Equazioni Differenziali 2 — Foglio 3

Nome: — Consegna: 12 Maggio 2010

Exercise 1 Compute the solution to the Cauchy Problem

$$\begin{cases} y'' + y = \dfrac{1}{\cos x}, \quad x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$$

Exercise 2 For $0 < T \leq +\infty$ consider the vector space $X = \big\{ b \in C([0,T); \mathbb{R}^n) : b \text{ is bounded} \big\}$, $n \geq 1$, endowed with the norm

$$\|b\|_\infty = \sup_{x \in [0,T)} |b(x)|.$$

Let $A \in M_n(\mathbb{R})$ and $y_0 \in \mathbb{R}^n$, and consider the Cauchy Problem

$$\begin{cases} y' = Ay + b \quad \text{in } [0,T) \\ y(0) = y_0. \end{cases}$$

Define $F : X \to C^1([0,T); \mathbb{R}^n)$ on letting $F(b) = y$ if and only if $y \in C^1([0,T); \mathbb{R}^n)$ is the solution of the Cauchy problem with datum b.

i) Prove that for any $0 < T < +\infty$ there exists a constant $0 < C_T < +\infty$ such that

$$\|F(b_1) - F(b_2)\|_\infty \leq C_T \|b_1 - b_2\|_\infty \tag{1}$$

for all $b_1, b_2 \in X$;

ii) What can we say in the case $T = +\infty$?

Exercise 3 Let $a, b \in C(\mathbb{R})$ be continuous functions and let $y_1 \in C^2(\mathbb{R})$, $y_1(x) \neq 0$ for all $x \in \mathbb{R}$, be a solution to the differential equation

$$y'' + ay' + by = 0.$$

Determine a solution $y_2 \in C^2(\mathbb{R})$ which is linearly independent from y_1.

Exercise 4 Consider the differential equation

$$x^3 y' - 2y + 2x = 0.$$

Prove that:

i) Any solution $y \in C^1(\mathbb{R} \setminus \{0\})$ can be continued to a function in $C^1(\mathbb{R})$;

ii) There is no analytic solution to the equation in any neighborhood of $x = 0$.