Equazioni Differenziali 2

Foglio 1

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Esercizio 1 Compute the solution $y \in C^1(\mathbb{R})$ to the Cauchy Problem

$$\begin{cases} y' = \dfrac{y}{1+e^x} + e^{-x} \\ y(0) = 0. \end{cases}$$

Esercizio 2 Compute the general solution of the differential equation

$$y' = y - \frac{x^2}{y}.$$

Esercizio 3 Consider the Cauchy Problem

$$\begin{cases} y'(x) = \sqrt{x} + \sqrt{|y(x)|}, \quad x \geq 0 \\ y(0) = 0. \end{cases}$$

i) Prove that any solution y satisfies $y(x) \geq \frac{2}{3}x^{3/2}$ for $x \geq 0$.

ii) Using the contraction principle, prove that there exists a unique local solution of the problem.

iii) Show that the local solution can be continued to a global solution on $[0, +\infty)$.

iv) Prove that

$$\lim_{x \to +\infty} \frac{y(x)}{x^2} = \frac{1}{4}, \quad \text{and} \quad \lim_{x \to 0^+} \frac{y(x)}{x^{3/2}} = \frac{2}{3}.$$

Esercizio 4 Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping such that $d(T(x), T(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Prove that there exists a unique $x \in X$ such that $x = T(x)$.