Differential Equations 1 - Second Part
The Heat Equation

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Università di Padova

Roberto Monti

CHAPTER 1

Heat Equation

1. Introduction

In $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, $n \geq 1$, let us consider the coordinates $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The differential operator in $\mathbb{R}^{n+1}$

$$H = \frac{\partial}{\partial t} - \Delta, \quad \text{where} \quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is called the *heat operator*. The three most important problems concerning the heat operator are the Cauchy Problem, the Dirichlet Problem, and the Neumann Problem.

Cauchy Problem in $\mathbb{R}^n$. The problem consists in finding a function $u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$ such that

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $f \in C(\mathbb{R}^n)$ is an initial distribution of temperature.

Dirichlet Problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The problem consists in finding a function $u \in C^2(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))$ such that

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = g(x,t), & x \in \partial\Omega,\ t > 0, \\ u(x,0) = f(x), & x \in \Omega. \end{cases} \tag{1.2}$$

The problem describes the evolution of the temperature of a body Ω having prescribed temperature $g \in C(\partial\Omega \times (0,\infty))$ at the boundary of Ω (for any positive time) and having an initial distribution of temperature $f \in C(\Omega)$ at time $t = 0$.

Neumann Problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1. We search for a function u defined in the cylinder $\Omega \times (0,\infty)$ (with gradient defined up to the boundary) such that

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu}(x,t) = g(x,t), & x \in \partial\Omega,\ t > 0, \\ u(x,0) = f(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\dfrac{\partial u}{\partial \nu}$ is the normal derivative of u at the boundary of Ω. In this case, prescribed is the variation g of the temperature on the boundary.

2. The foundamental solution and its properties

We derive a representation formula for the (a) solution of the Cauchy Problem using a formal argument.

2.1. Preliminaries on the Fourier transform. For a given function $f \in L^1(\mathbb{R}^n)$, we define its Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi i \langle \xi, x\rangle} f(x)dx, \quad \xi \in \mathbb{R}^n. \tag{2.4}$$

We shall also write $\mathfrak{F}(f)(\xi) = \widehat{f}(\xi)$. Let us recall some properties of the Fourier transform.

1) If $f, g \in L^1(\mathbb{R}^n)$ are integrable functions, then also their convolution

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} g(x-y)f(y)dy$$

is in $L^1(\mathbb{R}^n)$ and there holds

$$\mathfrak{F}(f * g) = \mathfrak{F}(f)\mathfrak{F}(g). \tag{2.5}$$

2) If $f, \widehat{f} \in L^1(\mathbb{R}^n)$ are both integrable functions then we have the *inversion formula*:

$$\mathfrak{F}(\mathfrak{F}(f))(x) = \mathfrak{F}^2(f)(x) = f(-x) \quad \text{for almost every } x \in \mathbb{R}^n. \tag{2.6}$$

3) If $f \in L^1(\mathbb{R}^n)$ and also $\dfrac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$ for some $j = 1, ..., n$, then

$$\mathfrak{F}\Big(\frac{\partial f}{\partial x_j}\Big)(\xi) = 2\pi i \xi_j \widehat{f}(\xi). \tag{2.7}$$

4) Consider the Gaussian function $f_s(x) = \mathrm{e}^{-s|x|^2}$, where $s > 0$ is a parameter. The Fourier transform of f_s is the function

$$\widehat{f_s}(\xi) = \Big(\frac{\pi}{s}\Big)^{\frac{n}{2}} \mathrm{e}^{-\frac{\pi^2|\xi|^2}{s}}. \tag{2.8}$$

2.2. Euristic computation of the foundamental solution. We transform the Cauchy Problem (1.1) with a Fourier transform in the spatial variables $x \in \mathbb{R}^n$. Assuming that the Fourier transform commutes with the partial derivative in t we obtain

$$\widehat{\frac{\partial u}{\partial t}}(\xi, t) = \frac{\partial \widehat{u}}{\partial t}(\xi, t).$$

From the rule (2.7) – we assume that the rule can be applied to all second derivatives in x of u, – we obtain

$$\mathfrak{F}(\Delta u)(\xi, t) = -4\pi^2 |\xi|^2 \widehat{u}(\xi, t).$$

Finally, if the initial datum $f \in L^1(\mathbb{R}^n)$ is integrable, then we also have $\widehat{u}(\xi,0) = \widehat{f}(\xi)$. Thus, we obtain the transformed problem

$$\begin{cases} \dfrac{\partial \widehat{u}}{\partial t}(\xi,t) = -4\pi^2|\xi|^2\widehat{u}(\xi,t), & \xi \in \mathbb{R}^n,\ t>0 \\ \widehat{u}(\xi,0) = \widehat{f}(\xi), & \xi \in \mathbb{R}^n. \end{cases}$$

The solution of the problem is the function

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\mathrm{e}^{-4\pi^2 t|\xi|^2}. \tag{2.9}$$

From the formula (2.8) with $s = 1/4t$ we obtain

$$\mathrm{e}^{-4\pi^2 t|\xi|^2} = \widehat{\Gamma}_t(\xi), \quad \text{dove} \quad \Gamma_t(x) = \Big(\frac{1}{4\pi t}\Big)^{n/2}\mathrm{e}^{-\frac{|x|^2}{4t}}.$$

By the convolution formula (2.5), identity (2.9) reads as follows:

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\widehat{\Gamma}_t(\xi) = \mathfrak{F}(f * \Gamma_t)(\xi).$$

Using the inversion formula (2.6), we obtain the *representation formula* for the solution

$$u(x,t) = f * \Gamma_t(x) = \Big(\frac{1}{4\pi t}\Big)^{n/2} \int_{\mathbb{R}^n} f(y)\mathrm{e}^{-\frac{|x-y|^2}{4t}}\,dy, \quad x \in \mathbb{R}^n. \tag{2.10}$$

DEFINITION 2.1. The function $\Gamma : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$\Gamma(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}}\mathrm{e}^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n,\ t>0, \\ 0 & t \le 0 \end{cases}$$

is called the *foundamental solution* of the heat equation.

THEOREM 2.2. *The function Γ has the following properties:*

1) $\Gamma \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$;
2) $\dfrac{\partial \Gamma(x,t)}{\partial t} = \Delta\Gamma(x,t)$ *for all* $(x,t) \in \mathbb{R}^{n+1} \setminus \{0\}$;
3) *For any $t>0$ we have*

$$\int_{\mathbb{R}^n} \Gamma(x,t)dx = 1. \tag{2.11}$$

4) *The function Γ verifies the equation $H\Gamma = \delta_0$ in $\mathbb{R}^{n+1}$ in the sense of distributions, where δ_0 is the Dirac mass in 0. Namely, for any test function $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ there holds*

$$\int_{\mathbb{R}^{n+1}} \Gamma(x,t)H^*\varphi(x,t)dxdt = -\varphi(0),$$

whre $H^* = \partial/\partial t + \Delta$ is the adjoint operator of H.

PROOF. Claim 1) follows from the fact that, for any $x \neq 0$, the function

$$t \mapsto \Big(\frac{1}{4\pi t}\Big)^{n/2}\mathrm{e}^{-\frac{|x|^2}{4t}}, \quad t>0,$$

can be continuously extended to $t=0$, is differentiable infinitely many times at $t=0$, and all derivatives vanish. Claim 2) can be verified by a short computation which is left as an exercise.

Identity (2.11) follows from the well known formula

$$\int_{-\infty}^{+\infty} e^{-s^2} ds = \sqrt{\pi}$$

and from Fubini-Tonelli theorem. In fact, we have:

$$\int_{\mathbb{R}^n} \Big(\frac{1}{4\pi t}\Big)^{n/2} \mathrm{e}^{-\frac{|x|^2}{4t}} dx = \Big(\frac{1}{4\pi t}\Big)^{n/2} \prod_{i=1}^{n} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{x_i^2}{4t}} dx_i = \frac{1}{\pi^{n/2}} \prod_{i=1}^{n} \int_{-\infty}^{+\infty} \mathrm{e}^{-x_i^2} dx_i = 1.$$

We prove Claim 4). For $\Gamma H^*\varphi \in L^1(\mathbb{R}^{n+1})$, by dominated convergence we have:

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} \Gamma(x,t) H^*\varphi(x,t) dx dt &= \int_0^\infty \int_{\mathbb{R}^n} \Gamma(x,t) H^*\varphi(x,t) dx\, dt \\ &= \lim_{\varepsilon\downarrow 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} \Gamma(x,t) H^*\varphi(x,t) dx\, dt.\end{aligned}$$

For any fixed $t > 0$, by an integration by parts we obtain

$$\int_{\mathbb{R}^n} \Gamma(x,t)\Delta\varphi(x,t) dx = \int_{\mathbb{R}^n} \Delta\Gamma(x,t)\varphi(x,t) dx.$$

There is no boundary contribution, because φ has compact support. Moreover, we have

$$\int_\varepsilon^\infty \Gamma(x,t)\frac{\partial\varphi(x,t)}{\partial t} dt = -\int_\varepsilon^\infty \frac{\partial\Gamma(x,t)}{\partial t}\varphi(x,t) dt - \Gamma(x,\varepsilon)\varphi(x,\varepsilon).$$

Summing up and using $H\Gamma = 0$, that holds on the set where $t > 0$, we obtain

$$\begin{aligned}\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Gamma(x,t) H^*\varphi(x,t) dx\, dt &= \int_\varepsilon^\infty \int_{\mathbb{R}^n} H\Gamma(x,t)\varphi(x,t) dx\, dt - \int_{\mathbb{R}^n} \Gamma(x,\varepsilon)\varphi(x,\varepsilon) dx \\ &= -\int_{\mathbb{R}^n} \Gamma(x,\varepsilon)\varphi(x,\varepsilon) dx \\ &= -\int_{\mathbb{R}^n} \Gamma(\xi,1)\varphi(2\sqrt{\varepsilon}\xi,\varepsilon) d\xi.\end{aligned}$$

Taking the limit as $\varepsilon \downarrow 0$, by dominated convergence we prove the claim. □

2.3. Cauchy Problem: existence of solutions.

THEOREM 2.3. *Let $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. The function u defined by the representation formula* (2.10) *solves the Cauchy Problem* (1.1)*, and namely:*

1) *$u \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and $u_t(x,t) = \Delta u(x,t)$ for all $x \in \mathbb{R}^n$ and $t > 0$;*
2) *For any $x_0 \in \mathbb{R}^n$ there holds*

$$\lim_{x\to x_0, t\downarrow 0} u(x,t) = f(x_0),$$

with uniform convergence for x_0 belonging to a compact set;

3) *Moreover, $\|u(\cdot,t)\|_\infty \le \|f\|_\infty$ for all $t > 0$.*

PROOF. Claim 1) follows from the fact that we can take partial derivatives of any order in x and t into the integral in the representation formula (2.10). We prove, for instance, that for any $x \in \mathbb{R}^n$ and for any $t > 0$ there holds

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^n} f(y)\mathrm{e}^{-\frac{|x-y|^2}{4t}} dy = \int_{\mathbb{R}^n} f(y)\frac{\partial}{\partial t}\mathrm{e}^{-\frac{|x-y|^2}{4t}} dy.$$

By the Corollary to the Dominated Convergence Theorem, it suffices to show that for any $0 < t_0 \le T < \infty$ there exists a function $g \in L^1(\mathbb{R}^n)$, in variable y, such that (for fixed $x \in \mathbb{R}^n$ and) for any $t \in [t_0, T]$ we have

$$\frac{|x-y|^2}{4t^2} \mathrm{e}^{-\frac{|x-y|^2}{4t}} \le g(y), \quad \text{for all } y \in \mathbb{R}^n.$$

This holds with the choice

$$g(y) = \frac{|x-y|^2}{4t_0^2} \mathrm{e}^{-\frac{|x-y|^2}{4T}}.$$

The case of derivatives in the variables x and the case of higher order derivatives is analogous and is left as an exercise.

By the previous argument, it follows that, for $t > 0$, we can take the heat operator into the integral:

$$\begin{aligned} u_t(x,t) - \Delta u(x,t) &= \int_{\mathbb{R}^n} f(y) \Big(\frac{\partial}{\partial t} - \Delta_x \Big) \Gamma(x-y,t) dy \\ &= \int_{\mathbb{R}^n} f(y) \big\{ \Gamma_t(x-y,t) - \Delta \Gamma(x-y,t) \big\} dy = 0. \end{aligned}$$

Thus, u solves the heat equation for positive times.

We prove claim 2). Let $K \subset \mathbb{R}^n$ be a compact set and let $x_0 \in K$. We may rewrite the representation formula (2.10) in the following way:

$$u(x,t) = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \Gamma(\xi, 1/4) f(2\sqrt{t}\xi + x) d\xi, \quad x \in \mathbb{R}^n, \, t > 0.$$

Hence, we have

$$|u(x,t) - f(x_0)| \le \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \Gamma(\xi, 1/4) |f(2\sqrt{t}\xi + x) - f(x_0)| d\xi.$$

Fix now $\varepsilon > 0$ and choose $R > 0$ such that

$$\frac{1}{\pi^{n/2}} \int_{|\xi|>R} \Gamma(\xi, 1/4) d\xi \le \varepsilon.$$

As f is uniformly continuous on compact sets, there exists a $\delta > 0$ such that for all $|\xi| \le R$ we have

$$|x - x_0| < \delta \text{ and } 0 < t < \delta \quad \Rightarrow \quad |f(2\sqrt{t}\xi - x) - f(x_0)| < \varepsilon.$$

The choice of δ is uniform in $x_0 \in K$. After all, we get

$$\begin{aligned} |u(x,t) - f(x_0)| &\le \frac{1}{\pi^{n/2}} \int_{|\xi|\le R} \Gamma(\xi, 1/4) |f(2\sqrt{t}\xi + x) - f(x_0)| d\xi \\ &\quad + \frac{1}{\pi^{n/2}} \int_{|\xi|>R} \Gamma(\xi, 1/4) |f(2\sqrt{t}\xi + x) - f(x_0)| d\xi \\ &\le \varepsilon + 2\|f\|_\infty \varepsilon. \end{aligned}$$

This proves claim 2). Claim 3) follows directly from the representation formula. □

2.4. Tychonov's counterexample. In general, the solution of the Cauchy Problem

$$\begin{cases} u_t(x,t) = \Delta u(x,t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.12}$$

even with $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, is not unique in the class of functions $C(\mathbb{R}^n \times [0,\infty)) \cap C^\infty(\mathbb{R}^n \cap (0,\infty))$.

In dimension $n = 1$, let us consider the problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{2.13}$$

The function $u = 0$ is a solution. We construct a second solution that is not identically zero.

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be the function

$$\varphi(z) = \begin{cases} \mathrm{e}^{-1/z^2}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

The function φ is holomorphic in $\mathbb{C} \setminus \{0\}$. Moreover, the function $t \mapsto \varphi(t)$ with $t \in \mathbb{R}$ is of class $C^\infty(\mathbb{R})$ and $\varphi^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Let us consider the series of functions

$$u(x,t) = \sum_{n=0}^{\infty} \varphi^{(n)}(t) \frac{x^{2n}}{(2n)!}, \quad t \geq 0,\ x \in \mathbb{R}.$$

We shall prove the following facts:

1) The sum defining u and the series of the derivatives of any order converge uniformly on any set of the form $[-R, R] \times [T, \infty)$ with $R, T > 0$;
2) u is a continuous function up to the boundary in the halfspace $t \geq 0$.

From 2) it follows that u attains the initial datum 0 at the time $t = 0$. By 1), we can interchange sum and partial derivatives. Then we can compute

$$\begin{aligned} u_{xx}(x,t) &= \sum_{n=1}^{\infty} \varphi^{(n)}(t) \frac{x^{2n-2}}{(2n-2)!} = \sum_{m=0}^{\infty} \varphi^{(m+1)}(t) \frac{x^{2m}}{(2m)!} \\ &= \frac{\partial}{\partial t} \sum_{m=0}^{\infty} \varphi^{(m)}(t) \frac{x^{2m}}{(2m)!} = u_t(x,t). \end{aligned}$$

Let us prove claim 1). For fixed $t > 0$, by the Cauchy formula for holomorphic functions we obtain

$$\varphi^{(n)}(t) = \frac{n!}{2\pi i} \int_{|z-t|=t/2} \frac{\varphi(z)}{(z-t)^{n+1}} dz.$$

On the circle $|z - t| = t/2$, we have $|\varphi(z)| \leq \mathrm{e}^{-\mathrm{Re}(1/z^2)} \leq \mathrm{e}^{-4/t^2}$ and thus

$$|\varphi^{(n)}(t)| \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{-4/t^2}}{(t/2)^{n+1}} \frac{t}{2} d\vartheta = n! 2^n \frac{\mathrm{e}^{-4/t^2}}{t^n}.$$

We shall use the following inequality, that can be proved by induction:

$$\frac{n! 2^n}{(2n)!} \leq \frac{1}{n!}.$$

Thus we get:

$$|u(x,t)| \le \sum_{n=0}^{\infty} |\varphi^{(n)}(t)| \frac{|x|^{2n}}{(2n)!} \le \sum_{n=0}^{\infty} n! 2^n \frac{\mathrm{e}^{-4/t^2}}{t^n} \frac{|x|^{2n}}{(2n)!}$$
$$\le \mathrm{e}^{-4/t^2} \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\frac{|x|^2}{t}\Big)^n = \mathrm{e}^{-4/t^2 + |x|^2/t},$$

where the last sum converges uniformly for $t \ge T > 0$ and $|x| \le R < \infty$. By Weierstrass' criterion, the sum defining u converges uniformly on the same set. In particular, by comparison we find

$$\lim_{t\to 0} \mathrm{e}^{-4/t^2+|x|^2/t} = 0 \quad \Rightarrow \quad \lim_{t\to 0} |u(x,t)| = 0$$

with uniform convergence for $|x| \le R$. This proves claim 2).

The study of convergence of the series of derivatives is analogous and is left as an exercise to the reader.

2.5. Nonhomogeneous problem. Let us consider the nonhomogeneous Cauchy problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = 0, & x \in \mathbb{R}^n, \end{cases} \tag{2.14}$$

where $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is a suitable function. We discuss the regularity of f later. A candidate solution of the problem can be obtained on using the "Duhamel's Principle". Fix $s > 0$ and assume there exists a (the) solution $v(\cdot; s)$ of the Cauchy Problem

$$\begin{cases} v_t(x,t;s) = \Delta v(x,t;s), & x \in \mathbb{R}^n,\ t > s, \\ v(x,s;s) = f(x,s), & x \in \mathbb{R}^n. \end{cases} \tag{2.15}$$

On integrating the solutions $v(x,t;s)$ for $s \in (0,t)$ we obtain the function

$$u(x,t) = \int_0^t v(x,t;s)ds. \tag{2.16}$$

When we formally insert $t = 0$ into this identity, we get $u(x,0) = 0$. If we formally differentiate the identity – taking derivatives into the integral is a idelicate issue, here, – we obtain

$$u_t(x,t) = v(x,t;t) + \int_0^t v_t(x,t;s)ds \quad \text{e} \quad \Delta u(x,t) = \int_0^t \Delta v(x,t;s)ds,$$

and thus $u_t(x,t) - \Delta u(x,t) = v(x,t;t) = f(x,t)$. If the previous computations are allowed, the function u is a solution to the problem (2.14).

Inserting the representation formula (2.10) for the solutions $v(x,t;s)$ into (2.16), we get the representation formula for the solution u

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x-y,t-s) f(y,s) dy\, ds, \quad x \in \mathbb{R}^n,\ t > 0. \tag{2.17}$$

In order the make rigorous the previuous argument, we need estimates for the solution to the Cauchy problem near time $t = 0$.

PROPOSIZIONE 2.4. *Let $f \in L^\infty(\mathbb{R}^n)$ and let $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$ be the function given by the representation formula (2.10). There exists a dimensional constant $C = C(n) > 0$ such that for all $x \in \mathbb{R}^n$ and $t > 0$ we have*

$$|\nabla u(x,t)| \leq \frac{C}{\sqrt{t}}\|f\|_\infty. \tag{2.18}$$

PROOF. We can take derivatives in x into the integral in formula (2.10). We obtain:

$$\nabla u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \frac{x-y}{-2t} \mathrm{e}^{-|x-y|^2/4t} f(y)dy,$$

and thus

$$|\nabla u(x,t)| \leq \frac{\|f\|_\infty}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \frac{|x-y|}{2t} \mathrm{e}^{-|x-y|^2/4t} dy = \frac{\|f\|_\infty}{(4\pi)^{n/2}\sqrt{t}} \int_{\mathbb{R}^n} |y|\mathrm{e}^{-|y|^2} dy.$$

□

PROPOSIZIONE 2.5. *Let $f \in L^\infty(\mathbb{R}^n)$ be a function in $C^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\alpha \in (0,1]$, i.e., for any compact set $K \subset \mathbb{R}^n$ there exists a constant $C_K > 0$ such that for all $x, y \in K$ we have*

$$|f(x) - f(y)| \leq C_K |x-y|^\alpha. \tag{2.19}$$

Let $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$ be the function given by the representation formula (2.10).

Then, for any $R > 0$ and $T > 0$ there exists a constant $C > 0$ depending on R, T, $\|f\|_\infty$, α, and $n \in \mathbb{N}$, such that for all $|x| \leq R$ and $t \in (0,T)$ we have

$$\Big|\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t)\Big| \leq \frac{C}{t^{1-\alpha/2}}, \tag{2.20}$$

for all indeces $i, j = 1, ..., n$.

PROOF. We compute second order derivatives in x in the identity:

$$\int_{\mathbb{R}^n} \Gamma(x-y,t)dy = 1, \quad x \in \mathbb{R}^n,\ t > 0.$$

We obtain, for any $i, j = 1, ..., n$,

$$\int_{\mathbb{R}^n} \Gamma_{ij}(x-y,t)dy = \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^n} \Gamma(x-y,t)dy = 0, \quad x \in \mathbb{R}^n,\ t > 0.$$

Here and hereafter, we let $\Gamma_{ij} = \frac{\partial^2 \Gamma}{\partial x_i \partial x_j}$. Taking derivatives into the integral is allowed. On using this piece of information, the second order derivatives of u may be written in the following way

$$u_{ij}(x,t) = \int_{\mathbb{R}^n} \Gamma_{ij}(x-y,t)\big(f(y) - f(x)\big)dy, \quad x \in \mathbb{R}^n,\ t > 0,$$

where a short computation shows that

$$\Gamma_{ij}(x,t) = \Big\{ -\frac{\delta_{ij}}{2t} + \frac{x_i x_j}{4t^2} \Big\} \Gamma(x,t).$$

Eventually, we obtain the estimate

$$|u_{ij}(x,t)| \le \int_{\mathbb{R}^n} \Big\{\frac{1}{2t} + \frac{|x-y|^2}{4t^2}\Big\}\Gamma(x-y,t)|f(y)-f(x)|dy,$$
$$= \int_{|y-x|\le R} \Big(\dots\Big)dy + \int_{|y-x|>R} \Big(\dots\Big)dy = A + B.$$

Let C_K be the constant in (2.19) relative to $K = \bar{B}_{2R}$. The term A can be estimated in the following way:

$$A \le C_K \int_{|y-x|\le R} \Big(\frac{1}{2t} + \frac{|x-y|^2}{4t^2}\Big)\Gamma(x-y,t)|x-y|^\alpha dy$$
$$\le 2^\alpha C_K t^{\alpha/2-1} \int_{\mathbb{R}^n} \Big(\frac{1}{2} + |\eta|^2\Big)\Gamma(\eta,1/4)|\eta|^\alpha d\eta.$$

We performed the change of variable $x - y = 2\sqrt{t}\eta$. The estimate for A holds for all $t > 0$ and for all $|x| \le R$.

Analogously, we can obtain the estimate

$$B \le \frac{2\|f\|_\infty}{t} \int_{|\eta|>r/2\sqrt{t}} \Big(\frac{1}{2} + |\eta|^2\Big)\Gamma(\eta,1/4)d\eta.$$

Now, for any $T > 0$ there exists a constant $C_T > 0$ such that for all $0 < t < T$ we have

$$\int_{|\eta|>r/2\sqrt{t}} \Big(\frac{1}{2} + |\eta|^2\Big)\Gamma(\eta,1/4)d\eta \le C_T t^{\alpha/2}.$$

The proof of this fact is left as an exercise. The claim of the theorem now follows. □

Definition 2.6. Let $U \subset \mathbb{R}^{n+1}$ be an open set. We denote by $C^{2,1}(U)$ the set of functions $u : U \to \mathbb{R}$ such that the following partial derivatives exist and are continuous

$$\frac{\partial u}{\partial t} \in C(U), \quad \frac{\partial^2 u}{\partial x_i \partial x_j} \in C(U), \quad i,j = 1,...,n.$$

Theorem 2.7. *Let $f \in L^\infty(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times (0,\infty))$ be a function such that $x \mapsto f(x,t)$ is in $C^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$, $0 < \alpha \le 1$, uniformly in $t > 0$. Then the function u in* (2.17) *satisfies:*

1) $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$;
2) $u_t(x,t) - \Delta u(x,t) = f(x,t)$ *for all* $x \in \mathbb{R}^n$, $t > 0$;
3) $\lim_{t\downarrow 0} u(x,t) = 0$ *uniformly in* $x \in \mathbb{R}^n$.

Proof. As in (2.17), letting

$$v(x,t;s) = \int_{\mathbb{R}^n} \Gamma(x-y,t-s)f(y,s)dyds,$$

the solution u may be written in the following way:

$$u(x,t) = \int_0^t v(x,t;s)ds, \quad x \in \mathbb{R}^n,\ t > 0.$$

By Proposition 2.4, it follows that there exists a constant $C > 0$ such that

$$|\nabla v(x,t;s)| \leq \frac{C\|f\|_\infty}{\sqrt{t-s}} \in L^1_s(0,t), \quad 0 < s < t,$$

and thus we can take derivatives in x into the integral in ds:

$$\nabla u(x,t) = \int_0^t \nabla v(x,t;s)ds = \int_0^t \int_{\mathbb{R}^n} \nabla\Gamma(x-y,t-s)f(y,s)dyds.$$

Analogously, by Proposition 2.5, for any $R > 0$ and $T > 0$ there exists a constant $C = C(R,T,\|f\|_\infty,\alpha)$ such that for $|x| \leq R$ and $0 < t < T$ we have, with $i,j = 1,...,n$,

$$|v_{ij}(x,t;s)| \leq \frac{C}{(t-s)^{1-\alpha/2}} \in L^1_s(0,t).$$

We can therefore take derivatives in x into the integral:

$$u_{ij}(x,t) = \int_0^t v_{ij}(x,t;s)ds. \tag{2.21}$$

It also follows that the function $(x,t) \mapsto u_{ij}(x,t)$ is continuous for $x \in \mathbb{R}^n$ and $t > 0$. The proof of this claim is left as an exercise.

In an analogous way, we can prove that the function $t \mapsto u(x,t)$ is differentiable and

$$u_t(x,t) = \frac{\partial}{\partial t}\int_0^t v(x,t;s)ds = v(x,t;t) + \int_0^t v_t(x,t;s)ds. \tag{2.22}$$

In order to prove this claim, notice that

$$|v_t(x,t;s)| = |\Delta v(x,t;s)| \leq \frac{C}{(t-s)^{1-\alpha/2}}.$$

Finally, the function $(x,t) \mapsto u_t(x,t)$ is also continuous (exercise).

Summing up (2.21) and (2.22), we obtain

$$u_t(x,t) - \Delta u(x,t) = v(x,t;t) + \int_0^t \big\{v_t(x,t;s) - \Delta v(x,t;s)\big\}ds = f(x,t).$$

Claim iii) follows from the inequalities:

$$|u(x,t)| \leq \int_0^t |v(x,t;s)|ds \leq \|f\|_\infty t.$$

□

3. Parabolic mean formula

DEFINITION 3.8. Let $r > 0$ and $(x,t) \in \mathbb{R}^{n+1}$. The set

$$E_r(x,t) = \Big\{(y,s) \in \mathbb{R}^{n+1} : s < t \text{ and } \Gamma(x-y,t-s) > \frac{1}{r^n}\Big\}$$

is called *parabolic ball* with radius r centered at (x,t). For $(x,t) = (0,0)$ we also let $E_r = E_r(0,0)$.

PROPOSIZIONE 3.9. *For all $r > 0$ and $(x,t) \in \mathbb{R}^{n+1}$ there holds:*

i) *$E_r(x,t) = (x,t) + E_r$;*

ii) *Letting* $\delta_\lambda(x,t)=(\lambda x,\lambda^2 t)$, $\lambda>0$, *we have* $\delta_\lambda(E_r(x,t))=E_{\lambda r}(\delta_\lambda(x,t))$.

PROOF. Claim i) follows from the fact that the definition of $E_r(x,t)$ depends only on the differences $x-y$ and $t-s$. Claim ii) follows from the fact that $(y,s)\in\delta_\lambda(E_r(x,t))$ is equivalent to

$$\frac{\mathrm{e}^{-\frac{|x-y/\lambda|^2}{t-s/\lambda^2}}}{[4\pi(t-s/\lambda^2)]^{n/2}}>\frac{1}{r^n}\quad\Leftrightarrow\quad\frac{\mathrm{e}^{-\frac{|\lambda x-y|^2}{\lambda^2 t-s}}}{[4\pi(\lambda^2 t-s)]^{n/2}}>\frac{1}{\lambda^n r^n},$$

that is equivalent with $(y,s)\in E_{\lambda r}(\lambda x,\lambda^2 t)$.

□

OSSERVAZIONE 3.10. The parabolic ball E_r is the set of points $(y,s)\in\mathbb{R}^{n+1}$ with $s<0$ such that $\Gamma(y,-s)>1/r^n$, condition that is equivalent to

$$|y|^2<4s\Big(\frac{n}{2}\log(-4\pi s)-n\log r\Big)=\vartheta(s). \tag{3.23}$$

In particular, the balls is contained in the strip $-r^2/4\pi<s<0$. The maximum value of ϑ is $nr^2/2\pi\mathrm{e}$.

The balls E_r has a size of order r in the spatial directions and of order r^2 in the time direction. The center of the ball is in fact the "north pole".

THEOREM 3.11. *let* $U\subset\mathbb{R}^{n+1}$ *be an open set and let* $u\in C^2(U)$ *be a function that satisfies* $u_t=\Delta u$ *in* U*. Then for any* $r>0$ *and for all* $(x,t)\in U$ *such that* $E_r(x,t)\subset U$ *there holds the mean formula*

$$u(x,t)=\frac{1}{c_n r^n}\int_{E_r(x,t)}u(y,s)\frac{|y-x|^2}{(t-s)^2}dyds, \tag{3.24}$$

where $c_n>0$ *is a dimensional constant (and in fact* $c_n=4$ *does not depend on* $n\in\mathbb{N}$*).*

PROOF. It sufficies to prove the theorem in the case $x=0$ and $t=0$. Consider the function

$$\varphi(r)=\frac{1}{r^n}\int_{E_r}u(y,s)\frac{|y|^2}{s^2}dyds,$$

for $r>0$ small enough. We claim that the function φ is constant. Formula (3.24) then follows from the limit

$$\lim_{r\downarrow 0}\frac{1}{r^n}\int_{E_r}u(y,s)\frac{|y|^2}{s^2}dyds=\lim_{r\downarrow 0}\int_{E_1}u(ry,r^2s)\frac{|y|^2}{s^2}dyds=c_n u(0),$$

where $c_n>0$ is the constant

$$c_n=\int_{E_1}\frac{|y|^2}{s^2}dyds.$$

The fact that c_n is finite and the computation of its value are left as exercises. In the change of variable, we used Proposition 3.9.

It suffices to show that $\varphi'(r)=0$ for $r>0$. We can take the derivative into the integral in the definition of φ, after the change of variable transforming the integration domain into E_1:

$$\begin{aligned}\varphi'(r) &= \int_{E_1} \Big\{ y\cdot\nabla u(ry,r^2s) + 2rsu_s(ry,r^2s)\Big\}\frac{|y|^2}{s^2}dyds\\ &= \frac{1}{r^{n+1}}\int_{E_r}\Big\{y\cdot\nabla u(y,s)+2su_s(y,s)\Big\}\frac{|y|^2}{s^2}dyds\\ &= \frac{1}{r^{n+1}}\int_{E_r} y\cdot\nabla u(y,s)\frac{|y|^2}{s^2}dyds + \frac{1}{r^{n+1}}\int_{E_r}2u_s(y,s)\frac{|y|^2}{s}dyds\\ &= \frac{1}{r^{n+1}}\big(A+B\big).\end{aligned}$$

Consider the function

$$\psi(y,s) = \frac{|y|^2}{4s} - \frac{n}{2}\log(-4\pi s) + n\log r.$$

The definition of ψ is suggested by condition (3.23) that characterizes the parabolic ball E_r. The function satisfies $\psi = 0$ on ∂E_r and, moreover,

$$\nabla\psi(y,s) = \frac{y}{2s}. \tag{3.25}$$

We use the last identity to transform B in the following way:

$$\begin{aligned}B &= \int_{E_r} 2u_s(y,s)\frac{|y|^2}{s}dyds = 4\int_{E_r} u_s(y,s)y\cdot\nabla\psi(y,s)dyds\\ &= -4\int_{E_r}\psi(y,s)\mathrm{div}(u_s(y,s)y)dyds\\ &= -4\int_{E_r}\psi(y,s)\big\{y\cdot\nabla u_s(y,s) + nu_s(y,s)\big\}dyds.\end{aligned}$$

We used the divergence theorem (integration by parts) in the variables y for fixed s (and, implicitly, also Fubini-Tonelli theorem). Now we integrate by parts in s for fixed y in the first term, and we use the differential equation $u_s = \Delta u$ in the second one. We get

$$\begin{aligned}B &= 4\int_{E_r}\big\{\psi_s(y,s)y\cdot\nabla u(y,s) - n\psi(y,s)\Delta u(y,s)\big\}dyds\\ &= 4\int_{E_r}\Big\{-\frac{|y|^2}{4s^2} - \frac{n}{2s}\Big\}y\cdot\nabla u(y,s)dyds + 4n\int_{E_r}\nabla\psi(y,s)\cdot\nabla u(y,s)dyds\\ &= -\int_{E_r}\frac{|y|^2}{s^2}y\cdot\nabla u(y,s)dyds = -A.\end{aligned}$$

We used again the divergence theorem and the properties of ψ.

We eventually obtain $A+B=0$ identically in $r>0$ and the theorem is proved. □

4. Parabolic maximum principles

Let $\Omega\subset\mathbb{R}^n$ be an open set and $T>0$. We denote by $\Omega_T = \Omega\times(0,T)$ the cylinder of height T over Ω. With abuse of notation, we define the *parabolic boundary* of Ω_T as the set $\partial\Omega_T\subset\mathbb{R}^{n+1}$ defined in the following way

$$\partial\Omega_T = \partial\Omega\times[0,T]\cup\Omega\times\{0\}.$$

THEOREM 4.12 (Weak maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u \in C^2(\Omega_T) \cap C(\bar{\Omega}_T)$ be a solution of the equation $u_t - \Delta u = 0$ in Ω_T. Then we have*

$$\max_{\bar{\Omega}_T} |u| = \max_{\partial\Omega_T} |u|.$$

The weak maximum principle is a corollary of the strong maximum principle. We postpone the proof.

THEOREM 4.13 (Strong maximum principle). *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $u \in C^2(\Omega_T)$ be a solution to the differential equation $u_t - \Delta u = 0$ in Ω_T. If there is a point $(x_0, t_0) \in \Omega_T$ such that*

$$|u(x_0, t_0)| = \max_{(x,t)\in\Omega_T} |u(x, t)|$$

then we have $u(x, t) = u(x_0, t_0)$ for all $(x, t) \in \Omega \times (0, t_0]$.

PROOF. Let $(x_0, t_0) \in \Omega_T$ be a point such that

$$u(x_0, t_0) = M := \max_{(x,t)\in\Omega_T} u(x, t).$$

Let $(x, t) \in \Omega_T$ be any point such that $t < t_0$ and such that the line segment S connecting (x_0, t_0) to (x, t), i.e.,

$$S = \big\{(x_\tau, t_\tau) = (1 - \tau)(x_0, t_0) + \tau(x, t) \in \mathbb{R}^{n+1} : 0 \le \tau \le 1\big\},$$

is entirely contained in Ω_T. Let

$$A = \big\{\tau \in [0, 1] : u(x_\tau, t_\tau) = M\big\}.$$

We have $A \neq \emptyset$ because $0 \in A$. We shall prove that if $\tau \in A$ then also $\tau + \delta \in A$ for all $0 < \delta < \delta_0$, for some $\delta_0 > 0$. Indeed, there exists $r > 0$ such that $E_r(x_\tau, t_\tau) \subset \Omega_T$, because Ω_T is open and thus, by the parabolic mean formula, we have

$$\begin{aligned} M = u(x_\tau, t_\tau) &= \frac{1}{4r^n} \int_{E_r(x_\tau,t_\tau)} u(y, s) \frac{|y - x_\tau|^2}{(s - t_\tau)^2} dyds \\ &\le \frac{M}{4r^n} \int_{E_r(x_\tau,t_\tau)} \frac{|y - x_\tau|^2}{(s - t_\tau)^2} dyds = M. \end{aligned}$$

It follows that $u = M$ in $E_r(x_\tau, t_\tau)$ and the existence of $\delta > 0$ is implied by the "shape" of parabolic balls. From the previous argument it follows that $A = [0, 1]$ and thus $u = M$ on S.

Let $(x, t) \in \Omega_T$ be any point such that $0 < t < t_0$. As Ω is a connected open set, then it is pathwise connected by polygonal arcs: there exist $m + 1$ points $x_0, x_1, ..., x_m = x$ contained Ω such that each segment $[x_{i-1}, x_i]$, $i = 1, ..., m$, is contained in Ω. Choose times $t_0 > t_1 > ... > t_m = t$. A successive application of the previous argument shows that $u = M$ on each segment $S_i = \big\{(1-\tau)(x_{i-1}, t_{i-1}) + \tau(x_i, t_i) \in \Omega_T : 0 \le \tau \le 1\big\}$ and thus $u(x, t) = M$. By continuity, the claim holds also for $t = t_0$. □

PROOF OF THEOREM 4.12. We prove for instance that

$$M = \max_{\bar{\Omega}_T} u = \max_{\partial\Omega_T} u.$$

Notice that the maximum on the left hand side is attained, beacause u is continuous in $\bar{\Omega}_T$, that is a compact set. Then there exists $(x_0, t_0) \in \bar{\Omega}_T$ such that $u(x_0, t_0) = M$.

If $(x_0, t_0) \in \partial\Omega_T$ the proof is finished. Let $(x_0, t_0) \in \Omega \times (0, T]$. Let $\Omega^{x_0} \subset \Omega$ denote the connected component of Ω containing x_0. From the strong maximum principle it follows that $u = M$ on $\Omega^{x_0} \times (0, t_0]$. This holds also in the case $t_0 = T$. Eventually, u attaines the maximum (also) on the parabolic boundary $\partial\Omega_T$. □

The weak maximum principle implies the uniqueness of the solution of the parabolic Dirichlet problem on a bounded domain with initial and boundary conditions.

THEOREM 4.14 (Uniqueness for the Dirichlet problem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, $T > 0$, $f \in C(\Omega_T)$ and $g \in C(\partial\Omega_T)$. Then the problem*

$$\begin{cases} u_t - \Delta u = f, & \text{in } \Omega_T, \\ u = g, & \text{su } \partial\Omega_T, \end{cases} \tag{4.26}$$

has at most one solution $u \in C^2(\Omega_T) \cap C(\bar{\Omega}_T)$.

PROOF. Indeed, if u, v are solutions then the function $w = u - v$ satisfies $w = 0$ on $\partial\Omega_T$ and $w_t - \Delta w = 0$ in Ω_T. From the weak maximum principle, it follows that $\max_{\bar{\Omega}_T} |w| = \max_{\partial\Omega_t} |w| = 0$ and thus $u = v$. □

The uniqueness for the Cauchy problem on $\mathbb{R}^n$ requires a global version of the maximum principle.

THEOREM 4.15. *Let $f \in C(\mathbb{R}^n)$ and let $u \in C^2(\mathbb{R}^n \times (0, T)) \cap C(\mathbb{R}^n \times [0, T])$ be a solution of the Cauchy problem*

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0, T), \\ u = f, & \text{su } \mathbb{R}^n \end{cases} \tag{4.27}$$

that satisfies for some constants $A, b > 0$

$$|u(x, t)| \leq A\mathrm{e}^{b|x|^2}, \quad x \in \mathbb{R}^n, \ t \in [0, T]. \tag{4.28}$$

Then we have

$$\sup_{x \in \mathbb{R}^n, \, t \in [0,T]} |u(x, t)| \leq \sup_{x \in \mathbb{R}^n} |f(x)|. \tag{4.29}$$

PROOF. We prove, for instance, that $u(x, t) \leq \sup_{\mathbb{R}^n} f$ for $x \in \mathbb{R}^n$ and $t \in [0, T]$. Assume that there also holds $4bT < 1$. This assumption will be removed at the end of the proof. Then there exists $\varepsilon > 0$ such that $4b(T + \varepsilon) < 1$ and thus $\frac{1}{4(T+\varepsilon)} = b + \gamma$ for some $\gamma > 0$. Let $\delta > 0$ be a positive parameter and consider the function

$$v(x, t) = u(x, t) - \frac{\delta}{(T + \varepsilon - t)^{n/2}} \mathrm{e}^{\frac{|x|^2}{4(T+\varepsilon-t)}}, \quad x \in \mathbb{R}^n, \, t \in [0, T].$$

An explicit computation, that is omitted, shows that $v_t = \Delta v$. Moreover, from (4.28) it follows that for $x \in \mathbb{R}^n$ and $t \in [0, T]$ we have

$$v(x, t) \leq A\mathrm{e}^{b|x|^2} - \frac{\delta}{(T + \varepsilon)^{n/2}} \mathrm{e}^{\frac{|x|^2}{4(T+\varepsilon)}} = A\mathrm{e}^{b|x|^2} - \frac{\delta}{(T + \varepsilon)^{n/2}} \mathrm{e}^{(b+\gamma)|x|^2}.$$

As $\delta > 0$, there exists $R > 0$ such that for $|x| \geq R$ and for all $t \in [0, T]$ we have

$$v(x, t) \leq \sup_{x \in \mathbb{R}^n} f(x).$$

On the other hand, letting $\Omega = \{|x| < R\}$, by the weak maximum principle we have

$$\max_{(x,t)\in\bar{\Omega}_T} v(x,t) = \max_{(x,t)\in\partial\Omega_T} v(x,t) \le \sup_{x\in\mathbb{R}^n} f(x).$$

After all, we obtain

$$u(x,t) - \frac{\delta}{(T+\varepsilon-t)^{n/2}} \mathrm{e}^{\frac{|x|^2}{4(T+\varepsilon-t)}} = v(x,t) \le \sup_{x\in\mathbb{R}^n} f(x), \quad x\in\mathbb{R}^n,\ t\in[0,T],$$

and letting $\delta \downarrow 0$ we obtain the claim.

The restriction $4bT < 1$ can be removed on dividing the interval $[0,T]$ into subintervals $[0,T_1]$, $[T_1, 2T_1]$, $[(k-1)T_1, kT_1]$ with $kT_1 = T$ and $4bT_1 < 1$, and then applying the previous argument to each subinterval.

□

THEOREM 4.16 (Uniqueness for the Cauchy problem). *Let $T > 0$, $f \in C(\mathbb{R}^n \times [0,T])$ and $g \in C(\mathbb{R}^n)$. Then the Cauchy problem*

$$\begin{cases} u_t - \Delta u = f, & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = g(x), & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{4.30}$$

has at most one solution $u \in C^2(\mathbb{R}^n \times (0,T)) \cap C(\mathbb{R}^n \times [0,T])$ within the class of functions that satisfies the growth condition

$$|u(x,t)| \le A\mathrm{e}^{b|x|^2}, \quad x\in\mathbb{R}^n,\ t\in[0,T], \tag{4.31}$$

for some constants $A, b > 0$.

The proof is an elementary exercise.