Differential Equations 1 - Second Part
The Heat Equation

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CHAPTER 1

Heat Equation

1. Introduction

In $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, $n \geq 1$, let us consider the coordinates $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The differential operator in $\mathbb{R}^{n+1}$

$$H = \frac{\partial}{\partial t} - \Delta, \quad \text{where} \quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is called the *heat operator*. The three most important problems concerning the heat operator are the Cauchy Problem, the Dirichlet Problem, and the Neumann Problem.

Cauchy Problem in $\mathbb{R}^n$. The problem consists in finding a function $u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$ such that

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $f \in C(\mathbb{R}^n)$ is an initial distribution of temperature.

Dirichlet Problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The problem consists in finding a function $u \in C^2(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$ such that

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = g(x,t), & x \in \partial\Omega,\ t > 0, \\ u(x,0) = f(x), & x \in \Omega. \end{cases} \tag{1.2}$$

The problem describes the evolution of the temperature of a body Ω having prescribed temperature $g \in C(\partial\Omega \times (0,\infty))$ at the boundary of Ω (for any positive time) and having an initial distribution of temperature $f \in C(\Omega)$ at time $t = 0$.

Neumann Problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1. We search for a function u defined in the cylinder $\Omega \times (0,\infty)$ (with gradient defined up to the boundary) such that

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu}(x,t) = g(x,t), & x \in \partial\Omega,\ t > 0, \\ u(x,0) = f(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\dfrac{\partial u}{\partial \nu}$ is the normal derivative of u at the boundary of Ω. In this case, prescribed is the variation g of the temperature on the boundary.

2. The foundamental solution and its properties

We derive a representation formula for the (a) solution of the Cauchy Problem using a formal argument.

2.1. Preliminaries on the Fourier transform. For a given function $f \in L^1(\mathbb{R}^n)$, we define its Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi i \langle \xi, x\rangle} f(x)dx, \quad \xi \in \mathbb{R}^n. \tag{2.4}$$

We shall also write $\mathfrak{F}(f)(\xi) = \widehat{f}(\xi)$. Let us recall some properties of the Fourier transform.

1) If $f, g \in L^1(\mathbb{R}^n)$ are integrable functions, then also their convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} g(x-y)f(y)dy$$

is in $L^1(\mathbb{R}^n)$ and there holds

$$\mathfrak{F}(f * g) = \mathfrak{F}(f)\mathfrak{F}(g). \tag{2.5}$$

2) If $f, \widehat{f} \in L^1(\mathbb{R}^n)$ are both integrable functions then we have the *inversion formula*:

$$\mathfrak{F}(\mathfrak{F}(f))(x) = \mathfrak{F}^2(f)(x) = f(-x) \quad \text{for almost every } x \in \mathbb{R}^n. \tag{2.6}$$

3) If $f \in L^1(\mathbb{R}^n)$ and also $\dfrac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$ for some $j = 1, ..., n$, then

$$\mathfrak{F}\Big(\frac{\partial f}{\partial x_j}\Big)(\xi) = 2\pi i \xi_j \widehat{f}(\xi). \tag{2.7}$$

4) Consider the Gaussian function $f_s(x) = \mathrm{e}^{-s|x|^2}$, where $s > 0$ is a parameter. The Fourier transform of f_s is the function

$$\widehat{f_s}(\xi) = \Big(\frac{\pi}{s}\Big)^{\frac{n}{2}} \mathrm{e}^{-\frac{\pi^2|\xi|^2}{s}}. \tag{2.8}$$

2.2. Euristic computation of the foundamental solution. We transform the Cauchy Problem (1.1) with a Fourier transform in the spatial variables $x \in \mathbb{R}^n$. Assuming that the Fourier transform commutes with the partial derivative in t we obtain

$$\widehat{\frac{\partial u}{\partial t}}(\xi, t) = \frac{\partial \widehat{u}}{\partial t}(\xi, t).$$

From the rule (2.7) – we assume that the rule can be applied to all second derivatives in x of u, – we obtain

$$\mathfrak{F}(\Delta u)(\xi, t) = -4\pi^2|\xi|^2\widehat{u}(\xi, t).$$

Finally, if the initial datum $f \in L^1(\mathbb{R}^n)$ is integrable, then we also have $\widehat{u}(\xi,0) = \widehat{f}(\xi)$. Thus, we obtain the transformed problem

$$\begin{cases} \dfrac{\partial \widehat{u}}{\partial t}(\xi,t) = -4\pi^2|\xi|^2\widehat{u}(\xi,t), & \xi \in \mathbb{R}^n,\ t>0 \\ \widehat{u}(\xi,0) = \widehat{f}(\xi), & \xi \in \mathbb{R}^n. \end{cases}$$

The solution of the problem is the function

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\mathrm{e}^{-4\pi^2 t|\xi|^2}. \tag{2.9}$$

From the formula (2.8) with $s = 1/4t$ we obtain

$$\mathrm{e}^{-4\pi^2 t|\xi|^2} = \widehat{\Gamma}_t(\xi), \quad \text{dove} \quad \Gamma_t(x) = \Big(\frac{1}{4\pi t}\Big)^{n/2}\mathrm{e}^{-\frac{|x|^2}{4t}}.$$

By the convolution formula (2.5), identity (2.9) reads as follows:

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\widehat{\Gamma}_t(\xi) = \mathfrak{F}(f * \Gamma_t)(\xi).$$

Using the inversion formula (2.6), we obtain the *representation formula* for the solution

$$u(x,t) = f * \Gamma_t(x) = \Big(\frac{1}{4\pi t}\Big)^{n/2}\int_{\mathbb{R}^n} f(y)\mathrm{e}^{-\frac{|x-y|^2}{4t}}\,dy, \quad x \in \mathbb{R}^n. \tag{2.10}$$

DEFINITION 2.1. The function $\Gamma : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$\Gamma(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}}\mathrm{e}^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n,\ t>0, \\ 0 & t \leq 0 \end{cases}$$

is called the *foundamental solution* of the heat equation.

THEOREM 2.2. *The function Γ has the following properties:*

1) $\Gamma \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$*;*
2) $\dfrac{\partial \Gamma(x,t)}{\partial t} = \Delta\Gamma(x,t)$ *for all* $(x,t) \in \mathbb{R}^{n+1}\setminus\{0\}$*;*
3) *For any* $t>0$ *we have*

$$\int_{\mathbb{R}^n} \Gamma(x,t)dx = 1. \tag{2.11}$$

4) *The function Γ verifies the equation $H\Gamma = \delta_0$ in $\mathbb{R}^{n+1}$ in the sense of distributions, where δ_0 is the Dirac mass in 0. Namely, for any test function $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ there holds*

$$\int_{\mathbb{R}^{n+1}} \Gamma(x,t)H^*\varphi(x,t)dxdt = -\varphi(0),$$

whre $H^* = \partial/\partial t + \Delta$ is the adjoint operator of H.

PROOF. Claim 1) follows from the fact that, for any $x \neq 0$, the function

$$t \mapsto \Big(\frac{1}{4\pi t}\Big)^{n/2}\mathrm{e}^{-\frac{|x|^2}{4t}}, \quad t>0,$$

can be continuously extended to $t=0$, is differentiable infinitely many times at $t=0$, and all derivatives vanish. Claim 2) can be verified by a short computation which is left as an exercise.

Identity (2.11) follows from the well known formula

$$\int_{-\infty}^{+\infty} e^{-s^2} ds = \sqrt{\pi}$$

and from Fubini-Tonelli theorem. In fact, we have:

$$\int_{\mathbb{R}^n} \Big(\frac{1}{4\pi t}\Big)^{n/2} \mathrm{e}^{-\frac{|x|^2}{4t}} dx = \Big(\frac{1}{4\pi t}\Big)^{n/2} \prod_{i=1}^{n} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{x_i^2}{4t}} dx_i = \frac{1}{\pi^{n/2}} \prod_{i=1}^{n} \int_{-\infty}^{+\infty} \mathrm{e}^{-x_i^2} dx_i = 1.$$

We prove Claim 4). For $\Gamma H^*\varphi \in L^1(\mathbb{R}^{n+1})$, by dominated convergence we have:

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} \Gamma(x,t) H^*\varphi(x,t) dx dt &= \int_0^\infty \int_{\mathbb{R}^n} \Gamma(x,t) H^*\varphi(x,t) dx\, dt \\ &= \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} \Gamma(x,t) H^*\varphi(x,t) dx\, dt. \end{aligned}$$

For any fixed $t > 0$, by an integration by parts we obtain

$$\int_{\mathbb{R}^n} \Gamma(x,t) \Delta\varphi(x,t) dx = \int_{\mathbb{R}^n} \Delta\Gamma(x,t)\varphi(x,t) dx.$$

There is no boundary contribution, because φ has compact support. Moreover, we have

$$\int_\varepsilon^\infty \Gamma(x,t) \frac{\partial\varphi(x,t)}{\partial t} dt = -\int_\varepsilon^\infty \frac{\partial\Gamma(x,t)}{\partial t} \varphi(x,t) dt - \Gamma(x,\varepsilon)\varphi(x,\varepsilon).$$

Summing up and using $H\Gamma = 0$, that holds on the set where $t > 0$, we obtain

$$\begin{aligned} \int_\varepsilon^\infty \int_{\mathbb{R}^n} \Gamma(x,t) H^*\varphi(x,t) dx\, dt &= \int_\varepsilon^\infty \int_{\mathbb{R}^n} H\Gamma(x,t)\varphi(x,t) dx\, dt - \int_{\mathbb{R}^n} \Gamma(x,\varepsilon)\varphi(x,\varepsilon) dx \\ &= -\int_{\mathbb{R}^n} \Gamma(x,\varepsilon)\varphi(x,\varepsilon) dx \\ &= -\int_{\mathbb{R}^n} \Gamma(\xi,1)\varphi(2\sqrt{\varepsilon}\xi,\varepsilon) d\xi. \end{aligned}$$

Taking the limit as $\varepsilon \downarrow 0$, by dominated convergence we prove the claim. □

2.3. Cauchy Problem: existence of solutions.

THEOREM 2.3. *Let $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. The function u defined by the representation formula* (2.10) *solves the Cauchy Problem* (1.1)*, and namely:*

1) *$u \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and $u_t(x,t) = \Delta u(x,t)$ for all $x \in \mathbb{R}^n$ and $t > 0$;*
2) *For any $x_0 \in \mathbb{R}^n$ there holds*

$$\lim_{x\to x_0, t\downarrow 0} u(x,t) = f(x_0),$$

with uniform convergence for x_0 belonging to a compact set;

3) *Moreover, $\|u(\cdot,t)\|_\infty \le \|f\|_\infty$ for all $t > 0$.*

PROOF. Claim 1) follows from the fact that we can take partial derivatives of any order in x and t into the integral in the representation formula (2.10). We prove, for instance, that for any $x \in \mathbb{R}^n$ and for any $t > 0$ there holds

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^n} f(y) \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, dy = \int_{\mathbb{R}^n} f(y) \frac{\partial}{\partial t} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, dy.$$

By the Corollary to the Dominated Convergence Theorem, it suffices to show that for any $0 < t_0 \leq T < \infty$ there exists a function $g \in L^1(\mathbb{R}^n)$, in variable y, such that (for fixed $x \in \mathbb{R}^n$ and) for any $t \in [t_0, T]$ we have

$$\frac{|x-y|^2}{4t^2} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \leq g(y), \quad \text{for all } y \in \mathbb{R}^n.$$

This holds with the choice

$$g(y) = \frac{|x-y|^2}{4t_0^2} \mathrm{e}^{-\frac{|x-y|^2}{4T}}.$$

The case of derivatives in the variables x and the case of higher order derivatives is analogous and is left as an exercise.

By the previous argument, it follows that, for $t > 0$, we can take the heat operator into the integral:

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_{\mathbb{R}^n} f(y) \Big(\frac{\partial}{\partial t} - \Delta_x \Big) \Gamma(x-y,t) dy \\ &= \int_{\mathbb{R}^n} f(y) \big\{ \Gamma_t(x-y,t) - \Delta\Gamma(x-y,t) \big\} dy = 0. \end{aligned}$$

Thus, u solves the heat equation for positive times.

We prove claim 2). Let $K \subset \mathbb{R}^n$ be a compact set and let $x_0 \in K$. We may rewrite the representation formula (2.10) in the folloperwing way:

$$u(x,t) = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \Gamma(\xi,1) f(2\sqrt{t}\xi + x) d\xi, \quad x \in \mathbb{R}^n,\ t > 0.$$

Hence, we have

$$|u(x,t) - f(x_0)| \leq \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \Gamma(\xi,1) |f(2\sqrt{t}\xi + x) - f(x_0)| d\xi.$$

Fix now $\varepsilon > 0$ and choose $R > 0$ such that

$$\frac{1}{\pi^{n/2}} \int_{|\xi|>R} \Gamma(\xi,1) d\xi \leq \varepsilon.$$

As f is uniformly continuous on compact sets, there exists a $\delta > 0$ such that for all $|\xi| \leq R$ we have

$$|x - x_0| < \delta \text{ and } 0 < t < \delta \quad \Rightarrow \quad |f(2\sqrt{t}\xi - x) - f(x_0)| < \varepsilon.$$

The choice of δ is uniform in $x_0 \in K$. After all, we get

$$\begin{aligned} |u(x,t) - f(x_0)| &\leq \frac{1}{\pi^{n/2}} \int_{|\xi|\leq R} \Gamma(\xi,1) |f(2\sqrt{t}\xi + x) - f(x_0)| d\xi \\ &\quad + \frac{1}{\pi^{n/2}} \int_{|\xi|>R} \Gamma(\xi,1) |f(2\sqrt{t}\xi + x) - f(x_0)| d\xi \\ &\leq \varepsilon + 2\|f\|_\infty \varepsilon. \end{aligned}$$

This proves claim 2). Claim 3) follows directly from the representation formula. □