Calculus of Variations
Preliminary program of the course - 2020

(1) Classic functionals of the Calculus of Variations
   (a) Euler-Lagrange's equations
   (b) Du Bois-Reymond's equation
   (c) Indirect methods and convexity methods
   (d) Fermat's principle for the geometrical optics
   (e) Brachistochrone's problem
(2) Semi-direct methods
   (a) Functionals depending on the gradient
   (b) Bounded slope condition
(3) Direct method of the Calculus of Variations
(4) Functional in Sobolev spaces
   (a) Brief review on Sobolev spaces
   (b) Convexity and lower-semicontinuity in $W^{1,p}$
   (c) Existence of minimizers in $W^{1,p}$
   (d) Examples
   (e) Interior Sobolev regularity
   (f) Schauder regularity and De Giorgi-Nash theorem (without proofs)
(5) Plateau's problem
   (a) The parametric formulation of Douglas-Radò
   (b) Brief introductions to BV functions and sets with finite perimeter
   (c) Plateau's problem for sets with finite perimeter
(6) Γ-convergence and applications to phase transition
   (a) Relaxation and Γ-limits
   (b) Convergence of minima and minimizers
   (c) Applications: Modica-Mortola functional and perimeter
   (d) De Giorgi's conjecture on the equation $\Delta u = -u(u^2 - 1)$
(7) The isoperimetric property of the sphere and its applications
   (a) Steiner's rearrangement
   (b) Isoperimetric property of the sphere
   (c) Quantitative version of the isoperimetric inequality
   (d) The best shape of a drum: minimum eigenvalue for $\Delta u = \lambda u$
   (e) Schwarz's rearrangement and sharp Sobolev inequalities
   (f) The Yamabe's equation $\Delta u = -u^{\frac{n+2}{n-2}}$
(8) Introduction into the theory of optimal transportation
   (a) Monge's problem
   (b) Kantorovic's formulation
   (c) Brenier's theorem
   (d) Another proof of the isoperimetric inequality

(9) Introduction to the theory of currents
   (a) Brief review on exterior algebras. Currents, mass and boundary
   (b) The Plateau's problem for rectifiable currents
   (c) Overview on the regularity of minimal surfaces. Simon's cone
   (d) Holomorphic manifolds are area minimizers

Bibliography

General books on Calculus of Variations:

1) B. Dacorogna, Introduction to the Calculus of Variations, Imperial College Press 2015. This is an excellent introductory book. There are many exercises with solutions.
2) B. Dacorogna, Direct Methods in the Calculus of Variations, Springer 2007. Advanced book. It deals functionals on vector valued functions, that are not considered in our lecture.
3) G. Buttazzo & M. Giaquinta & S. Hildebrandt, One-dimensional Variational Problems, Oxford University Press 2008. This is a complete book on functionals acting on functions of one variable.
4) E. Giusti, Direct Methods in the Calculus of Variations, Italian and English editions. I am following this book for the semi-classical theory.

Geometric measure theory:

1) F. Maggi, Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory, Cambridge 2012. Nice introduction to the theory of sets with finite perimeter, including regularity.
2) H. Federer, Geometric Measure Theory, Springer. The bible of GMT, very difficult.
3) F. Morgan, Geometric Measure Theory, Academic Press 2008. This is an introduction to Federer's book.
4) S. G. Krantz & H. R. Parks, Geometric Integration Theory, Birkhäuser 2008. A reasonable introduction to the theory of currents.

Plateau's problem:

1) M. Struwe, Plateau's Problem and the Calculus of Variations, Princeton 1988.

Sobolev and BV functions:

1) L. C. Evans & R. F. Gariepy, Measure Theory and Fine Properties of Functions, CRC press. This is the most coincise and yet rigorous introduction to Sobolev and BV functions.
2) L. Ambrosio & N. Fusco & D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press.

Γ-convergence, Optimal transportation, Regularity theory:

1) G. Dal Maso, An Introduction to Γ-Convergence, Birkhäuser 1993.
2) C. Villani, Topics in Optimal Transportation, AMS.
3) L. C. Evans, Partial Differential Equations, AMS.
4) M. Giaquinta, L. Martinazzi, An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs, Edizioni SNS.