^n}{\sinh t}\, e^{-xt} \qquad (n \ge 1).$$

For $x \ge 0$ we have

$$|\partial_x^n f(x, t)| = \frac{t^n}{\sinh t}\, |e^{-xt}| \le \frac{t^n}{\sinh t},$$

with the function $t \mapsto t^n/\sinh t$ in $L^1_m([0, \infty[)$, for every $n \ge 1$: in fact at $t = 0$ this function is continuous, and at ∞ it is dominated by a function such as $e^{-t/2}$. This proves that $F \in C^\infty([0, \infty[)$.

(ii) The series has positive terms, so that it can certainly be integrated termwise (monotone convergence). We get:

$$\int_0^\infty (1-e^{-xt})\,e^{-(2n+1)t}\,dt = \int_0^\infty e^{-(2n+1)t}\,dt - \int_0^\infty e^{-(2n+1+x)t}\,dt = \frac{1}{2n+1} - \frac{1}{2n+1+x}.$$

Then

$$F(x) = \sum_{n=0}^\infty \left(\frac{1}{2n+1} - \frac{1}{2n+1+x}\right) = \sum_{n=0}^\infty \frac{x}{(2n+1+x)(2n+1)} \qquad (x \ge 0).$$

(iii) The integrand is continuous at $t=0$, for every x, so that there are no problems at 0. For $t\to\infty$ the integrand is asymptotic to $1/\sinh t \sim 2e^{-t}$ for $x\ge 0$, so that (as asserted) trivially $t\mapsto f(x,t)$ belongs to $L^1_m([0,\infty[)$ is $x\ge 0$. If $x<0$ the integrand is asymptotic to $e^{-xt}/\sinh t \sim 2e^{-(x+1)t}$ as $t\to\infty$, so that the integrand is in $L^1_m([0,\infty[)$ iff $x>-1$. In other words we have $D =]-1,\infty[$. We still have $F\in C^\infty(D)$. In fact, given $x>-1$, pick a with $-1<a<x$; in the neighborhood $[a,\infty[$ of x we have:

$$|\partial_x^n f(x,t)| = \frac{t^n}{\sinh t}\,|e^{-xt}| \le \frac{t^n}{\sinh t}\,e^{-at},$$

Remark. Even if it is not required, we observe that the series representation is valid also for $x>-1$, i.e. for every $x\in D$. In fact, the terms of the series become all negative if $x<0$: for every $t>0$ and $x<0$ we have $1-e^{-xt}<0$.

□

Exercise 20. Let $(X,\mathcal{M},\mu)$ be a measure space.

(i) Prove that the following are equivalent:
  (a) There exists a sequence $E_n\in\mathcal{M}$ with $\lim_{n\to\infty}\mu(E_n)=0$ and $0<\mu(E_n)$ for every n.
  (b) There is a sequence $A_k\in\mathcal{M}$ with $0<\mu(A_k)\le 1/2^k$ for every k.
  (c) There is a function $f\in L^1(\mu)\smallsetminus L^\infty(\mu)$.
  (d) There is a disjoint sequence $B_k\in\mathcal{M}$ with $0<\mu(B_k)\le 1/2^k$ for every k.
  ((a) implies (b) easy; for (b) implies (c) prove that the formula $f(x)=\sum_{k=0}^\infty k\,\chi_{A_k}$ defines a.e. a function $f\in L^1(\mu)\smallsetminus L^\infty(\mu)$; for (c) implies (d) consider a suitable subsequence of the sequence $E_n=\{n<|f|\le n+1\}$, with $f\in L^1(\mu)\smallsetminus L^\infty(\mu)\,\dots$).

(ii) [3] Given a sequence $B_k\in\mathcal{M}$ as in (d) above ($0<\mu(B_k)\le 1/2^k$), set $b_k=\mu(B_k)$, and for $\alpha>0$ define the measurable function $g_\alpha : X\to\mathbb{R}$ by $g_\alpha=\sum_{k=0}^\infty b_k^{-\alpha}\,\chi_{B_k}$. Given $0<p<q<\infty$, prove that if $1/q<\alpha<1/p$ we have $g_\alpha\in L^p(\mu)\smallsetminus L^q(\mu)$.

Solution. (i) That (a) implies (b) is trivial: if a sequence of strictly positive numbers tends to 0, then there is a subsequence $(\mu(E_{n(k)}))_{k\in\mathbb{N}}$ such that $\mu(E_{n(k)})\le 1/2^k$; simply set $A_k=E_{n(k)}$.

(b) implies (c) The series $\sum_{k=0}^\infty k\,\chi_{A_k}$ is a series of positive measurable functions, so that we have

$$\int_X f = \sum_{k=0}^\infty k\,\mu(A_k) \le \sum_{k=0}^\infty \frac{k}{2^k} < \infty.$$

Then $\{f=\infty\}$ has measure 0, and $f\in L^1(\mu)$ (to be more precise for the punctilious: f coincides a.e. with a function in $L^1(\mu)$, which we still call f). And $f\notin L^\infty(\mu)$: since all terms are positive, we have $f\ge k\,\chi_{A_k}$, so that $\{f\ge k\}\supseteq A_k$, hence $\mu(\{f\ge k\})\ge\mu(A_k)>0$, for every $k\in\mathbb{N}$, and hence $\|f\|_\infty=\infty$.

(c) implies (d) Since $f\notin L^\infty(\mu)$, infinitely many E_n have strictly positive measure. Moreover $\lim_{n\to\infty}\mu(E_n)=0$, since by Čebičeff's inequality we have $\mu(E_n)\le(1/n)\|f\|_1$; and the E_n are pairwise disjoint. Some subsequence $B_k=E_{n(k)}$ will then be such that $\mu(B_k)\le 1/2^k$.

That (d) implies (a) is trivial.

(ii) We have

$$\int_X g_\alpha^p = \sum_{k=0}^\infty b_k^{-\alpha p}\,b_k = \sum_{k=0}^\infty b_k^{1-\alpha p} \le \sum_{k=0}^\infty \frac{1}{2^{k\beta}},$$

where $\beta=1-\alpha p>0$, by the hypothesis $\alpha<1/p$. Since the series $\sum_{k=0}^\infty 1/(2^\beta)^k$ is convergent, we have $g_\alpha\in L^p(\mu)$. And

$$\int_X g_\alpha^q = \sum_{k=0}^\infty b_k^{-\alpha q}\,b_k = \sum_{k=0}^\infty b_k^{1-\alpha q} = \infty,$$

because $1-\alpha q<0$, so that $\lim_{k\to\infty} b_k^{1-\alpha q}=\infty$. □

Analisi reale – Recupero – 12 luglio 2013

Exercise 21. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = x^2\,\chi_{]-\infty,0[}(x) + \operatorname{frac}(x)\,\chi_{[0,3[}(x) + (2 - e^{3-x})\,\chi_{[3,\infty[}(x),$$

where $\operatorname{frac}(x) = x - [x]$ is the fractional part of x.

(i) Plot F; find the points of discontinuity of F; is F right–continuous?

Define $\mu = dF(= \mu_F)$ the Radon–Stieltjes signed measure associated to F.

(ii) Find a Hahn decomposition for μ, and find the decomposition for μ^+ and μ^- into absolutely continuous and singular part with respect to Lebesgue measure m.

(iii) Find right continuous functions A, B with $A(0) = B(0) = 0$ such that $\mu^+ = dA$ and $\mu^- = dB$; plot A and B.

(iv) Compute the integrals

$$\int_{\mathbb{R}} e^{-|x|}\,d\mu^+(x), \qquad \int_{\mathbb{R}} e^{-|x|}\,d\mu^-(x)$$

(v) If $T = \{(x, y) \in \mathbb{R}^2 : y \le x\}$ compute

$$\mu^+ \otimes m(T).$$

Solution. (i) The possible discontinuities for f are $0, 1, 2, 3$; but it's easy to see that F is continuous at 0 and 3, so that the only discontinuities are 1 and 2, with jumps $\sigma_F(1) = \sigma_F(2) = -1$. Plainly F is right–continuous, because so is $x \mapsto \operatorname{frac}(x)$. The plot of F is immediate:

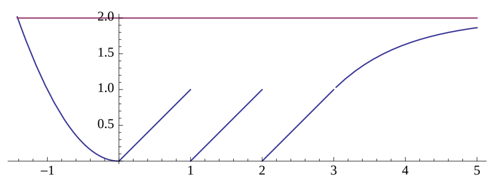

Figure 13. Plot of F.

(ii) A positive set for μ is $P =]0, 1[\cup]1, 2[\cup]2, \infty[$, with negative complement $] - \infty, 0] \cup \{1\} \cup \{2\}$. The derivative $F'(x)$ exists in $\mathbb{R} \setminus \{0, 1, 2\}$, and we have

$$F'(x) = 2x \quad x < 0;\ F'(x) = 1 \quad x \in]0, 3[\setminus\{1, 2\};\ F'(x) = e^{3-x} \quad x \ge 3$$

(it is easy to check that $F'(3)$ exists and that $F'(3) = 1$). The measure μ^+ is absolutely continuous with respect to m and we have

$$d\mu^+ = F'\,\chi_{]0,\infty[}\,dm \quad \text{so that}\ \mu^+(E) = m(E\cap]0, 3[) + \int_{E\cap[3,\infty[} e^{3-x}\,dx,$$

The absolutely continuous part of μ^- is $-2x\,\chi_{]-\infty,0]}\,dx$, the singular part is $\delta_1 + \delta_2$, so that

$$\mu^-(E) = \int_{E\cap]-\infty,0]} (-2x)\,dx + \chi_E(1) + \chi_E(2).$$

(iii) Clearly $A(x) = 0$ for $x \le 0$, and $A(x) = x$ for $0 \le x < 3$, while for $x \ge 3$:

$$A(x) = \mu^+(]0, x]) = \mu^+(]0, 3]) + \mu^+(]3, x]) = 3 + \int_3^x e^{3-t}\,dt = 3 + \left[-e^{3-t}\right]_{t=3}^{t=x} = 4 - e^{3-x}.$$

For B we get

$$B(x) = -\mu^-([x, 0[) = -\int_x^0 (-2x)\,dx = -x^2 \quad (x < 0);\ B(x) = 0 \quad 0 \le x < 1;\ B(x) = 1 \quad 0 \le x < 2;$$

$$B(x) = 2 \quad x \ge 2.$$

(iv) We have

$$\int_{\mathbb{R}} e^{-|x|}\,d\mu^+(x) = \int_0^\infty e^{-|x|}\,F'(x)\,dx = \int_0^3 e^{-x}\,dx + \int_3^\infty e^{3-2x}\,dx =$$

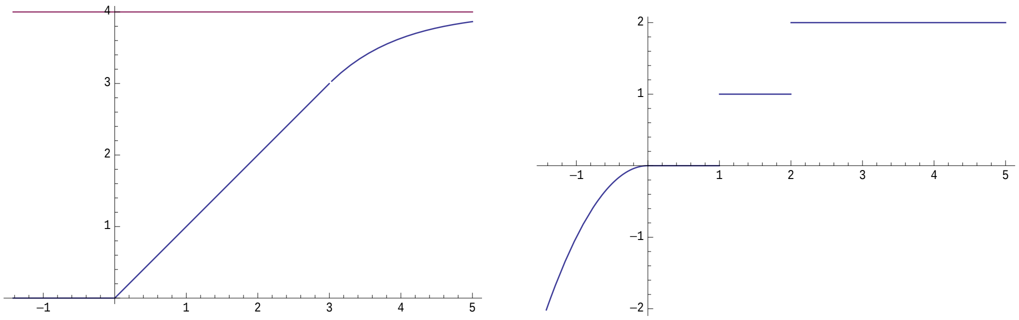

FIGURE 14. Plot of the functions A, B.

$$= [-e^{-x}]_0^3 + \frac{1}{-2}\left[e^{3-2x}\right]_0^{\infty} = 1 - e^{-3} + \frac{e^{-3}}{2} = 1 - \frac{1}{2e^3}.$$

$$\int_{\mathbb{R}} e^{-|x|}\, d\mu^-(x) = \int_{-\infty}^{0} e^{-|x|}\,(-2x)\, dx + \delta_1(e^{-|x|}) + \delta_2(e^{-|x|}) =$$

$$= [e^x(-2x)]_{-\infty}^0 + 2\int_{-\infty}^{0} e^x\, dx + e^{-1} + e^{-2} = 2 + e^{-1} + e^{-2}.$$

(v) μ^+ is finite and m is $\sigma-$finite, so Tonelli's theorem is applicable. Given $x \in \mathbb{R}$ the $x-$section $T(x)$ of T is of course $]-\infty, x]$, with Lebesgue measure ∞. Then

$$\mu^+ \otimes m(T) = \int_{\mathbb{R}} m(]-\infty, x])\, d\mu^+(x) = \int_{\mathbb{R}} \infty\, d\mu^+ = \infty.$$

REMARK. To confirm the result, we can integrate with respect to dm the μ^+-measure of the $y-$sections; For every $y \neq 0$ the $y-$section $T(y) = [y, \infty[$ of T has measure $\mu^+(T(y)) = \mu^+([0,\infty[\setminus[0,y[) = 4 - \mu^+([0,y[) = 4 - A(y)$, so that

$$\mu^+ \otimes m(T) = \int_{-\infty}^{0} 4\, dy + \int_0^{\infty} (4 - A(y))\, dy = \infty + \int_0^{\infty} (4 - A(y))\, dy = \infty$$

Some people interpreted T as contained in the first quadrant, that is they took

$$S = \{(x,y) \in \mathbb{R}^2 :\ 0 \le y \le x\}$$

instead of T (because other times it was so!). I accepted this change. The $\mu^+ \otimes m-$measure of S is then

$$\mu^+ \otimes m(S) = \int_0^{\infty} (4 - A(y))\, dy = \int_0^{3} (4 - y)\, dy + \int_3^{\infty} e^{3-y}\, dy =$$

$$= 12 - \frac{9}{2} - [e^{3-y}]_3^{\infty} = \frac{24-9}{2} + 1 = \frac{17}{2}.$$

We can of course also integrate on $[0,\infty[$ the Lebesgue measure of the $x-$sections of S with respect to μ^+; the x section is $[0, x]$ with Lebesgue measure x, so that

$$\mu^+ \otimes m(S) = \int_0^{\infty} x\, d\mu^+(x) = \int_0^3 x\, dx + \int_3^{\infty} x\, e^{3-x}\, dx = \frac{9}{2} + \left[-xe^{3-x}\right]_3^{\infty} + \int_3^{\infty} e^{3-x}\, dx =$$

$$= \frac{9}{2} + 3 + \left[-e^{3-x}\right]_3^{\infty} = \frac{15}{2} + 1 = \frac{17}{2}.$$

□

EXERCISE 22. Let $\mathcal{A}$ be an algebra of parts of X, and let $\mu : \mathcal{A} \to [0,\infty]$ be a (positive) premeasure. Let $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be the outer measure associated to μ in the usual way.

(i) Give the precise definition of $\mu^*(E)$ for every $E \subseteq X$, and prove that $\mu^*(A) = \mu(A)$ for every $A \in \mathcal{A}$. Where does countable additivity of μ enter the proof?

If $\phi : \mathcal{P}(X) \to [0,\infty]$ is an outer measure, and A, $E \subseteq X$, we say that A splits E additively (with respect to ϕ) if $\phi(E) = \phi(E \cap A) + \phi(E \setminus A)$.

(ii) With μ and μ^* as above, prove that $B \subseteq X$ is μ^*-measurable if and only if B splits additively every $A \in \mathcal{A}$ with $\mu(A) < \infty$. Deduce from this that every $B \in \mathcal{A}$ is μ^*-measurable.

Solution. (i) We have

$$\mu^*(E) = \inf\left\{\sum_{n=0}^{\infty}\mu(A_n) :\ A_n \in \mathcal{A},\ A \subseteq \bigcup_{n=0}^{\infty} A_n\right\},$$

infimum taken over all countable covers $(A_n)_{n\in\mathbb{N}}$ of E by elements of $\mathcal{A}$. Given $A \in \mathcal{A}$ since $(A, \emptyset, \emptyset, \dots)$ is a cover of A we have $\mu^*(A) \le \mu(A)$ for every $A \in \mathcal{A}$. And if $(A_n)_{n\in\mathbb{N}}$ is a countable cover of A by elements of $\mathcal{A}$, then $A = \bigcup_{n=0}^{\infty} A \cap A_n$ so that, by countable subadditivity and monotonicity of μ:

$$\mu(A) \le \sum_{n=0}^{\infty}\mu(A\cap A_n) \le \sum_{n=0}^{\infty}\mu(A_n),$$

which implies $\mu(A) \le \mu^*(A)$. We know that for positive finitely additive functions countable additivity is equivalent to countable subadditivity; countable additivity has then just been used in the proof.

(ii) If B is μ^*-measurable, then it splits additively *every* subset of X, and not only the sets of $\mathcal{A}$ with μ finite. For the converse, assuming that B splits additively every $A \in \mathcal{F}(\mu) = \{A \in \mathcal{A} :\ \mu(A) < \infty\}$, we have to prove that for every $E \subseteq X$ with $\mu^*(E) < \infty$ we have

$$\mu^*(E) \ge \mu^*(E \cap B) + \mu^*(E \setminus B).$$

Given $\varepsilon > 0$ pick a cover $(A_n)_{n\in\mathbb{N}}$ of E by elements of $\mathcal{A}$ such that $\sum_{n=0}^{\infty}\mu(A_n) \le \mu^*(E) + \varepsilon$. Then

$$\mu^*(E) + \varepsilon \ge \sum_{n=0}^{\infty}\mu(A_n);$$

of course $\mu(A_n) < \infty$ for every $n \in \mathbb{N}$ so that by the hypothesis we have

$$\mu(A_n) = \mu(A_n \cap B) + \mu(A_n \setminus B),$$

and the preceding inequality yields

$$\mu^*(E) + \varepsilon \ge \sum_{n=0}^{\infty}\mu(A_n) = \sum_{n=0}^{\infty}\mu(A_n \cap B) + \sum_{n=0}^{\infty}\mu(A_n \setminus B);$$

by countable subadditivity , setting $A = \bigcup_{n=0}^{\infty} A_n$ we now get

$$\sum_{n=0}^{\infty}\mu^*(A_n \cap B) \ge \mu^*(A\cap B); \qquad \sum_{n=0}^{\infty}\mu(A_n \setminus B) \ge \mu^*(A \setminus B);$$

and by monotonicity

$$\mu^*(A\cap B) \ge \mu^*(E\cap B); \qquad \mu^*(A \setminus B) \ge \mu^*(E \setminus B).$$

We have proved that for every $\varepsilon > 0$:

$$\mu^*(E) + \varepsilon \ge \mu^*(E \cap B) + \mu^*(E \setminus B),$$

and since $\varepsilon > 0$ is arbitrary we conclude.

Finally, if $B \in \mathcal{A}$ then we have, for every $A \in \mathcal{A}$

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B),$$

by (finite) additivity of μ on $\mathcal{A}$. By (i) the preceding relation may be also written

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \setminus B),$$

thus proving that B splits additively with respect to μ^* all elements of $\mathcal{A}$, so that is μ^*-measurable.

Remark. The argument used to prove (ii) is of course exactly the one that shows μ^*-measurability of elements of $\mathcal{A}$.

□

Exercise 23. (i) State the theorem on continuity and differentiability of parameter depending integrals (the version with general measure spaces).

(ii) Using the preceding theorem prove that the formula:

(*) $$\varphi(x) = \int_0^{\infty} e^{-xt}\,\frac{\sin t}{t}\,dt$$

defines a function $\varphi \in C^1(]0,\infty[,\mathbb{R})$.

(iii) Give an esplicit formula for $\varphi'(x)$, not containing integrals, and deduce from it an analogous expression for $\varphi(x)$.

Solution. (i) See Lecture Notes, 7.6.

(ii) The derivative with respect to x of the integrand is $-e^{-xt}\sin t$. Given $x > 0$, let $a = x/2$ (or simply pick any a with $0 < a < x$), and let $U = [a, \infty[$. For $y \in U$ we have

$$|-e^{-yt}\sin t| = e^{-yt}\,|\sin t| \le e^{-yt} \le e^{-at};$$

of course $t \mapsto e^{-at}$ belongs to $L^1([0,\infty[)$, since $a > 0$. Then $\varphi \in C^1(]0,\infty[$, and

(iii) (see formula for the primitive of $e^{-xt}\sin t$):

$$\varphi'(x) = \int_0^\infty (-e^{-xt}\sin t)\,dt = \left[\frac{e^{-xt}}{1+x^2}(\sin t + \cos t)\right]_{t=0}^{t=\infty} = \frac{-1}{1+x^2}.$$

Then we get

$$\varphi(x) = \operatorname{arccotan} x + k \qquad (x > 0);$$

but one easily sees that $\lim_{x\to\infty}\varphi(x) = 0$ (e.g., by dominated convergence; or simply because $|\varphi(x)| \le \int_0^\infty e^{-xt}\,dt = 1/x$), so that

$$\varphi(x) = \operatorname{arccotan} x \qquad (x > 0).$$

Remark. Nobody seems to be able to verify the hypotheses of the theorem in this particular case, and apparently many have not even understood the statement.

□

Exercise 24. Let $(X, \mathcal{M}, \mu)$ be a measure space..

(i) Assume that $L^1(\mu)$ is contained in $L^\infty(\mu)$. Prove that then we also have $L^p(\mu) \subseteq L^\infty(\mu)$, for every $p > 0$

(ii) Prove that the hypothesis $L^1(\mu) \subseteq L^\infty(\mu)$ implies that the spaces $L^p(\mu)$ increase with p (that is, if $0 < p < q$ then $L^p(\mu) \subseteq L^q(\mu)$).

(iii) Assume that there is $f \in L^1(\mu) \setminus L^\infty(\mu)$. Prove that then there is a disjoint sequence $E_n \in \mathcal{M}$ with $0 < \mu(E_n) < \infty$ and $\lim_{n\to\infty}\mu(E_n) = 0$. Conversely, the existence of such a sequence implies the existence of a function $f \in L^1(\mu) \setminus L^\infty(\mu)$.

Solution. (i) Recall that $f \in L^p(\mu)$ is equivalent to $|f|^p \in L^1(\mu)$, if $0 < p < \infty$. But clearly $|f|^p \in L^\infty(\mu)$ holds if and only if $f \in L^\infty(\mu)$ ($|f|^p \le M \iff |f| \le M^{1/p}$).

(ii) If $0 < p < q < \infty$ we have $|f|^q = |f|^{q-p}\,|f|^p \le \|f\|_\infty^{q-p}\,|f|^p$; integrating both sides we have

$$\|f\|_q^q \le \|f\|_\infty^{q-p}\,\|f\|_p^p \Longrightarrow \|f\|_q \le \|f\|_\infty^{1-p/q}\,\|f\|_p^{p/q}.$$

If $f \in L^p(\mu)$ then also $f \in L^\infty(\mu)$ by the hypothesis made and (i), so that the right–hand side is finite, forcing finiteness of the left–hand side. That is $f \in L^p(\mu)$ implies $f \in L^q(\mu)$, as desired.

(iii) See the exam of February 25, 2013. Everybody ought to look at previous exams!

□

Analisi Reale per Matematica – Recupero – 3 settembre 2013

Exercise 25. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = e^x\,\chi_{]-\infty,0[}(x) + [x]\,\chi_{[0,3[}(x) + (3 - e^{3-x})\,\chi_{[3,\infty[}(x),$$

where $[x]$ is the integer part of x.

(i) Plot F; find the points of discontinuity of F; is F right–continuous?

Define $\mu = dF(= \mu_F)$ the Radon–Stieltjes signed measure associated to F.

(ii) Find a Hahn decomposition for μ, and find the decomposition for μ^+ and μ^- into absolutely continuous and singular part with respect to Lebesgue measure m.

(iii) Find right continuous functions A, $B : \mathbb{R} \to \mathbb{R}$ with $A(-\infty) = B(-\infty) = 0$ such that $\mu^+ = dA$ and $\mu^- = dB$; plot A and B.

(iv)] Compute the integrals

$$\int_{\mathbb{R}} e^{i\alpha x}\,d\mu^+(x), \quad \int_{\mathbb{R}} e^{i\alpha x}\,d\mu^-(x)$$

($\alpha \in \mathbb{R}$ is a constant).

(v) If $T = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le |x|\}$ compute

$$|\mu| \otimes m(T)$$

in two ways, by integrating the measures of both $x-$ and $y-$sections.

Solution. (i) Characteristic functions of upper half–open intervals are right–continuous, and $x \mapsto [x]$ is right–continuous, so F is also right–continuous, as the sum of three right–continuous functions. We have $F(0^-) = 1$, $F(0)^+ = 0$, so $\mu_F(\{0\}) = -1$; $F(1^-) = 0$, $F(1^+) = F(1) = 1$, so $\mu_F(\{1\}) = 1$; $F(2^-) = 1$, $F(2^+) = 2$ and $\mu_F(\{2\}) = 1$; there are no other points of discontinuity besides $\{0,1,2\}$. The plot is easy.

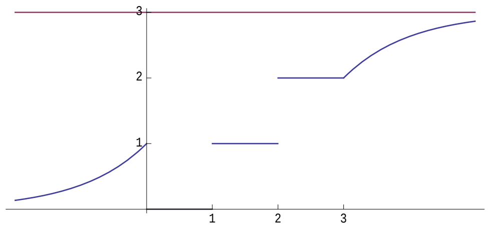

FIGURE 15. Plot of F.

(ii) La derivata di F esiste in $\mathbb{R} \setminus \{0,1,2,3\}$ ed in tale insieme coincide con

$$F'(x) = e^x \, \chi_{]-\infty,0[}(x) + e^{3-x} \, \chi_{]3,\infty[}(x);$$

Posto $P =]-\infty, 0[\cup]0,\infty[$ ed $N = \{0\}$, la coppia P, N è una decomposizione di Hahn per μ. Si ha

$$\mu^+ = F'(x)\,dx + (\delta_1 + \delta_2); \qquad \mu^- = \delta_0,$$

dove ovviamente $F'(x)\,dx$ è la parte assolutamente continua e $\delta_1 + \delta_2$ quella singolare; μ^- ha parte assolutamente continua nulla.

(iii) Since $A(x) = \mu^+(]-\infty, x])$ we get

$$A(x) = e^x \quad x < 0;\ A(x) = 1 \quad 0 \le x < 1; \quad A(x) = 2 \quad 1 \le x < 2;\ A(x) = 3 \quad 2 \le x < 3;$$
$$A(x) = 4 - e^{3-x} \quad 3 < x.$$

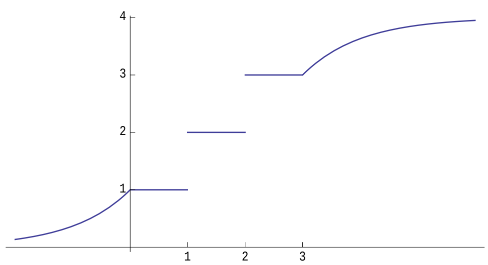

FIGURE 16. Plot of A.

And $B(x) = \mu^-(]-\infty, x])$ coincides with $\chi_{[0,\infty[}$, the Heaviside step.

(iv) We have

$$\int_{\mathbb{R}} e^{i\alpha x}\, d\mu^-(x) = \int_{\mathbb{R}} e^{i\alpha x}\, d\delta_0 = e^{i\alpha 0} = 1.$$

And

$$\int_{\mathbb{R}} e^{i\alpha x}\, d\mu^+(x) = \int_{\mathbb{R}} e^{i\alpha x}(F'(x)\,dx + d(\delta_1 + \delta_2)) = \int_{-\infty}^{0} e^{(i\alpha x+1)x}\,dx + e^{i\alpha} + e^{2i\alpha} + \int_3^\infty e^{3+(i\alpha - x)}\,dx =$$

$$\left[\frac{e^{(i\alpha x+1)x}}{i\alpha+1}\right]_{-\infty}^{0} + e^{i\alpha} + e^{2i\alpha} + e^3 \left[\frac{e^{(i\alpha-1)x}}{i\alpha - 1}\right]_3^\infty = \frac{1}{i\alpha+1} + e^{i\alpha} + e^{2i\alpha} - \frac{e^{3i\alpha}}{i\alpha-1}.$$

(v) The $x-$section $[0, |x|]$ has Lebesgue measure $|x|$, so that

$$|\mu| \otimes m(T) = \int_{\mathbb{R}} |x|\, d|\mu| = \int_{\mathbb{R}} |x|\, dA(x) + \int_{\mathbb{R}} |x|\, d\delta_0(x) = \int_{-\infty}^{0} (-x)\, e^x\, dx + 1 + 2 + \int_3^\infty x\, e^{3-x}\, dx =$$

$$[-x\,e^x]_{-\infty}^0 + \int_{-\infty}^0 e^x\,dx + 3 - [x\,e^{3-x}]_3^\infty + \int_3^\infty e^{3-x}\,dx = 0 + 1 + 3 + 3 + 1 = 8.$$

The $y-$section is empty for $y < 0$, and is $T(y) =]-\infty, -y] \cup [y, \infty[$ if $y \geq 0$, with measure $|\mu|(T(0)) = |\mu|(\mathbb{R}) = 4 + 1 = 5$, whereas for $y > 0$ we get:

$$|\mu|(T(y)) = \mu^+(T(y)) = \mu^+(]-\infty, -y]) + \mu^+([y, \infty[) = A(-y) + (4 - A(y^-));$$

then

$$|\mu| \otimes m(T) = \int_0^\infty \mu^+(T(y))\,dy = \int_0^\infty A(-y)\,dy + \int_0^\infty (4 - A(y^-))\,dy =$$
$$\int_0^\infty e^y\,dy + \int_0^1 (4-1)\,dy + \int_1^2 (4-2)\,dy + \int_2^3 (4-3)\,dy + \int_0^\infty (4 - (4 - e^{3-y})\,dy =$$
$$1 + 3 + 2 + 1 + 1 = 8.$$

□

Exercise 26. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, where $f_0(x) = (1/2)\chi_{[0,1/2[} - \chi_{[1/2,1[}$ and $f_n(x) = (1/n)\,f_0(x/n)$ for $n \geq 1$. Plot f_0, f_2, f_3, f_7, evaluate $f(x) = \lim_{n\to\infty} f_n(x)$, compute the integrals $\int_\mathbb{R} f_n(x)\,dx$ and notice that

$$\int_\mathbb{R} f(x)\,dx > \lim_{n\to\infty} \int_\mathbb{R} f_n(x)\,dx;$$

why does this not contradict Fatou's lemma (3 points)? Let now $(X, \mathcal{M}, \mu)$ be a measure space.

(i) Assume that $f_n : X \to \mathbb{R}$ is a sequence of functions in $L^1(\mu)$ that converges uniformly to $f \in L^1(\mu)$. Is it true that $\lim_n \int_X f_n = \int_X f$? if not, can you give a counterexample? what hypothesis can be added to μ to ensure that this holds?

(ii) Let u_n be a sequence in $L^1_\mu(X, \mathbb{R})$ which converges pointwise a.e. to $u \in L^1_\mu(X, \mathbb{R})$, and is such that $\lim_n \int_X u_n = \int_X u$; let $f_n \in L^1_\mu(X, \mathbb{R})$ be a sequence with $u_n \leq f_n$ a.e., for every $n \in \mathbb{N}$. Prove that

$$\int_X \liminf f_n \leq \liminf \int_X f_n.$$

Solution. Notice that the plot of f_n is obtained from the plot of f_0 by a dilation of ratio n in the direction of the $x-$axis, and one of ratio $1/n$ in the direction y, so that the plots are as in the following figure.

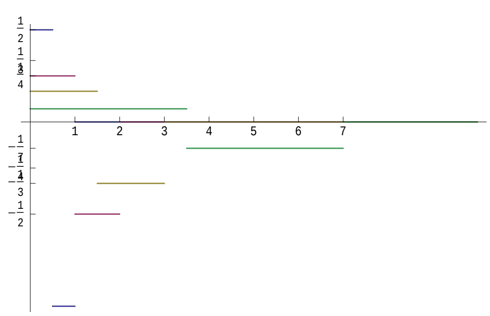

Figure 17. Plots of some f_n.

Since $\|f_n\|_\infty = 1/n$ for $n \geq 1$ the sequence f_n converges uniformly to the identically zero function, whose integral is 0. The integral of f_0 is clearly $1/4 - 1/2 = -1/4$; and setting $x = nt$ we have

$$\int_\mathbb{R} f_n(x)\,dx = \int_\mathbb{R} f_n(nt)\,ndt = \int_\mathbb{R} \frac{1}{n}\,f_0(nt/n)\,ndt = \int_\mathbb{R} f_0(t)\,dt = -\frac{1}{4}.$$

Fatou's lemma is not violated because it concerns positive functions.

(i) Not true: the above is a counterexample. If $\mu(X) < \infty$ then uniform convergence of L^1 functions implies convergence in $L^1(\mu)$, according to the inequality:

$$\|f - f_n\|_1 = \int_X |f - f_n|\,d\mu \leq \int_X \|f - f_n\|_\infty\,d\mu = \|f - f_n\|_\infty\,\mu(X).$$

(ii) We have $f_n - u_n \geq 0$, so that Fatou's lemma may be applied to the sequence $f_n - u_n$; we get:

$$(*) \qquad \int_X \liminf_n (f_n - u_n)\, d\mu \leq \liminf_n \int_X (f_n - u_n)\, d\mu;$$

But since $\lim_n u_n(x) = u(x)$ exists for a.e. $x \in X$ we get a.e in X:

$$\liminf_n (f_n(x) - u_n(x)) = \liminf_n f_n(x) - u(x),$$

and since $\lim_n \int_X u_n$ exists and coincides with $\int_X u$ we also have

$$\liminf_n \int_X (f_n - u_n) = \liminf_n \int_X f_n - \int_X u,$$

so that (*) is

$$\int_X (\liminf_n f_n - u) \leq \liminf_n \int_X f_n - \int_X u \iff \int_X \liminf_n f_n - \int_X u \leq \liminf_n \int_X f_n - \int_X u,$$

ad cancelling $-\int_X u$ we conclude. □

Exercise 27. (i) Using the theorem on differentiability of parameter depending integrals prove that the formula:

$$(*) \qquad \varphi(x) = \int_{\mathbb{R}} e^{-t^2 - xt}\, dt$$

defines a function $\varphi \in C^1(\mathbb{R}, \mathbb{R})$, whose derivative is

$$\varphi'(x) = \int_{\mathbb{R}} (-t) e^{-t^2} e^{-xt}\, dt.$$

(ii) Integrating φ' by parts, find a differential equation verified by φ, and from it deduce an explicit expression of φ, not containing integrals.

(iii) The explicit formula for $\varphi(x)$ can also be easily obtained directly (complete the square ...).

Solution. (i) It is clear that for every given $x \in \mathbb{R}$ the integrand is in $L^1_m(\mathbb{R})$, so that φ is defined. We have $\partial_x(e^{-t^2 - xt}) = (-t)\, e^{-t^2 - xt}$. Given $x \in \mathbb{R}$, the function $\gamma(t) = e^{-t^2} e^{(|x|+1)|t|}$ is in $L^1(\mathbb{R})$ and dominates $e^{-t^2 - yt}$ for $y \in [x-1, x+1]$ and $t \in \mathbb{R}$. Then $\varphi \in C^1(\mathbb{R})$, and

$$\varphi'(x) = \int_{\mathbb{R}} (-t)\, e^{-t^2} e^{-xt}\, dt.$$

(ii) Integrating by parts in the preceding formula we get

$$\varphi'(x) = \left[\frac{e^{-t^2}}{2} e^{-xt}\right]_{t=-\infty}^{t=\infty} + \frac{x}{2} \int_{-\infty}^{+\infty} e^{-t^2} 2\, e^{-xt}\, dt = \frac{x}{2} \varphi(x);$$

Then φ' satisfies the differential equation $\varphi'(x) = (x/2)\, \varphi(x)$; since $\varphi(0) = \sqrt{\pi}$ we have

$$\varphi(x) = \sqrt{\pi}\, e^{x^2/4} \qquad (x \in \mathbb{R}).$$

(iii) We have $-t^2 - xt = -(t^2 + xt) = -(t^2 + xt + x^2/4 - x^2/4) = -(t + x/2)^2 + x^2/4$, so that (recalling also translation invariance of the Lebesgue integral)

$$\varphi(x) = \int_{\mathbb{R}} e^{-t^2 - xt}\, dt = \int_{\mathbb{R}} e^{-(t+x)^2 + x^2/4}\, dt = e^{x^2/4} \int_{\mathbb{R}} e^{-(t+x)^2}\, dt = \sqrt{\pi}\, e^{x^2/4}.$$

□

Exercise 28. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) If $g \in L^1_\mu(X, \mathbb{C})$ give a careful proof of the fact that

$$\int_X g\, d\mu = \int_X |g|\, d\mu$$

holds if and only if $g(x) = |g(x)|$ for a.e. $x \in X$.

(ii) Given $g \in L^1_\mu(X, \mathbb{C})$ find a bounded measurable function $u : X \to \mathbb{C}$ such that

$$\int_X g\, u\, d\mu = \int_X |g|\, d\mu.$$

Let $p, q > 1$ be conjugate exponents, $1/p + 1/q = 1$, and fix a nonzero $g \in L^q(\mu)$.

(iii) Prove that the formula

$$\varphi_g(f) = \int_X f g \, d\mu$$

defines φ_g as a (trivially linear) continuous map of $L^p(\mu)$ into $\mathbb{C}$, of (operator) norm $\|\varphi_g\|$ not larger than $\|g\|_q$.

(iv) Accepting the following fact: if $f, g \in L(X)$ and $0 < \|f\|_p < \infty$, $0 < \|g\|_q < \infty$ then $\|fg\|_1 = \|f\|_p \|g\|_q$ holds if and only if there is a constant $k > 0$ such that $|g(x)|^q = k|f(x)|^p$ for a.e. $x \in X$, prove that $\|\varphi_g\| = \|g\|_q$, and find $a \in L^p(\mu)$ with $\|a\|_p = 1$ and $\varphi_g(a) = \|g\|_q$.

Solution. (i) Sufficiency is trivial. For necessity, write $g = u + i\,v$, with $u = \operatorname{Re} g$ and $v = \operatorname{Im} g$. We get

$$\int_X g := \int_X u + i \int_X v = \int_X |g|$$

Since $\int_X |g|$ is real, we have $\int_X v = 0$, so that the preceding equality writes

$$\int_X u = \int_X |g|;$$

now of course we have $u \le |u| \le |g|$ so that the equality implies

$$\int_X (|g| - u) = 0 \qquad \text{and since } |g| - u \ge 0, \text{ this holds iff } |g(x)| = u(x) (= \operatorname{Re} g(x)) \text{ for a.e. } x \in X;$$

and since the modulus of a complex number equals its real part iff this number is real and positive, we are done.

(ii) To ensure equality we simply take u in such a way that $g(x)\,u(x) = |g(x)|$ for every $x \in X$; since $g(x) = \operatorname{sgn} g(x)\,|g(x)|$, we have $g(x)\,\overline{\operatorname{sgn} g(x)} = |g(x)|$; so we set $u(x) = \overline{\operatorname{sgn} g(x)}$, recalling that $|\overline{\operatorname{sgn} g(x)}| = 1$ or 0, so that u is bounded. Measurability of u follows from the fact that the sign function $\operatorname{sgn} : \mathbb{C} \to \mathbb{C}$, although not continuous, is Borel measurable, as we have seen.

(iii) Simply use Hölder's inequality:

$$|\varphi_g(f)| = \left| \int_X f g \, d\mu \right| \le \int_X |fg| \, d\mu = \|fg\|_1 \le \|f\|_p \, \|g\|q = (\|g\|_q) \, \|f\|_p.$$

this shows that $\|g\|_q$ is a Lipschitz constant for φ_g; the operator norm is the smallest such constant.

(iv) To get $\|f\,g\|_1 = \|f\|_p \, \|g\|_q$ we have to use f such that $|f|^p = k|g|^q$, hence $|f| = k^{1/p}|g|^{q/p} = \rho\,|g|^{q-1}$ with $\rho > 0$ a constant. We have to make such an f of L^p–norm 1, so that:

$$1 = \left(\int_X \rho^p \, |g|^{(q-1)p} \, d\mu \right)^{1/p} \iff \rho = \left(\int_X |g|^q \, d\mu \right)^{-1/p} = 1/\|g\|_q^{q-1}.$$

Finally to make $\varphi_g(f) = \|fg\|_1$ we have to make

$$\int_X fg = \int_X |f\,g|;$$

so we take $f(x) = \overline{\operatorname{sgn} g(x)} |g(x)|^{q-1}/\|g\|_q^{q-1}$.

□

Analisi Reale per Matematica – III Recupero – 24 settembre 2013

Exercise 29. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x) = \frac{\chi_{]-\infty,0[}(x)}{1 - x^3} + \frac{x + \psi(x)}{2} \, \chi_{[0,1[}(x) + \frac{\chi_{[1,\infty[}(x)}{1 + (x-1)^3},$$

where ψ is the Cantor function with $\delta_n = (2/3)^n$.

(i) Plot F; find the points of discontinuity of F; is F right–continuous? Plot $T(x) = VF(]-\infty, x])$

Define $\mu = dF (= \mu_F)$ the Radon–Stieltjes signed measure associated to F.

(ii) Find right continuous functions A, $B : \mathbb{R} \to \mathbb{R}$ with $A(-\infty) = B(-\infty) = 0$ such that $\mu^+ = dA$ and $\mu^- = dB$; plot A and B.

(iii) For μ^+ and μ^- write the decomposition into absolutely continuous and singular part.

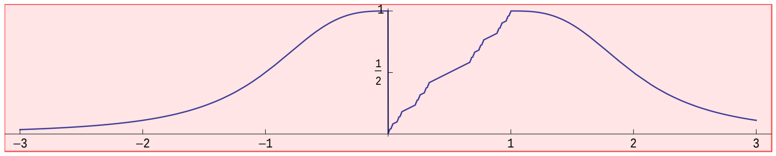

Figure 18. Plot of F.

Solution. (i) The plot is easy. It is clear that 0 is the only jump point, and that F is right–continuous. By piecewise monotonicity of F, it is clear that we have

$$T(x) = F(x) = \frac{1}{1-x^3} \quad \text{for} \quad x < 0;$$
$$T(x) = 2 + \frac{x+\psi(x)}{2} \quad \text{for} \quad 0 \le x < 1;$$
$$T(x) = 4 - \frac{1}{1+(x-1)^3} \quad \text{for} \quad 1 \le x.$$

Notice that $T(0^-) = 1$, $T(0^+) = 1 + 1 = 2$. The plot is easy:

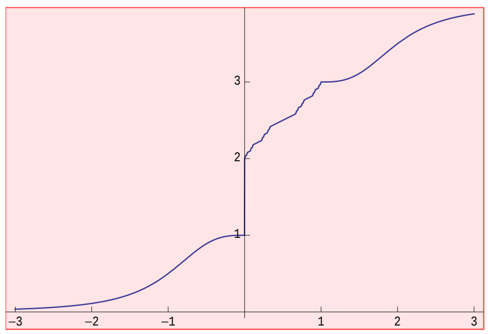

Figure 19. Plot of T.

(iii) It is boring but easy to plot $A = (T+F)/2$ and $B = (T-F)/2$, for which $\mu^+ = dA$ and $\mu^- = dB$; we do not give the expressions

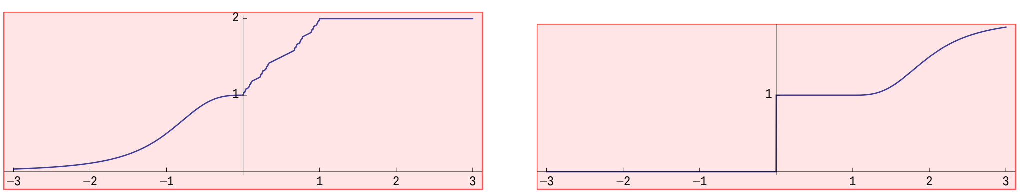

Figure 20. Plot of A (left) and B.

(iv) The absolutely continuous part of $\mu^+ = dA$ is

$$A'(x)\,dx = \left(\frac{3x^2}{(1-x^3)^2}\chi_{]-\infty,0[}(x) + \frac{1}{2}\chi_{]0,1[}(x)\right)\,dx;$$

the singular part is $d\psi/2$. For μ^- the singular part is δ_0, the absolutely continuous part is

$$B'(x)\,dx = \frac{3(x-1)^2}{(1+(x-1)^3)^2}\chi_{[1,\infty[}(x)\,dx.$$

□

Exercise 30. In this problem $L^p = L^p_m([0,1])$, with m Lebesgue measure. For $n = 3, 4, 5, \dots$ set $f_n = (n/\log n)\,\chi_{]0,1/n]}$.

(i) Plot f_3, f_4, f_7 and prove that f_n converges everywhere on $[0,1]$ to a function f; find f.

(ii) Find all $p \in [1,\infty]$ such that f_n converges in L^p.

(iii) Find all $p \in [1, \infty]$ such that the series

$$\sum_{n=3}^{\infty} \frac{n}{\log n} \chi_{]1/(n+1),1/n]}$$

converges in L^p; prove first that this series converges pointwise everywhere on $[0, 1]$ to a function g; plot g.

(iv) Deduce from the above that a sequence of positive functions can converge pointwise and in L^1 without being dominated by a function in L^1.

Solution. (i) The plots are easy.

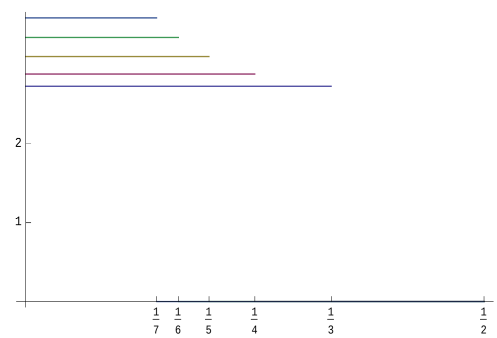

FIGURE 21. Plot of some f_n (not on scale).

Given $x \in [0, 1]$, if $x = 0$ we have $f_n(x) = 0$ for all $n \geq 3$, and if $x > 0$ for $n > 1/x$ we have $f_n(x) = 0$; the sequence converges everywhere to the zero function, $f(x) = 0$ for every $x \in [0, 1]$.

(ii) We have

$$\|f_n\|_p^p = \int_0^1 \frac{n^p}{\log^p n} \chi_{]0,1/n]}\, dx = \frac{n^p}{\log^p n} \frac{1}{n} = \frac{n^{p-1}}{\log^p n};$$

if $p > 1$ we clearly have $\lim_{n\to\infty} n^{p-1}/\log^p n = \infty$, for $p = 1$ we have $\lim_{n\to\infty}(1/\log n) = 0$. For $p = \infty$ we have $\|f_n\|_\infty = n/\log n$, with limit ∞ for $n \to \infty$. Then the sequence converges in L^p only for $p = 1$, to the zero function.

(iii) The important fact is that the intervals $]1/(n + 1), 1/n]$ are pairwise disjoint so that the series is pointwise convergent to the function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = (n/\log n)$ if $1/(n + 1) < x \leq 1/n$, for $n \geq 3$, and $g(x) = 0$ for all other x. We also have

$$(g(x))^p = \sum_{n=3}^{\infty} \frac{n^p}{\log^p n} \chi_{]1/(n+1),1/n]},$$

(for every given $x \in [0, 1]$ there is at most one term in the sum which is nonzero!) so that, by the theorem on termwise integration of series of positive functions:

$$\|g\|_p^p = \sum_{n=3}^{\infty} \frac{n^p}{\log^p n} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=3}^{\infty} \frac{n^{p-1}}{(n+1)\, \log^p n}.$$

For $p = 1$ we have the series

$$\sum_{n=3}^{\infty} \frac{1}{(n+1)\, \log n},$$

which is not convergent (use the integral test: $1/((n+1)\, \log n) \sim 1/(n\, \log n)$ and the integral $\int_2^\infty dx/(x \log x)$ diverges); for $p > 1$ we have, if n is large enough:

$$\frac{n^{p-1}}{(n+1)\, \log^p n} \geq \frac{1}{n+1} \quad \left(\text{in fact} \quad \lim_{n\to\infty} \frac{n^{p-1}}{\log^p n} = \infty\right),$$

so that the series is divergent. So the series never converges in L^p, for no p with $1 \leq p < \infty$; and since the sum is not in L^∞, it does not converge in L^∞, either.

We observe next that we have, for every $x \in [0, 1]$

$$g(x) = \sup\{f_n(x) : n \geq 3\};$$

in fact the sequence $n \mapsto n/\log n$ is increasing for $n \geq 3$:

$$\frac{n}{\log n} < \frac{n+1}{\log(n+1)} \iff n\,\log(n+1) < (n+1)\,\log n \iff \log(n+1)^n < \log n^{n+1} \iff$$
$$\iff (n+1)^n < \log n^{n+1} \iff \left(1+\frac{1}{n}\right)^n < n,$$

certainly true for $n \geq 3$, since $(1+1/n)^n < e < 3$. Then there is no function $h \in L^1$ such that $f_n(x) \leq h(x)$ for every $n \geq 3$.

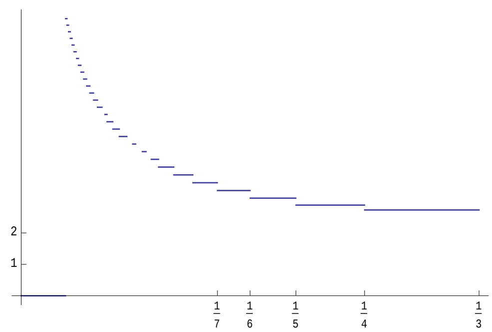

FIGURE 22. Plot of g (not on scale).

□

EXERCISE 31. (i) Using the theorem on differentiability of parameter depending integrals prove that the formula:

$$(*) \qquad \varphi(x) = \int_1^\infty \frac{e^{-xt}}{t}\,dt$$

defines a function $\varphi \in C^1(]0,\infty[,\mathbb{R})$.

(ii) Find for φ' an expression not containing integrals.

(iii) What are the limits

$$\lim_{x\to 0^+} \varphi(x); \quad \lim_{x\to\infty} \varphi(x)?$$

(iv) Prove that for every $a > 0$ we have $\varphi \in L^1([a,\infty[)$ and espress the integral $\int_a^\infty \varphi(x)\,dx$ by means of $\varphi(a)$ (use Fubini – Tonelli's theorem …).

Solution. (i) Clearly $t \mapsto e^{-xt}/t$ belongs to $L^1([1,\infty[)$ for every $x > 0$, so φ is defined for $x > 0$. We have $\partial_x(e^{-xt}/t) = -e^{-xt}$. Given $x > 0$, let $U = [x/2,\infty[$; the function $\gamma(t) = e^{-(x/2)t)}$ is in $L^1([1,\infty[)$ and $e^{-yt}(= |-e^{-yt}|) \leq \gamma(t)$ for every $y \in U$ and $t \geq 1$. By the theorem on differentiability we get

$$\varphi'(x) = \int_1^\infty \partial_x(e^{-xt}/t)\,dt = \int_1^\infty (-e^{-xt})\,dt = \left[\frac{e^{-xt}}{x}\right]_{t=1}^{t=\infty} = -\frac{e^{-x}}{x}.$$

We have also solved (ii).

(iii) Notice that for fixed $t \geq 1$ the function $x \mapsto e^{-xt}/t$ is decreasing on $]0,\infty[$ (trivially). If $x_j \downarrow 0$ we then have that the sequence $f_j(t) = e^{-x_j t}/t$ is increasing and converges to $t \mapsto 1/t$. By the monotone convergence theorem we then have

$$\int_1^\infty f_j(t)\,dt \uparrow \int_1^\infty \frac{dt}{t} = \infty \quad \text{in other words} \quad \lim_{x\to 0^+} \varphi(x) = \infty.$$

And if $x_j \uparrow \infty$ then $f_j(t) = e^{-x_j t}/t$ is dominated by $f_0 \in L^1([1,\infty[)$ and converges pointwise to 0 so that, by dominated convergence:

$$\lim_{j\to\infty} \int_1^\infty f_j(t)\,dt = 0 \quad \text{in other words} \quad \lim_{x\to\infty} \varphi(x) = 0.$$

Of course we can also argue like that: $e^{-xt}/t \leq e^{-xt}$ for $t \geq 1$, so that

$$0 < \varphi(x) \leq \int_1^\infty e^{-xt}\,dt = \frac{e^{-x}}{x},$$

and $e^{-x}/x \to 0$ as $x \to +\infty$.

(iv) We have to compute

$$\int_a^\infty \varphi(x)\,dx = \int_a^\infty \left(\int_1^\infty \frac{e^{-xt}}{t}\,dt\right)\,dx;$$

all spaces have $\sigma-$finite measure and the integrand is measurable and positive; so the iterated integral obtained by exchanging the order of integration coincides with the given one:

$$\int_a^\infty \left(\int_1^\infty \frac{e^{-xt}}{t}\,dt\right)\,dx = \int_1^\infty \left(\int_a^\infty \frac{e^{-xt}}{t}\,dx\right)\,dt = \int_1^\infty \left[\frac{e^{-xt}}{-t^2}\right]_{x=a}^{x=\infty} dt =$$
$$= \int_1^\infty \frac{e^{-at}}{t^2}\,dt = \text{(by parts)} = \left[-\frac{e^{-at}}{t}\right]_{t=1}^{t=\infty} + a\int_1^\infty \frac{e^{-at}}{t}\,dt = e^{-a} + a\,\varphi(a);$$

we have obtained:

$$\int_a^\infty \varphi(x)\,dx = e^{-a} + a\,\varphi(a).$$

□

EXERCISE 32. In $\mathbb{R}^n$ let x_k be a sequence converging to $x \in \mathbb{R}^n$, and let $r_k > 0$ converge to $r > 0$.

(i) If $\chi_k = \chi_{B(x_k,r_k]}$ then χ_k converges a.e. to $\chi = \chi_{B(x,r]}$.

(ii) Prove that if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ then

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f\,\chi_k\,dm = \int_{\mathbb{R}^n} f\,\chi\,dm$$

(hint: if $R = \sup_k\{r_k + |x - x_k|\}$ then $B(x_k, r_k] \subseteq B(x, R]$ for every $k \in \mathbb{N}\dots$)

(iii) Assume now that $f \in L^1(\mathbb{R}^n)$, and define $g : \mathbb{R}^n \to \mathbb{K}$ by

$$g(x) = \int_{B(x,1]} f\,dm.$$

Then g is continuous, and $\lim_{|x|\to\infty} g(x) =?$

Solution. (i) and (ii) are Exercise 7.1.1.1 of the Lecture Notes, and the solution shan't be repeated here.

(iii) Continuity of g is clear from (ii), keeping $r = 1$ fixed. Clearly the limit is 0: if x_k is a sequence in $\mathbb{R}^n$ with $\lim_k |x_k| = \infty$, the sequence $f_k = f\,\chi_{B(x_k,1]}$ is dominated by $|f| \in L^1(\mathbb{R}^n)$ and converges to 0 a.e. (given $x \in \mathbb{R}^n$, if $|x_k| > |x| + 1$ then $f_k(x) = 0$). By dominated convergence the limit of $g(x_k)$ is 0. □