Differential Equations 1 Sheet 3

11th March 2011

Problem 1 Let $u \in C^\infty(\mathbb{R}^{n+1})$ be a solution of the heat equation $u_t = \Delta u$ in $\mathbb{R}^{n+1}$.

i) Assume that
$$\int_{\mathbb{R}^{n+1}} |u(x,t)| dxdt < \infty.$$
Prove that u is a constant, and in fact $u = 0$.

ii) Assume that
$$\int_{\mathbb{R}^{n+1}} \big\{ |\nabla u(x,t)| + |u_t(x,t)| \big\} dxdt < \infty.$$
Prove that u is a constant.

Problem 2 Let $f \in C(\mathbb{R})$ be a function that is odd and 2π-periodic. Consider the Dirichlet problem
$$\begin{cases} u_t = u_{xx} & \text{in } (0,\pi) \times (0,\infty), \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0, \\ u(x,0) = f(x) & \text{for } x \in [0,\pi]. \end{cases}$$

i) By separation of variables and superposition, construct a "formal solution" of the form
$$u(x,t) = \sum_{k=1}^{\infty} c_k v_k(x) w_k(t).$$
Determine the functions v_k and w_k. Determine the coefficients $c_k \in \mathbb{R}$ using the Fourier expansion of f in sine functions.

ii) Prove that the function u is in $C^\infty\big([0,\pi] \times (0,\infty)\big)$ and solves the differential equation.

iii) Assume, in addition, that $f \in C^2(\mathbb{R})$. Prove that $u \in C([0,\pi] \times [0,\infty)$ and $u(\cdot, 0) = f$.

Problem 3 Let $U \subset \mathbb{R}^{n+1}$ be an open set and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions $u_k \in C^\infty(U)$, $k \in \mathbb{N}$, that solve the heat equation $\partial u_k/\partial t = \Delta u_k$ in U. Assume that for any compact set $K \subset U$ there exists a constant C_K such that for any $k \in \mathbb{N}$ there holds
$$\sup_{(x,t)\in K} |u_k(x,t)| \le C_K.$$

i) Prove that there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ that converges uniformly on compact sets to a function $u \in C(U)$;

ii) Prove that $u \in C^\infty(U)$ and $u_t = \Delta u$ in U.