Equazioni Differenziali 2 — Foglio 5

Nome: Consegna: 28 Maggio 2010

Exercise 1 Let $Y = C([0,1];\mathbb{R})$ be the space of continuous functions on $[0,1]$ with the norm $\|y\| = \max\limits_{x\in[0,1]} |y(x)|$.

i) Prove that the set

$$K = \Big\{ y \in Y : y(0) = 0,\, y \in C^1([0,1]),\, \int_0^1 |y'(x)|^2 dx \le 1 \Big\}$$

is precompact in Y, i.e., it is equibounded and equicontinuous.

ii) Prove that the set

$$K = \Big\{ y \in Y : y(0) = 0,\, y \in C^1([0,1]),\, \int_0^1 |y'(x)| dx \le 1 \Big\}$$

is not precompact in Y, i.e., there is a sequence in K which has no subsequence converging in the norm $\|\cdot\|$.

Exercise 2 Let $h : [0,1] \to \mathbb{R}$ be a measurable function such that $h(x) \ge 0$ for all $x \in [0,1]$. Let $K \subset L^1(0,1)$ be the set

$$K = \big\{ f \in L^1(0,1) : |f(x)| \le h(x) \text{ for a.e. } x \in [0,1] \big\}.$$

Prove that:

i) K is closed;

ii) If K is compact in $L^1(0,1)$ then $h \in L^1(0,1)$;

iii) If $h \in L^1(0,1)$ then K is sequentially weakly compact in $L^1(0,1)$ (but not necessarily compact).

Exercise 3 Provare l'implicazione B)⇒A) del Teorema di Riesz-Kolmogorov, seguendo le indicazioni date in classe. Anche soluzioni parziali sono benvenute.