

Introduction

to ordinary differential equations

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Introduction

Let $\Omega \subset \mathbb{R}^{n+2}$, $n \in \mathbb{N}$, be an open set and let $F : \Omega \rightarrow \mathbb{R}$ be a continuous function. An equation of the form

$$F(x, y, y', \dots, y^{(n)}) = 0 \tag{0.0.1}$$

is called ordinary differential equation of order n . Here, x is a real variable, y is a real valued unknown function, and $y', \dots, y^{(n)}$ are its derivatives.

A function $\varphi \in C^n(I)$ is a solution of the differential equation if:

- i) $I \subset \mathbb{R}$ is an open interval;
- ii) $(x, y(x), \dots, y^{(n)}(x)) \in \Omega$ for all $x \in I$;
- iii) $F(x, y(x), \dots, y^{(n)}(x)) = 0$ for all $x \in I$.

The main problems concerning ordinary differential equations are:

- 1) Existence of solutions;
- 2) Uniqueness of solutions (with suitable initial conditions or boundary value data);
- 3) Regularity and stability of solutions (e.g. dependence on the initial conditions, large time stability, higher regularity depending on F or on parameters);
- 4) Computation of solutions.

The existence of solutions can be proved by fixed point theorems, by approximation and compactness, by variational methods (minimization and critical point theorems), by the implicit function theorem in Banach spaces, by Functional Analysis techniques. The problem of uniqueness is typically more difficult. Only in very special cases, it is possible to compute the solutions in some explicit form.

CHAPTER 1

Some methods of resolution

1. First order linear equations

1.1. First order linear equations. Let $I \subset \mathbb{R}$ be an open interval and let $a, b \in C(I)$ be two continuous functions. A first order differential equation of the form

$$y' + a(x)y = b(x), \quad x \in I, \quad (1.1.2)$$

is called linear. In the case $b = 0$, the equation is said to be homogeneous

$$y' + a(x)y = 0, \quad x \in I. \quad (1.1.3)$$

We solve the homogeneous equation, first. Assuming $y \neq 0$, e.g. $y > 0$, the differential equation (1.1.3) has the form $y'/y = -a(x)$. A primitive of y'/y is $\log y$. Then, denoting by A a primitive of a , i.e., $A'(x) = a(x)$ for all $x \in I$, we have

$$-A = \log y + d,$$

for some constant $d \in \mathbb{R}$. It follows that $y = \exp(-d - A)$ and letting $c = e^{-d}$ we find the solution

$$y(x) = ce^{-A(x)}, \quad x \in I. \quad (1.1.4)$$

This function is a solution to the homogeneous equation for any $c \in \mathbb{R}$ (i.e., the restriction $y > 0$ can be dropped).

Now we look for a solution of the form (1.1.4) for the non homogeneous equation (1.1.2), where now $c \in C^1(I)$ is a function that has to be determined (this method is called “Variation of constants”). Plugging $y' = c'e^{-A} - ace^{-A}$ into (1.1.2) we get

$$c'e^{-A} = b, \quad \text{that is } c' = be^A.$$

Integrating this equation on some interval $(x_0, x) \subset I$ we get

$$c(x) = c(x_0) + \int_{x_0}^x b(t)e^{A(t)} dt,$$

and we find

$$y(x) = \left(c(x_0) + \int_{x_0}^x b(t)e^{A(t)} dt \right) e^{-A(x)}, \quad x \in I, \quad (1.1.5)$$

where $c(x_0) \in \mathbb{R}$ is a real number and $x_0 \in I$.

PROPOSITION 1.1.1. *Let $x_0 \in I$ and A be a primitive of a . Then the function in (1.1.5) is a solution to (1.1.2). Moreover, any solution of (1.1.2) is of the form (1.1.5) for some $c(x_0) \in \mathbb{R}$.*

PROOF. The first statement is a computation. Let $z \in C^1(I)$ be a solution to (1.1.2) and let

$$w(x) = e^{A(x)}z(x) - \int_{x_0}^x b(t)e^{A(t)}dt.$$

For we have

$$w' = (az + z')e^A - be^A = 0,$$

the function w is constant on I , and the second claim is proved. \square

2. Separation of variables

Let $I, J \subset \mathbb{R}$ be two open intervals and let $f \in C(I)$ and $g \in C(J)$ be two continuous functions. We look for solutions to the first order differential equation

$$y' = f(x)g(y). \quad (1.2.6)$$

Let $x_0 \in I$ and $y_0 \in J$. If $g(y_0) = 0$ for some $y_0 \in J$, then the constant function $y(x) = y_0$, $x \in I$, is a solution to the differential equation (1.2.6). Assume that $g(y_0) \neq 0$. Then it is $g \neq 0$ in a neighborhood of y_0 and we can divide the equation by $g(y)$ (separation of variables). We find

$$\frac{y'(x)}{g(y(x))} = f(x). \quad (1.2.7)$$

Let $G \in C^1(J_1)$ be the primitive of $1/g(y)$ (in the variable y), defined in some interval J_1 containing y_0 . The function G is strictly monotonic, because $G'(y) \neq 0$, and thus invertible. Moreover, let $F \in C^1(I)$ be a primitive of f . Upon integrating (1.2.7), we get

$$G(y(x)) = F(x) + C, \quad x \in I_1, \quad (1.2.8)$$

for some interval $I_1 \subset I$. Here $C \in \mathbb{R}$ is a real constant. The general solution of the differential equation is then

$$y(x) = G^{-1}(F(x) + C), \quad x \in I_1, \quad (1.2.9)$$

where $G^{-1} : G(J_1) \rightarrow J_1$ is the inverse function of G . The constant C is uniquely determined by the initial condition $y(x_0) = y_0$, i.e., $C = G(y_0) - F(x_0)$.

This argument identifies two kinds of solutions to the equation (1.2.6): constant solutions and solutions such that $g(y) \neq 0$. There could be other solutions (see Section 5). If e.g. $g \in C^1(J)$, however, there are no other solutions (see Chapter 8.1.3).

EXAMPLE 1.2.1. We look for the solution to the Cauchy Problem

$$\begin{cases} y' = \frac{1+2x}{\cos y} \\ y(0) = \pi. \end{cases} \quad (1.2.10)$$

The differential equation is of the form $y' = f(x)g(y)$ with $f(x) = 1 + 2x$ and $g(y) = 1/\cos y$. ular, g is defined for $\cos y \neq 0$, i.e., for $y \neq \pi/2 + k\pi$ with $k \in \mathbb{Z}$.

Separating the variables we get $y' \cos y = 1 + 2x$, and integrating we find the general solution in implicit form

$$\sin y = x + x^2 + C,$$

where $C \in \mathbb{R}$ is a constant, which is determined by the initial condition $y(0) = \pi$, i.e., $C = \sin y(0) = 0$. The function

$$z(x) = \arcsin(x + x^2)$$

is not, however, the solution to (1.2.10) because $z(0) = \arcsin(0) = 0$. In order to determine the correct solution, notice the arcsin is the inverse function of sin when restricted to $[-\pi/2, \pi/2]$, whereas y takes values in a neighborhood of π . Letting $w(x) = y(x) - \pi$, we have $w(0) = y(0) - \pi = 0$ and $\sin w = \sin(y - \pi) = -\sin y = -(x + x^2)$. Now we can invert the sine function, obtaining $w = -\arcsin(x + x^2)$ and thus

$$y(x) = \pi - \arcsin(x + x^2).$$

The solution y is defined in a suitable neighborhood of the origin.

3. Equations of homogeneous type

A differential equation of the form

$$y' = f\left(\frac{y}{x}\right)$$

is called of homogeneous type. Here $f : I \rightarrow \mathbb{R}$ is a (continuous) function in some interval $I \subset \mathbb{R}$. With the change of variable $y = xz$, where z is the new unknown function, we get $y' = z + xz'$ and the differential equation transforms into

$$xz' + z = f(z).$$

This equation can be solved on separating the variables.

For instance, the following differential equation is of homogeneous type (see Exercise 8)

$$y' = \frac{x^2 + y^2}{xy} = f\left(\frac{y}{x}\right), \quad \text{with} \quad f(t) = \frac{1}{t} + t.$$

4. Bernoulli's equations

A differential equation of the form

$$y' + a(x)y = b(x)y^\alpha, \quad x \in I, \tag{1.4.11}$$

where α is a real parameter such that $\alpha \neq 0, 1$ is said to be of Bernoulli type. Letting

$$y = z^{\frac{1}{1-\alpha}}, \quad y' = \frac{1}{1-\alpha} z^{\frac{\alpha}{1-\alpha}} z',$$

the equations transforms into

$$z' + (1 - \alpha)a(x)z = (1 - \alpha)b(x).$$

This is a linear equation.

5. Exact equations

Let $\Omega \subset \mathbb{R}^2$ be an open set and $f, g \in C(\Omega)$ be continuous function. We look for a solution $y \in C^1(I)$, $I \subset \mathbb{R}$ open interval, of the differential equation

$$f(x, y) + g(x, y)y' = 0, \quad (1.5.12)$$

satisfying the condition $y(x_0) = y_0$ for some $(x_0, y_0) \in \Omega$ with $x_0 \in I$.

To this aim, consider the differential form ω in Ω

$$\omega = f(x, y)dx + g(x, y)dy, \quad (1.5.13)$$

where dx, dy is the dual basis of the basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

DEFINITION 1.5.1 (Exact forms). The differential form ω is exact if there exists a function $F \in C^1(\Omega)$ such that

$$\omega = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \text{in } \Omega.$$

The function F is called a potential of ω . In this case, the differential equation (1.5.12) is called exact.

THEOREM 1.5.2. *Assume that ω is an exact form with potential F such that*

$$\frac{\partial F(x_0, y_0)}{\partial y} \neq 0. \quad (1.5.14)$$

Then the equation $F(x, y) = F(x_0, y_0)$ implicitly defines a function $y \in C^1(I)$ for some open interval I containing x_0 solving the differential equation (1.5.12) along with the condition $y(x_0) = y_0$. This solution is unique on the interval I .

PROOF. Assume w.l.g. that $F(x_0, y_0) = 0$. By the implicit function theorem, there exist $\delta, \eta > 0$ and $y \in C^1(x_0 - \delta, x_0 + \delta)$ such that

$$\{(x, y) \in \Omega : |x - x_0| < \delta, |y - y_0| < \eta, F(x, y) = 0\} = \{(x, y(x)) \in \Omega : |x - x_0| < \delta\}. \quad (1.5.15)$$

Differentiating the identity $F(x, y(x)) = 0$ we get

$$\begin{aligned} 0 &= \frac{d}{dx}F(x, y(x)) = \frac{\partial F(x, y(x))}{\partial x} + \frac{\partial F(x, y(x))}{\partial y}y'(x) \\ &= f(x, y(x)) + g(x, y(x))y'(x), \end{aligned} \quad (1.5.16)$$

i.e., y is a solution of the differential equation, and moreover $y(x_0) = y_0$.

On the other hand, if $z \in C^1(I)$ is a solution to the equation (1.5.12) such that $z(x_0) = y_0$, then the same argument as in (1.5.16) shows that

$$\frac{d}{dx}F(x, z(x)) = 0,$$

and therefore $F(x, z(x)) = F(x_0, z(x_0)) = F(x_0, y_0) = 0$. By (1.5.15) it must be $z = y$. \square

DEFINITION 1.5.3 (Closed forms). Assume that $f, g \in C^1(\Omega)$. The differential form $\omega = f dx + g dy$ is closed in Ω if

$$\frac{\partial f(x, y)}{\partial y} = \frac{\partial g(x, y)}{\partial x} \quad \text{for all } (x, y) \in \Omega.$$

An exact differential form in an open set Ω with a potential $F \in C^2(\Omega)$ is closed in Ω because mixed derivatives are equal by Schwarz theorem

$$\frac{\partial^2 F}{\partial x \partial y}(x, y) = \frac{\partial^2 F}{\partial y \partial x}(x, y), \quad (x, y) \in \Omega.$$

The converse is also true if Ω is simply connected.

THEOREM 1.5.4. *If $\Omega \subset \mathbb{R}^2$ is a simply connected open set, then any closed differential form in Ω is exact.*

Convex and starshaped open sets are simply connected. In particular, closed forms always have a potential locally.

If $\varphi \in C(\Omega)$ is a function such that $\varphi \neq 0$ in Ω , then the differential equation (1.5.12) and the differential equation

$$\varphi(x, y) \{ f(x, y) + g(x, y)y' \} = 0, \quad (1.5.17)$$

have the same solutions. For a suitable choice of φ , the differential equation (1.5.17) may happen to be exact, even though (1.5.12) is not exact. The function φ is then called integrating factor (or multiplier). If $f, g \in C^1(\Omega)$, a necessary condition for a function $\varphi \in C^1(\Omega)$ to be a multiplier is

$$\frac{\partial}{\partial y} \varphi f = \frac{\partial}{\partial x} \varphi g \quad \text{in } \Omega. \quad (1.5.18)$$

EXAMPLE 1.5.5. The differential equation

$$xy^2 + y - xy' = 0 \quad (1.5.19)$$

is not exact. In fact, with $f = xy^2 + y$ and $g = -x$, we have

$$\frac{\partial f(x, y)}{\partial y} = 2xy + 1 \quad \text{and} \quad \frac{\partial g(x, y)}{\partial x} = -1.$$

We look for a function φ such that (1.5.18) holds. We try with the ansatz $\varphi = \varphi(y)$, i.e., φ depends only on y . We get the necessary condition

$$\varphi'(y)(xy^2 + y) + \varphi(y)(2xy + 1) = -\varphi(y),$$

that is implied by $y\varphi' + 2\varphi = 0$ (this equation does not depend on x). A solution for this linear equation is $\varphi(y) = 1/y^2$. Now the differential equation

$$x + \frac{1}{y} - \frac{xy'}{y^2} = 0$$

is exact, where $y \neq 0$. A potential F for this exact differential equation can be found on solving

1

An integration yields

$$F(x, y) = \frac{x^2}{2} + \frac{x}{y} + C, \quad C \in \mathbb{R}.$$

The equation $F(x, y) = 0$ implicitly defines solutions to the differential equation (1.5.19) (for $x \neq 0$ and $y \neq 0$).

6. Second order linear equations with constant coefficients

Let $f : I \rightarrow \mathbb{R}$ be a continuous function on the interval $I \subset \mathbb{R}$ and let $a, b, c \in \mathbb{R}$ be real numbers such that $a \neq 0$. The differential equation

$$ay'' + by' + cy = f(x), \quad x \in I, \quad (1.6.20)$$

is a second order linear differential equation with constant coefficients. When $f = 0$ the equation is called homogeneous. The general solution of the homogeneous equation

$$ay'' + by' + cy = 0, \quad x \in \mathbb{R}, \quad (1.6.21)$$

is of the form $y_{GH} = C_1y_1 + C_2y_2$ where $C_1, C_2 \in \mathbb{R}$ are real numbers and y_1, y_2 are two solutions of (1.6.21) which are linearly independent, i.e., such that for real numbers $\alpha, \beta \in \mathbb{R}$

$$\alpha y_1 + \beta y_2 = 0 \text{ in } \mathbb{R} \quad \Rightarrow \quad \alpha = \beta = 0.$$

The general solution of the inhomogeneous equation (1.6.20) is a function $y \in C^2(I)$ of the form $y = y_{GH} + y_P$, where $y_P \in C^2(I)$ is a particular solution of the inhomogeneous equations. We describe some practical methods to compute y_{GH} and y_P . The general theory is dealt with in Chapter 3.

6.1. Homogeneous equation. The solutions to the homogeneous equation

$$ay'' + by' + cy = 0 \quad (1.6.22)$$

are a real vector space, i.e., any linear combination of solutions is still a solution. We shall prove in Chapter 3 that this vector space has dimension 2. It is therefore sufficient to find two linearly independent solutions to the equation. We look for solutions of the form $y(x) = e^{\lambda x}$ for some complex number $\lambda \in \mathbb{C}$. Inserting y, y', y'' into (1.6.22) we get $e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$. Since $e^{\lambda x} \neq 0$, the complex number λ must solve the characteristic equation

$$a\lambda^2 + b\lambda + c = 0. \quad (1.6.23)$$

According to the sign of $\Delta = b^2 - 4ac$ we distinguish three cases.

Case 1: $\Delta > 0$. In this case the characteristic equation has two real (simple) solutions

$$\lambda_1 = \frac{-b + \sqrt{\Delta}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{\Delta}}{2a}.$$

¹MANCA TESTO

The general solution of the homogeneous equation (1.6.22) is

$$y_{GH}(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x},$$

where $C_1, C_2 \in \mathbb{R}$ are real constant.

Case 2: $\Delta = 0$. In this case the characteristic equation $a\lambda^2 + b\lambda + c = 0$ has one real double solution $\lambda = -b/2a$. The ansatz yields only the solution $y_1(x) = e^{\lambda x}$. A direct computation shows that the function $y_2(x) = x e^{\lambda x}$ is also a solution which is linearly independent from the first one. The general solution of the homogeneous equation (1.6.22) is then

$$y_{GH}(x) = e^{\lambda x}(C_1 + C_2 x),$$

where $C_1, C_2 \in \mathbb{R}$ are real constants.

Case 3: $\Delta < 0$. In this case the characteristic equation has two complex conjugate solutions

$$\lambda_1 = \alpha + i\beta \quad \text{and} \quad \lambda_2 = \alpha - i\beta, \quad \text{where} \quad \alpha = -\frac{b}{2a}, \quad \beta = \frac{\sqrt{|\Delta|}}{2a}.$$

We get the complex valued solutions

$$z_1(x) = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos(\beta x) + i \sin(\beta x)),$$

$$z_2(x) = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos(\beta x) - i \sin(\beta x)),$$

and the real valued solutions

$$y_1(x) = \frac{1}{2}(z_1(x) + z_2(x)) = e^{\alpha x} \cos(\beta x),$$

$$y_2(x) = \frac{1}{2i}(z_1(x) - z_2(x)) = e^{\alpha x} \sin(\beta x).$$

The general solution of the homogeneous equation is

$$y_{GH}(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)),$$

where $C_1, C_2 \in \mathbb{R}$ are real constants.

6.2. Inhomogeneous equation. Similar solutions. Consider the inhomogeneous equation (1.6.20), where the right hand side is a function $f : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$f(x) = e^{\alpha x}(P_0(x) \cos(\beta x) + Q_0(x) \sin(\beta x)),$$

for some $\alpha, \beta \in \mathbb{R}$ and real polynomials P_0 and Q_0 .

We describe a practical method to find a particular solution y_P to the differential equation

$$ay'' + by' + cy = f(x). \tag{1.6.24}$$

A more systematic method is given in the next section.

Consider the real or complex number $\lambda = \alpha + i\beta$ and denote by $m \in \{0, 1, 2\}$ the multiplicity of λ as a solution of the characteristic equation (1.6.23). The case $m = 0$ means that λ is not a solution, the case $m = 1$ means that λ is a simple solution, the case $m = 2$ means that λ is a double solution.

It is always possible to find a particular solution y_P for the inhomogeneous equation (1.6.24) of the form

$$y_P(x) = x^m e^{\alpha x} (P(x) \cos(\beta x) + Q(x) \sin(\beta x)),$$

where P and Q are unknown polynomials with degree equal to the maximum of the degrees of P_0 and Q_0 . These polynomials can be determined inserting y_P, y'_P, y''_P into (1.6.24) and comparing the coefficients of the trigonometric functions in the left and right hand side.

6.3. Inhomogeneous equation. Variation of constants. We look for a particular solution of the inhomogeneous equation (1.6.20) when $f : I \rightarrow \mathbb{R}$ is any continuous function on some interval $I \subset \mathbb{R}$.

The general solution of the homogeneous equation (1.6.21) is of the form

$$y = C_1 y_1 + C_2 y_2, \quad (1.6.25)$$

where C_1, C_2 are two real constants and y_1 and y_2 are two linearly independent solutions of the homogeneous equation. These solutions are known.

The method of the variation of constants consists in letting C_1, C_2 be functions of the variable x . We look for a particular solution y of (1.6.20) of the form (1.6.25), where now C_1 and C_2 are functions. We have to determine C_1 and C_2 . On differentiating y , we get

$$y' = C'_1 y_1 + C_1 y'_1 + C'_2 y_2 + C_2 y'_2. \quad (1.6.26)$$

We impose on C'_1 and C'_2 the condition

$$C'_1 y_1 + C'_2 y_2 = 0. \quad (1.6.27)$$

On differentiating y' we find

$$y'' = C'_1 y'_1 + C_1 y''_1 + C'_2 y'_2 + C_2 y''_2. \quad (1.6.28)$$

Plugging (1.6.25), (1.6.26), (1.6.28) into the inhomogeneous equation (1.6.20) we find

$$a(C'_1 y'_1 + C'_2 y'_2) = f(x). \quad (1.6.29)$$

We also used (1.6.27) and the fact that y_1, y_2 are solution to the homogeneous equation.

As y_1 and y_2 are linearly independent, it is

$$\det \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \neq 0. \quad (1.6.30)$$

This fact follows from Exercise 12. Then it is possible to solve the system of equations

$$\begin{cases} C'_1 y_1 + C'_2 y_2 = 0 \\ C'_1 y'_1 + C'_2 y'_2 = f(x)/a. \end{cases} \quad (1.6.31)$$

Finally, the functions C_1 and C_2 can be computed on integrating C'_1 and C'_2 .

7. Euler's second order equations

Let $f : I \rightarrow \mathbb{R}$ be a continuous function on the interval $I \subset \mathbb{R}^+$ and let $a, b, c \in \mathbb{R}$ be real numbers such that $a \neq 0$. The differential equation

$$ax^2y'' + bxy' + cy = f(x), \quad x \in I, \quad (1.7.32)$$

is a second order differential equation of Euler's type. We consider only the case $f = 0$, i.e., the homogeneous equation

$$ax^2y'' + bxy' + cy = 0, \quad x \in \mathbb{R}^+. \quad (1.7.33)$$

The differential equation is singular at $x = 0$ because the coefficient of y'' vanishes. We look for solutions on the half line $\mathbb{R}^+ = (0, +\infty)$. For the differential equation is linear, solutions are a two dimensional vector space. We look for two linearly independent solutions of the form

$$y(x) = x^\lambda = e^{\lambda \log(x)} = e^{(\alpha+i\beta) \log x} = x^\alpha (\cos(\beta \log x) + i \sin(\beta \log x)),$$

where $\lambda = \alpha + i\beta$ is a complex parameter. Plugging y , $y' = \lambda x^{\lambda-1}$, and $y'' = \lambda(\lambda-1)x^{\lambda-2}$ into (1.7.33) we get $x^\lambda(a\lambda(\lambda-1) + b\lambda + c) = 0$. Because $x^\lambda \neq 0$, λ must solve the characteristic equation

$$a\lambda^2 + (b-a)\lambda + c = 0. \quad (1.7.34)$$

According to the sign of $\Delta = (b-a)^2 - 4ac$ we distinguish three cases.

Case 1: $\Delta > 0$. In this case the characteristic equation has two real (simple) solutions $\lambda_1, \lambda_2 \in \mathbb{R}$ and the general solution of the homogeneous equation (1.7.33) is

$$y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2},$$

where $C_1, C_2 \in \mathbb{R}$ are real constant.

Case 2: $\Delta = 0$. In this case the characteristic equation has one real double solution $\lambda \in \mathbb{R}$ and we get the solution $y_1(x) = x^\lambda$. A direct computation shows that the function $y_2(x) = x^\lambda \log x$ is also a solution which is linearly independent from the first one. The general solution of the homogeneous equation (1.7.33) is then

$$y(x) = x^\lambda (C_1 + C_2 \log x),$$

where $C_1, C_2 \in \mathbb{R}$ are real constants.

Case 3: $\Delta < 0$. In this case the characteristic equation has two complex conjugate solutions

$$\lambda_1 = \alpha + i\beta \quad \text{and} \quad \lambda_2 = \alpha - i\beta.$$

We get the complex valued solutions

$$\begin{aligned} z_1(x) &= x^{\alpha+i\beta} = x^\alpha (\cos(\beta \log x) + i \sin(\beta \log x)), \\ z_2(x) &= x^{\alpha-i\beta} = x^\alpha (\cos(\beta \log x) - i \sin(\beta \log x)), \end{aligned}$$

and the real valued solutions

$$y_1(x) = \frac{1}{2}(z_1(x) + z_2(x)) = x^\alpha \cos(\beta \log x),$$
$$y_2(x) = \frac{1}{2i}(z_1(x) - z_2(x)) = x^\alpha \sin(\beta \log x).$$

The general solution of the homogeneous equation is

$$y(x) = x^\alpha (C_1 \cos(\beta \log x) + C_2 \sin(\beta \log x)),$$

where $C_1, C_2 \in \mathbb{R}$ are real constants.

CHAPTER 2

Existence and uniqueness in the Lipschitz case

1. Banach fixed point theorem

DEFINITION 2.1.1 (Contraction). Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is a *contraction* if there exists $0 < \lambda < 1$ such that $d(T(x), T(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.

THEOREM 2.1.2. *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a contraction. Then there exists a unique $x \in X$ such that $x = T(x)$.*

PROOF. Let $x_0 \in X$ be a given point and define $x_n = T^n(x_0) = T \circ \dots \circ T(x_0)$, n -times. The sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy. In fact, by the triangle inequality we have for all $n, k \in \mathbb{N}$

$$\begin{aligned} d(x_{n+k}, x_n) &\leq \sum_{h=1}^k d(x_{n+h}, x_{n+h-1}) = \sum_{h=1}^k d(T^{n+h}(x_0), T^{n+h-1}(x_0)) \\ &\leq d(T(x_0), x_0) \sum_{h=1}^k \lambda^{n+h-1} \leq \lambda^n d(T(x_0), x_0) \sum_{h=1}^{\infty} \lambda^{h-1}. \end{aligned}$$

The series converges and $\lambda^n \rightarrow 0$ as $n \rightarrow \infty$, because $\lambda < 1$. For X is complete, there exists $x \in X$ such that $x = \lim_{n \rightarrow \infty} T^n(x_0)$.

We show that $x = T(x)$. The mapping T is continuous and so we have

$$x = \lim_{n \rightarrow \infty} T^n(x_0) = \lim_{n \rightarrow \infty} T(T^{n-1}(x_0)) = T(\lim_{n \rightarrow \infty} T^{n-1}(x_0)) = T(x).$$

Finally, we prove that the fixed point is unique. Let $\bar{x} \in X$ be such that $\bar{x} = T(\bar{x})$. Then we have

$$d(x, \bar{x}) = d(T(x), T(\bar{x})) \leq \lambda d(x, \bar{x}) \quad \Rightarrow \quad d(x, \bar{x}) = 0,$$

because $\lambda < 1$, and thus $x = \bar{x}$. □

THEOREM 2.1.3. *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a mapping such that T^n is a contraction, for some $n \in \mathbb{N}$. Then there exists a unique $x \in X$ such that $x = T(x)$.*

PROOF. There exists a unique $x \in X$ such that $T^n(x) = x$. Then we have for some $0 \leq \lambda < 1$

$$d(x, T(x)) = d(T^n(x), T(T^n(x))) = d(T^n(x), T^n(T(x))) \leq \lambda d(x, T(x)),$$

and thus $d(x, T(x)) = 0$, which is equivalent to $T(x) = x$.

Now assume that for some $y \in X$ it is $y = T(y)$. Then we also have $y = T^n(y)$ and thus $x = y$, because the fixed point of T^n is unique. \square

2. Excursus. Other fixed point theorems

Some fixed point theorems are based on the notion of convexity.

THEOREM 2.2.1 (Brouwer). *Let $K \subset \mathbb{R}^n$ be a closed ball and let $T : K \rightarrow K$ be continuous. Then there exists $x \in K$ such that $T(x) = x$.*

For an analytical proof, see Evans, *Partial Differential Equations*, p.441. Brouwer's theorem extends to the infinite dimensional case.

THEOREM 2.2.2 (Schauder I). *Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be a nonempty, convex, and compact set. Any continuous mapping $T : K \rightarrow K$ has at least one fixed point in K , i.e., there exists $x \in K$ tale che $T(x) = x$.*

See Evans, *Partial Differential Equations*, p.502. The assumption on K to be compact can be transferred to the mapping T .

THEOREM 2.2.3 (Schauder II). *Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be a nonempty, convex, closed and bounded set. Let $T : K \rightarrow K$ be a mapping such that:*

- (i) T is continuous;
- (ii) T is compact, i.e., $T(K) \subset K$ is precompact.

Then there exists $x \in K$ such that $T(x) = x$.

Tarki's Fixed Point theorem relies upon the notion of partial order.

THEOREM 2.2.4 (Tarski). *Let (X, \leq) be a partially ordered set such that any subset $Y \subset X$ has a supremum. Let $T : X \rightarrow X$ be an order preserving mapping, i.e., a mapping such that*

$$x \leq y \Rightarrow T(x) \leq T(y).$$

Then there exists $x \in X$ such that $x = T(x)$.

The proof of Tarki's Lemma is an exercise.

3. Cauchy Problem. Introduction

In $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$, $n \geq 1$, we introduce the coordinates $x \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let $f \in C(\Omega; \mathbb{R}^n)$ be a continuous function. Given a point $(x_0, y_0) \in \Omega$ we consider the Cauchy Problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0. \end{cases} \quad (2.3.35)$$

A function $y \in C^1(I; \mathbb{R}^n)$ is a solution to the problem if:

- i) $I \subset \mathbb{R}$ is an interval such that $x_0 \in I$;

- ii) $(x, y(x)) \in \Omega$ for all $x \in I$;
- iii) $y'(x) = f(x, y(x))$ for all $x \in I$ (the differential equations is solved);
- iv) $y(x_0) = y_0$ (the initial datum is attained).

We are interested in the following questions:

- a) Existence of solutions;
- b) Uniqueness of solutions;
- c) Dependence of solutions from x_0, y_0 , and f (regularity and stability).

Integrating the differential equation $y' = f(x, y)$ on the interval with end-points x_0 and x we get the integral equation

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt = Ty(x), \tag{2.3.36}$$

where $y \mapsto Ty$ is a mapping defined on a suitable functional space. A solution to the Cauchy Problem is then a fixed point of the mapping T . On the other hand, if a continuous function y solves the fixed point equation (2.3.36) then y is of class C^1 and solves the Cauchy Problem (2.3.35).

We fix the functional space. For a $\delta > 0$ consider the real vector space

$$V = C([x_0 - \delta, x_0 + \delta]; \mathbb{R}^n). \tag{2.3.37}$$

Endowed with the norm

$$\|y\| = \max_{x \in [x_0 - \delta, x_0 + \delta]} |y(x)|, \quad y \in V, \tag{2.3.38}$$

the vector space V is a Banach space, because any Cauchy sequence in V converges to a function in V . For any $\varepsilon > 0$, the subset X of V

$$X = \{y \in V : y(x_0) = y_0, \|y - y_0\| \leq \varepsilon\} \tag{2.3.39}$$

is closed because both conditions $y(x_0) = y_0$ and $\|y - y_0\| \leq \varepsilon$ are preserved by the uniform (pointwise) convergence. Then the metric space (X, d) is complete w.r.t. the metric $d(y, z) = \|y - z\|$.

We shall see that for a suitable choice of δ and ε the mapping $T : X \rightarrow X$

$$Ty(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt \tag{2.3.40}$$

is well defined, i.e., it is $Ty \in X$ for all $y \in X$.

4. Local existence and uniqueness under the Lipschitz condition

DEFINITION 2.4.1. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. We say that a function $f \in C(\Omega; \mathbb{R}^n)$ has the local Lipschitz property in y if for any compact set $K \subset \Omega$ there exists a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \tag{2.4.41}$$

for all $(x, y_1), (x, y_2) \in K$.

THEOREM 2.4.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, $(x_0, y_0) \in \Omega$, and let $f \in C(\Omega; \mathbb{R}^n)$ be a function with the local Lipschitz property in y . Then there exists $\delta > 0$ such that the Cauchy Problem (2.3.35) has a unique solution $y \in C^1(I; \mathbb{R}^n)$ in the interval $I = [x_0 - \delta, x_0 + \delta]$.*

PROOF. Let $\delta > 0$ and $\varepsilon > 0$ be such that $K = [x_0 - \delta, x_0 + \delta] \times \{y \in \mathbb{R}^n : |y - y_0| \leq \varepsilon\} \subset \Omega$. Let $H \subset \Omega$ be any compact set such that $K \subset \text{int}(H)$. For f is continuous on H , the number

$$M = \sup_{(x,y) \in H} |f(x,y)| < +\infty$$

is finite. Let X be the set introduced in (2.3.39) and let T be the mapping (2.3.40). For any $y \in X$ we have for $x \in I$

$$|Ty(x) - y_0| \leq \left| \int_{x_0}^x |f(t, y(t))| dt \right| \leq M|x - x_0| \leq \delta M.$$

In fact it is $(t, y(t)) \in K$ for all $t \in I$. Possibly choosing a smaller $\delta > 0$ (this does not affect M), we can assume that $\delta M \leq \varepsilon$. With such a choice, it is $Ty \in X$ for all $y \in X$. The choice of $\delta > 0$ is independent from x_0 and y_0 as long as $K \subset \text{int}(H)$.

We prove that the mapping $T : X \rightarrow X$ has a unique fixed point. It is enough to show that, for some $k \in \mathbb{N}$, the iterated mapping T^k is a contraction. Let $y, \bar{y} \in X$ and $x \in I$. We have (with e.g. $x \geq x_0$)

$$\begin{aligned} |Ty(x) - T\bar{y}(x)| &= \left| \int_{x_0}^x (f(t, y(t)) - f(t, \bar{y}(t))) dt \right| \\ &\leq \int_{x_0}^x |f(t, y(t)) - f(t, \bar{y}(t))| dt \\ &\leq L \int_{x_0}^x |y(t) - \bar{y}(t)| dt \leq L|x - x_0| \cdot \|y - \bar{y}\|. \end{aligned}$$

Here, L is the Lipschitz constant for f relative to the compact set H . Analogously, it is (e.g. $x \geq x_0$)

$$\begin{aligned} |T^2y(x) - T^2\bar{y}(x)| &= \left| \int_{x_0}^x (f(t, Ty(t)) - f(t, T\bar{y}(t))) dt \right| \\ &\leq L \int_{x_0}^x |Ty(t) - T\bar{y}(t)| dt \\ &\leq L^2 \|y - \bar{y}\| \int_{x_0}^x (t - x_0) dt \leq L^2 \frac{(x - x_0)^2}{2} \|y - \bar{y}\|. \end{aligned}$$

By induction, we get for any $k \in \mathbb{N}$ and $x \in I$

$$|T^k y(x) - T^k \bar{y}(x)| \leq \frac{L^k |x - x_0|^k}{k!} \|y - \bar{y}\|,$$

which implies

$$\|T^k y - T^k \bar{y}\| \leq \frac{(L\delta)^k}{k!} \|y - \bar{y}\|.$$

Because

$$\lim_{k \rightarrow +\infty} \frac{(L\delta)^k}{k!} = 0,$$

there exists $k \in \mathbb{N}$ such that $\frac{(L\delta)^k}{k!} < 1$. For such a k , the mapping T^k is a contraction. Then T has a unique fixed point $y \in X$. Then it is $y \in C^1([x_0 - \delta, x_0 + \delta]; \mathbb{R}^n)$ and y solves the Cauchy Problem (2.3.35). \square

5. Peano's example

Consider the Cauchy problem

$$\begin{cases} y'(x) = 2\sqrt{|y(x)|}, & x \in \mathbb{R}, \\ y(0) = 0 \end{cases} \quad (2.5.42)$$

The function $f(x, y) = 2\sqrt{|y|}$ is not locally Lipschitz in the variable y . The Lipschitz property (2.4.41) fails in a neighborhood of $y = 0$ and the assumptions of Theorem 2.4.2 are not fulfilled. The Cauchy Problem could have more than one solution.

In fact, a solution is the constant function $y = 0$. A second solution can be found separating the variables: $2 = y'/\sqrt{|y|}$. Integrating this equation on the interval between 0 and $x \in \mathbb{R}$ we get

$$2x = \int_0^x \frac{y'(t)}{\sqrt{|y(t)|}} dt = \int_0^{y(x)} \frac{1}{\sqrt{|z|}} dz = \begin{cases} 2\sqrt{y(x)}, & \text{if } y(x) > 0 \\ -2\sqrt{-y(x)}, & \text{if } y(x) < 0. \end{cases}$$

In the change of variable $z = y(t)$ we used the initial datum $y(0) = 0$. Then we find the solution $y \in C^1(\mathbb{R})$

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

On the other hand, for all real numbers $\alpha \leq 0 \leq \beta$, the function

$$y_{\alpha\beta}(x) = \begin{cases} (x - \beta)^2 & \text{if } x \geq \beta, \\ 0 & \text{if } \alpha < x < \beta, \\ -(x - \alpha)^2 & \text{if } x \leq \alpha \end{cases}$$

is of class $C^1(\mathbb{R})$ and solves the Cauchy Problem (2.5.42). So there is a continuum of solutions.

6. Maximal solutions

Let $f \in C(\Omega; \mathbb{R}^n)$ be a function satisfying the local Lipschitz condition (2.4.41) and let $(x_0, y_0) \in \Omega$.

PROPOSITION 2.6.1. *Under the hypotheses of Theorem 2.4.2, let I_1 and I_2 be two open intervals containing x_0 and assume that $y_1 \in C^1(I_1; \mathbb{R}^n)$ and $y_2 \in C^1(I_2; \mathbb{R}^n)$ are solutions to the Cauchy Problem (2.3.35). Then it is $y_1 = y_2$ on $I_1 \cap I_2$.*

PROOF. The set $A = \{x \in I_1 \cap I_2 : y_1(x) = y_2(x)\}$ is relatively closed in $I_1 \cap I_2$ because y_1 and y_2 are continuous. We show that A is also open in $I_1 \cap I_2$. Since $I_1 \cap I_2$ is connected it then follows that $A = I_1 \cap I_2$.

Let $\bar{x}_0 \in A$ and $\bar{y}_0 = y_1(\bar{x}_0) = y_2(\bar{x}_0)$. By Theorem 2.4.2 there exists $\delta > 0$ such that the Cauchy Problem

$$\begin{cases} y' = f(x, y) \\ y(\bar{x}_0) = \bar{y}_0 \end{cases} \quad (2.6.43)$$

has a unique solution $y \in C^1(I; \mathbb{R}^n)$ with $I = [\bar{x}_0 - \delta, \bar{x}_0 + \delta]$. For a small $\delta > 0$ it is $I \subset I_1 \cap I_2$. It then follows that $y = y_1 = y_2$ in I , and thus $I \subset A$. \square

Consider the family \mathcal{A} of all pairs (J, y_J) where $J \subset \mathbb{R}$ is an open interval containing x_0 and $y_J \in C^1(J; \mathbb{R}^n)$ is a solution to the Cauchy Problem (2.3.35). By Theorem 2.4.2, it is $\mathcal{A} \neq \emptyset$.

Let $I \subset \mathbb{R}$ be the interval $I = \bigcup J$, where the union is over all intervals J such that $(J, y_J) \in \mathcal{A}$. Let $y \in C^1(I; \mathbb{R}^n)$ be the function defined by

$$y(x) = y_J(x) \quad \text{if } x \in J. \quad (2.6.44)$$

The function y is well defined because by Proposition 2.6.1 it is $y_J = y_{J'}$ on $J \cap J'$. Moreover, y is a solution to the Cauchy Problem (2.3.35).

DEFINITION 2.6.2 (Maximal solution). The function y defined in (2.6.44) is called maximal solution to the Cauchy Problem (2.3.35).

THEOREM 2.6.3 (Continuation criterion). *Let $I = (a_0, b_0) \subset \mathbb{R}$ be an open interval with $-\infty < a_0 < b_0 < +\infty$, $\Omega = I \times \mathbb{R}^n$, and $f \in C(\Omega; \mathbb{R}^n)$ be a function satisfying the local Lipschitz property in y . If $y \in C^1((a, b); \mathbb{R}^n)$ is the maximal solution to the Cauchy Problem (2.3.35), for some interval $(a, b) \subset (a_0, b_0)$, then we have either*

- i) $b = b_0$; or,
- ii) $\lim_{x \uparrow b} |y(x)| = +\infty$.

There is an analogous statement for a .

PROOF. Assume by contradiction that $b < b_0$ and there exists a sequence $x_k \in (a, b)$, $k \in \mathbb{N}$, such that

$$\lim_{k \rightarrow +\infty} x_k = b \quad \text{and} \quad \sup_{k \in \mathbb{N}} |y(x_k)| \leq M_0,$$

for some constant $M_0 < +\infty$. Letting $\bar{y}_k = y(x_k) \in \mathbb{R}^n$, possibly taking a subsequence we can assume that

$$\lim_{k \rightarrow +\infty} \bar{y}_k = \bar{y}_0$$

for some $\bar{y}_0 \in \mathbb{R}^n$.

We study the Cauchy Problem

$$\begin{cases} z'(x) = f(x, z(x)) \\ z(x_k) = \bar{y}_k. \end{cases} \quad (2.6.45)$$

Fix a compact set $H \subset \Omega$ such that $(b, \bar{y}_0) \in \text{int}(H)$ and let

$$M = \max_{(x,y) \in H} |f(x,y)| < +\infty.$$

For some $\varepsilon > 0$ and for some large enough $k \in \mathbb{N}$, the compact set

$$K = [x_k, 2b - x_k] \times \{y \in \mathbb{R}^n : |y - \bar{y}_k| \leq \varepsilon\}$$

is contained in H . Let us introduce the functional space

$$X = \{z \in C([x_k, 2b - x_k]; \mathbb{R}^n) : z(x_k) = \bar{y}_k, \|z - \bar{y}_k\| \leq \varepsilon\}.$$

For large enough $k \in \mathbb{N}$ we also have $2(b - x_k)M \leq \varepsilon$. Then, the integral operator

$$Tz(x) = \bar{y}_k + \int_{x_k}^x f(t, z(t)) dt$$

maps X into itself, i.e., $T : X \rightarrow X$.

As in the proof of Theorem 2.4.2, some iterated of T is a contraction on X and therefore by Theorem 2.1.3 there exists a unique solution $z \in C^1([x_k, 2b - x_k]; \mathbb{R}^n)$ to the Cauchy Problem (2.6.45).

On the other hand, the function y solves the same Cauchy Problem on the interval $[x_k, b)$ and by uniqueness it is $y = z$ on $[x_k, b)$. This shows that y can be continued as a solution to the Cauchy Problem (2.3.35) beyond b . This contradicts the maximality of y . \square

7. Gronwall's Lemma

LEMMA 2.7.1. *Let $I \subset \mathbb{R}$ be an interval, $x_0 \in I$, and $\varphi \in C(I)$ be a non negative $\varphi \geq 0$ continuous function. If there exist $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$, such that*

$$\varphi(x) \leq \alpha + \beta \int_{x_0}^x \varphi(t) dt, \quad \text{for all } x \in I \text{ with } x \geq x_0, \quad (2.7.46)$$

then

$$\varphi(x) \leq \alpha e^{\beta(x-x_0)} \quad \text{for all } x \in I \text{ with } x \geq x_0. \quad (2.7.47)$$

PROOF. Let $\Phi : I \rightarrow \mathbb{R}$ be the function

$$\Phi(x) = \alpha + \beta \int_{x_0}^x \varphi(t) dt.$$

It is $\Phi \in C^1(I)$ and moreover, $\Phi'(x) = \beta\varphi(x)$ for all $x \in I$, by the Fundamental Theorem of Calculus. From (2.7.46) it follows that $\Phi'(x) \leq \beta\Phi(x)$ for $x \in I$, because $\beta \geq 0$. The function $\Psi(x) = e^{-\beta(x-x_0)}\Phi(x)$ satisfies

$$\Psi'(x) = -\beta e^{-\beta(x-x_0)}\Phi(x) + e^{-\beta(x-x_0)}\Phi'(x) = e^{-\beta(x-x_0)}(-\beta\Phi(x) + \Phi'(x)) \leq 0$$

and $\Psi(x_0) = \Phi(x_0) = \alpha$. It follows that $\Psi(x) \leq \alpha$ for $x \geq x_0$, i.e.,

$$\Phi(x) \leq \alpha e^{\beta(x-x_0)}$$

for all $x \in I$ with $x \geq x_0$. This implies (2.7.47), because $\varphi(x) \leq \Phi(x)$, by (2.7.46). \square

8. Existence in the large

THEOREM 2.8.1 (Global solutions). *Let $I = (a_0, b_0)$ with $-\infty \leq a_0 < b_0 \leq +\infty$, $\Omega = I \times \mathbb{R}^n$, and $f \in C(\Omega; \mathbb{R}^n)$ be a continuous function satisfying the local Lipschitz assumption (2.4.41). Assume that for any compact set $K \subset I$ there exists a constant $C \geq 0$ such that*

$$|f(x, y)| \leq C(1 + |y|), \quad \text{for all } x \in K \text{ and } y \in \mathbb{R}^n. \quad (2.8.48)$$

Then the Cauchy Problem (2.3.35), with $x_0 \in I$ and $y_0 \in \mathbb{R}^n$, has a (unique) global solution defined on I .

PROOF. Let $y \in C^1(J; \mathbb{R}^n)$ be the maximal solution to the Cauchy Problem (2.3.35), with $J = (a, b) \subset I$. Assume by contradiction that $b < b_0$. By Theorem 2.6.3 it is

$$\lim_{x \uparrow b} |y(x)| = +\infty. \quad (2.8.49)$$

Let $K = [x_0, b]$ and $C > 0$ such that (2.8.48) holds. From

$$y(x) = y_0 + \int_{x_0}^x f(t, y(t)) dt, \quad x \in J,$$

we get for $x \in J$ with $x \geq x_0$

$$|y(x)| \leq |y_0| + C \int_{x_0}^x (1 + |y(t)|) dt \leq |y_0| + C(b - x_0) + C \int_{x_0}^x |y(t)| dt.$$

By the Gronwall's Lemma it follows that

$$|y(x)| \leq \{|y_0| + C(b - x_0)\} e^{C(x-x_0)}, \quad x \in (x_0, b),$$

and therefore (2.8.49) cannot hold. □

CHAPTER 3

Linear systems

1. Introduction

Denote by $M_n(\mathbb{R})$, $n \geq 1$, the vector space of $n \times n$ matrices with real entries. If $A = (a_{ij})_{i,j=1,\dots,n} \in M_n(\mathbb{R})$ and $y \in \mathbb{R}^n$, we denote by Ay the standard matrix-vector product where y is thought of as a column vector, and precisely

$$Ay = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1j}y_j \\ \vdots \\ \sum_{j=1}^n a_{nj}y_j \end{pmatrix}.$$

The matrix norm of A is

$$\|A\| = \max_{|y|=1} |Ay|.$$

The matrix norm has the following properties:

- i) $|Ay| \leq \|A\||y|$ for all $y \in \mathbb{R}^n$;
- ii) $\|A + B\| \leq \|A\| + \|B\|$ for all $A, B \in M_n(\mathbb{R})$;
- iii) $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in M_n(\mathbb{R})$.

Let $I = (a, b) \subset \mathbb{R}$ be an interval. A function $A : I \rightarrow M_n(\mathbb{R})$ is continuous if $A(x) = (a_{ij}(x))_{i,j=1,\dots,n}$ for $x \in I$ and $a_{ij} \in C(I)$ for all $i, j = 1, \dots, n$.

Let $A : I \rightarrow M_n(\mathbb{R})$ be continuous and let $b : I \rightarrow \mathbb{R}^n$ be a continuous mapping. A system of differential equations of the form

$$y' = A(x)y + b(x) \tag{3.1.50}$$

is called linear. The function $f : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$f(x, y) = A(x)y + b(x).$$

has the following properties:

- 1) $f \in C(I \times \mathbb{R}^n; \mathbb{R}^n)$;
- 2) f has the local Lipschitz property in y ;
- 3) for any compact set $K \subset I$ there is a constant $C > 0$ such that

$$|f(x, y)| \leq C(1 + |y|), \quad \text{for all } x \in K, y \in \mathbb{R}^n;$$

In fact, for any compact set $K \subset I$ it is $L = \max_{x \in K} \|A(x)\| < +\infty$ and thus

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= |A(x)y_1 - A(x)y_2| = |A(x)(y_1 - y_2)| \\ &\leq \|A(x)\||y_1 - y_2| \leq L|y_1 - y_2| \end{aligned}$$

for all $x \in K$ and $y_1, y_2 \in \mathbb{R}^n$. This shows 2). Moreover, let $M = \max_{x \in K} |b(x)|$ and $C = \max\{L, M\}$. Then we have

$$|f(x, y)| \leq |A(x)y| + |b(x)| \leq C(|y| + 1), \quad x \in K, y \in \mathbb{R}^n.$$

By Theorem 2.4.2, the Cauchy problem

$$\begin{cases} y' = A(x)y + b(x) \\ y(x_0) = y_0 \end{cases} \quad (3.1.51)$$

has a unique local solution, for any $x_0 \in I$ and $y_0 \in \mathbb{R}^n$. On the other hand, by Theorem 2.8.1 the maximal solution of the Cauchy Problem (3.1.51) is defined on the whole interval I . In the following, by solution of the differential equation (3.1.50) we mean a maximal solution.

2. Homogeneous equations

A differential equation of the form (3.1.50) with $b = 0$ is called homogeneous.

THEOREM 3.2.1. *Let $A : I \rightarrow M_n(\mathbb{R})$ be continuous. The set of solutions of the differential equation*

$$y' = A(x)y, \quad x \in I, \quad (3.2.52)$$

is a real vector space of dimension $n \in \mathbb{N}$.

PROOF. Let $S = \{y \in C^1(I; \mathbb{R}^n) : y \text{ is a solution of (3.2.52)}\}$ be the set of solutions. If $y, z \in S$, then $\alpha y + \beta z \in C^1(I; \mathbb{R}^n)$ is also a solution, for any $\alpha, \beta \in \mathbb{R}$:

$$(\alpha y + \beta z)' = \alpha y' + \beta z' = \alpha A(x)y + \beta A(x)z = A(x)(\alpha y + \beta z), \quad x \in I.$$

Then S is a linear subspace of $C^1(I; \mathbb{R}^n)$.

We show that the dimension of S is n . For some fixed $x_0 \in I$, define the mapping $T : S \rightarrow \mathbb{R}^n$

$$T(y) = y(x_0). \quad (3.2.53)$$

T is linear: $T(\alpha y + \beta z) = \alpha y(x_0) + \beta z(x_0) = \alpha T(y) + \beta T(z)$. T is injective, i.e., $T(y) = 0$ implies $y = 0$. In fact, y solves equation (3.2.52) with initial condition $y(x_0) = 0$. The solution to this problem is unique and 0 is a solution. Then it is $y = 0$. Finally, T is surjective because for any $y_0 \in \mathbb{R}^n$ the differential equation (3.2.52) with initial datum $y(x_0) = y_0$ has a solution $y \in C^1(I; \mathbb{R}^n)$. \square

PROPOSITION 3.2.2. *Let $S \subset C^1(I; \mathbb{R}^n)$ be the space of solutions to (3.2.52) and let $y_1, \dots, y_n \in S$. The following are equivalent:*

- i) y_1, \dots, y_n are a basis of S ;
- ii) $\det[y_1(x_0), \dots, y_n(x_0)] \neq 0$ for all $x_0 \in I$;
- iii) $\det[y_1(x_0), \dots, y_n(x_0)] \neq 0$ for some $x_0 \in I$.

By $[y_1, \dots, y_n]$ we mean the $n \times n$ matrix with columns $y_1, \dots, y_n \in \mathbb{R}^n$.

DEFINITION 3.2.3 (Fundamental matrix). If one of the three equivalent conditions of Proposition 3.2.2 holds, then the functions y_1, \dots, y_n are called a fundamental system of solutions of the differential equation $y' = Ay$. The matrix $Y = [y_1, \dots, y_n]$ is then called a fundamental matrix for the equation.

PROOF OF PROPOSITION 3.2.2. i) \Rightarrow ii) Let $x_0 \in I$, and let $T : S \rightarrow \mathbb{R}^n$ be the isomorphism defined in (3.2.53). Then $y_1(x_0) = Ty_1, \dots, y_n(x_0) = Ty_n$ form a basis for \mathbb{R}^n . This is equivalent with ii).

iii) \Rightarrow i) Let $x_0 \in I$ be such that iii) holds and let $T : S \rightarrow \mathbb{R}^n$ be the isomorphism (3.2.53) relative to x_0 . Then $T^{-1} : \mathbb{R}^n \rightarrow S$ is also an isomorphism. It follows that $y_1 = T^{-1}(y_1(x_0)), \dots, y_n = T^{-1}(y_n(x_0))$ is a basis of S . \square

DEFINITION 3.2.4 (Wronski determinant). Let $y_1, \dots, y_n \in S$ be solutions to the differential equations (3.2.52). The function $w \in C^1(I; \mathbb{R}^n)$

$$w(x) = \det[y_1(x), \dots, y_n(x)], \quad x \in I, \quad (3.2.54)$$

is called Wronski determinant of y_1, \dots, y_n .

THEOREM 3.2.5. The Wronski determinant w of $y_1, \dots, y_n \in S$ solves the differential equation

$$w' = \operatorname{tr}A(x)w, \quad x \in I, \quad (3.2.55)$$

where $\operatorname{tr}A(x) = \sum_{i=1}^n a_{ii}(x)$ is the trace of the matrix $A(x) = (a_{ij}(x))_{i,j=1,\dots,n}$.

PROOF. If y_1, \dots, y_n are linearly dependent then $w(x) = 0$ for all $x \in I$ and equation (3.2.55) trivially holds. Assume that y_1, \dots, y_n are linearly independent, i.e., $w(x) \neq 0$ for all $x \in I$. Denote by $Y : I \rightarrow M_n(\mathbb{R})$ the fundamental matrix having as columns the solutions y_1, \dots, y_n . Letting $y_j = (y_{1j}, \dots, y_{nj})^T$, $j = 1, \dots, n$, we have

$$Y(x) = (y_{ij}(x))_{i,j=1,\dots,n}, \quad x \in I.$$

We check equation (3.2.55) at the point $x_0 \in I$, i.e., we show that $w'(x_0) = \operatorname{tr}A(x_0)w(x_0)$. To this aim, let $z_j \in C^1(I; \mathbb{R}^n)$ be the solution to the Cauchy problem

$$\begin{cases} z' = A(x)z \\ z(x_0) = e_j, \end{cases} \quad (3.2.56)$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the j th position. The functions z_1, \dots, z_n are a basis for the space of solutions to the differential equation $z' = Az$. Letting, as above,

$$Z(x) = (z_{ij}(x))_{i,j=1,\dots,n}, \quad x \in I,$$

there exists an invertible matrix $C \in GL_n(\mathbb{R})$ such that

$$Y(x) = CZ(x), \quad x \in I.$$

We show that the function $v(x) = \det Z(x)$ solves $v'(x_0) = \operatorname{tr}A(x_0)v(x_0)$. In fact, we have

$$v'(x) = \frac{d}{dx} \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}\sigma} \prod_{i=1}^n z_{i\sigma(i)}(x) = \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}\sigma} \sum_{j=1}^n z'_{j\sigma(j)}(x) \prod_{i \neq j} z_{i\sigma(i)}(x),$$

where

$$\prod_{i \neq j} z_{i\sigma(i)}(x_0) = 0 \quad \text{unless } \sigma = \text{Id},$$

and

$$z'_{jj}(x_0) = (A(x_0)z_j(x_0))_j = \sum_{k=1}^n a_{jk}(x_0)z_{kj}(x_0) = \sum_{k=1}^n a_{jk}(x_0)\delta_{kj}(x_0) = a_{jj}(x_0).$$

Then it is $v'(x_0) = \text{tr}A(x_0)$. Now the general result follows on differentiating the identity

$$w = \det Y = \det(CZ) = \det C \det Z = \det Cv.$$

In fact,

$$w'(x_0) = \det Cv'(x_0) = \det C \text{tr}A(x_0) = \text{tr}A(x_0)w(x_0),$$

because $v(x_0) = 1$. □

3. Inhomogeneous equations

Consider an inhomogeneous linear differential equation of the form

$$y' = A(x)y + b(x), \tag{3.3.57}$$

with $A \in C(I; M_n(\mathbb{R}))$ and $b \in C(I; \mathbb{R}^n)$ for some open interval $I \subset \mathbb{R}$.

Let Y be a fundamental matrix for the homogeneous equation $y' = A(x)y$, i.e., $Y' = AY$ and $\det Y \neq 0$ on I . Then, any solution y to this equation is of the form

$$y(x) = Y(x)c, \quad x \in I, \tag{3.3.58}$$

for some (column) vector $c \in \mathbb{R}^n$. We look for a solution to (3.3.57) of the form (3.3.58) with $c \in C^1(I; \mathbb{R}^n)$. This method is called “variation of constants”. In this case,

$$y' = Y'c + Yc' = AYc + Yc' = Ay + Yc'.$$

Plugging this identity into (3.3.57), we get $Yc' = b$. Being Y invertible, by an integration over an interval $[x_0, x]$ we find

$$c(x) = c_0 + \int_{x_0}^x Y(t)^{-1}b(t)dt,$$

for some $c_0 \in \mathbb{R}^n$. Thus we find the solution

$$y(x) = Y(x) \left(c_0 + \int_{x_0}^x Y(t)^{-1}b(t)dt \right). \tag{3.3.59}$$

THEOREM 3.3.1. *Let Y be a fundamental matrix for the homogeneous equation $y' = Ay$. For any $c_0 \in \mathbb{R}^n$ the function y in (3.3.59) is a solution to (3.3.57). Moreover, any solution to (3.3.57) is of the form (3.3.59) for some $c_0 \in \mathbb{R}^n$.*

PROOF. The first statement is an easy computation. Let y be the function (3.3.59) and let $z \in C^1(I; \mathbb{R}^n)$ be a solution to (3.3.57). Then

$$(z - y)' = z' - y' = Az + b - (Ay + b) = A(z - y).$$

It follows that $z - y = Yc_1$ for some $c_1 \in \mathbb{R}^n$ and the claim follows. \square

4. Exponential of a matrix

For a matrix $A \in M_n(\mathbb{C})$ define the exponential matrix $e^A \in M_n(\mathbb{C})$ on letting

$$e^A = \sum_{k=0}^{+\infty} \frac{A^k}{k!}.$$

In order to prove that the series converges, we show that the sequence of matrices $(B_k)_{k \in \mathbb{N}} \subset M_n(\mathbb{C})$

$$B_k = \sum_{h=0}^k \frac{A^h}{h!}, \quad k \in \mathbb{N},$$

is a Cauchy sequence in the norm $\|\cdot\|$. In fact, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$ and for all $p \in \mathbb{N}$ we have

$$\|B_{k+p} - B_k\| = \left\| \sum_{h=k+1}^{k+p} \frac{A^h}{h!} \right\| \leq \sum_{h=k+1}^{k+p} \frac{\|A\|^h}{h!} \leq \varepsilon.$$

Notice that the normed space $(M_n(\mathbb{C}), \|\cdot\|)$ is complete.

We list some properties of the exponential matrix.

4.1. Exponential of the sum. If $A, B \in M_n(\mathbb{C})$ and $AB = BA$, then

$$e^{A+B} = e^A e^B. \quad (3.4.60)$$

The proof of this fact is left as an exercise.

4.2. Diagonal matrix. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. The exponential matrix of a diagonal matrix $A \in M_n(\mathbb{C})$ of the form

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{is} \quad e^A = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}.$$

This follows directly from the formula for the exponential.

4.3. Block matrix. Let $A_j \in M_{k_j}(\mathbb{C})$ for $j = 1, \dots, p$, with $k_1 + \dots + k_p = n$. The exponential matrix of a block matrix $A \in M_n(\mathbb{C})$ of the form

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix} \quad \text{is} \quad e^A = \begin{pmatrix} e^{A_1} & & 0 \\ & \ddots & \\ 0 & & e^{A_p} \end{pmatrix}.$$

This also follows directly from the formula for the exponential.

4.4. Fundamental Jordan block. Consider a matrix $A \in M_n(\mathbb{C})$ of the form

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}, \quad A = \lambda I_n + J, \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix},$$

and $\lambda \in \mathbb{C}$. The matrix A is called fundamental Jordan block of order n relative to $\lambda \in \mathbb{C}$. Later, we shall use the notation $A = J_n(\lambda)$.

We show that for any $t \in \mathbb{R}$ we have

$$e^{tA} = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2! & \dots & t^{n-1}/(n-1)! \\ & 1 & t & & \vdots \\ & & 1 & \ddots & t^2/2! \\ & & & \ddots & t \\ 0 & & & & 1 \end{pmatrix}. \quad (3.4.61)$$

The matrix $J^k = J \dots J$ k -times, $k = 0, 1, \dots, n-1$, has 1 on the $(k+1)$ -th left-right downwards diagonal and 0 otherwise. Moreover, it is $J^k = 0$ for $k \geq n$. Then we have

$$\begin{aligned} \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!} &= \sum_{k=0}^{+\infty} \frac{t^k}{k!} \sum_{h=0}^k \binom{k}{h} \lambda^h J^{k-h} \\ &= \sum_{h=0}^{+\infty} \lambda^h \sum_{k=h}^{h+n-1} \frac{t^k}{k!} \binom{k}{h} J^{k-h} \\ &= \sum_{h=0}^{+\infty} \frac{\lambda^h t^h}{h!} \sum_{p=0}^{n-1} \frac{t^p}{p!} J^p. \end{aligned}$$

4.5. Conjugation and exponentiation. Let $A, B \in M_n(\mathbb{C})$ and $C \in GL_n(\mathbb{C})$ be matrices such that $A = CBC^{-1}$. Then we have

$$e^A = Ce^B C^{-1}.$$

In fact

$$e^A = \sum_{k=0}^{+\infty} \frac{A^k}{k!} = \sum_{k=0}^{+\infty} \frac{(CBC^{-1})^k}{k!} = \sum_{k=0}^{+\infty} \frac{CB^k C^{-1}}{k!} = Ce^B C^{-1}.$$

5. Linear systems with constant coefficients

Let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix and consider the differential equation

$$y' = Ay, \quad x \in \mathbb{R}. \quad (3.5.62)$$

This is a linear, homogeneous system of differential equations with constant coefficients. The solutions are defined on \mathbb{R} and the set of solutions is a real vector space

of dimension n . For some $x_0 \in \mathbb{R}$, fix the initial data $y(x_0) = y_0 \in \mathbb{R}^n$. The solution to the differential equation with this initial data is a fixed point of the mapping $T : X \rightarrow X$

$$Ty(x) = y_0 + \int_{x_0}^x Ay(t) dt = y_0 + A \int_{x_0}^x y(t) dt, \quad x \in \mathbb{R}, \quad (3.5.63)$$

where $X = \{y \in C(\mathbb{R}; \mathbb{R}^n) : y(x_0) = y_0\}$. We can interchange integral and A , because A has constant coefficients.

The fixed point is unique and can be obtained as the limit of $T^k y$ for $k \rightarrow +\infty$, for any $y \in X$. In particular, we can choose the constant function $y = y_0$. In this case we have

$$Ty(x) = y_0 + (x - x_0)Ay_0,$$

and, in general, we find for any $k \in \mathbb{N}$

$$T^k y(x) = \sum_{h=0}^k \frac{(x - x_0)^h}{h!} A^h y_0.$$

This formula can be checked by induction. It holds for $k = 0, 1$, with the convention $A^0 = I_n$, the identity matrix. Assume it holds for k . Then we have

$$\begin{aligned} T^{k+1}y(x) &= T(T^k y)(x) = y_0 + A \int_{x_0}^x T^k y(t) dt \\ &= y_0 + A \sum_{h=0}^k A^h y_0 \int_{x_0}^x \frac{(t - x_0)^h}{h!} dt \\ &= y_0 + \sum_{h=0}^k \frac{(x - x_0)^{h+1}}{(h+1)!} A^{h+1} y_0 = \sum_{h=0}^{k+1} \frac{(x - x_0)^h}{h!} A^h y_0. \end{aligned}$$

For any compact set $K \subset \mathbb{R}$, the sequence of matrices

$$B_k(x) = \sum_{h=0}^k \frac{(x - x_0)^h}{h!} A^h, \quad k \in \mathbb{N},$$

converges uniformly for $x \in K$. From the theory of power series, it follows that the function $\varphi : \mathbb{R} \rightarrow M_n(\mathbb{R})$

$$\varphi(x) = e^{(x-x_0)A} = \sum_{h=0}^{+\infty} \frac{(x - x_0)^h}{h!} A^h$$

is of class C^∞ , and in fact it is analytic.

PROPOSITION 3.5.1. *Let $A \in M_n(\mathbb{R})$. For any $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^n$, the function $y \in C^\infty(\mathbb{R}; \mathbb{R}^n)$*

$$y(x) = e^{(x-x_0)A} y_0$$

is the unique solution to the Cauchy Problem $y' = Ay$ and $y(x_0) = y_0$.

PROOF. The function y is the unique fixed point of the mapping T in (3.5.63). Alternatively, the function y can be differentiated term by term, because the series of the derivatives converges uniformly on compact sets. Then we find

$$y'(x) = \sum_{h=1}^{+\infty} \frac{(x-x_0)^{h-1}}{(h-1)!} A^h y_0 = A \sum_{h=1}^{+\infty} \frac{(x-x_0)^{h-1}}{(h-1)!} A^{h-1} y_0 = Ay(x), \quad x \in \mathbb{R}.$$

Moreover, $y(x_0) = y_0$. □

DEFINITION 3.5.2 (Jordan block). A matrix $A \in M_n(\mathbb{C})$ of the form

$$A = \begin{pmatrix} J_{k_1}(\lambda) & & 0 \\ & \ddots & \\ 0 & & J_{k_p}(\lambda) \end{pmatrix},$$

where $\lambda \in \mathbb{C}$, $k_1 + \dots + k_p = n$, and $J_{k_1}(\lambda), \dots, J_{k_p}(\lambda)$ are fundamental Jordan blocks, is called Jordan block of orders k_1, \dots, k_p relative to $\lambda \in \mathbb{C}$. We denote $A = J_{k_1 \dots k_p}(\lambda)$.

The exponential of a Jordan block can be computed using the rules of Section 4.

By known results from Linear Algebra, for any matrix $A \in M_n(\mathbb{R})$ with complex eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ there exists a matrix $C \in GL_n(\mathbb{C})$ such that $A = CBC^{-1}$, where B is the Jordan normal form of A , i.e.,

$$B = \begin{pmatrix} J_{k_1^1 \dots k_{p_1}^1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{k_1^m \dots k_{p_m}^m}(\lambda_m) \end{pmatrix}, \quad (3.5.64)$$

with $k_1^1 + \dots + k_{p_1}^1 + \dots + k_1^m + \dots + k_{p_m}^m = n$ and $J_{k_1^1 \dots k_{p_1}^1}(\lambda_1), \dots, J_{k_1^m \dots k_{p_m}^m}(\lambda_m)$ are Jordan blocks relative to the eigenvalues $\lambda_1, \dots, \lambda_m$.

PROPOSITION 3.5.3. Let $A \in M_n(\mathbb{R})$, $A = CBC^{-1}$ where $C \in GL_n(\mathbb{C})$ and B is as in (3.5.64). A fundamental system of solutions of the homogeneous linear equation $y' = Ay$ is given by the columns of the (real) matrix

$$e^{xA} = C \begin{pmatrix} e^{xJ_{k_1^1 \dots k_{p_1}^1}(\lambda_1)} & & 0 \\ & \ddots & \\ 0 & & e^{xJ_{k_1^m \dots k_{p_m}^m}(\lambda_m)} \end{pmatrix} C^{-1}, \quad x \in \mathbb{R}.$$

PROOF. This follows from Proposition 3.5.1 and by the computation rules of Section 4. □

6. Higher order linear equations

Let $f, a_k \in C(I)$, $k = 0, 1, \dots, n-1$, be continuous functions in some interval $I \subset \mathbb{R}$. We transform the linear n -th order differential equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = f(x), \quad x \in I, \quad (3.6.65)$$

into a linear system. Consider the mapping $T : C^n(I) \rightarrow C^1(I; \mathbb{R}^n)$

$$Ty = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}.$$

The vector of functions $z = Ty$ satisfies the system of equations

$$\begin{cases} z'_i = z_{i+1}, & i = 1, \dots, n-1, \\ z'_n = f(x) - \sum_{k=0}^{n-1} a_k(x) z_{k+1}, \end{cases}$$

which can be written in the following way

$$z' = Az + F, \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} & \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix}. \quad (3.6.66)$$

PROPOSITION 3.6.1. *Let $y \in C^n(I)$ and $z = Ty \in C^1(I; \mathbb{R}^n)$. Then y solves equation (3.6.65) if and only if z solves system (3.6.66). Moreover, the set of solutions $y \in C^n(I)$ of equation (3.6.65) with $f = 0$ is a real vector space of dimension n .*

The proof of this proposition is straightforward.

7. Higher order linear equations with constant coefficients

We solve the differential equation (3.6.65) in the homogeneous case $f = 0$ and with constant coefficients $a_0, a_1, \dots, a_k \in \mathbb{R}$. Equivalently, we solve the linear system

$$z' = Az, \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} & \end{pmatrix} \in M_n(\mathbb{R}). \quad (3.7.67)$$

We establish some algebraic properties of the matrix A . The characteristic polynomial in the variable $\lambda \in \mathbb{C}$ of the matrix A is

$$p(\lambda) = \det(A - \lambda I) = (-1)^n \sum_{k=0}^n a_k \lambda^k,$$

with $a_n = 1$. In fact, we can develop the determinant in the last row:

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} -\lambda & 1 & & & \\ & -\lambda & 1 & & \\ & & -\lambda & \ddots & \\ & & & \ddots & 1 \\ -a_0 & -a_1 & & \dots & -a_{n-1} - \lambda \end{pmatrix} \\ &= -(-1)^{n+1}a_0 - (-1)^{n+2}a_1(-\lambda)^1 - \dots - (-1)^{2n}(a_{n-1} + \lambda)(-\lambda)^{n-1} \\ &= (-1)^n(a_0 + a_1\lambda + \dots + (a_{n-1} + \lambda)\lambda^{n-1}). \end{aligned}$$

The geometric multiplicity (i.e., the dimension of the eigenspace) of any eigenvalue $\lambda \in \mathbb{C}$ of A is 1 and a corresponding eigenvector is

$$v_\lambda = \begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{pmatrix}. \quad (3.7.68)$$

Indeed, let $v = (v^1, \dots, v^n)^t$ be an eigenvector of A , $Av = \lambda v$. If $v^1 = 0$ then it follows that $v = 0$. We can then assume $v^1 = 1$ and from $v^{i+1} = \lambda v^i$ we deduce that $v = v_\lambda$ as in (3.7.68).

A Jordan chain of vectors v_0, v_1, \dots, v_{r-1} relative to the eigenvector v with eigenvalue λ of algebraic multiplicity $r \geq 1$ is defined through the recursive relations $v_0 = v$ and $(A - \lambda)v_{i+1} = v_i$, $i = 0, 1, \dots, r-2$. Jordan chains will be used to transform A into its Jordan normal form.

In our case, a Jordan chain relative to the eigenvalue $\lambda \in \mathbb{C}$ of algebraic multiplicity r_λ is given by the vectors

$$v_{\lambda,i} = \frac{1}{i!} D_\lambda^i v_\lambda, \quad i = 0, 1, \dots, r_\lambda - 1,$$

where D_λ^i is the i -th derivative operator w.r.t. λ . Explicitly, we have

$$v_{\lambda,0} = \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{pmatrix}, \quad v_{\lambda,1} = \begin{pmatrix} 0 \\ 1 \\ 2\lambda \\ \vdots \\ (n-1)\lambda^{n-2} \end{pmatrix}, \quad v_{\lambda,2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ \frac{(n-1)(n-2)}{2}\lambda^{n-3} \end{pmatrix}, \quad \text{etc.} \quad (3.7.69)$$

We check that $(A - \lambda)v_{\lambda,i+1} = v_{\lambda,i}$ for all $i = 0, 1, \dots, r_\lambda - 1$. Let us introduce the following notation:

$$v_{\lambda,i} = (v_i^1, \dots, v_i^n)^t, \quad v_i^j = \frac{1}{i!} D^i \lambda^{j-1}.$$

Then we have to check that

$$v_{i+1}^{j+1} - \lambda v_{i+1}^j = v_i^j, \quad j = 1, \dots, n-1, \quad - \sum_{k=0}^{n-1} a_k v_{i+1}^{k+1} - \lambda v_{i+1}^n = v_i^n, \quad i+1 \leq r_\lambda - 1.$$

The last equation is equivalent with

$$\begin{aligned} 0 &= \sum_{k=0}^{n-1} a_k D^{i+1} \lambda^k + \lambda D^{i+1} \lambda^{n-1} + (i+1) D^i \lambda^{n-1} \\ &= \sum_{k=0}^{n-1} a_k D^{i+1} \lambda^k + D^{i+1} \lambda^n \\ &= \sum_{k=0}^n a_k D^{i+1} \lambda^k = (-1)^n D^{i+1} p(\lambda). \end{aligned}$$

The equation $D^{i+1} p(\lambda) = 0$ is satisfied as soon as $i+1 \leq r_\lambda - 1$, because it is $p(\lambda) = 0$ with (algebraic) multiplicity r_λ .

Now we determine the Jordan normal form of the matrix A . Let $\lambda_1, \dots, \lambda_p \in \mathbb{C}$ be the eigenvalues of A and $v_{\lambda_1}, \dots, v_{\lambda_p}$ the corresponding eigenvectors. Denote by r_{λ_j} the algebraic multiplicity of λ_j , for $j = 1, \dots, p$. Finally, let $v_{\lambda_j}^k$ with $k = 0, 1, \dots, r_{\lambda_j} - 1$ be a Jordan chain relative to $v_{\lambda_j} = v_{\lambda_j}^0$.

Let $C \in GL_n(\mathbb{C})$ be the matrix

$$C = [v_{\lambda_1}^0 \dots v_{\lambda_1}^{r_{\lambda_1}-1} \dots v_{\lambda_p}^0 \dots v_{\lambda_p}^{r_{\lambda_p}-1}].$$

Then A has the Jordan normal form

$$A = C \begin{pmatrix} J_{r_{\lambda_1}}(\lambda_1) & & \\ & \ddots & \\ & & J_{r_{\lambda_p}}(\lambda_p) \end{pmatrix} C^{-1},$$

where $J_{r_{\lambda_1}}(\lambda_1), \dots, J_{r_{\lambda_p}}(\lambda_p)$ are fundamental Jordan blocks. The exponential of A is then

$$e^{xA} = C \begin{pmatrix} e^{xJ_{r_{\lambda_1}}(\lambda_1)} & & \\ & \ddots & \\ & & e^{xJ_{r_{\lambda_p}}(\lambda_p)} \end{pmatrix} C^{-1},$$

where the exponential of a fundamental Jordan block is computed in (3.4.61).

The column of the matrix $e^{xA}C$ are a fundamental system of complex valued solutions for the system of equations $z' = Az$. The n functions appearing in the first row of the matrix $e^{xA}C$ are thus n linearly independent complex valued solutions of equation (3.6.65) with $f = 0$. Then the following functions are a system of n linearly independent complex valued solution to the equation

$$e^{\lambda_1 x}, x e^{\lambda_1 x}, \dots, x^{r_{\lambda_1}-1} e^{\lambda_1 x}, \dots, e^{\lambda_p x}, x e^{\lambda_p x}, \dots, x^{r_{\lambda_p}-1} e^{\lambda_p x}. \quad (3.7.70)$$

In order to get real valued solutions notice that $\lambda \in \mathbb{C}$ is an eigenvalue for A if and only if $\bar{\lambda}$ is an eigenvalue, because A has real coefficients. Complex valued solutions are thus coupled, and by linear combinations we obtain real valued solutions.

THEOREM 3.7.1. *Let $a_k \in \mathbb{R}$, $k = 0, 1, \dots, n-1$, and $a_n = 1$. Let $\mu_1, \dots, \mu_q \in \mathbb{R}$ and $\lambda_1 = \alpha_1 + i\beta_1, \dots, \lambda_p = \alpha_p + i\beta_p, \bar{\lambda}_1, \dots, \bar{\lambda}_p \in \mathbb{C} \setminus \mathbb{R}$ be the real respectively complex*

solutions of the equation

$$\sum_{k=0}^n a_k \lambda^k = 0.$$

Let $r_{\mu_i} \geq 1$ be the algebraic multiplicity of μ_i , and let $r_{\lambda_j} \geq 1$ be the algebraic multiplicity of λ_j (and so also of $\bar{\lambda}_j$). A basis of solutions to the differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = 0, \quad x \in \mathbb{R}, \quad (3.7.71)$$

is given by the functions

$$\begin{aligned} e^{\mu_1 x}, xe^{\mu_1 x}, \dots, x^{r_{\mu_1}-1}e^{\mu_1 x} \\ \vdots \\ e^{\mu_q x}, xe^{\mu_q x}, \dots, x^{r_{\mu_q}-1}e^{\mu_q x} \end{aligned}$$

along with

$$\begin{aligned} e^{\alpha_1 x} \sin(\beta_1 x), xe^{\alpha_1 x} \sin(\beta_1 x), \dots, x^{r_{\lambda_1}-1}e^{\alpha_1 x} \sin(\beta_1 x) \\ e^{\alpha_1 x} \cos(\beta_1 x), xe^{\alpha_1 x} \cos(\beta_1 x), \dots, x^{r_{\lambda_1}-1}e^{\alpha_1 x} \cos(\beta_1 x) \\ \vdots \\ e^{\alpha_p x} \sin(\beta_p x), xe^{\alpha_p x} \sin(\beta_p x), \dots, x^{r_{\lambda_p}-1}e^{\alpha_p x} \sin(\beta_p x) \\ e^{\alpha_p x} \cos(\beta_p x), xe^{\alpha_p x} \cos(\beta_p x), \dots, x^{r_{\lambda_p}-1}e^{\alpha_p x} \cos(\beta_p x). \end{aligned}$$

CHAPTER 4

Regularity of solutions

1. Higher differentiability of solutions

PROPOSITION 4.1.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let $f : \Omega \rightarrow \mathbb{R}^n$ be locally Lipschitz continuous, i.e., for any compact set $K \subset \Omega$ there exists a constant $L > 0$ such that for all $(x, y), (\bar{x}, \bar{y}) \in K$*

$$|f(x, y) - f(\bar{x}, \bar{y})| \leq L(|x - \bar{x}| + |y - \bar{y}|). \quad (4.1.1)$$

Then any solution $y \in C^1(I; \mathbb{R}^n)$, with $I \subset \mathbb{R}$ compact interval, of the differential equation $y' = f(x, y)$ is in $C^{1,1}(I; \mathbb{R}^n)$, i.e., y' exists and is Lipschitz continuous on I .

PROOF. The graph of y is a compact subset of Ω . Then we have

$$M = \max_{x \in I} |f(x, y(x))| < +\infty.$$

It follows the Lipschitz estimate for y

$$|y(x) - y(\bar{x})| \leq \left| \int_{\bar{x}}^x f(t, y(t)) dt \right| \leq M|x - \bar{x}|,$$

for all $x, \bar{x} \in I$. Using (4.1.1) we obtain

$$\begin{aligned} |y'(x) - y'(\bar{x})| &\leq |f(x, y(x)) - f(\bar{x}, y(\bar{x}))| \\ &\leq L(|x - \bar{x}| + |y(x) - y(\bar{x})|) \\ &\leq L(1 + M)|x - \bar{x}|, \end{aligned}$$

for all $x, \bar{x} \in I$. □

THEOREM 4.1.2. *If $f \in C^k(\Omega; \mathbb{R}^n)$, $k \geq 0$, then any solution of the differential equation $y' = f(x, y)$ is of class C^{k+1} .*

PROOF. The proof is by induction, the case $k = 0$ being clear. If $f \in C^k(\Omega)$ then y is at least of class C^k , by the inductive assumption. Then the function $x \mapsto f(x, y(x)) = y'(x)$ is also of class C^k . The function y is then of class C^{k+1} . □

2. Analytic solutions

THEOREM 4.2.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. If $f \in C^\infty(\Omega; \mathbb{R}^n)$ is a real analytic function then any solution of the differential equation $y' = f(x, y)$ is also real analytic.*

PROOF. Without loss of generality we assume that $(0, 0) \in \Omega$. We show that a solution $y \in C^\infty([-\delta, \delta]; \mathbb{R}^n)$ to the differential equation $y' = f(x, y)$ with $y(0) = 0$ is real analytic, provided that $\delta > 0$ is small enough.

Because $f = (f_1, \dots, f_n)$ is analytic (i.e., each component is analytic), there exist $\eta > 0$ and $\gamma > 0$ such that

$$f(x, y) = \sum_{m=0}^{+\infty} \sum_{p+|q|=m} \frac{\partial_x^p \partial_y^q f(0)}{p!q!} x^p y^q, \quad \text{for } |x| < 2\eta, |y| < 2\gamma. \quad (4.2.2)$$

We are using the following notation: $p \in \mathbb{N}_0$, $q = (q_1, \dots, q_n) \in \mathbb{N}_0^n$, $|q| = q_1 + \dots + q_n$, $q! = q_1! \dots q_n!$, $\partial_y^q = \partial_{y_1}^{q_1} \dots \partial_{y_n}^{q_n}$, and $y^q = y_1^{q_1} \dots y_n^{q_n}$.

We have to prove that there exists $\delta > 0$ such that for $|x| < \delta$

$$y(x) = \sum_{k=1}^{+\infty} c_k x^k, \quad \text{with } c_k = \frac{y^{(k)}(0)}{k!} \in \mathbb{R}^n, k \in \mathbb{N}. \quad (4.2.3)$$

The coefficients $c_k \in \mathbb{R}^n$ can be recursively determined by f and its derivatives at the origin, e.g. c_0, c_1 , and c_2 are given by

$$\begin{aligned} c_0 &= y(0) = 0, \\ c_1 &= y'(0) = f(0), \\ c_2 &= \frac{1}{2!} y''(0) = \frac{1}{2!} (f_x(0) + f_y(0)y'(0)) = \frac{1}{2!} (f_x(0) + f_y(0)f(0)), \quad \text{etc.} \end{aligned} \quad (4.2.4)$$

Here, $f_y(0)f(0)$ is a matrix-vector multiplication.

Assume that the series

$$\varphi(x) = \sum_{k=1}^{+\infty} c_k x^k \quad (4.2.5)$$

converges for $|x| < \delta$. Then φ is an analytic function in the interval $(-\delta, \delta)$. The function $\psi(x) = \varphi'(x) - f(x, \varphi(x))$ is thus also analytic and moreover $\psi^{(k)}(0) = 0$ for any $k \in \mathbb{N}_0$. This can be recursively proved using (4.2.4). It follows that $\psi = 0$, i.e., φ is a solution of the differential equation $y' = f(x, y)$ with $\varphi(0) = 0$. By the uniqueness of the solution to the Cauchy problem it follows that $\varphi = y$.

In order to prove the theorem, it is enough to show that the series (4.2.5) converges in $(-\delta, \delta)$ for some $\delta > 0$. To this aim, assume there exists an analytic function $F \in C^\infty(U; \mathbb{R})$, where $U \subset \mathbb{R}^{n+1}$ is a neighborhood of 0, such that

$$|\partial_x^p \partial_y^q f_i(0)| \leq \partial_x^p \partial_y^q F(0) \quad \text{for all } p \in \mathbb{N}_0 \text{ and } q \in \mathbb{N}_0^n, \quad i = 1, \dots, n. \quad (4.2.6)$$

Moreover, assume that the solution $Y = (Y_1, \dots, Y_n)$ to the Cauchy Problem

$$\begin{cases} Y_i'(x) = F(x, Y(x)), & i = 1, \dots, n, \\ Y(0) = 0 \end{cases} \quad (4.2.7)$$

is analytic in $(-\delta, \delta)$, i.e.,

$$Y_i(x) = \sum_{k=1}^{+\infty} \frac{Y_i^{(k)}(0)}{k!} x^k, \quad |x| < \delta, \quad i = 1, \dots, n. \quad (4.2.8)$$

From (4.2.4) and (4.2.6) it follows that $|y_i^{(k)}(0)| \leq Y^{(k)}(0)$ for all $k \in \mathbb{N}_0$. The convergence of the series (4.2.8) implies the absolute convergence of the series (4.2.5) in the interval $(-\delta, \delta)$.

We look for a function F satisfying (4.2.6). Because the series (4.2.2) converges absolutely for $|x| \leq \eta$ and $|y| \leq \gamma$, there is $M > 0$ such that

$$\sup_{|x| \leq \eta, |y| \leq \gamma} \left| \frac{\partial_x^p \partial_y^q f(0)}{p!q!} x^p y^q \right| \leq M \quad \text{for all } p \in \mathbb{N}_0, q \in \mathbb{N}_0^n,$$

and then

$$|\partial_x^p \partial_y^q f(0)| \leq \frac{p!q!}{\eta^p \gamma^{|q|}} M, \quad \text{for all } p \in \mathbb{N}_0, q \in \mathbb{N}_0^n.$$

The function

$$F(x, y) = \frac{M}{\left(1 - \frac{x}{\eta}\right) \left(1 - \frac{y_1}{\gamma}\right) \dots \left(1 - \frac{y_n}{\gamma}\right)} = M \sum_{p \in \mathbb{N}_0, q \in \mathbb{N}_0^n} \left(\frac{x}{\eta}\right)^p \left(\frac{y}{\gamma}\right)^q$$

is analytic in $|x| < \eta$ and $|y| < \gamma$ and moreover $\partial_x^p \partial_y^q F(0) = \frac{p!q!}{\eta^p \gamma^{|q|}} M$. In other words, F satisfies (4.2.6). The solution of the Cauchy Problem (4.2.7) can be computed with the Ansatz $Y_1 = \dots = Y_n = Z$, i.e.,

$$Z' = \frac{M}{\left(1 - \frac{x}{\eta}\right) \left(1 - \frac{Z}{\gamma}\right)^n}, \quad Z(0) = 0.$$

On separating the variables, we find

$$Z(x) = \gamma \left(1 - \sqrt[n+1]{1 + \frac{(n+1)M\eta}{\gamma} \log \left(1 - \frac{x}{\eta}\right)} \right).$$

The function Z is analytic in an interval $(-\delta, \delta)$ for some $\delta > 0$. □

3. Continuity w.r.t. the initial data

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and $f \in C(\Omega; \mathbb{R}^n)$ be a function which is locally Lipschitz in y . For $(\xi, \eta) \in \Omega$ consider the Cauchy Problem

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(\xi) = \eta. \end{cases} \quad (4.3.9)$$

The problem has a unique solution in some interval $I_{\xi\eta}$ containing ξ . We denote this solution by $y_{\xi\eta} \in C^1(I_{\xi\eta}; \mathbb{R}^n)$. Now fix a point $(x_0, y_0) \in \Omega$. For some $\delta > 0$, the solution $y_{x_0 y_0}$ is defined on the interval $I = [x_0 - \delta, x_0 + \delta]$. If $(\xi, \eta) \in \Omega$ is a point such that $|\xi - x_0| + |\xi - y_0| < r$ for some small enough $r > 0$, the solution $y_{\xi\eta}$ is also defined on the same interval $I = [x_0 - \delta, x_0 + \delta]$. I.e., we can assume that $I_{\xi,\eta} = I$ for all such ξ and η , and for some small enough $\delta > 0$.

THEOREM 4.3.1. *With the notation and the assumptions stated above, let $y_{\xi\eta} \in C^1(I; \mathbb{R}^n)$ be the solution to the Cauchy Problem (4.3.9) and let $y_{x_0y_0} \in C^1(I; \mathbb{R}^n)$ be the solution with initial data $y(x_0) = y_0$. Then we have*

$$\lim_{\xi \rightarrow x_0, \eta \rightarrow y_0} \max_{x \in I} |y_{\xi\eta}(x) - y_{x_0y_0}(x)| = 0. \quad (4.3.10)$$

PROOF. There is $h > 0$ such that $|y_{\xi\eta}(x) - y_0| \leq h$ for all $x \in I$ and for all $(\xi, \eta) \in \Omega$ such that $|\xi - x_0| + |\eta - y_0| < r$, for some small $r > 0$. We can also assume that $K = I \times \{y \in \mathbb{R}^n : |y - y_0| \leq h\} \subset \Omega$. Let

$$M = \max_{(x,y) \in K} |f(x, y)|,$$

and let L a Lipschitz constant for f relative to K , as in (2.4.41). Then we have

$$\begin{aligned} y_{\xi\eta}(x) - y_{x_0y_0}(x) &= \eta - y_0 + \int_{\xi}^x f(t, y_{\xi\eta}(t)) dt - \int_{x_0}^x f(t, y_{x_0y_0}(t)) dt \\ &= \eta - y_0 + \int_{\xi}^{x_0} f(t, y_{\xi\eta}(t)) dt + \int_{x_0}^x \{f(t, y_{\xi\eta}(t)) - f(t, y_{x_0y_0}(t))\} dt, \end{aligned}$$

and by the triangle inequality (with $\xi \leq x_0 \leq x$) we get

$$\begin{aligned} |y_{\xi\eta}(x) - y_{x_0y_0}(x)| &\leq |\eta - y_0| + \int_{\xi}^{x_0} |f(t, y_{\xi\eta}(t))| dt + \int_{x_0}^x |f(t, y_{\xi\eta}(t)) - f(t, y_{x_0y_0}(t))| dt \\ &\leq |\eta - y_0| + M|\xi - x_0| + L \int_{x_0}^x |y_{\xi\eta}(t) - y_{x_0y_0}(t)| dt. \end{aligned}$$

Now Gronwall's Lemma implies

$$|y_{\xi\eta}(x) - y_{x_0y_0}(x)| \leq (|\eta - y_0| + M|\xi - x_0|)e^{L|x-x_0|},$$

for all $x \in I$, and the uniform convergence follows. \square

REMARK 4.3.2. Let $I = [x_0 - \delta, x_0 + \delta]$ and $B = \{y \in \mathbb{R}^n : |y - y_0| \leq \delta\}$. Define the mapping $\Phi : I \times I \times B \rightarrow \mathbb{R}^n$ on letting $\Phi(x, \xi, \eta) = y_{\xi\eta}(x)$. We show that Φ is continuous. In fact, fix $(x_0, \xi_0, \eta_0) \in I \times I \times B$ and let $\varepsilon > 0$. Then we have

$$|\Phi(x, \xi, \eta) - \Phi(x_0, \xi_0, \eta_0)| \leq |\Phi(x, \xi, \eta) - \Phi(x, \xi_0, \eta_0)| + |\Phi(x, \xi_0, \eta_0) - \Phi(x_0, \xi_0, \eta_0)|,$$

where $|\Phi(x, \xi, \eta) - \Phi(x, \xi_0, \eta_0)| \leq \varepsilon/2$ for all ξ, η such that $|\xi - \xi_0| \leq \delta_1$ and $|\eta - \eta_0| \leq \delta_1$. Here, $\delta_1 > 0$ is a suitable number which does not depend on $x \in I$, by Theorem 4.3.1. Moreover, we have $|\Phi(x, \xi_0, \eta_0) - \Phi(x_0, \xi_0, \eta_0)| \leq \varepsilon/2$ as soon as $|x - x_0| \leq \delta_2$, because $x \mapsto \Phi(x, \xi_0, \eta_0)$ is continuous.

4. Higher regularity

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let $f \in C(\Omega; \mathbb{R}^n)$ be a function such that

$$\frac{\partial f(x, y)}{\partial y_i} \in C(\Omega; \mathbb{R}^n), \quad i = 1, \dots, n. \quad (4.4.11)$$

In particular, f is locally Lipschitz continuous in y . For a given $(\xi, \eta) \in \Omega$ let $y_{\xi\eta}$ be the (unique) solution of the Cauchy Problem (4.3.9). We assume that $y_{\xi\eta}$ is defined in the interval $I = [x_0 - \delta, x_0 + \delta]$, for some $x_0 \in \mathbb{R}$ and $\delta > 0$ independent from (ξ, η) . In this section, we show that the mapping $(\xi, \eta) \mapsto y_{\xi\eta}(x)$ is of class C^1 , under the assumption (8.1.4).

Before stating the result, we compute the derivatives of $y_{\xi\eta}$ formally. First, we have

$$\frac{\partial}{\partial x} \frac{\partial}{\partial \xi} y_{\xi\eta}(x) = \frac{\partial}{\partial \xi} \frac{\partial}{\partial x} y_{\xi\eta}(x) = \frac{\partial}{\partial \xi} f(x, y_{\xi\eta}(x)) = \frac{\partial f}{\partial y}(x, y_{\xi\eta}(x)) \frac{\partial}{\partial \xi} y_{\xi\eta}(x).$$

In this formal computation, we also assumed that we can interchange $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial \xi}$.

Now we compute, $\frac{\partial y_{\xi\eta}}{\partial \xi}(x)$ at the point $x = \xi$. From the fact that $y_{\xi\eta}(\xi) = \eta$ for all $\xi \in I$, it follows that the derivative of the function $\xi \mapsto y_{\xi\eta}(\xi)$ vanishes. Thus, by the chain rule

$$0 = \frac{\partial y_{\xi\eta}}{\partial \xi}(\xi) + \left. \frac{\partial y_{\xi\eta}(x)}{\partial x} \right|_{x=\xi} = \frac{\partial y_{\xi\eta}}{\partial \xi}(\xi) + f(\xi, y_{\xi\eta}(\xi)).$$

In other words, the function $\psi_{\xi\eta} : I \rightarrow \mathbb{R}^n$

$$\psi_{\xi\eta}(x) = \frac{\partial y_{\xi\eta}(x)}{\partial \xi}, \quad x \in I,$$

is the solution of the linear Cauchy Problem

$$\begin{cases} \psi'(x) = F_{\xi\eta}(x)\psi(x) \\ \psi(\xi) = -f(\xi, y_{\xi\eta}(\xi)), \end{cases} \quad (4.4.12)$$

where $F_{\xi\eta} \in C(I; M_n(\mathbb{R}))$ is the matrix valued function

$$F_{\xi\eta}(x) = \frac{\partial f}{\partial y}(x, y_{\xi\eta}(x)). \quad (4.4.13)$$

Problem (4.4.12) has (always) a unique solution.

Now we compute formally the derivatives of $y_{\xi\eta}$ w.r.t. η . For $i = 1, \dots, n$ we have

$$\frac{\partial}{\partial x} \frac{\partial}{\partial \eta_i} y_{\xi\eta}(x) = \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial x} y_{\xi\eta}(x) = \frac{\partial}{\partial \eta_i} f(x, y_{\xi\eta}(x)) = \frac{\partial f}{\partial y}(x, y_{\xi\eta}(x)) \frac{\partial}{\partial \eta_i} y_{\xi\eta}(x).$$

Moreover, from $y_{\xi\eta}(\xi) = \eta$ for all $\xi \in I$ it follows that

$$\frac{\partial y_{\xi\eta}}{\partial \eta_i}(\xi) = e_i = (0, \dots, 1, \dots, 0).$$

In other words, the function $\varphi_{\xi\eta,i} : I \rightarrow \mathbb{R}^n$

$$\varphi_{\xi\eta,i}(x) = \frac{\partial y_{\xi\eta}}{\partial \eta_i}(x)$$

is the solution of the linear Cauchy Problem

$$\begin{cases} \varphi'_i(x) = F_{\xi\eta}(x)\varphi_i(x), & x \in I, \\ \varphi_i(\xi) = e_i. \end{cases} \quad (4.4.14)$$

THEOREM 4.4.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, $(x_0, y_0) \in \Omega$ and $f \in C(\Omega; \mathbb{R}^n)$ be a function satisfying (8.1.4). For $\delta > 0$ let $I = [x_0 - \delta, x_0 + \delta]$ and $B = \{y \in \mathbb{R}^n : |y - y_0| \leq \delta\}$. Then there exists $\delta > 0$ such that the mapping $\Phi : I \times I \times B \rightarrow \mathbb{R}^n$*

$$\Phi(x, \xi, \eta) = y_{\xi\eta}(x),$$

where $y_{\xi\eta} \in C^1(I; \mathbb{R}^n)$ is the solution of the Cauchy Problem (4.3.9), is of class $C^1(I \times I \times B; \mathbb{R}^n)$. Moreover,

$$\frac{\partial \Phi(x, \xi, \eta)}{\partial \xi} = \psi_{\xi\eta}(x) \quad \text{and} \quad \frac{\partial \Phi(x, \xi, \eta)}{\partial \eta_i} = \varphi_{\xi\eta, i}(x), \quad i = 1, \dots, n,$$

where $\psi_{\xi\eta}$ and $\varphi_{\xi\eta, i}$ are the solutions of the Cauchy Problems (4.4.12) and (4.4.14).

PROOF. If $\delta > 0$ is small enough, then the map Φ is well defined and it is continuous, by Theorem 4.3.1 and Remark 4.3.2.

We prove that Φ is continuously differentiable in η . It is enough to consider the case $n = 1$, i.e., η is one dimensional. For $x, \xi \in I$, $\eta \in B$ and $h \in \mathbb{R}$ with $0 < |h| \leq h_0$ small enough

$$\begin{aligned} \frac{y_{\xi, \eta+h}(x) - y_{\xi\eta}(x)}{h} &= \frac{1}{h} \left[\eta + h + \int_{\xi}^x f(t, y_{\xi, \eta+h}(t)) dt - \eta - \int_{\xi}^x f(t, y_{\xi\eta}(t)) dt \right] \\ &= 1 + \int_{\xi}^x \frac{f(t, y_{\xi, \eta+h}(t)) - f(t, y_{\xi\eta}(t))}{h} dt \\ &= 1 + \int_{\xi}^x \frac{y_{\xi, \eta+h}(t) - y_{\xi\eta}(t)}{h} \frac{\partial f}{\partial y}(t, \bar{y}_h(t)) dt. \end{aligned} \tag{4.4.15}$$

In the last line, we used the mean value theorem, providing us some $\bar{y}_h(t) \in (y_{\xi, \eta+h}(t), y_{\xi\eta}(t))$ such that

$$f(t, y_{\xi, \eta+h}(t)) - f(t, y_{\xi\eta}(t)) = (y_{\xi, \eta+h}(t) - y_{\xi\eta}(t)) \frac{\partial f}{\partial y}(t, \bar{y}_h(t)).$$

Let $\varphi \in C^1(I; \mathbb{R})$ be the solution of the Cauchy Problem (4.4.14). We drop the index i , because $n = 1$. We also drop the dependence on ξ and η . The initial data reads $\varphi(\xi) = 1$. Then φ solves the integral equation

$$\varphi(x) = 1 + \int_{\xi}^x \varphi(t) \frac{\partial f}{\partial y}(t, y_{\xi\eta}(t)) dt. \tag{4.4.16}$$

Subtracting (4.4.16) from (4.4.15) we obtain

$$\begin{aligned} R(x, h) &:= \frac{y_{\xi, \eta+h}(x) - y_{\xi\eta}(x)}{h} - \varphi(x) \\ &= \int_{\xi}^x \left(\frac{y_{\xi, \eta+h}(t) - y_{\xi\eta}(t)}{h} \frac{\partial f}{\partial y}(t, \bar{y}_h(t)) - \varphi(t) \frac{\partial f}{\partial y}(t, y_{\xi\eta}(t)) \right) dt, \end{aligned} \tag{4.4.17}$$

where we dropped the reference to ξ and η .

We claim that there exists a constant $C > 0$ such that for any $\varepsilon > 0$ there exists $\bar{h} > 0$ such that $|R(x, h)| \leq C\varepsilon$ for all $0 < |h| \leq \bar{h}$ and for all $x \in I$. The constant C does not depend on x, ξ, η . This will show that

$$\lim_{h \rightarrow 0} \frac{y_{\xi, \eta+h}(x) - y_{\xi\eta}(x)}{h} = \varphi(x), \quad (4.4.18)$$

with convergence uniform in x, ξ, η . The uniform convergence implies in particular that

$$\frac{\partial \Phi(x, \xi, \eta)}{\partial \eta} \quad \text{exists and is continuous.}$$

Indeed, adding and subtracting $\varphi(t) \frac{\partial f}{\partial y}(t, \bar{y}_h(t))$ inside the integral in the right hand side of (4.4.17), we get

$$R(x, h) = \int_{\xi}^x \frac{\partial f}{\partial y}(t, \bar{y}_h(t)) R(t, h) dt + \int_{\xi}^x \varphi(t) \left(\frac{\partial f}{\partial y}(t, \bar{y}_h(t)) - \frac{\partial f}{\partial y}(t, y_{\xi\eta}(t)) \right) dt.$$

There exists a constant $M > 0$, which is uniform in a neighborhood of (ξ, η) , such that

$$\sup_{t \in I} |\varphi(t)| \leq M \quad \text{and} \quad \sup_{|h| \leq h_0, t \in I} \left| \frac{\partial f}{\partial y}(t, \bar{y}_h(t)) \right| \leq M.$$

Moreover, for $\partial f / \partial y$ is continuous in Ω , it is uniformly continuous on compact subsets of Ω . Thus there exists $\sigma > 0$ depending on ε such that

$$\left| \frac{\partial f}{\partial y}(t, \bar{y}_h(t)) - \frac{\partial f}{\partial y}(t, y_{\xi\eta}(t)) \right| \leq \varepsilon$$

as soon as $|y_{\xi, \eta+h}(t) - y_{\xi\eta}(t)| \leq \sigma$. By Theorem 4.3.1, this estimate holds for all $t \in I$ as soon as $|h| \leq \bar{h}$ for some $\bar{h} > 0$ depending on σ .

Eventually, for all $|h| \leq \bar{h}$ and $x \in I$ there holds

$$|R(x, h)| \leq 2\varepsilon\delta M + M \left| \int_{\xi}^x |R(t, h)| dt \right|,$$

and by Gronwall's Lemma it follows that $|R(x, h)| \leq 2\varepsilon\delta M e^{M|x-\xi|}$. This finishes the proof of (4.4.18).

Now we show that

$$\lim_{h \rightarrow 0} \frac{y_{\xi+h, \eta}(x) - y_{\xi\eta}(x)}{h} = \psi(x), \quad (4.4.19)$$

where ψ is the solution to the Cauchy Problem (4.4.12). We have

$$\begin{aligned} \frac{y_{\xi+h, \eta}(x) - y_{\xi\eta}(x)}{h} &= \frac{1}{h} \left[\eta + \int_{\xi+h}^x f(t, y_{\xi+h, \eta}(t)) dt - \eta - \int_{\xi}^x f(t, y_{\xi\eta}(t)) dt \right] \\ &= \int_{\xi}^x \frac{f(t, y_{\xi+h, \eta}(t)) - f(t, y_{\xi\eta}(t))}{h} dt - \frac{1}{h} \int_{\xi}^{\xi+h} f(t, y_{\xi+h, \eta}(t)) dt \\ &= \int_{\xi}^x \frac{y_{\xi+h, \eta}(t) - y_{\xi\eta}(t)}{h} \frac{\partial f}{\partial y}(t, \bar{y}_h(t)) dt - \frac{1}{h} \int_{\xi}^{\xi+h} f(t, y_{\xi+h, \eta}(t)) dt, \end{aligned} \quad (4.4.20)$$

for some (new) $\bar{y}_h(t) \in (y_{\xi+h,\eta}(t), y_{\xi\eta}(t))$. Let $\psi \in C^1(I; \mathbb{R})$ be the solution of the Cauchy Problem (4.4.12). Then ψ solves the integral equation

$$\psi(x) = -f(\xi, y_{\xi\eta}(\xi)) + \int_{\xi}^x \psi(t) \frac{\partial f}{\partial y}(t, y_{\xi\eta}(t)) dt. \quad (4.4.21)$$

Subtracting (4.4.21) from (4.4.20) we obtain

$$\begin{aligned} S(x, h) &:= \frac{y_{\xi+h,\eta}(x) - y_{\xi\eta}(x)}{h} - \psi(x) \\ &= \int_{\xi}^x \left(\frac{y_{\xi+h,\eta}(t) - y_{\xi\eta}(t)}{h} \frac{\partial f}{\partial y}(t, \bar{y}_h(t)) - \psi(t) \frac{\partial f}{\partial y}(t, y_{\xi\eta}(t)) \right) dt + \\ &\quad - \frac{1}{h} \int_{\xi}^{\xi+h} \{f(t, y_{\xi+h,\eta}(t)) - f(\xi, y_{\xi\eta}(\xi))\} dt \\ &= \int_{\xi}^x S(t, h) \frac{\partial f}{\partial y}(t, \bar{y}_h(t)) - \psi(t) \left(\frac{\partial f}{\partial y}(t, y_{\xi\eta}(t)) - \frac{\partial f}{\partial y}(t, \bar{y}_h(t)) \right) dt + \\ &\quad - \frac{1}{h} \int_{\xi}^{\xi+h} \{f(t, y_{\xi+h,\eta}(t)) - f(\xi, y_{\xi\eta}(\xi))\} dt. \end{aligned}$$

Now, using the uniform continuity of f and $\frac{\partial f}{\partial y}$, we have as above that for any $\varepsilon > 0$ there is $\bar{h} > 0$ such that for all $|h| \leq \bar{h}$ and $x \in I$ there holds

$$|S(x, h)| \leq 2\varepsilon\delta(M+1) + M \left| \int_{\xi}^x |S(t, h)| dt \right|,$$

where M is now a bound for ψ and $\frac{\partial f}{\partial y}$. The claim follows. \square

5. Flow of a vector field

In this section we change our notation. We denote by $t \in \mathbb{R}$ the “time variable” and by $x \in \mathbb{R}^n$ the “space variable”. By $\dot{\gamma}$ we mean the derivative of γ w.r.t. t .

A vector field in \mathbb{R}^n is a mapping $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The vector field is Lipschitz continuous if there exists a constant $L > 0$ such that

$$|F(x_1) - F(x_2)| \leq L|x_1 - x_2| \quad \text{for all } x_1, x_2 \in \mathbb{R}^n. \quad (4.5.22)$$

In this case, the Cauchy Problem

$$\begin{cases} \dot{\gamma} = F(\gamma) \\ \gamma(0) = x. \end{cases} \quad (4.5.23)$$

has a unique (local) solution $\gamma \in C^1$ for any $x \in \mathbb{R}^n$, by Theorem 2.4.2. By Theorem 2.8.1, the solution is actually defined for all $t \in \mathbb{R}$ because $|F(x)| \leq |F(0)| + L|x|$ for all $x \in \mathbb{R}^n$. We denote by $\gamma_x \in C^1(\mathbb{R}; \mathbb{R}^n)$ the unique global solution to the Cauchy Problem (4.5.23).

DEFINITION 4.5.1 (Flow). The flow of a Lipschitz continuous vector field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the mapping $\Phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $\Phi(t, x) = \gamma_x(t)$, where $\gamma_x \in C^1(\mathbb{R}; \mathbb{R}^n)$ is the solution of (4.5.23). For any $t \in \mathbb{R}$, we define the mapping $\Phi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\Phi_t(x) = \Phi(t, x)$.

PROPOSITION 4.5.2. *Let Φ be the flow of a Lipschitz continuous vector field F . Then:*

- i) Φ is locally Lipschitz continuous, i.e., for any compact set $K \subset \mathbb{R} \times \mathbb{R}^n$ there is $L > 0$ such that for $(t_1, x_1), (t_2, x_2) \in K$

$$|\Phi(t_1, x_1) - \Phi(t_2, x_2)| \leq L(|t_1 - t_2| + |x_1 - x_2|). \quad (4.5.24)$$

Moreover, if $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ then $\Phi \in C^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$.

- ii) It is $\Phi(0, x) = x$ for all $x \in \mathbb{R}^n$, i.e., $\Phi_0 = \text{Id}$.
 iii) The flow has the group property $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $s, t \in \mathbb{R}$. In particular, we have $\Phi_t^{-1} = \Phi_{-t}$.

PROOF. The first statement in i) follows from Gronwall's Lemma. We leave the details to the reader. If $F \in C^1$ then $\Phi \in C^1$ by Theorem 4.4.1.

The group property iii) follows from the uniqueness of the Cauchy Problem. \square

DEFINITION 4.5.3. The Jacobi matrix of a mapping $\Phi = (\Phi_1, \dots, \Phi_n) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is the $n \times n$ matrix

$$J\Phi(x) = \left(\frac{\partial \Phi_i(x)}{\partial x_j} \right)_{i,j=1,\dots,n}.$$

The divergence of a vector field $F = (F_1, \dots, F_n) \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is

$$\text{div}F(x) = \text{tr}JF(x) = \frac{\partial F_1(x)}{\partial x_1} + \dots + \frac{\partial F_n(x)}{\partial x_n}.$$

PROPOSITION 4.5.4. *Let Φ be the flow of a (Lipschitz) vector field $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$. Then the Jacobi determinant $w_x(t) = \det(J\Phi_t(x))$ solves for any $x \in \mathbb{R}^n$ the differential equation*

$$\dot{w}_x(t) = \text{div}F(\Phi_t(x))w_x(t), \quad t \in \mathbb{R}. \quad (4.5.25)$$

PROOF. We prove (4.5.25) in the case $t = 0$ first. In this case, $\Phi_0(x) = x$ and thus $w_x(0) = \det(J\Phi_0(x)) = 1$. We have to show that

$$\left. \frac{\partial}{\partial t} \det(J\Phi_t(x)) \right|_{t=0} = \text{div}F(x), \quad x \in \mathbb{R}^n, \quad (4.5.26)$$

where

$$\det(J\Phi_t(x)) = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^n \frac{\partial \Phi_i(t, x)}{\partial x_{\sigma(i)}},$$

and therefore

$$\frac{\partial}{\partial t} \det(J\Phi_t(x)) = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} \sum_{j=1}^n \left(\frac{\partial}{\partial t} \frac{\partial \Phi_j(t, x)}{\partial x_{\sigma(j)}} \right) \prod_{i \neq j} \frac{\partial \Phi_i(t, x)}{\partial x_{\sigma(i)}}.$$

For $\Phi(x, 0) = x$, we have

$$\prod_{i \neq j} \frac{\partial \Phi_i(0, x)}{\partial x_{\sigma(i)}} = 0 \text{ if } \sigma \in S_n \text{ is not the identity.}$$

Moreover, for any $j = 1, \dots, n$ the function

$$\frac{\partial \Phi_j(t, x)}{\partial x_j} = 1 + \int_0^t \nabla F_j(\Phi(s, x)) \cdot \frac{\partial \Phi(s, x)}{\partial x_j} ds$$

is differentiable in t and

$$\left. \frac{\partial}{\partial t} \frac{\partial \Phi(t, x)}{\partial x_j} \right|_{t=0} = \nabla F_j(\Phi(t, x)) \left. \frac{\partial \Phi(t, x)}{\partial x_j} \right|_{t=0} = \frac{\partial F_j(x)}{\partial x_j}.$$

The claim (4.5.26) follows.

Now let $t \in \mathbb{R}$. Using the group property for the flow $\Phi_{t+s}(x) = \Phi_s(\Phi_t(x))$ we get $J\Phi_{t+s}(x) = J\Phi_s(\Phi_t(x))J\Phi_t(x)$ and thus

$$\begin{aligned} \dot{w}_x(t) &= \lim_{s \rightarrow 0} \frac{\det(J\Phi_{t+s}(x)) - \det(J\Phi_t(x))}{s} \\ &= \det(J\Phi_t(x)) \lim_{s \rightarrow 0} \frac{\det(J\Phi_s(\Phi_t(x))) - 1}{s} \\ &= \det(J\Phi_t(x)) \operatorname{div} F(\Phi_t(x)). \end{aligned}$$

In the last equality we used (4.5.26). □

Existence of solutions under the continuity assumption

In this chapter, we prove some results concerning ordinary differential equations which rely on Ascoli-Arzelà theorem.

1. Existence of solutions by polygonal approximation

THEOREM 5.1.1. *Let $f \in C(I \times \mathbb{R}^n; \mathbb{R}^n)$, $I = [a, b] \subset \mathbb{R}$, be a continuous function such that*

$$M = \sup_{(x,y) \in I \times \mathbb{R}^n} |f(x, y)| < +\infty. \quad (5.1.27)$$

For any $x_0 \in I$ and $y_0 \in \mathbb{R}^n$ there exists a solution $y \in C^1(I; \mathbb{R}^n)$ to the Cauchy Problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0. \end{cases} \quad (5.1.28)$$

PROOF. Without loss of generality we assume that $I = [0, 1]$ and $x_0 = 0$. For any $k \in \mathbb{N}$ and $i = 0, 1, \dots, k - 1$ let

$$x_k^i = \frac{i}{k} \quad \text{and} \quad I_k^i = [x_k^i, x_k^{i+1}].$$

Then we have $I = I_k^0 \cup \dots \cup I_k^{k-1}$ and the intervals meet only at consecutive end-points. We define recursively $\varphi_k^i : I_k^i \rightarrow \mathbb{R}^n$ in the following way

$$\begin{aligned} \varphi_k^0(x) &= y_0 + (x - x_0)f(x_0, y_0) && \text{with } x \in I_k^0 \\ \varphi_k^i(x) &= \varphi_k^{i-1}(x_k^i) + (x - x_k^i)f(x_k^i, \varphi_k^{i-1}(x_k^i)) && \text{with } x \in I_k^i, \quad i = 1, \dots, k - 1, \end{aligned}$$

and then we let $\varphi_k : I \rightarrow \mathbb{R}^n$ be the polygonal function

$$\varphi_k(x) = \varphi_k^i(x) \quad \text{if and only if } x \in I_k^i.$$

Let $\psi_k : [0, 1] \rightarrow \mathbb{R}^n$ be the function

$$\begin{cases} \psi_k(\xi) = f(x_0, y_0) & \text{if } \xi \in I_k^0, \\ \psi_k(\xi) = f(x_k^i, \varphi_k^{i-1}(x_k^i)) & \text{if } \xi \in [x_k^i, x_k^{i+1}), \quad i = 1, \dots, k - 1, \end{cases}$$

with $[x_k^{k-1}, 1]$ replacing $[x_k^{k-1}, 1)$. Then we have the identity

$$\varphi_k(x) = y_0 + \int_{x_0}^x \psi_k(\xi) d\xi, \quad x \in I. \quad (5.1.29)$$

This identity can be checked recursively.

By (5.1.27), it is $|\psi_k(\xi)| \leq M$ for all ξ and for all $k \in \mathbb{N}$. It follows that the sequence of functions $(\varphi_k)_{k \in \mathbb{N}}$ is equi-Lipschitz. In fact,

$$|\varphi_k(x) - \varphi_k(\bar{x})| \leq \left| \int_{\bar{x}}^x \psi_k(\xi) d\xi \right| \leq M|x - \bar{x}|.$$

Because $\varphi_k(x_0) = y_0$ for all $k \in \mathbb{N}$, the sequence is also equibounded.

By Ascoli-Arzelà Theorem, there exists a subsequence - which is still denoted by $(\varphi_k)_{k \in \mathbb{N}}$ - which converges uniformly to a function $y \in C(I; \mathbb{R}^n)$. We claim that

$$\lim_{k \rightarrow +\infty} \psi_k(x) = f(x, y(x)) \quad (5.1.30)$$

with uniform convergence on $[0, 1)$. Taking the limit in the integral identity (5.1.29) we then get

$$y(x) = y_0 + \int_{x_0}^x f(\xi, y(\xi)) d\xi,$$

showing that $y \in C^1(I; \mathbb{R}^n)$ is a solution to the Cauchy Problem (5.1.28).

We preliminarily notice that

$$|\varphi_k(x) - y_0| \leq \left| \int_{x_0}^x \psi_k(\xi) d\xi \right| \leq M,$$

and then, with $K = I \times \bar{B}(y_0, M)$, we have $(x, \varphi_k(x)) \in K$ for all $x \in I$ and $k \in \mathbb{N}$. The set K is compact and f is continuous on K . It is therefore uniformly continuous on K , i.e., for any $\varepsilon > 0$ there exists $\sigma > 0$ such that

$$|x - \bar{x}| \leq \sigma, |y - \bar{y}| \leq \sigma \quad \Rightarrow \quad |f(x, y) - f(\bar{x}, \bar{y})| \leq \varepsilon.$$

Now fix \bar{k} such that $1/\bar{k} \leq \sigma$, $M/\bar{k} \leq \sigma$, and $|\varphi_k(x) - y(x)| \leq \sigma$ for all $x \in I$ and for all $k \geq \bar{k}$. This is possible thanks to the uniform convergence. Take $x \in I$ and assume that $x \in I_k^i$. Then we have

$$\begin{aligned} |f(x, y(x)) - \psi_k(x)| &= |f(x, y(x)) - f(x_k^i, \varphi_k^{i-1}(x_k^i))| \\ &\leq |f(x, y(x)) - f(x_k^i, \varphi_k^i(x))| + |f(x_k^i, \varphi_k^i(x)) - f(x_k^i, \varphi_k^{i-1}(x_k^i))|, \end{aligned}$$

where $|x - x_k^i| \leq 1/k \leq \sigma$ and

$$|\varphi_k^i(x) - \varphi_k^{i-1}(x_k^i)| = |\varphi_k^i(x) - \varphi_k^i(x_k^i)| \leq M|x - x_k^i| \leq \frac{M}{k} \leq \sigma.$$

It follows that $|f(x, y(x)) - \psi_k(x)| \leq 2\varepsilon$ for all $k \geq \bar{k}$ and for all $x \in [0, 1)$. □

THEOREM 5.1.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let $f \in C(\Omega; \mathbb{R}^n)$ be a continuous function. For any $(x_0, y_0) \in \Omega$ there exist $\delta > 0$ and $y \in C^1([x_0 - \delta, x_0 + \delta]; \mathbb{R}^n)$ solution to the Cauchy Problem*

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0. \end{cases} \quad (5.1.31)$$

PROOF. Fix $\eta > 0$ such that $[x_0 - \eta, x_0 + \eta] \times \{y \in \mathbb{R}^n : |y - y_0| \leq \eta\} \subset \Omega$. Let $I = [x_0 - \eta, x_0 + \eta]$. We define $g : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ in the following way

$$g(x, y) = \begin{cases} f(x, y) & \text{if } |y - y_0| \leq \eta \\ f\left(x, y_0 + \eta \frac{y - y_0}{|y - y_0|}\right) & \text{if } |y - y_0| \geq \eta. \end{cases}$$

The function g is continuous and bounded. The Cauchy Problem $y' = g(x, y)$ and $y(x_0) = y_0$ has a solutions $y \in C^1(I; \mathbb{R}^n)$. For $\delta > 0$ small enough and $x \in [x_0 - \delta, x_0 + \delta]$, it is $|y(x) - y_0| \leq \eta$. Then y solves (5.1.31) on the interval $[x_0 - \delta, x_0 + \delta]$. \square

2. Maximal and minimal solution

In this section, we consider only the case $n = 1$.

THEOREM 5.2.1. *Let $f \in C(I \times \mathbb{R}; \mathbb{R})$, $I = [a, b] \subset \mathbb{R}$, be a continuous function such that*

$$\sup_{(x,y) \in I \times \mathbb{R}} |f(x, y)| \leq M < +\infty, \quad (5.2.32)$$

and let $x_0 \in I$ and $y_0 \in \mathbb{R}$. Then there exist two solutions $y^+, y^- \in C^1(I; \mathbb{R})$ to the Cauchy Problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \quad (5.2.33)$$

with the following properties:

- i) Any solution $y \in C^1(I; \mathbb{R})$ to the Cauchy Problem satisfies $y^-(x) \leq y(x) \leq y^+(x)$ for all $x \in I$.
- ii) For any point $(x_1, y_1) \in I \times \mathbb{R}$ such that $y^-(x_1) \leq y_1 \leq y^+(x_1)$, there exists a solution $y \in C^1(I; \mathbb{R})$ to the Cauchy Problem such that $y(x_1) = y_1$.

PROOF. The proof of the theorem is based on the following construction. Assume w.l.g. that $I = [0, 1]$, $x_0 = y_0 = 0$, and $M = 1$. Then the graph of any solution $y \in C^1([0, 1]; \mathbb{R})$ of the Cauchy Problem is contained in the rectangle $[0, 1] \times [-1, 1]$.

For $n \in \mathbb{N}$, $i \in \{0, 1, \dots, 2^n - 1\}$ and $j \in \mathbb{Z}$ define the squares

$$Q_{ij}^n = \left[\frac{i}{2^n}, \frac{i+1}{2^n} \right) \times \left[\frac{j}{2^n}, \frac{j+1}{2^n} \right).$$

We agree that for $i = 2^n - 1$ (resp. $j = 2^n - 1$) the half open interval in x (resp. in y) is replaced by a closed interval.

Then we have

$$Q = [0, 1] \times [-1, 1] = \bigcup_{i=0}^{2^n-1} \bigcup_{j=-2^n}^{2^n-1} Q_{ij}^n.$$

We also define the rectangles $R_{ij}^n = Q_{i,j-1}^n \cup Q_{ij}^n \cup Q_{i,j+1}^n$ and we define the numbers

$$m_{ij}^n = \inf_{(x,y) \in R_{ij}^n} f(x, y), \quad M_{ij}^n = \sup_{(x,y) \in R_{ij}^n} f(x, y).$$

These numbers satisfy $-1 \leq m_{ij}^n \leq M_{ij}^n \leq 1$. The function f is uniformly continuous on compact sets. Then there is an increasing function $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\omega(t) \rightarrow 0$ as $t \rightarrow 0^+$ and

$$M_{ij}^n - m_{ij}^n \leq \omega(1/2^n) \quad (5.2.34)$$

for all $n \in \mathbb{N}$, $i = 0, 1, \dots, 2^n - 1$ and $j = -2^n, \dots, 2^n - 1$.

Now choose numbers $\mu_{ij}^n \in [m_{ij}^n, M_{ij}^n]$ and define the functions $f_n : Q \rightarrow [-1, 1]$ on letting

$$f_n(x, y) = \mu_{ij}^n \quad \text{if and only if} \quad (x, y) \in Q_{ij}^n.$$

From (5.2.34) it follows that

$$\lim_{n \rightarrow +\infty} \sup_{(x,y) \in Q} |f_n(x, y) - f(x, y)| = 0. \quad (5.2.35)$$

Now consider the Cauchy problem

$$\begin{cases} y_n' = f_n(x, y_n), & x \in [0, 1) \\ y_n(0) = 0. \end{cases} \quad (5.2.36)$$

This problem has a unique solution. The solution $y_n : [0, 1] \rightarrow \mathbb{R}$ is a function which is piecewise of class C^1 . The solution is actually a polygonal curve. The differential equations is solved for all $x \in [0, 1)$ but for a finite number of points. In fact, $f_n(x, y_n(x))$ is a step function. Integrating the differential equation we get the integral equation

$$y_n(x) = \int_0^x f_n(t, y_n(t)) dt, \quad n \in \mathbb{N}. \quad (5.2.37)$$

Because $|f_n(x, y)| \leq 1$, the solution y_n is 1-Lipschitz continuous, i.e., $|y_n(x_1) - y_n(x_2)| \leq |x_1 - x_2|$ for all $x_1, x_2 \in [0, 1)$.

The sequence of functions $(y_n)_{n \in \mathbb{N}}$ is equibounded and equi-Lipschitz on $[0, 1]$. Then there exists a subsequence – by abuse of notation we denote it by $(y_n)_{n \in \mathbb{N}}$ – which converges uniformly to a 1-Lipschitz function $y : [0, 1] \rightarrow \mathbb{R}$, i.e.,

$$\lim_{n \rightarrow +\infty} \sup_{x \in [0, 1]} |y_n(x) - y(x)| = 0. \quad (5.2.38)$$

By (5.2.35) and (5.2.38), we can pass to the limit in the integral in (5.2.37) and we obtain

$$y(x) = \int_0^x f(t, y(t)) dt, \quad x \in [0, 1].$$

Then $y \in C^1([0, 1]; \mathbb{R})$ is a solution to the Cauchy Problem. The solution depends on the choice of the numbers μ_{ij}^n .

We denote by y_n^+ the sequence obtained with the choice $\mu_{ij}^n = M_{ij}^n$, and by y^+ the corresponding limit functions. We shall see later that this limit function is unique, i.e., it does not depend on the subsequence given by Ascoli-Arzelà theorem. We denote by y_n^- the sequence obtained with the choice $\mu_{ij}^n = m_{ij}^n$. The limit function is denoted by y^- .

We claim that for any solution $y \in C^1([0, 1]; \mathbb{R})$ to the Cauchy problem, there exists a choice of the numbers μ_{ij}^n such that the corresponding sequence $(y_n)_{n \in \mathbb{N}}$ converges uniformly to y .

For a fixed $n \in \mathbb{N}$ consider the points $p_i = (x_i, y_i) \in Q$ with $x_i = i/2^n$ and $y_i = y(x_i)$, $i = 0, 1, \dots, 2^n$. Notice that

$$p_i \in Q_{ij}^n \Rightarrow p_{i+1} \in R_{ij}^n = Q_{i,j-1}^n \cup Q_{ij}^n \cup Q_{i,j+1}^n.$$

More precisely, in this case we have $\{(x, y(x)) \in Q : x_i \leq x < x_{i+1}\} \subset R_{ij}^n$. This follows from $y' = f(x, y)$ with $|f(x, y)| \leq 1$.

Assume that $p_i \in Q_{ij}^n$. By Lagrange theorem there exists $\xi_i \in [x_i, x_{i+1}]$ such that

$$\frac{y(x_{i+1}) - y(x_i)}{x_{i+1} - x_i} = y'(\xi_i) = f(\xi_i, y(\xi_i)), \quad i = 0, 1, \dots, 2^n - 1.$$

Moreover, we have $(\xi_i, y(\xi_i)) \in R_{ij}^n$ and thus we can choose

$$\mu_{ij}^n = y'(\xi_i) = f(\xi_i, y(\xi_i)) \in [m_{ij}^n, M_{ij}^n].$$

We also let $\mu_{i,j-1}^n = \mu_{i,j+1}^n = \mu_{ij}^n$. This choice is also admissible. The choice of μ_{ij}^n for $j' \neq j-1, j, j+1$ is free.

The polygonal curve $y_n : [0, 1] \rightarrow \mathbb{R}$ has then the property $y_n(i/2^n) = y(i/2^n)$ for all $n \in \mathbb{N}$ and for all $i = 0, 1, \dots, 2^n$. Then

- a) $y_n(i/2^m) \rightarrow y(i/2^m)$ as $n \rightarrow +\infty$, for all $m \in \mathbb{N}$ and $i = 0, 1, \dots, 2^m$. In fact, it holds $y_n(i/2^m) = y(i/2^m)$ for $n \geq m$.
- b) The sequence $(y_n)_{n \in \mathbb{N}}$ has a subsequence which converges uniformly.

It follows that $y_n \rightarrow y$ uniformly as $n \rightarrow +\infty$.

Now we have $y_n^- \leq y_n \leq y_n^+$ for all $n \in \mathbb{N}$. This implies that $y^- \leq y \leq y^+$. This argument also shows that y^- and y^+ are unique. This finishes the proof of statement i).

In order to prove ii), consider the (left) Cauchy Problem

$$\begin{cases} y' = f(x, y), & x_0 \leq x \leq x_1 \\ y(x_1) = y_1. \end{cases} \quad (5.2.39)$$

A solution y must intersect y^- or y^+ . A gluing argument provides the desired solution. \square

3. Comparison theorem

Let $f, F : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be two continuous functions and consider the Cauchy problems

$$\begin{cases} y' = f(x, y) \\ y(0) = y_0, \end{cases} \quad (5.3.40)$$

and

$$\begin{cases} Y' = F(x, Y) \\ Y(0) = Y_0, \end{cases} \quad (5.3.41)$$

where $y_0, Y_0 \in \mathbb{R}$.

THEOREM 5.3.1. *Let $y, Y \in C^1([0, 1])$ be solutions to the Cauchy problems (5.3.40) and (5.3.41). Assume that:*

- i) *The solution to the problem (5.3.40) is unique;*
- ii) *$F(x, y) \geq 0$ for all $x \in [0, 1]$ and $y \in \mathbb{R}$;*
- iii) *$|f(x, y)| \leq F(x, |y|)$ for all $x \in [0, 1]$ and $y \in \mathbb{R}$;*
- iv) *$F(x, y_1) \leq F(x, y_2)$ for all $0 \leq y_1 \leq y_2$ and $x \in [0, 1]$;*
- v) *$|y_0| \leq Y_0$.*

Then it is $|y(x)| \leq Y(x)$ for all $x \in [0, 1]$.

PROOF. For $k \in \mathbb{N}$ define recursively the function $y_k : [0, 1] \rightarrow \mathbb{R}$

$$y_k(x) = \begin{cases} y_0 & \text{if } x \in [0, 1/k], \\ y_0 + \int_0^{x-\frac{1}{k}} f(\xi, y_k(\xi)) d\xi & \text{if } x \in [i/k, (i+1)/k], i = 1, \dots, k-1. \end{cases}$$

We show that $|y_k(x)| \leq Y(x)$ for all $x \in [0, 1]$ and $k \in \mathbb{N}$. The proof is by induction on $i = 0, 1, \dots, k-1$, with $x \in [i/k, (i+1)/k]$. For $i = 0$, we have by ii) and v)

$$|y_k(x)| = |y_0| \leq Y_0 \leq Y_0 + \int_0^x F(\xi, Y(\xi)) d\xi = Y(x).$$

Assume the claim holds for all $x \in [0, i/k]$ and let $x \in [i/k, (i+1)/k]$. Then, by ii)–v) we get

$$\begin{aligned} |y_k(x)| &= \left| y_0 + \int_0^{x-\frac{1}{k}} f(\xi, y_k(\xi)) d\xi \right| \leq |y_0| + \int_0^{x-\frac{1}{k}} |f(\xi, y_k(\xi))| d\xi \\ &\leq Y_0 + \int_0^{x-\frac{1}{k}} F(\xi, |y_k(\xi)|) d\xi \leq Y_0 + \int_0^{x-\frac{1}{k}} F(\xi, Y(\xi)) d\xi \\ &\leq Y_0 + \int_0^x F(\xi, Y(\xi)) d\xi = Y(x). \end{aligned}$$

The sequence of functions $(y_k)_{k \in \mathbb{N}}$ is uniformly bounded. Moreover, there exists $M > 0$ such that $|f(x, y_k(x))| \leq M$ for any $x \in [0, 1]$ and all $k \in \mathbb{N}$. The sequence is therefore equi-Lipschitz. By Ascoli-Arzelà theorem there exists a subsequence that converges uniformly to a continuous function $y : [0, 1] \rightarrow \mathbb{R}$. This function satisfies $|y(x)| \leq Y(x)$. We assume that $(y_k)_{k \in \mathbb{N}}$ converges to y .

The function y is the unique solution of the problem (5.3.40). In fact, we have $y(0) = y_0$ and using the uniform continuity of f on compact sets, we obtain

$$y(x) = \lim_{k \rightarrow \infty} y_k(x) = y_0 + \lim_{k \rightarrow \infty} \int_0^{x-\frac{1}{k}} f(\xi, y_k(\xi)) d\xi = y_0 + \int_0^x f(\xi, y(\xi)) d\xi,$$

i.e., $y \in C^1([0, 1])$ solves the differential equation. □

4. Periodic solutions

DEFINITION 5.4.1. A function $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is periodic in the variable x if there exists $T > 0$ such that

$$f(x + T, y) = f(x, y)$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}^n$. The number T is said to be (a) period of f in the variable x .

THEOREM 5.4.2. Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that:

- i) f is continuous and locally Lipschitz continuous in y ;
- ii) f is periodic in the variable x .

Then, the differential equation $y' = f(x, y)$ has periodic solutions if and only if it has bounded solutions.

PROOF. Periodic solutions are clearly bounded. Assume that there exists a function $y \in C^1(\mathbb{R})$ which solves the differential equation and such that $|y(x)| \leq M$ for all $x \in \mathbb{R}$ and for some $M < +\infty$. Define the sequence of function $(y_n)_{n \in \mathbb{N}}$ on letting $y_n(x) = y(x + nT)$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Here, $T > 0$ is the period of f in x . Thanks to the periodicity of f , we have

$$y'_n(x) = y'(x + nT) = f(x + nT, y(x + nT)) = f(x, y_n(x)).$$

Consider the functions y and y_1 . There are two cases:

- 1) there is $x_0 \in \mathbb{R}$ such that $y(x_0) = y_1(x_0) = y_0$;
- 2) we have $y(x) \neq y_1(x)$ for all $x \in \mathbb{R}$.

In the first case, both y and y_1 are solutions to the Cauchy Problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0. \end{cases}$$

Because of i), the solution is unique and thus $y = y_1$. It follows that y is T -periodic, i.e., $y(x + T) = y(x)$ for $x \in \mathbb{R}$.

In the second case, we have $y_1 < y$ or $y < y_1$ on the whole real line. Assume e.g. that $y_1(x) < y(x)$ for all $x \in \mathbb{R}$. Then we have for any $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$y_{n+1}(x) = y_1(x + nT) < y(x + nT) = y_n(x).$$

Because $|y_n(x)| \leq M$ for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$, we can define the function $y_\infty : \mathbb{R} \rightarrow \mathbb{R}$

$$y_\infty(x) = \lim_{n \rightarrow \infty} y_n(x).$$

For any $k \in \mathbb{N}$ and $|x| \leq k$ we have

$$|y'_n(x)| = |f(x, y_n(x))| \leq \max_{|x| \leq k, |z| \leq M} |f(x, z)| < +\infty,$$

because f is continuous.

The sequence $(y_n)_{n \in \mathbb{N}}$ is then equicontinuous and equibounded on the interval $[-k, k]$. By Ascoli–Arzelà Theorem, the sequence $(y_n)_{n \in \mathbb{N}}$ has a subsequence converging uniformly on $[-k, k]$. As $(y_n)_{n \in \mathbb{N}}$ is already converging pointwise, it follows that $y_n \rightarrow y_\infty$ as $n \rightarrow \infty$ uniformly on compact sets. In particular, it is $y_\infty \in C(\mathbb{R})$.

Each y_n , $n \in \mathbb{N}$, solves the integral equation

$$y_n(x) = y_n(0) + \int_0^x f(\xi, y_n(\xi)) d\xi, \quad x \in \mathbb{R}. \quad (5.4.42)$$

The function $f : [-k, k] \times [-M, M] \rightarrow \mathbb{R}$ is continuous, and thus uniformly continuous. It follows that all $\varepsilon > 0$ there is $\delta > 0$ such that for $(x, y_1), (x, y_2) \in [-k, k] \times [-M, M]$ we have

$$|y_1 - y_2| \leq \delta \quad \Rightarrow \quad |f(x, y_1) - f(x, y_2)| \leq \varepsilon$$

By the uniform convergence of $(y_n)_{n \in \mathbb{N}}$ there exists $N \in \mathbb{N}$ such that for $n \geq N$ and $|x| \leq k$ we have $|y_n(x) - y_\infty(x)| \leq \delta$ and thus

$$\left| \int_0^x \{f(\xi, y_n(\xi)) - f(\xi, y_\infty(\xi))\} d\xi \right| \leq \varepsilon k, \quad \text{for } |x| \leq k \text{ and } n \geq N.$$

Taking the limit as $n \rightarrow \infty$ in (5.4.42) we obtain

$$y_\infty(x) = y_\infty(0) + \int_0^x f(\xi, y_\infty(\xi)) d\xi, \quad x \in \mathbb{R}. \quad (5.4.43)$$

Then it is $y_\infty \in C^1(\mathbb{R})$ and $y'_\infty = f(x, y_\infty)$.

Finally, we show that the function y_∞ is periodic with period T :

$$y_\infty(x + T) = \lim_{n \rightarrow \infty} y_n(x + T) = \lim_{n \rightarrow \infty} y_{n+1}(x) = y_\infty(x).$$

□

EXAMPLE 5.4.3 (Small perturbations of periodic solutions). Let $f \in C^1(\mathbb{R}^{n+2}; \mathbb{R}^n)$ be a function such that for some $T > 0$

$$f(x + T, \varepsilon, y) = f(x, \varepsilon, y), \quad x \in \mathbb{R}, \varepsilon \in \mathbb{R}, y \in \mathbb{R}^n, \quad (5.4.44)$$

and for $\eta \in \mathbb{R}^n$ consider the Cauchy Problem

$$\begin{cases} y' = f(x, \varepsilon, y) \\ y(0) = \eta. \end{cases} \quad (5.4.45)$$

Here, ε is a real parameter which is “small”. Denote by $y_{\varepsilon\eta}$ the solution of the Problem depending on ε and η .

Assume that:

- i) For some $y_0 \in \mathbb{R}^n$ we have $y_{0y_0}(T) = y_{0y_0}(0) = y_0$, i.e., for y_0 and $\varepsilon = 0$ the solution of the Cauchy Problem is periodic with period T . This follows from (5.4.44). In particular, the solution is defined for all $x \in \mathbb{R}$.
- ii) For some $r_0 > 0$ we have $y_{\varepsilon\eta} \in C^1([0, T]; \mathbb{R}^n)$ for all $|\varepsilon| < r_0$ and for all $\eta \in \mathbb{R}^n$ with $|\eta - y_0| < r_0$, i.e., the solutions are defined on the whole interval $[0, T]$.

Then we can define the function $F : \{|\varepsilon| < r_0\} \times \{|\eta| < r_0\} \rightarrow \mathbb{R}^n$ on letting $F(\varepsilon, \eta) = y_{\varepsilon\eta}(T) - \eta$. The function F has the following properties:

- a) $F(0, y_0) = 0$, by ii).
- b) $F \in C^1(\{|\varepsilon| < r_0\} \times \{|\eta| < r_0\}; \mathbb{R}^n)$. This follows from Theorem 4.4.1. The differentiability with respect to the parameter ε is left as an exercise. In particular, $\varphi_{\varepsilon\eta} = \partial y_{\varepsilon\eta} / \partial \eta$ is the solution of the Cauchy Problem

$$\begin{cases} \varphi'_{\varepsilon\eta} = \frac{\partial f(x, \varepsilon, y_{\varepsilon\eta})}{\partial \eta} \varphi_{\varepsilon\eta} \\ \varphi_{\varepsilon\eta}(0) = I_n. \end{cases} \quad (5.4.46)$$

Then we have,

$$\frac{\partial F(\varepsilon, \eta)}{\partial \eta} = \frac{\partial y_{\varepsilon\eta}(T)}{\partial \eta} - I_n = \varphi_{\varepsilon\eta}(T) - I_n.$$

Now assume that:

- iii) The matrix $\varphi_{0y_0}(T) - I_n$ is invertible.

Then, by the implicit function theorem there exist $\delta > 0$ and $g \in C^1((-\delta, \delta); \mathbb{R}^n)$, with $g(0) = y_0$, such that $F(\varepsilon, g(\varepsilon)) = 0$ for all $\varepsilon \in (-\delta, \delta)$.

Conclusion: under the assumptions i), ii), and iii), for small perturbations of the function, there still exist periodic solutions of the differential equation.

First order partial differential equations

1. Quasilinear differential equations

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and let $a_i, b \in C(\Omega)$, $i = 1, \dots, n$, be continuous functions. A first order partial differential equation of the form

$$\mathcal{L}u = \sum_{i=1}^n a_i(x, u) \frac{\partial u}{\partial x_i} = b(x, u) \quad (6.1.1)$$

is called quasilinear. The differential operator \mathcal{L} is called quasilinear, as well.

A function $u \in C^1(U)$ is a solution of the quasilinear equation (6.1.1) if:

- i) $U \subset \mathbb{R}^n$ is an open set;
- ii) $\text{gr}(u) = \{(x, u(x)) \in \mathbb{R}^{n+1} : x \in U\} \subset \Omega$;
- iii) $\mathcal{L}u(x) = b(x, u(x))$ for all $x \in U$.

The differential equation (6.1.1) has the following geometric interpretation. Denote by $a = (a_1, \dots, a_n) \in C(\Omega; \mathbb{R}^n)$ the vector of the coefficients of \mathcal{L} . Then equation (6.1.1) can be written in the following way

$$\langle (\nabla u, -1), (a(x, u), b(x, u)) \rangle = 0, \quad (6.1.2)$$

where ∇u is the gradient of u in the variable x . The function $g \in C^1(U \times \mathbb{R})$, $g(x, y) = u(x) - y$, is a defining function for the graph of u , i.e.,

$$\text{gr}(u) = \{(x, y) \in U \times \mathbb{R} : g(x, y) = 0\}.$$

Then, the gradient of g in the variables x, y , i.e., the vector $N = (\nabla u, -1)$, is orthogonal to the graph of u . In fact, a basis of tangent vectors to the graph of u is given by

$$T_i = \left(\underbrace{0, \dots, 1, \dots, 0}_{ith}, \frac{\partial u}{\partial x_i} \right), \quad i = 1, \dots, n,$$

and we have $\langle N, T_i \rangle = 0$ for all $i = 1, \dots, n$. By (6.1.2), the vector field $X \in C(U \times \mathbb{R}; \mathbb{R}^{n+1})$, defined by

$$X = (a(x, u), b(x, u)) \quad (6.1.3)$$

is tangent to the graph of u . We call X the characteristic vector field of the equation $\mathcal{L}u = b$.

DEFINITION 6.1.1 (Characteristic curves). The integral curves of the vector field $X = (a(x, u), b(x, u))$, i.e., the solutions of the ordinary differential equation $\dot{\gamma} = X(\gamma)$, are called *characteristic curves* of the differential equation $\mathcal{L}u = b$.

DEFINITION 6.1.2 (Integral surface). A C^1 hypersurface $S \subset \Omega$ of the form $S = \text{gr}(u) = \{(x, u(x)) \in \Omega : x \in U\}$ for some $u \in C^1(U)$, $U \subset \mathbb{R}^n$ open set, is called integral surface for the equation $\mathcal{L}u = b$ if the vector $(a(x, u(x)), b(x, u(x))) \in \mathbb{R}^{n+1}$ is tangent to S at the point $(x, u(x)) \in S$ for all $x \in U$.

PROPOSITION 6.1.3. *Let $u \in C^1(U)$, $U \subset \mathbb{R}^n$ open set. The following statements are equivalent:*

- i) *The function u solves the differential equation $\mathcal{L}u = b$;*
- ii) *The hypersurface $S = \{(x, u(x)) \in \Omega : x \in U\}$ is an integral surface for the equation $\mathcal{L}u = b$;*
- iii) *For any $x_0 \in U$ there is $\delta > 0$ such that any characteristic curve $\gamma \in C^1((-\delta, \delta); \mathbb{R}^{n+1})$ of X with $\gamma(0) = (x_0, u(x_0)) \in S$ is contained in S .*

PROOF. i) \Rightarrow ii). This is the discussion made above. ii) \Rightarrow iii) The Cauchy problem $\dot{\gamma} = X(\gamma)$ with $\gamma(0) = (x_0, u(x_0))$ has a solution defined on the interval $(-\delta, \delta)$. Let us write $\gamma(s) = (x(s), z(s)) \in \mathbb{R}^n \times \mathbb{R}$. Then we have

$$\begin{cases} \dot{x}(s) = a(\gamma(s)), & x(0) = x_0 \\ \dot{z}(s) = b(\gamma(s)), & z(0) = u(x_0). \end{cases}$$

Consider the function $\varphi \in C^1(-\delta, \delta)$, $\varphi(s) = u(x(s)) - z(s)$. We have $\varphi(0) = 0$ and moreover

$$\dot{\varphi}(s) = \langle \nabla u(x(s)), \dot{x}(s) \rangle - \dot{z}(s) = \langle (\nabla u(x(s)), -1), (a(\gamma(s)), b(\gamma(s))) \rangle = 0,$$

because $N = (\nabla u(x(s)), -1)$ and $X = (a(\gamma(s)), b(\gamma(s)))$ are by assumption orthogonal. It follows that $\varphi(s) = \varphi(0) = 0$ for all $s \in (-\delta, \delta)$ and this implies $\gamma(s) \in S$ for all $s \in (-\delta, \delta)$.

iii) \Rightarrow ii) For any $x_0 \in U$, let $\gamma = (x, z)$ be a curve as in iii). Then we have $u(x(s)) - z(s) = 0$ for all $s \in (-\delta, \delta)$ and differentiating this identity at $s = 0$ we get $\mathcal{L}u(x_0) = b(x_0, u(x_0))$. \square

2. Cauchy Problem for quasilinear equations

Let $\Gamma \subset \mathbb{R}^n$ be a hypersurface of class C^1 . By definition, for any $x_0 \in \Gamma$ there are an open set $V \subset \mathbb{R}^{n-1}$, an open set $U \subset \mathbb{R}^n$ with $x_0 \in U$ and a bijective mapping $\Phi \in C^1(V; U \cap \Gamma)$ such that the Jacobi matrix $J\Phi(v)$ has maximal rank for all $v \in V$. Then the vectors

$$\frac{\partial \Phi(v)}{\partial v_1}, \dots, \frac{\partial \Phi(v)}{\partial v_{n-1}} \in \mathbb{R}^n$$

are linearly independent and are a basis for the tangent space of Γ at the point $\Phi(v) \in \Gamma$. Φ is called C^1 -parameterization of $U \cap \Gamma$.

Let $\varphi : \Gamma \rightarrow \mathbb{R}$ be a function of class C^1 . This means that the function $\varphi \circ \Phi : V \rightarrow \mathbb{R}$ is of class C^1 for any (local) parameterization Φ of Γ . We write $\varphi \in C^1(\Gamma)$. We are interested in the Cauchy Problem

$$\begin{cases} \mathcal{L}u(x) = b(x, u) & x \in U \\ u(x) = \varphi(x) & x \in \Gamma \cap U \end{cases} \quad (6.2.4)$$

where U is an open subset of \mathbb{R}^n .

DEFINITION 6.2.1 (Noncharacteristic manifold). Let $\Gamma \subset \mathbb{R}^n$ be a C^1 hypersurface and let $\varphi \in C^1(\Gamma)$. The pair (Γ, φ) is noncharacteristic for the quasilinear differential operator $\mathcal{L} = \sum_{i=1}^n a_i(x, u) \frac{\partial}{\partial x_i}$ at the point $x_0 \in \Gamma$, if the vector $a(x_0, \varphi(x_0)) = (a_1(x_0, \varphi(x_0)), \dots, a_n(x_0, \varphi(x_0))) \in \mathbb{R}^n$ is not tangent at Γ at the point x_0 .

Equivalently, the pair (Γ, φ) is non characteristic at $x_0 = \Phi(v_0)$ for \mathcal{L} if

$$\det \left[\frac{\partial \Phi(v_0)}{\partial v_1} \dots \frac{\partial \Phi(v_0)}{\partial v_{n-1}}, a(x_0, \varphi(x_0)) \right] \neq 0, \quad (6.2.5)$$

where the vectors in $[\dots]$ are thought of as column vectors.

THEOREM 6.2.2. *Let $\mathcal{L} = \sum_{i=1}^n a_i(x, u) \frac{\partial}{\partial x_i}$ be a quasilinear differential operator in the open set $\Omega \subset \mathbb{R}^{n+1}$, let $\Gamma \subset \mathbb{R}^n$ be a hypersurface of class C^1 and $\varphi \in C^1(\Gamma)$ with $(x_0, \varphi(x_0)) \in \Omega$. Assume that:*

- i) $b, a_1, \dots, a_n \in C^1(\Omega)$;
- ii) *The pair (Γ, φ) is non characteristic at $x_0 \in \Gamma$ for \mathcal{L} .*

Then there exists $u \in C^1(U)$, $U \subset \mathbb{R}^n$ open neighborhood of x_0 , solution to the Cauchy Problem (6.2.4). The solution is unique in a neighborhood of x_0 .

PROOF. Let $X = (a(x, u), b(x, u))$ be the characteristic vector field of the equation $\mathcal{L}u = b$, and let $\Phi : V \rightarrow U \cap \Gamma$ be a parameterization of Γ in a neighborhood of $x_0 \in U$. We can w.l.g. assume that $V = (-\eta, \eta)^{n-1}$ for some $\eta > 0$ and $\Phi(0) = x_0$. For any p in a neighborhood of $(x_0, \varphi(x_0)) \in \mathbb{R}^{n+1}$ consider the Cauchy Problem

$$\begin{cases} \dot{\gamma}(s) = X(\gamma(s)) \\ \gamma(0) = p. \end{cases} \quad (6.2.6)$$

The solution γ_p is (uniquely) defined on the interval $(-\delta, \delta)$ for some $\delta > 0$ independent from p . Because $a \in C^1(\Omega; \mathbb{R}^n)$ and $b \in C^1(\Omega)$, the mapping $(p, s) \mapsto \gamma_p(s)$ is of class C^1 , by Theorem 4.4.1.

By abuse of notation, we define γ_v to be the curve γ_p with $p = (\Phi(v), \varphi(\Phi(v))) \in \Gamma \times \mathbb{R}$ for $v \in V$. The mapping $(v, s) \mapsto \gamma_v(s)$ is of class C^1 , because it is the composition of C^1 mappings. The curve γ_v is of the form $\gamma_v(s) = (x_v(s), z_v(s)) \in \mathbb{R}^n \times \mathbb{R}$, where

$$\begin{aligned} \dot{x}_v(s) &= a(\gamma_v(s)), & x_v(0) &= \Phi(v) \\ \dot{z}_v(s) &= b(\gamma_v(s)), & z_v(0) &= \varphi(\Phi(v)). \end{aligned}$$

Define the function $F : V \times (-\delta, \delta) \rightarrow \mathbb{R}^n$

$$F(v, s) = x_v(s).$$

Then we have $F \in C^1(V \times (-\delta, \delta); \mathbb{R}^n)$ and moreover

$$\frac{\partial F}{\partial s}(v, s) = \dot{x}_v(s) = a(\gamma_v(s)),$$

and for $s = 0$ and $i = 1, \dots, n - 1$

$$\frac{\partial F}{\partial v_i}(v, 0) = \frac{\partial x_v(0)}{\partial v_i} = \frac{\partial \Phi(v)}{\partial v_i}.$$

In particular, at the point $v = 0$ and $s = 0$ the Jacobi matrix of F satisfies

$$\det J_F(0) = \det \left[\frac{\partial \Phi}{\partial v_1}(0), \dots, \frac{\partial \Phi}{\partial v_{n-1}}(0), a(x_0, \varphi(x_0)) \right] \neq 0,$$

because the pair (Γ, φ) is non characteristic for \mathcal{L} at the point $x_0 \in \Gamma$.

By the Inverse Function Theorem, the mapping $F : V \times (-\delta, \delta) \rightarrow U = F(V \times (-\delta, \delta)) \subset \mathbb{R}^n$ is a diffeomorphism of class C^1 , provided that $\delta > 0$ and $\eta > 0$ are small enough. The inverse function $F^{-1} : U \rightarrow V \times (-\delta, \delta)$ is of the form $F^{-1}(x) = (v(x), s(x))$, where the functions $x \mapsto v(x), s(x)$ are both of class C^1 .

Define the function $u : U \rightarrow \mathbb{R}$ on letting $u(x) = z_{v(x)}(s(x))$, $x \in U$. Then:

- i) We have $u \in C^1(U)$, because it is a composition of C^1 functions.
- ii) If $x = \Phi(v) \in \Gamma$ then $s(x) = 0$ and $v(x) = v \in V$. Thus

$$u(x) = z_v(0) = \varphi(\Phi(v)) = \varphi(x).$$

- iii) The graph $S = \{(x, u(x)) \in \mathbb{R}^{n+1} : x \in U\}$ is the union of integral curves of the characteristic vector field X . In fact, we have

$$x = F(F^{-1}(x)) = F(v(x), s(x)) = x_{v(x)}(s(x)),$$

and thus

$$(x, u(x)) = (x_{v(x)}(s(x)), z_{v(x)}(s(x))) = \gamma_{v(x)}(s(x)).$$

This shows that for any $x \in U$, the integral curve of X passing through the point $(x, u(x))$ lies entirely in the graph of u . Then, by Proposition 6.1.3 we have $\mathcal{L}u(x) = b(x, u(x))$ for all $x \in U$.

We proved that u is a solution to the Cauchy problem (6.2.4). The solution is unique, because of the uniqueness of the characteristic curves of the equation $\mathcal{L}u = b$. \square

EXAMPLE 6.2.3. We look for a solution $u \in C^1(U)$ for some open set $U \subset \mathbb{R}^2$ with $0 \in U$ (and possibly the largest one) of the Cauchy Problem

$$\begin{cases} u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, & (x, y) \in U \\ u(x, 0) = x, & (x, 0) \in U. \end{cases}$$

The quasi-linear differential operator

$$\mathcal{L}u = u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$$

is known as Burgers' operator. The vector of the coefficients is $a(x, y, u) = (u, 1)$, we have $b(x, y, u) = 0$, and the characteristic vector field of the equation is $X = (u, 1, 0)$.

We have $\Gamma = \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ and $\varphi(x) = x$. A normal vector to Γ is the constant vector $N = (0, 1)$. For any $(x, 0) \in \Gamma$ we have

$$\langle N, a((x, 0), \varphi(x)) \rangle = 1 \neq 0.$$

Then the pair (Γ, φ) is non characteristic for the equation $\mathcal{L}u = 0$ at any point of Γ . By Theorem 6.2.2 we can find a (unique) local solution.

For a fixed $(x_0, 0) \in \Gamma$ consider the Cauchy problem

$$\dot{\gamma} = X(\gamma), \quad \gamma(0) = (x_0, 0, \varphi(x_0)).$$

The system of differential equations is

$$\dot{\gamma}_1 = \gamma_3, \quad \dot{\gamma}_2 = 1, \quad \dot{\gamma}_3 = 0,$$

which yields, along with the initial conditions,

$$\gamma_1(s) = s\varphi(x_0) + x_0 = (s+1)x_0, \quad \gamma_2(s) = s\gamma_3(s) = \varphi(x_0) = x_0,$$

where $s \in \mathbb{R}$. The solution u to the Cauchy problem satisfies

$$u(\gamma_1(s), \gamma_2(s)) = \gamma_3(s) \quad \Leftrightarrow \quad u((s+1)x_0, s) = \varphi(x_0) = x_0.$$

Letting $x = (s+1)x_0$ and $y = s$ we find

$$u(x, y) = \frac{x}{y+1}.$$

The “maximal” solution u of the Cauchy problem is defined in the half plane $U = \{(x, y) \in \mathbb{R}^2 : y > -1\}$.

CHAPTER 7

Transport equation

In this Chapter, ∇u denotes the gradient of the function u in the space variables $x \in \mathbb{R}^n$.

1. Classical solutions and a priori estimates

THEOREM 7.1.1. *Let $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ be a bounded vector field and $g \in C^1(\mathbb{R}^n)$. The solution of the Cauchy Problem*

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + \nabla u(x, t) \cdot f(x) = 0, & (x, t) \in \mathbb{R}^n \times \mathbb{R} \\ u(x, 0) = g(x), & x \in \mathbb{R}^n, \end{cases} \quad (7.1.7)$$

is $u(x, t) = g(\Phi_{-t}(x))$, where $\Phi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the flow of f .

PROOF. By Proposition 4.5.2, the flow Φ is defined for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We show that $u(x, t) = g(\Phi_{-t}(x))$ is a solution of the problem (the solution is unique by Theorem 6.1.3). For any $\psi \in C_c^1(\mathbb{R}^n)$ we have:

$$\begin{aligned} \int_{\mathbb{R}^n} \psi(x) \frac{\partial u(x, t)}{\partial t} dx &= \frac{d}{dt} \int_{\mathbb{R}^n} \psi(x) u(x, t) dx \\ &= \frac{d}{dt} \int_{\mathbb{R}^n} \psi(x) g(\Phi_{-t}(x)) dx \\ &= \frac{d}{dt} \int_{\mathbb{R}^n} \psi(\Phi_t(x')) g(x') \det J\Phi_t(x') dx'. \end{aligned}$$

In the last integral, we performed the change of variable $x = \Phi_t(x')$. Now by Proposition 4.5.4 we obtain

$$\begin{aligned} \frac{d}{dt} \psi(\Phi_t(x)) \det J\Phi_t(x) &= \nabla \psi(\Phi_t(x)) \dot{\Phi}_t(x) \det J\Phi_t(x) + \psi(\Phi_t(x)) \frac{\partial}{\partial t} \det J\Phi_t(x) \\ &= \left\{ \nabla \psi(\Phi_t(x)) f(\Phi_t(x)) + \psi(\Phi_t(x)) \operatorname{div} f(\Phi_t(x)) \right\} \det J\Phi_t(x). \end{aligned}$$

With a new change of variable we get

$$\begin{aligned} \int_{\mathbb{R}^n} \psi(x) \frac{\partial u(x, t)}{\partial t} dx &= \int_{\mathbb{R}^n} \left\{ \nabla \psi(x) f(x) + \psi(x) \operatorname{div} f(x) \right\} g(\Phi_{-t}(x)) dx \\ &= \int_{\mathbb{R}^n} \operatorname{div}(\psi f)(x) u(x, t) dx \\ &= - \int_{\mathbb{R}^n} \psi(x) \nabla u(x, t) \cdot f(x) dx. \end{aligned}$$

In the last line we performed an integration by parts. Because $\psi \in C_c^1(\mathbb{R}^n)$ is arbitrary, we deduce that

$$\frac{\partial u(x, t)}{\partial t} = -\nabla u(x, t) \cdot f(x)$$

for all $x \in \mathbb{R}^n$ and for all $t \in \mathbb{R}$. \square

LEMMA 7.1.2. *For any $u \in C^1(\mathbb{R}^n)$, the function $|u|$ is differentiable at a.e. $x \in \mathbb{R}^n$ and moreover $\nabla|u|(x) = 0$ for a.e. $x \in \{x \in \mathbb{R}^n : u(x) = 0\}$.*

PROOF. The proof is left to the reader as an exercise. \square

PROPOSITION 7.1.3 (A priori estimate). *Let $g \in C_c^1(\mathbb{R}^n)$ and $f \in C^1(\mathbb{R}^n; \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be such that $\operatorname{div} f \in L^\infty(\mathbb{R}^n)$. Let $u \in C^1(\mathbb{R}^n \times [0, T])$, $T > 0$, be the solution of the Cauchy Problem*

$$\begin{cases} u_t + \nabla u \cdot f = 0, & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, 0) = g(x), & x \in \mathbb{R}^n. \end{cases} \quad (7.1.8)$$

Then for any $t \in [0, T]$ and for any $1 \leq p < \infty$ we have

$$\|u(\cdot, t)\|_\infty \leq \|g\|_\infty \quad \text{and} \quad \|u(\cdot, t)\|_p \leq e^{t\|\operatorname{div} f\|_\infty/p} \|g\|_p. \quad (7.1.9)$$

PROOF. According to Theorem 7.1.1, we have $u(x, t) = g(\Phi_{-t}(x))$, where Φ_t is the flow of f . The estimate $\|u(\cdot, t)\|_\infty \leq \|g\|_\infty$ follows.

Now let $1 \leq p < \infty$. For fixed $t \in [0, T]$, on the set $\{u(\cdot, t) \neq 0\}$ we have

$$\frac{\partial}{\partial t} |u|^p + \nabla |u|^p \cdot f = 0. \quad (7.1.10)$$

In the case $p > 1$, this identity holds for all $x \in \mathbb{R}^n$. We integrate identity (7.1.10) over the set $\{u(\cdot, t) \neq 0\}$, we use Lemma 7.1.2, and the divergence theorem (integration by parts). We obtain

$$\begin{aligned} \int_{\{u(\cdot, t) \neq 0\}} \frac{\partial}{\partial t} |u|^p dx &= - \int_{\{u(\cdot, t) \neq 0\}} \nabla |u|^p \cdot f dx \\ &= - \int_{\mathbb{R}^n} \nabla |u|^p \cdot f dx \\ &= \int_{\mathbb{R}^n} |u|^p \operatorname{div} f dx \leq \|\operatorname{div} f\|_\infty \int_{\mathbb{R}^n} |u|^p dx. \end{aligned}$$

We claim that

$$\int_{\{u(\cdot, t) \neq 0\}} \frac{\partial}{\partial t} |u|^p dx = \frac{d}{dt} \int_{\mathbb{R}^n} |u|^p dx. \quad (7.1.11)$$

In fact, the set

$$K = \bigcup_{t \in [0, T]} \Phi_{-t}(\operatorname{supp} g)$$

is bounded, because $\operatorname{supp} g$ compact, and moreover,

$$M = \sup_{x \in K, t \in [0, T]} |\nabla |u|^p \cdot f| < \infty.$$

Then, by (7.1.10) we have for any $t \in [0, T]$ and $x \in \mathbb{R}^n$

$$\left| \frac{\partial}{\partial t} |u(x, t)|^p \right| \leq M \chi_K(x) \in L^1(\mathbb{R}^n),$$

and we can take the t derivative into the integral in (7.1.11).

The function

$$\varphi(t) = \int_{\mathbb{R}^n} |u(x, t)|^p dx, \quad t \in [0, T],$$

satisfies the differential inequality $\varphi' \leq \|\operatorname{div} f\|_\infty \varphi$ and thus $\varphi(t) \leq e^{t\|\operatorname{div} f\|_\infty} \varphi(0)$ for $t \in [0, T]$, i.e.,

$$\int_{\mathbb{R}^n} |u(x, t)|^p dx \leq e^{t\|\operatorname{div} f\|_\infty} \int_{\mathbb{R}^n} |g(x)|^p dx.$$

□

2. Existence of weak solutions

Let $L^\infty(0, T; L^p(\mathbb{R}^n))$, $T > 0$ and $1 \leq p < \infty$, denote the set of all measurable functions $u : \mathbb{R}^n \times (0, T) \rightarrow [-\infty, \infty]$ such that

$$\|u\|_{L^\infty(0, T; L^p(\mathbb{R}^n))} := \operatorname{ess\,sup}_{t \in (0, T)} \left(\int_{\mathbb{R}^n} |u(x, t)|^p dx \right)^{1/p} < \infty.$$

Analogously, let $L^1(0, T; L^p(\mathbb{R}^n))$ denote the set of all measurable functions $v : \mathbb{R}^n \times (0, T) \rightarrow [-\infty, \infty]$ such that

$$\|v\|_{L^1(0, T; L^p(\mathbb{R}^n))} := \int_0^T \left(\int_{\mathbb{R}^n} |v(x, t)|^p dx \right)^{1/p} dt < \infty.$$

The definitions can be generalized to the case $p = \infty$.

In the following, let $q \in [1, \infty]$ denote the conjugate Hölder exponent of p , i.e.,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

A function $u \in L^\infty(0, T; L^p(\mathbb{R}^n))$ can be naturally identified with a linear and continuous functional over $L^1(0, T; L^q(\mathbb{R}^n))$ via the duality

$$\langle u, v \rangle = \int_0^T \int_{\mathbb{R}^n} u(x, t) v(x, t) dx dt.$$

The operator norm of u is $\|u\|_{L^\infty(0, T; L^p(\mathbb{R}^n))}$. In fact, we have the canonical isometric identification

$$L^\infty(0, T; L^p(\mathbb{R}^n)) = L^1(0, T; L^q(\mathbb{R}^n))^*, \quad 1 < p \leq \infty.$$

We do not prove this fact, here. As a special case of Alaoglu's Theorem, we have the following compactness theorem.

THEOREM 7.2.1 (weak* compactness). *Let $p > 1$ and $u_\varepsilon \in L^\infty(0, T; L^p(\mathbb{R}^n))$, $\varepsilon > 0$, be a family of functions such that*

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^\infty(0, T; L^p(\mathbb{R}^n))} < \infty.$$

Then there exist a sequence $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ and a function $u \in L^\infty(0, T; L^p(\mathbb{R}^n))$ such that $u_{\varepsilon_j} \rightharpoonup u$ as $j \rightarrow \infty$ in the weak topology, and namely*

$$\lim_{j \rightarrow \infty} \langle u_{\varepsilon_j}, v \rangle = \langle u, v \rangle,$$

for any $v \in L^1(0, T; L^q(\mathbb{R}^n))$.

PROOF. The space $L^1(0, T; L^q(\mathbb{R}^n))$ with $q < \infty$ is separable. Then, the weak* topology of its dual space $L^1(0, T; L^q(\mathbb{R}^n))^* = L^\infty(0, T; L^p(\mathbb{R}^n))$ is metrizable (see Dunford-Schwarz, *Linear Operators*, p. 426). From Alaoglu's theorem it follows that bounded sets in the dual space are weakly* sequentially compact. \square

DEFINITION 7.2.2. Let $g \in L^p_{\text{loc}}(\mathbb{R}^n)$ and let $f \in L^q_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ be a vector field such that $\text{div} f \in L^q_{\text{loc}}(\mathbb{R}^n)$ in distributional sense. We say that a function $u \in L^\infty(0, T; L^p(\mathbb{R}^n))$ is a weak solution of the Cauchy problem

$$\begin{cases} u_t(x, t) + \nabla u(x, t) \cdot f(x) = 0, & (x, t) \in \mathbb{R}^n \times (0, T) \\ u(x, 0) = g(x), & x \in \mathbb{R}^n, \end{cases} \quad (7.2.12)$$

if for any $v \in C^1_c(\mathbb{R}^n \times [0, T])$ we have

$$\int_{\mathbb{R}^n} v(x, 0)g(x)dx + \int_0^T \int_{\mathbb{R}^n} u(x, t) \{v_t(x, t) + \text{div}(v(x, t)f(x))\} dx dt = 0. \quad (7.2.13)$$

REMARK 7.2.3. Notice that for any fixed $t \in (0, T)$, we have

$$\text{div}(v(\cdot, t)f) = v(\cdot, t)\text{div} f + \langle \nabla v(\cdot, t), f \rangle \in L^q(\mathbb{R}^n),$$

with bound for the $L^q(\mathbb{R}^n)$ norm independent of $t \in (0, T)$. It follows that the integrals in (7.2.13) are well defined.

REMARK 7.2.4. If $u \in C^1(\mathbb{R}^n \times [0, T]) \cap L^\infty(0, T; L^p(\mathbb{R}^n))$ is a classical solution of the problem (7.2.12) for some function $g \in C^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and for some $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ then we can integrate the equation $vu_t + v\langle \nabla u, f \rangle = 0$ and get

$$\int_0^T \int_{\mathbb{R}^n} vu_t dx dt + \int_0^T \int_{\mathbb{R}^n} v\langle \nabla u, f \rangle dx dt = 0,$$

that is equivalent to

$$\int_{\mathbb{R}^n} \int_0^T vu_t dt dx - \int_0^T \int_{\mathbb{R}^n} u \text{div}(vf) dx dt = 0.$$

On the other hand,

$$\int_0^T v(x, t)u_t(x, t) dt = -v(x, 0)u(x, 0) - \int_0^T v_t(x, t)u(x, t) dt,$$

and using the initial condition $u(x, 0) = g(x)$ we obtain identity (7.2.13).

THEOREM 7.2.5. *Let $T > 0$, $1 < p \leq \infty$, $1 \leq q < \infty$ with $1/p + 1/q = 1$, $g \in L^p(\mathbb{R}^n)$ and $f \in L^q_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ be such that $\text{div} f \in L^\infty(\mathbb{R}^n)$. Then the Cauchy Problem*

$$\begin{cases} u_t(x, t) + \nabla u(x, t) \cdot f(x) = 0, & (x, t) \in \mathbb{R}^n \times (0, T) \\ u(x, 0) = g(x), & x \in \mathbb{R}^n, \end{cases} \quad (7.2.14)$$

has a weak solution $u \in L^\infty(0, T; L^p(\mathbb{R}^n))$.

PROOF. There exist functions $g_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ and $f_\varepsilon \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $\varepsilon > 0$, such that:

- i) $\|g_\varepsilon\|_p \leq \|g\|_p$ and $g_\varepsilon \rightarrow g$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$. When $p = \infty$, the convergence has to be understood as $g_\varepsilon(x) \rightarrow g(x)$ for a.e. $x \in \mathbb{R}^n$;
- ii) $\|\text{div} f_\varepsilon\|_\infty \leq \|\text{div} f\|_\infty$, $\text{div} f_\varepsilon(x) \rightarrow \text{div} f(x)$ for a.e. $x \in \mathbb{R}^n$, and $f_\varepsilon \rightarrow f$ in $L^q_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$.

The construction of such functions is explained in the next section. Let $u_\varepsilon \in C^\infty(\mathbb{R}^n \times [0, T])$ be the solution of the Cauchy Problem

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} + \nabla u_\varepsilon \cdot f_\varepsilon = 0, & \text{in } \mathbb{R}^n \times (0, T) \\ u_\varepsilon(x, 0) = g_\varepsilon(x), & x \in \mathbb{R}^n. \end{cases} \quad (7.2.15)$$

By Proposition 7.1.3, we have for $1 \leq p < \infty$ and $0 \leq t \leq T$

$$\int_{\mathbb{R}^n} |u_\varepsilon(x, t)|^p dx \leq e^{t\|\text{div} f_\varepsilon\|_\infty} \int_{\mathbb{R}^n} |g_\varepsilon(x)|^p dx \leq e^{T\|\text{div} f\|_\infty} \int_{\mathbb{R}^n} |g(x)|^p dx.$$

Then the family $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0, T; L^p(\mathbb{R}^n))$. By Theorem 7.2.1, there exist a sequence $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ and a function $u \in L^\infty(0, T; L^p(\mathbb{R}^n))$ such that $u_{\varepsilon_j} \rightharpoonup u$ in the weakly* topology as $j \rightarrow \infty$.

We claim that u is a weak solution of the Cauchy Problem. We start from the identity

$$\int_{\mathbb{R}^n} v(x, 0)g_\varepsilon(x)dx + \int_0^T \int_{\mathbb{R}^n} u_\varepsilon(x, t) \{v_t(x, t) + \text{div}(v(x, t)f_\varepsilon(x))\} dx dt = 0, \quad (7.2.16)$$

that holds for any $v \in C^1_c(\mathbb{R}^n \times [0, T])$. We have

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} v(x, 0)g_\varepsilon(x)dx = \int_{\mathbb{R}^n} v(x, 0)g(x)dx, \quad (7.2.17)$$

$$\lim_{j \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} u_{\varepsilon_j} v_t dx dt = \int_0^T \int_{\mathbb{R}^n} u v_t dx dt, \quad (7.2.18)$$

$$\lim_{j \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} u_{\varepsilon_j} v \text{div} f_{\varepsilon_j} dx dt = \int_0^T \int_{\mathbb{R}^n} u v \text{div} f dx dt, \quad (7.2.19)$$

$$\lim_{j \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} u_{\varepsilon_j} \nabla v \cdot f_{\varepsilon_j} dx dt = \int_0^T \int_{\mathbb{R}^n} u \nabla v \cdot f dx dt. \quad (7.2.20)$$

The claims (7.2.17)–(7.2.20) follows from i), ii), and the weak* convergence. We check e.g. (7.2.19). We start from

$$\int_0^T \int_{\mathbb{R}^n} u_{\varepsilon_j} v \operatorname{div} f_{\varepsilon_j} dx dt = \int_0^T \int_{\mathbb{R}^n} u_{\varepsilon_j} v \operatorname{div} f dx dt + \int_0^T \int_{\mathbb{R}^n} u_{\varepsilon_j} v (\operatorname{div} f_{\varepsilon_j} - \operatorname{div} f_{\varepsilon_j}) dx dt,$$

where, by the weak* convergence,

$$\lim_{j \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} u_{\varepsilon_j} v \operatorname{div} f dx dt + \int_0^T \int_{\mathbb{R}^n} uv \operatorname{div} f dx dt,$$

and, by ii),

$$\lim_{j \rightarrow \infty} \int_0^T \int_{\mathbb{R}^n} u_{\varepsilon_j} v (\operatorname{div} f_{\varepsilon_j} - \operatorname{div} f_{\varepsilon_j}) dx dt = 0.$$

(Apply Hölder inequality and then use the dominated convergence theorem.)

Passing to the limit in (7.2.16) we obtain, for any $v \in C_c^1(\mathbb{R}^n \times [0, T])$,

$$\int_{\mathbb{R}^n} v(x, 0)g(x)dx + \int_0^T \int_{\mathbb{R}^n} u(x, t)\{v_t(x, t) + \operatorname{div}(v(x, t)f(x))\}dxdt = 0, \quad (7.2.21)$$

and the theorem is proved. \square

3. Approximation of weak solutions

Let $\chi \in C^\infty(\mathbb{R}^n)$ be a mollifying kernel, i.e., a non negative radial function that is supported in the unit ball with $\int_{\mathbb{R}^n} \chi(x)dx = 1$. For $\varepsilon > 0$ let

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon^n} \chi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^n.$$

For any function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, let

$$u_\varepsilon(x) = u * \chi_\varepsilon(x) = \int_{\mathbb{R}^n} u(y)\chi_\varepsilon(x - y)dy, \quad x \in \mathbb{R}^n. \quad (7.3.22)$$

If u depends on $x \in \mathbb{R}^n$ and $t \in [0, T]$, let $u_\varepsilon(x, t)$ denote the mollification in x only, and precisely

$$u_\varepsilon(x, t) = u(\cdot, t) * \chi_\varepsilon(x) = \int_{\mathbb{R}^n} u(y, t)\chi_\varepsilon(x - y)dy, \quad x \in \mathbb{R}^n. \quad (7.3.23)$$

THEOREM 7.3.1 (Stability of weak solutions). *Let $1 \leq p, q \leq \infty$ be such that $1/p + 1/q = 1$. Let $f \in W^{1,q}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, and let $u \in L^\infty(0, T; L^p(\mathbb{R}^n))$ be a weak solution of the differential equation*

$$u_t + \nabla u \cdot f = 0, \quad \text{in } \mathbb{R}^n \times (0, T). \quad (7.3.24)$$

Then the function u_ε in (7.3.23) satisfies in distributional sense the differential equation

$$\frac{\partial u_\varepsilon}{\partial t} + \nabla u_\varepsilon \cdot f = R_\varepsilon, \quad (7.3.25)$$

where $R_\varepsilon \in L^1(0, T; L^1_{\text{loc}}(\mathbb{R}^n))$ is given by

$$R_\varepsilon = f \cdot \nabla u_\varepsilon - (f \cdot \nabla u)_\varepsilon. \quad (7.3.26)$$

Moreover, for any $R > 0$ we have

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \|R_\varepsilon(\cdot, t)\|_{L^1(B_R)} dt = 0. \quad (7.3.27)$$

PROOF. We first check that the function R_ε in (7.3.26) is well-defined in distributional sense. In fact, we have

$$\begin{aligned} (f \cdot \nabla u)_\varepsilon(x, t) &= \int_{\mathbb{R}^n} f(y) \cdot \nabla u(y, t) \chi_\varepsilon(x - y) dy \\ &= - \int_{\mathbb{R}^n} u(y, t) \operatorname{div}(f(y) \chi_\varepsilon(x - y)) dy \\ &= \int_{\mathbb{R}^n} u(y, t) \left\{ f(y) \cdot \nabla \chi_\varepsilon(x - y) - \chi_\varepsilon(x - y) \operatorname{div} f(y) \right\} dy, \end{aligned}$$

and

$$f \cdot \nabla u_\varepsilon(x, t) = \int_{\mathbb{R}^n} u(y, t) f(x) \cdot \nabla \chi_\varepsilon(x - y) dy.$$

Taking the difference we get

$$R_\varepsilon(x, t) = \int_{\mathbb{R}^n} u(y, t) \left\{ (f(x) - f(y)) \cdot \nabla \chi_\varepsilon(x - y) + \chi_\varepsilon(x - y) \operatorname{div} f(y) \right\} dy. \quad (7.3.28)$$

Using this formula, we can prove that for any $R > 0$

$$\int_0^T \|R_\varepsilon(\cdot, t)\|_{L^1(B_R)} dt < \infty.$$

We skip the details, which are left to the reader. In fact, the proof of (7.3.27) will give a stronger statement.

Now we check formula (7.3.26). To this end, let $v \in C_c^1(\mathbb{R}^n \times (0, T))$ be a test function and set

$$A = \int_0^T \int_{\mathbb{R}^n} R_\varepsilon(x, t) v(x, t) dx dt.$$

We show that, if the differential equation (7.3.25) holds in distributional sense, then R_ε has the form (7.3.28). We have

$$\begin{aligned} A &= \int_0^T \int_{\mathbb{R}^n} v(x, t) \left\{ \frac{\partial u_\varepsilon(x, t)}{\partial t} + f(x) \cdot \nabla u_\varepsilon(x, t) \right\} dx dt \\ &= \int_0^T \int_{\mathbb{R}^n} \left\{ v(x, t) f(x) \cdot \nabla u_\varepsilon(x, t) - v_t(x, t) u_\varepsilon(x, t) \right\} dx dt \\ &= \int_0^T \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left\{ v(x, t) f(x) \cdot \nabla \chi_\varepsilon(x - y) - v_t(x, t) \chi_\varepsilon(x - y) \right\} u(y, t) dy dx dt \\ &= \int_0^T \int_{\mathbb{R}^n} u(y, t) \int_{\mathbb{R}^n} \left\{ v(x, t) f(x) \cdot \nabla \chi_\varepsilon(x - y) - v_t(x, t) \chi_\varepsilon(x - y) \right\} dx dy dt \end{aligned}$$

Now we use the distributional form of equation (7.2.13) for u to obtain the identity

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} u(y, t) \int_{\mathbb{R}^n} v_t(x, t) \chi_\varepsilon(x - y) dx dy dt &= \int_0^T \int_{\mathbb{R}^n} u(y, t) \partial_t v_\varepsilon(y, t) dy dt \\ &= - \int_0^T \int_{\mathbb{R}^n} u(y, t) \operatorname{div}(v_\varepsilon f)(y, t) dy dt. \end{aligned}$$

Substituting this identity above we get

$$A = \int_0^T \int_{\mathbb{R}^n} u(y, t) \left\{ \operatorname{div}(v_\varepsilon f)(y, t) - \operatorname{div}(v f)_\varepsilon(y, t) \right\} dy dt. \quad (7.3.29)$$

The curly bracket in (7.3.29) is

$$\begin{aligned} \{ \dots \} &= f(y) \cdot \nabla v_\varepsilon(y, t) + v_\varepsilon(y, t) \operatorname{div} f(y) - \int_{\mathbb{R}^n} \operatorname{div}(v f)(x, t) \chi_\varepsilon(y - x) dx \\ &= \int_{\mathbb{R}^n} v(x, t) \left\{ (f(y) - f(x)) \cdot \nabla \chi_\varepsilon(y - x) + \chi_\varepsilon(y - x) \operatorname{div} f(y) \right\} dx. \end{aligned}$$

Comparing with (7.3.28), our claim (7.3.26) is now proved.

Finally, we prove statement (7.3.27). We first notice that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} u(y, t) \operatorname{div} f(y) \chi_\varepsilon(x - y) dy = u(x, t) \operatorname{div} f(x),$$

in $L^1(0, T; L^1_{\text{loc}}(\mathbb{R}^n))$, because $u \operatorname{div} f \in L^1(0, T; L^1_{\text{loc}}(\mathbb{R}^n))$. It is thus enough to prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} u(y, t) (f(y) - f(x)) \cdot \nabla \chi_\varepsilon(x - y) dy = u(x, t) \operatorname{div} f(x), \quad (7.3.30)$$

in $L^1(0, T; L^1_{\text{loc}}(\mathbb{R}^n))$.

Consider the ‘‘bilinear form’’ in u and f

$$\Delta_\varepsilon[u, f](x, t) := \int_{\mathbb{R}^n} u(y, t) (f(y) - f(x)) \cdot \nabla \chi_\varepsilon(x - y) dy.$$

Step 1. In a first step, we show that the convergence (7.3.30) holds pointwise when u and f are of class C^1 . In this case, we have

$$\Delta_\varepsilon[u, f](x, t) = \int_{\mathbb{R}^n} \operatorname{div}(u(y, t) (f(y) - f(x))) \chi_\varepsilon(x - y) dy,$$

and

$$\lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon[u, f](x, t) = u(x, t) \operatorname{div} f(x).$$

The convergence clearly holds also in $L^1(0, T; L^1_{\text{loc}}(\mathbb{R}^n))$, if u and f are of class C^1 .

Step 2. Now let $0 < \varepsilon < 1$. By Hölder's inequality,

$$\begin{aligned} \int_{B_R} |\Delta_\varepsilon[u, f](x, t)| dx &\leq \int_{\mathbb{R}^n} |u(y, t)| \int_{B_R} |f(y) - f(x)| |\nabla \chi_\varepsilon(x - y)| dx dy \\ &\leq \|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \left(\int_{B_R} |f(y) - f(x)| |\nabla \chi_\varepsilon(x - y)| dx \right)^q dy \right)^{1/q} \\ &\leq \frac{\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \|\nabla \chi\|_\infty}{\varepsilon^n} \left(\int_{B_{R+1}} \left(\int_{|x-y|<\varepsilon} \frac{|f(y) - f(x)|}{\varepsilon} dx \right)^q dy \right)^{1/q}. \end{aligned}$$

By the integral Minkowski inequality,

$$\begin{aligned} \left(\int_{B_{R+1}} \left(\int_{|x-y|<\varepsilon} \frac{|f(y) - f(x)|}{\varepsilon} dx \right)^q dy \right)^{1/q} &\leq \varepsilon^n \left(\int_{B_{R+1}} \left(\int_{|z|<1} \int_0^1 |\nabla f(x + t\varepsilon z)| dt dz \right)^q dy \right)^{1/q} \\ &\leq \varepsilon^n \int_{|z|<1} \int_0^1 \left(\int_{B_{R+1}} |\nabla f(x + t\varepsilon z)|^q dx \right)^{1/q} dt dz \\ &\leq \omega_n \varepsilon^n \left(\int_{B_{R+2}} |\nabla f(x)|^q dx \right)^{1/q}. \end{aligned}$$

We eventually find the estimate

$$\int_{B_R} |\Delta_\varepsilon[u, f](x, t)| dx \leq \omega_n \|\nabla \chi\|_\infty \|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \|\nabla f\|_{L^q(B_{R+2})}. \quad (7.3.31)$$

The computation is rigorous if f is of class C^1 . By approximation, estimate (7.3.31) also holds when $f \in W_{\text{loc}}^{1,q}(\mathbb{R}^n; \mathbb{R}^n)$.

Step 3. For $\sigma > 0$ let $u^{(\sigma)}$ and $f^{(\sigma)}$ be of class C^1 such that:

- i) $\|u^{(\sigma)}\|_{L^\infty(0,T;L^p(\mathbb{R}^n))} \leq \|u\|_{L^\infty(0,T;L^p(\mathbb{R}^n))}$ and $\|u^{(\sigma)} - u\|_{L^p(0,T;\mathbb{R}^n)} \leq \sigma$;
- ii) $\|\nabla f^{(\sigma)}\|_{L^q(B_{R+2})} \leq \|\nabla f\|_{L^q(B_{R+2})}$ and $\|\nabla f^{(\sigma)} - \nabla f\|_{L^q(B_{R+2})} \leq \sigma$;
- iii) $\|u^{(\sigma)} \operatorname{div} f^{(\sigma)} - u \operatorname{div} f\|_{L^1(0,T;L^1_{\text{loc}}(\mathbb{R}^n))} \leq \sigma$.

Denoting by $\|\cdot\|$ the relevant norm, we have

$$\begin{aligned} \|\Delta_\varepsilon[u, f] - u \operatorname{div} f\| &\leq \|\Delta_\varepsilon[u - u^{(\sigma)}, f]\| + \|\Delta_\varepsilon[u^{(\sigma)}, f - f^{(\sigma)}]\| + \\ &\quad + \|\Delta_\varepsilon[u^{(\sigma)}, f^{(\sigma)}] - u^{(\sigma)} \operatorname{div} f^{(\sigma)}\| + \|u^{(\sigma)} \operatorname{div} f^{(\sigma)} - u \operatorname{div} f\| \\ &\leq C\sigma + \|\Delta_\varepsilon[u^{(\sigma)}, f^{(\sigma)}] - u^{(\sigma)} \operatorname{div} f^{(\sigma)}\| \end{aligned}$$

where C is a constant independent of ε . Now our claim follows from the Step 2. \square

4. Uniqueness of solutions

THEOREM 7.4.1. *Let $1 \leq p, q \leq \infty$ be such that $1/p + 1/q = 1$. Let $f \in W_{\text{loc}}^{1,q}(\mathbb{R}^n; \mathbb{R}^n)$ be a vector field such that $\operatorname{div} f \in L^\infty(\mathbb{R}^n)$ and*

$$\frac{|f(x)|}{1 + |x|} \in L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n).$$

Let $u \in L^\infty(0, T; L^p(\mathbb{R}^n))$ be a weak solution of the Cauchy problem

$$\begin{cases} u_t + \nabla u \cdot f = 0, & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, 0) = 0, & x \in \mathbb{R}^n. \end{cases} \quad (7.4.32)$$

Then we have $u = 0$.

PROOF. Let u_ε , $\varepsilon > 0$, be an approximating function of the solution u as in (7.3.23) and let $\beta \in \text{Lip}(\mathbb{R})$ be a non negative, bounded Lipschitz function. Precisely, let

$$\beta(t) = \min\{|t|^p, M\}, \quad t \in \mathbb{R},$$

where $M > 0$ is an arbitrary constant.

Because the function u_ε is a solution to equation (7.3.25), the function $\beta(u_\varepsilon)$ is a distributional solution to the differential equation

$$\partial_t \beta(u_\varepsilon) + \nabla \beta(u_\varepsilon) \cdot f = \beta'(u_\varepsilon) R_\varepsilon, \quad (7.4.33)$$

where R_ε is as in 7.3.26. In particular, we have (7.3.27). As β' stays bounded, taking the limit as $\varepsilon \rightarrow 0$ in (7.4.33) we deduce that $\beta(u)$ is a distributional solution to the differential equation

$$\partial_t \beta(u) + \nabla \beta(u) \cdot f = 0. \quad (7.4.34)$$

We take a test function of the form

$$v(x, t) = \psi(t) \varphi_R(x), \quad t \in [0, T], \quad x \in \mathbb{R}^n,$$

where $\psi \in C_c^1(0, T)$ and $\varphi_R \in C^1(\mathbb{R}^n)$ is a function such that $\varphi_R(x) = 1$ for $|x| \leq R$, $\varphi_R(x) = 0$ for $|x| \geq 2R$, and $|\nabla \varphi_R| \leq 2/R$, $R > 0$. Using such a test function in the distributional equation (7.4.34), we get

$$\int_0^T \psi'(t) \int_{\mathbb{R}^n} \beta(u) \varphi_R dx dt = - \int_0^T \psi(t) \int_{\mathbb{R}^n} \beta(u) \text{div}(\varphi_R f) dx dt.$$

In particular, the function

$$t \mapsto \int_{\mathbb{R}^n} \beta(u(\cdot, t)) \varphi_R dx$$

is absolutely continuous and

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} \beta(u(\cdot, t)) \varphi_R dx &= \int_{\mathbb{R}^n} \beta(u(\cdot, t)) \text{div}(\varphi_R f) dx = \\ &= \int_{\mathbb{R}^n} \beta(u(\cdot, t)) (\varphi_R \text{div} f + \nabla \varphi_R \cdot f) dx \\ &\leq \|\text{div} f\|_\infty \int_{\mathbb{R}^n} \beta(u(\cdot, t)) \varphi_R dx + \frac{2}{R} \int_{R \leq |x| \leq 2R} \beta(u(\cdot, t)) |f| dx. \end{aligned}$$

Now we have for $R > 1$

$$\begin{aligned} \frac{2}{R} \int_{R \leq |x| \leq 2R} \beta(u) |f| dx &\leq 6 \int_{|x| \geq R} \frac{|f|}{1 + |x|} \beta(u) dx \\ &\leq 6 \int_{|x| \geq R} g_1 \beta(u) dx + 6 \int_{|x| \geq R} g_2 \beta(u) dx, \end{aligned}$$

where by assumption

$$g_1 \in L^1(\mathbb{R}^n) \quad \text{and} \quad g_2 \in L^\infty(\mathbb{R}^n).$$

Because $\beta(u(\cdot, t)) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we let $R \rightarrow \infty$ and obtain

$$\int_{\mathbb{R}^n} \beta(u(x, t)) dx \leq \|\operatorname{div} f\|_\infty \int_0^t \int_{\mathbb{R}^n} \beta(u(x, s)) dx ds.$$

Letting $M \rightarrow \infty$ we obtain

$$\int_{\mathbb{R}^n} |u(x, t)|^p dx \leq \|\operatorname{div} f\|_\infty \int_0^t \int_{\mathbb{R}^n} |u(x, s)|^p dx ds.$$

This integral inequality and the initial condition

$$\int_{\mathbb{R}^n} |u(x, 0)|^p dx = 0,$$

imply that

$$\int_{\mathbb{R}^n} |u(x, t)|^p dx = 0,$$

for any t . This finally yields $u = 0$. □

Boundary value problems. Examples

1. Linear boundary value problem

1.1. The equation $y'' = f(x)$.

PROPOSITION 8.1.1. *Let $f \in C([0, 1])$ be a continuous function and let $a, b \in \mathbb{R}$ be real numbers. The boundary value problem*

$$\begin{cases} y''(x) = f(x), & x \in [0, 1], \\ y(0) = a, \\ y(1) = b, \end{cases} \quad (8.1.1)$$

has a unique solution $y \in C^2([0, 1])$.

PROOF. Integrating the differential equation we get

$$y'(x) = y'(0) + \int_0^x f(t) dt, \quad x \in [0, 1],$$

where $y'(0) \in \mathbb{R}$. Integrating again, we obtain for $x \in [0, 1]$

$$y(x) = y(0) + \int_0^x \left\{ y'(0) + \int_0^t f(s) ds \right\} dt = a + y'(0)x + \int_0^x \int_0^t f(s) ds dt.$$

The number $y'(0) \in \mathbb{R}$ is uniquely determined by the condition $y(1) = b$. Precisely,

$$y'(0) = b - a - \int_0^1 \int_0^t f(s) ds dt.$$

Then we find

$$y(x) = a + \left\{ b - a - \int_0^1 \int_0^t f(s) ds dt \right\} x + \int_0^x \int_0^t f(s) ds dt. \quad (8.1.2)$$

This function is the unique solution of (8.1.1). \square

REMARK 8.1.2. Consider the normed spaces

$$Y = C^2([0, 1]) \quad \text{with} \quad \|y\|_Y = \max_{[0,1]} |y| + \max_{[0,1]} |y'| + \max_{[0,1]} |y''|, \quad (8.1.3)$$

and

$$Z = C([0, 1]) \times \mathbb{R}^2 \quad \text{with} \quad \|(f, a, b)\|_Z = \max_{[0,1]} |f| + |a| + |b|. \quad (8.1.4)$$

Both $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are Banach spaces.

Consider the mapping $G : Z \rightarrow Y$ such that $G(f, a, b) = y$ if and only if y is the (unique) solution of the boundary value problem (8.1.1). Then G is linear (easy exercise). Moreover, G is bounded (i.e., continuous), and precisely

$$\|G(f, a, b)\|_Y = \|y\|_Y \leq 5 \left(\max_{[0,1]} |f| + |a| + |b| \right). \quad (8.1.5)$$

In fact, we have $|y(x)| \leq |a| + |b| + 2\|f\|_\infty$,

$$|y'(x)| = \left| b - a - \int_0^1 \int_0^t f(s) ds dt + \int_0^x f(s) ds \right| \leq |a| + |b| + 2\|f\|_\infty,$$

and $|y''(x)| \leq \|f\|_\infty$.

1.2. The equation $y'' = \lambda y$. Let $\lambda \in \mathbb{R}$. We look for a solution $y \in C^2([0, \pi])$ to the boundary value problem

$$\begin{cases} -y'' = \lambda y, & \text{in } [0, \pi], \\ y(0) = 0, \\ y(\pi) = 0. \end{cases} \quad (8.1.6)$$

The function $y = 0$ is a solution, for any $\lambda \in \mathbb{R}$. Moreover, if y is a solution, then αy is also a solution for any $\alpha \in \mathbb{R}$. We look for non trivial solution.

Case 1: $\lambda < 0$. The general solution of the differential equation is

$$y(x) = C_1 e^{-\sqrt{|\lambda|x}} + C_2 e^{\sqrt{|\lambda|x}}, \quad C_1, C_2 \in \mathbb{R}.$$

The boundary value $y(0) = 0$ implies $C_1 = -C_2$. The boundary value $y(\pi) = 0$ implies then $C_1(e^{-\sqrt{|\lambda|\pi}} - e^{\sqrt{|\lambda|\pi}}) = 0$. With $\lambda < 0$, this implies $C_1 = 0$. Then $y = 0$ is the unique solution of the boundary value problem.

Case 2: $\lambda > 0$. The general solution of the differential equation is

$$y(x) = C_1 \sin(\sqrt{\lambda}x) + C_2 \cos(\sqrt{\lambda}x), \quad C_1, C_2 \in \mathbb{R}.$$

The boundary value $y(0) = 0$ implies $C_2 = 0$. The boundary value $y(\pi) = 0$ implies then $C_1 \sin(\lambda\pi) = 0$. This equation has non trivial solutions when $\sqrt{\lambda}\pi = n\pi$ with $n \in \mathbb{Z}$. It is sufficient to consider the case $n \in \mathbb{N}$. The numbers

$$\lambda_n = n^2, \quad n \in \mathbb{N},$$

are the eigenvalues of the differential operator $\mathcal{L} : C^2([0, \pi]) \rightarrow C([0, \pi])$, $\mathcal{L}y = -y''$, with the so called Dirichlet boundary condition $y(0) = y(\pi) = 0$.

2. Linearization methods

In this section we consider the boundary value problem

$$\begin{cases} y'' + g(x, y, y') = f, & x \in [0, 1], \\ y(0) = a, \\ y(1) = b, \end{cases} \quad (8.2.7)$$

where $a, b \in \mathbb{R}$, $f \in C([0, 1])$, and $g \in C([0, 1] \times \mathbb{R}^2)$. We denote the variables of g by x, p, q , respectively, i.e., $x \in [0, 1]$ and $(p, q) \in \mathbb{R}^2$.

THEOREM 8.2.1. Assume that $g \in C^1([0, 1] \times \mathbb{R}^2)$ and the partial derivatives

$$\frac{\partial g(x, 0, 0)}{\partial p} = \frac{\partial g(x, 0, 0)}{\partial q} = 0, \quad x \in [0, 1]. \quad (8.2.8)$$

Then there exist $\varepsilon > 0$ and $\delta > 0$ such that for all $|a|, |b| < \varepsilon$ and $f \in C([0, 1])$ with $\|f\|_\infty \leq \varepsilon$ the boundary value problem (8.2.7) has a solution $y \in C^2([0, 1])$. This solution is unique in the ball $B_Y(0, \delta)$ of the Banach space Y introduced in (8.1.3).

The proof of the Theorem is based on the Inverse Function Theorem in Banach spaces.

THEOREM 8.2.2. Let Y and Z be Banach spaces, $F : Y \rightarrow X$ and $y_0 \in Y$. Assume that:

- i) $F \in C^1(Y; Z)$, i.e., the Frechet derivative $DF(y) \in L(Y; X)$ exists for all $y \in Y$ and $y \mapsto DF(y)$ is continuous;
- ii) $DF(y_0) \in L(Y; Z)$ is invertible and $DF(y_0)^{-1} : Z \rightarrow Y$ is continuous.

Then there exists $\delta > 0$ such that $F : B_Y(0, \delta) \rightarrow F(B_Y(0, \delta))$ is a C^1 -diffeomorphism.

PROOF OF THEOREM 8.2.1. Let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be the Banach space introduced in (8.1.3) and (8.1.4). Define $F : Y \rightarrow Z$

$$F(y) = (y'' + g(x, y, y'), y(0), y(1)).$$

We prove that:

- i) $F \in C^1(Y; Z)$;
- ii) $DF(0) : Y \rightarrow Z$, the Frechet derivative of F at $y = 0$, is invertible with a continuous inverse.

The claim of the theorem follows from the Inverse Function Theorem.

The directional derivative of F at the point $y \in Y$ in the direction $v \in Y$ is

$$\begin{aligned} \frac{\partial F(y)}{\partial v} &= \lim_{t \rightarrow 0} \frac{F(y + tv) - F(y)}{t} \\ &= \lim_{t \rightarrow 0} \left(z'' + \frac{g(x, y + tv, y' + tv') - g(x, y, y')}{t}, v(0), v(1) \right) \\ &= (v'' + g_p(x, y, y')v + g_q(x, y, y')v', v(0), v(1)). \end{aligned}$$

Here, g_p and g_q are the partial derivatives of g w.r.t. p and q , respectively. The convergence of the first component is uniform in $x \in [0, 1]$. Then the limit is actually in Z . In fact, by Lagrange theorem

$$g(x, y + tv, y' + tv') - g(x, y, y') = g_p(x, y^*(t), y' + tv')tv + g_q(x, y, y'^*(t))tv',$$

with $y^*(t) \in [y, y + tv]$ and $y'^*(t) \in [y', y' + tv']$. Then we have $y^*(t) \rightarrow y$ and $y'^*(t) \rightarrow y'$ as $t \rightarrow 0$ and the limits are uniform in $x \in [0, 1]$. The functions g_p and g_q are continuous, and they may be assumed to be uniformly continuous. It follows that

$$\lim_{t \rightarrow 0} \max_{x \in [0, 1]} \left| \frac{g(x, y + tv, y' + tv') - g(x, y, y')}{t} - g_p(x, y, y')v - g_q(x, y, y')v' \right| = 0.$$

For any fixed $y \in Y$, the mapping $T_y : Y \rightarrow Z$ defined by $T_y z = \partial F(y)/\partial z$ is linear and continuous. Moreover, a computation analogous to the one made above shows that

$$\lim_{z \rightarrow y} \frac{\|F(z) - F(y) - T(y)(z - y)\|_Z}{\|z - y\|_Y} = 0.$$

Thus F is differentiable at $y \in Y$ and $DF(y) = T_y$.

It remains to show that $y \mapsto T_y$ is continuous from Y to $L(Y; Z)$ with the operator norm. This follows from the estimate

$$\begin{aligned} \|T_{y_1} - T_{y_2}\| &:= \sup_{\|v\|_Y \leq 1} \|T_{y_1}v - T_{y_2}v\|_Z \\ &\leq \max_{x \in [0,1]} \{|g_p(x, y_1, y'_1) - g_p(x, y_2, y'_2)| + |g_q(x, y_1, y'_1) - g_q(x, y_2, y'_2)|\}, \end{aligned}$$

which is left as an exercise. This ends the proof of *i*.

At the point $y = 0$, we have by (8.2.8)

$$DF(0)z = (z'', z(0), z(1)).$$

As proved in Proposition 8.1.1 and Remark 8.1.2, the linear map $DF(0) : Y \rightarrow Z$ is invertible and its inverse $G : Z \rightarrow Y$ satisfies (8.1.5), i.e., it is continuous.

□

CHAPTER 9

Compactness

1. Introduction

A metric space (X, d) is compact if any open covering of X has a finite sub-covering. A metric space X is separable if there is a subset $X_0 \subset X$ such that $\text{Card}(X_0) \leq \aleph_0$ and $\bar{X}_0 = X$. A metric space is complete if any Cauchy sequence is convergent.

PROPOSITION 9.1.1. *Compact metric spaces are separable.*

PROOF. By compactness, for any $k \in \mathbb{N}$ there is a finite number of points $x_1^k, \dots, x_{n_k}^k \in X$ such that

$$X = \bigcup_{i=1}^{n_k} B(x_i^k, 1/k).$$

The set $X_0 = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{n_k} \{x_i^k\}$ is at most countable and it is dense in X . \square

2. Characterization of compact metric spaces

DEFINITION 9.2.1 (Totally bounded metric space). A metric space (X, d) is totally bounded, if for any $\varepsilon > 0$ there are points $x_1, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^n B(x_i, \varepsilon)$.

THEOREM 9.2.2. *Let (X, d) be a metric space. The following statements are equivalent:*

- i) X is compact.
- ii) If $A \subset X$ is a set such that $\text{Card}(A) \geq \aleph_0$ then

$$\{x \in X : B_r(x) \setminus \{x\} \cap A \neq \emptyset \text{ for all } r > 0\} \neq \emptyset. \quad (9.2.9)$$

- iii) Any sequence $(x_n)_{n \in \mathbb{N}}$ in X has a convergent sub-sequence $(x_{n_j})_{j \in \mathbb{N}}$.
- iv) X is complete and totally bounded.

PROOF. i) \Rightarrow ii). Let X be compact and $A \subset X$ be a subset which is not finite. Assume by contradiction that (9.2.9) does not hold. Then for all $x \in X$ there exists $r_x > 0$ such that

$$B(x, r_x) \setminus \{x\} \cap A = \emptyset.$$

Because $X = \bigcup_{x \in X} B(x, r_x)$, by the compactness of X there exist $x_1, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^n B(x_i, r_{x_i})$. It follows that

$$A = A \cap X = \bigcup_{i=1}^n A \cap B(x_i, r_{x_i}) \subset \bigcup_{i=1}^n \{x_i\},$$

and A must be a finite set.

ii) \Rightarrow iii). If the cardinality of the set $A = \{x_n \in X : n \in \mathbb{N}\}$ is finite, then the sequence $(x_n)_{n \in \mathbb{N}}$ has a constant subsequence. If the cardinality of A is not finite, then there exists a point $x \in \{x \in X : B_r(x) \setminus \{x\} \cap A \neq \emptyset \text{ for all } r > 0\}$. Then for all $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $x_{n_k} \in B(x, 1/k)$. Moreover, the choice of n_k can be made in such a way that the sequence $k \mapsto n_k$ is increasing. The subsequence $(x_{n_k})_{k \in \mathbb{N}}$ converges to x .

iii) \Rightarrow iv). We show that X is a complete metric space. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence, i.e. for all $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that $d(x_n, x_m) \leq \varepsilon$ for all $n, m \geq \bar{n}$. There exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ which converges to $x \in X$, i.e. $d(x, x_{n_k}) \leq \varepsilon$ for $k \geq \bar{k}$. Then,

$$d(x, x_n) \leq d(x, x_{n_k}) + d(x_{n_k}, x_n) \leq 2\varepsilon$$

as soon as $k \geq \bar{k}$ and $n, n_k \geq \bar{n}$. This shows that $x_n \rightarrow x$.

We show that X is totally bounded. By contradiction there is $\varepsilon > 0$ such that there is no finite covering of X with balls with radius ε . Take $x_1 \in X$, $x_2 \in X \setminus B(x_1, \varepsilon)$ and by induction $x_n \in X \setminus \bigcup_{i=1}^{n-1} B(x_i, \varepsilon)$. The sequence $(x_n)_{n \in \mathbb{N}}$ satisfies $d(x_n, x_m) \geq \varepsilon$ for all $n \neq m$, and it has non converging subsequence.

iv) \Rightarrow i). Assume by contradiction that X is not compact. Then there is an open covering \mathcal{A} of X , $X = \bigcup_{\Omega \in \mathcal{A}} \Omega$, which has no finite subcovering. There are closed balls $B_1^1, \dots, B_{n_1}^1$ with radius 1 such that $X = \bigcup_{i=1}^{n_1} B_i^1$. In particular, there exists a ball $B_{i_1}^1$, $1 \leq i_1 \leq n_1$ which is not covered by a finite number of open sets $\Omega \in \mathcal{A}$. The ball $B_{i_1}^1$ is totally bounded, and then there exist closed balls $B_1^2, \dots, B_{n_2}^2$, relative to $B_{i_1}^1$, with radius $1/2$ such that $B_{i_1}^1 = \bigcup_{i=1}^{n_2} B_i^2$. In particular, there exists a ball $B_{i_2}^2$ which can not be covered by a finite number of open sets $\Omega \in \mathcal{A}$.

By induction, for any $k \in \mathbb{N}$ we find a closed ball $B_{i_k}^k$, relative to $B_{i_{k-1}}^{k-1}$, with radius $1/k$ which can not be covered by a finite number of open sets $\Omega \in \mathcal{A}$.

Because X is complete, the decreasing sequence of closed sets $(B_{i_k}^k)_{k \in \mathbb{N}}$ has non empty intersection, i.e. there exists $x \in \bigcap_{k=1}^{\infty} B_{i_k}^k$. On the other hand, $x \in \Omega$ for some $\Omega \in \mathcal{A}$ and there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \Omega$. With $k \in \mathbb{N}$ such that $1/k < \varepsilon/2$ it is $B_{i_k}^k \subset B(x, \varepsilon) \subset \Omega$. This is a contradiction, because $B_{i_k}^k$ cannot be covered with a finite number of open sets from \mathcal{A} . \square

3. Ascoli-Arzelà theorem

Let (X, d) be a compact metric space and let $Y = C(X; \mathbb{R}^n)$ be the vector space of continuous real valued functions on X . The vector space Y endowed with the norm

$$\|y\| = \max_{x \in X} |y(x)|$$

is a Banach space.

DEFINITION 9.3.1. i) A subset $K \subset C(X; \mathbb{R}^n)$ is said to be *equibounded* if there exists a constant $0 \leq M < +\infty$ such that

$$\sup_{y \in K} \|y\| \leq M.$$

ii) A subset $K \subset C(X; \mathbb{R}^n)$ is said to be *equicontinuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x_1, x_2) \leq \delta \Rightarrow \sup_{y \in K} |y(x_1) - y(x_2)| \leq \varepsilon.$$

THEOREM 9.3.2 (Ascoli-Arzelà). *Let (X, d) be a compact metric space and $K \subset C(X; \mathbb{R}^n)$. Then the following are equivalent:*

- i) K is compact in $C(X; \mathbb{R}^n)$;
- ii) K is closed, equibounded and equicontinuous.

PROOF. i) \Rightarrow ii) Assume that K is compact in $C(X; \mathbb{R}^n)$. Then K is closed. By Theorem 9.2.2, K is totally bounded and in particular it is (equi)bounded. We show that K is equicontinuous.

Let $\varepsilon > 0$. For K is totally bounded, there exist $y_1, \dots, y_k \in K$ such that

$$K \subset \bigcup_{i=1}^k B_Y(y_i, \varepsilon),$$

where $B_Y(y_i, \varepsilon)$ are balls in the distance of $C(X; \mathbb{R}^n)$. Each function y_i , $i = 1, \dots, k$, is continuous on X , which is compact. Then each y_i is uniformly continuous on X , and there exists $\delta > 0$ such that

$$d(x_1, x_2) \leq \delta \Rightarrow \sup_{i=1, \dots, k} |y_i(x_1) - y_i(x_2)| \leq \varepsilon.$$

For any $y \in K$ there is $i \in \{1, \dots, k\}$ such that $y \in B_Y(y_i, \varepsilon)$. It follows that for $d(x_1, x_2) \leq \delta$ we have

$$|y(x_1) - y(x_2)| \leq |y(x_1) - y_i(x_1)| + |y_i(x_1) - y_i(x_2)| + |y_i(x_2) - y(x_2)| \leq 3\varepsilon.$$

This shows that K is equicontinuous.

ii) \Rightarrow i) Assume that K is closed, equibounded, and equicontinuous. We show that any sequence in K has a sub-sequence converging in K . By Theorem 9.2.2, it follows that K is compact.

The metric space X is separable, because it is compact. Then there exists a countable subset $X_0 = \{x_k \in X : k \in \mathbb{N}\}$ such that $\bar{X}_0 = X$.

Let $(y_h)_{h \in \mathbb{N}}$ be a sequence in K . From the bound

$$\sup_{h \in \mathbb{N}} |y_h(x_1)| \leq M < +\infty$$

it follows that there exist $\alpha_1 \in \mathbb{R}^n$ and a subsequence $(y_h^1)_{h \in \mathbb{N}}$ of $(y_h)_{h \in \mathbb{N}}$ such that $y_h^1(x_1) \rightarrow \alpha_1$. Analogously, because it is

$$\sup_{h \in \mathbb{N}} |y_h^1(x_2)| \leq M < +\infty,$$

there exist $\alpha_2 \in \mathbb{R}^n$ and a subsequence $(y_h^2)_{h \in \mathbb{N}}$ of $(y_h^1)_{h \in \mathbb{N}}$ such that $y_h^2(x_2) \rightarrow \alpha_2$. By induction on $k \in \mathbb{N}$, we find $\alpha_k \in \mathbb{R}^n$ and $(y_h^k)_{h \in \mathbb{N}}$ subsequence of $(y_h^{k-1})_{h \in \mathbb{N}}$ such that $y_h^k(x_k) \rightarrow \alpha_k$.

The sequence $(z_h)_{h \in \mathbb{N}} = (y_h^k)_{h \in \mathbb{N}}$ is a subsequence of $(y_h)_{h \in \mathbb{N}}$ and moreover it is definitively a subsequence of any $(y_h^k)_{h \in \mathbb{N}}$. Thus

$$\lim_{h \rightarrow \infty} z_h(x_k) = \alpha_k, \quad \text{for all } k \in \mathbb{N}.$$

We claim that $(z_h)_{h \in \mathbb{N}}$ is a Cauchy sequence in $C(X; \mathbb{R}^n)$. By completeness, it converges uniformly to some $y \in K$, because K is closed and $z_h \in K$ for all $h \in \mathbb{N}$.

Let $\varepsilon > 0$. By the equicontinuity of K there exists $\delta > 0$ such that

$$d(x_1, x_2) \leq \delta \quad \Rightarrow \quad \sup_{h \in \mathbb{N}} |z_h(x_1) - z_h(x_2)| \leq \varepsilon.$$

As X_0 is dense in X , it is $X = \bigcup_{i=1}^{\infty} B_X(x_i, \delta)$, and by compactness there is $p \in \mathbb{N}$ such that

$$X = \bigcup_{i=1}^p B_X(x_i, \delta).$$

For any $i = 1, \dots, p$ the real sequences $(z_h(x_i))_{h \in \mathbb{N}}$ are Cauchy sequences in \mathbb{R}^n , because they converge. Then there exists $\bar{n} \in \mathbb{N}$ such that

$$|y_h(x_i) - y_k(x_i)| \leq \varepsilon \quad \text{for all } h, k \geq \bar{n} \text{ and for all } i = 1, \dots, p.$$

For any $x \in X$ there exists $i \in \{1, \dots, p\}$ such that $x \in B_X(x_i, \delta)$, and thus

$$|z_h(x) - z_k(x)| \leq |z_h(x) - z_h(x_i)| + |z_h(x_i) - z_k(x_i)| + |z_k(x_i) - z_k(x)| \leq 3\varepsilon,$$

as soon as $h, k \geq \bar{n}$. The number \bar{n} does not depend on x . This proves that $\|z_h - z_k\| \leq 3\varepsilon$ for all $h, k \geq \bar{n}$. \square

The proof of Theorem 9.3.2 shows that the assumption $K = \bar{K}$ can be dropped. In this case, Ascoli-Arzelà theorem reads as follows:

COROLLARY 9.3.3. *Let (X, d) be a compact metric space and $K \subset C(X; \mathbb{R}^n)$. Then the following are equivalent:*

- i) *Any sequence in K has a converging subsequence (in $C(X; \mathbb{R}^n)$);*
- ii) *K is equibounded and equicontinuous.*

A set K in a metric space such that \bar{K} is compact is called precompact.

EXAMPLE 9.3.4. Let $X = [0, 1]$ and $Y = C([0, 1]; \mathbb{R}^n)$, $n \in \mathbb{N}$, with the norm

$$\|y\| = \max_{x \in [0, 1]} |y(x)|.$$

Fix a point $y_0 \in \mathbb{R}^n$, $M \geq 0$ and consider the set

$$K = \{y \in Y : y(0) = y_0, y \in C^1([0, 1]; \mathbb{R}^n), \|y'\| \leq M\}.$$

We show that K is equibounded and equicontinuous. In fact, for any $x \in [0, 1]$ we have

$$|y(x) - y_0| \leq \int_0^x |y'(t)| dt \leq Mx \leq M.$$

And thus $y \in B_Y(y_0, M)$ for all $y \in K$. Moreover,

$$|y(x_1) - y(x_2)| \leq M|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in [0, 1].$$

The functions in K are equi-Lipschitz continuous with Lipschitz constant M . K is thus equicontinuous.

By Corollary 9.3.3, any sequence in K has a converging subsequence. In general, the limit of the subsequence is not in K , because K is not closed.

4. Compattezza negli spazi di Lebesgue

Un insieme K in uno spazio metrico si dice precompatto se la sua chiusura \bar{K} è compatta.

THEOREM 9.4.1 (Riesz-Kolmogorov). *Un insieme $K \subset L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, è precompatto se e solo se:*

- i) $\sup_{u \in K} \int_{\mathbb{R}^n} |u(x)|^p dx < \infty$ (uniforme limitatezza);
- ii) $\lim_{h \rightarrow 0} \sup_{u \in K} \int_{\mathbb{R}^n} |u(x+h) - u(x)|^p dx = 0$ (uniforme continuità in media);
- iii) $\lim_{R \rightarrow \infty} \sup_{u \in K} \int_{|x| \geq R} |u(x)|^p dx = 0$ (convergenza uniforme a zero all'infinito).

DIM. Osserviamo in primo luogo che $L^p(\mathbb{R}^n)$ è uno spazio di Banach e dunque la precompattezza di un suo sottoinsieme equivale alla sua totale limitatezza.

Supponiamo dunque che $K \subset L^p(\mathbb{R}^n)$ sia totalmente limitato, e controlliamo i), ii) e iii). Sia $\varepsilon > 0$ e supponiamo che risulti $K \subset \bigcup_{i=1}^k B_{L^p(\mathbb{R}^n)}(u_i, \varepsilon)$ con $u_1, \dots, u_k \in K$. Posto $M = \varepsilon + \sup_{i \in \{1, \dots, k\}} \|u_i\|_p$ è immediato verificare che $\sup_{u \in K} \|u\|_p \leq M$. In modo analogo si provano ii) e iii).

Supponiamo viceversa che siano verificate i), ii), iii) e proviamo la totale limitatezza di K . Per un generico $r > 0$, si introducono le regolarizzazioni di media

$$u_r(x) = \int_{B(x,r)} u(y) dy = \frac{1}{\omega_n r^n} \int_{B(x,r)} u(y) dy,$$

dove $B(x, r)$ sono palle Euclidee.

Proveremo i seguenti fatti:

- 1) La famiglia delle medie $K_r = \{u_r : u \in K\}$ è equicontinua ed equilimitata. Dunque, fissato un insieme compatto $Q \subset \mathbb{R}^n$, la famiglia $K_r(Q)$ delle medie ristrette a Q è precompatta nella topologia della convergenza uniforme (per il Teorema di Ascoli-Arzelà) e dunque anche nella topologia di $L^p(Q)$. In particolare $K_r(Q)$ è totalmente limitato in $L^p(Q)$.
- 2) Si ha la convergenza uniforme

$$\lim_{r \rightarrow 0} \sup_{u \in K} \|u_r - u\|_p = 0.$$

- 3) Usando l'ipotesi iii) e la convergenza uniforme si prova che K_r è totalmente limitato in $L^p(\mathbb{R}^n)$.
- 4) Di conseguenza, K è totalmente limitato, in quanto limite uniforme di insiemi totalmente limitati.

Iniziamo a provare la equilimitatezza delle medie. Ad esempio, nel caso $p > 1$ si ha per ogni $u \in K$ e $x \in \mathbb{R}^n$

$$|u_r(x)| \leq \int_{B(x,r)} |u(y)| dy \leq (\omega_n r^n)^{-1 + \frac{p-1}{p}} \|u\|_p \leq (\omega_n r^n)^{-\frac{1}{p}} M,$$

dove M è la costante che controlla le norme in K , che per l'ipotesi i) è finita.

L'equicontinuit  di K_r segue dall'ipotesi ii). Infatti per $x, y \in \mathbb{R}^n$:

$$\begin{aligned} |u_r(x) - u_r(y)| &= \left| \int_{B(x,r)} u(z) dz - \int_{B(y,r)} u(z) dz \right| \\ &= \left| \int_{B(x,r)} u(z) dz - \int_{B(x,r)} u(z+y-x) dz \right| \\ &\leq \int_{B(x,r)} |u(z) - u(z+y-x)| dz \\ &\leq (\omega_n r^n)^{-\frac{1}{p}} \left(\int_{\mathbb{R}^n} |u(z) - u(z+y-x)|^p dz \right)^{\frac{1}{p}}. \end{aligned} \tag{9.4.10}$$

Poich  fissato $\varepsilon > 0$ esiste $\delta > 0$ tale che per $|x - y| \leq \delta$ si ha

$$\left(\int_{\mathbb{R}^n} |u(z) - u(z+y-x)|^p dz \right)^{\frac{1}{p}} \leq \varepsilon$$

con stima uniforme per $u \in K$, l'affermazione sull'equicontinuit    provata.

Fissato un compatto $Q \subset \mathbb{R}^n$ la famiglia $K_r(Q)$ delle medie di K ristrette a Q   precompatta per il Teorema di Ascoli-Arzel  nella distanza della convergenza uniforme. Dunque $K_r(Q)$   compatta anche in $L^p(Q)$. In particolare, $K_r(Q)$   totalmente limitato per ogni $r > 0$.

Proviamo ora che le medie convergono uniformemente. Usando la disuguaglianza di Minkowski integrale si trova

$$\begin{aligned} \left(\int_{\mathbb{R}^n} |u_r(x) - u(x)|^p dx \right)^{\frac{1}{p}} &= \frac{1}{\omega_n r^n} \left(\int_{\mathbb{R}^n} \left| \int_{B(0,r)} (u(y+x) - u(x)) dy \right|^p dx \right)^{\frac{1}{p}} \leq \\ &\leq \frac{1}{\omega_n r^n} \int_{B(0,r)} \left(\int_{\mathbb{R}^n} |u(y+x) - u(x)|^p dx \right)^{\frac{1}{p}} dy \leq \varepsilon \end{aligned}$$

per $r \leq r(\varepsilon)$ uniformemente in K grazie all'ipotesi ii). Precisamente: per ogni $\varepsilon > 0$ esiste $r(\varepsilon) > 0$ tale che per ogni $0 < r \leq r(\varepsilon)$ e per ogni $u \in K$ vale la disuguaglianza precedente.

Proviamo ora che K_r   totalmente limitato in $L^p(\mathbb{R}^n)$. Fissato $\varepsilon > 0$ per l'ipotesi iii) esiste $R > 0$ tale che

$$\left(\int_{|x| \geq R} |u(x)|^p dx \right)^{\frac{1}{p}} \leq \varepsilon$$

per ogni $u \in K$. Scegliamo $Q = \{x \in \mathbb{R}^n : |x| \leq R\}$. Esistono $u_1, \dots, u_k \in K$ tali che

$$K_r(Q) \subset \bigcup_{i=1}^k B_{L^p(Q)}(u_{i,r}, \varepsilon),$$

dove $u_{i,r}$ indica la media di u_i . Dunque, data $u \in K$ si ha

$$\|u_r - u_{i,r}\|_p = \left(\int_{|x| \leq R} |u_r(x) - u_{i,r}(x)|^p dx \right)^{\frac{1}{p}} + \left(\int_{|x| \geq R} |u_r(x) - u_{i,r}(x)|^p dx \right)^{\frac{1}{p}}.$$

Il primo integrale è minore di ε su scelta di i per la totale limitatezza. Per quanto riguarda il secondo si ha

$$\begin{aligned} \left(\int_{|x| \geq R} |u_r(x) - u_{i,r}(x)|^p dx \right)^{\frac{1}{p}} &\leq \left(\int_{\mathbb{R}^n} |u_r(x) - u(x)|^p dx \right)^{\frac{1}{p}} + \\ &+ \left(\int_{|x| \geq R} |u(x) - u_i(x)|^p dx \right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |u_i(x) - u_{i,r}(x)|^p dx \right)^{\frac{1}{p}} \leq 4\varepsilon \end{aligned}$$

pur di prendere $r \leq r(\varepsilon)$. La stima è uniforme per $u \in K$.

La prova del punto 4) è ora un esercizio. \square

Sia $\Omega \subset \mathbb{R}^n$ un insieme aperto. Lo spazio di Sobolev $W^{1,p}(\Omega)$, $1 \leq p < \infty$, è l'insieme delle funzioni $u \in L^p(\Omega)$ che hanno gradiente distribuzionale $Du \in L^p(\Omega; \mathbb{R}^n)$. La norma in questo spazio è

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|Du\|_p.$$

Dire che l'inclusione $W^{1,p}(\Omega) \subset L^q(\Omega)$, $1 \leq p, q < \infty$, è compatta significa che un insieme limitato in $W^{1,p}(\Omega)$ è un sottoinsieme precompatto in $L^q(\Omega)$.

Un aperto $\Omega \subset \mathbb{R}^n$ si dice avere frontiera Lipschitziana se $\partial\Omega$ è localmente il grafico di una funzione Lipschitziana, a meno di una rotazione del sistema di coordinate.

THEOREM 9.4.2 (Rellich-Kondrachov). *Sia $\Omega \subset \mathbb{R}^n$ un aperto limitato con frontiera Lipschitziana. Siano $1 \leq p < n$ e $p^* = \frac{pn}{n-p}$. Allora l'immersione $W^{1,p}(\Omega) \subset L^q(\Omega)$ è compatta per ogni $1 \leq q < p^*$.*

Ricordiamo che la regolarizzazione di Friedrichs u_ε , $\varepsilon > 0$, di una funzione $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ è definita come

$$u_\varepsilon(x) = \varphi_\varepsilon * u(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y)u(y)dy,$$

dove $\varphi \in C_c^\infty(\mathbb{R}^n)$ è un nucleo di regolarizzazione di Friedrichs e $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right)$. Maggiori dettagli saranno dati a lezione.

LEMMA 9.4.3. *Sia $\Omega \subset \mathbb{R}^n$ un aperto limitato e sia $K \subset L^q(\Omega)$, $q \geq 1$, un insieme limitato. Sia $K_\varepsilon = \{u_\varepsilon : u \in K\}$, $\varepsilon > 0$, l'insieme regolarizzato di Friedrichs. Se*

$$\limsup_{\varepsilon \rightarrow 0} \sup_{u \in K} \|u_\varepsilon - u\|_q = 0,$$

allora K è precompatto in $L^q(\Omega)$.

DIM. Conveniamo di estendere le funzioni a 0 fuori da Ω . È noto che $u_\varepsilon \in C^\infty(\mathbb{R}^n)$, e inoltre, se $1/q + 1/q' = 1$,

$$|u_\varepsilon(x)| \leq \|\varphi_\varepsilon\|_{q'} \|u\|_q \leq C < \infty$$

per ogni $x \in \mathbb{R}^n$ e per ogni $u \in K$. Analogamente

$$|Du_\varepsilon(x)| \leq \|D\varphi_\varepsilon\|_{q'} \|u\|_q \leq C < \infty$$

per ogni $x \in \mathbb{R}^n$ e per ogni $u \in K$. In conclusione, per ogni $\varepsilon > 0$ fissato l'insieme $\{u_\varepsilon\}_{u \in K}$ è equilimitato in $C^1(\overline{\Omega})$ e quindi è totalmente limitato in $C(\overline{\Omega})$ per il Teorema di Ascoli-Arzelà. Dal fatto che Ω è limitato, segue anche la totale limitatezza di K_ε in $L^q(\Omega)$.

Proviamo che K è totalmente limitato in $L^q(\Omega)$. Fissato $\eta > 0$ esiste $\varepsilon > 0$ tale che $\sup_{u \in K} \|u_\varepsilon - u\|_q \leq \eta/2$. L'insieme K_ε è totalmente limitato, e quindi esistono $u_1, \dots, u_k \in K$ tali che

$$K_\varepsilon \subset \bigcup_{i=1}^k B_{L^q}(u_{i,\varepsilon}, \eta/2).$$

Dunque, per ogni $u \in K$ si avrà per un certo $i = 1, \dots, k$

$$\|u - u_i\|_q \leq \|u - u_\varepsilon\|_q + \|u_\varepsilon - u_{i,\varepsilon}\|_q \leq \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

In conclusione

$$K \subset \bigcup_{i=1}^k B_{L^q}(u_i, \eta),$$

e la totale limitatezza di K è provata. □

DIMOSTRAZIONE DEL TEOREMA DI RELlich-KONDRACHOV. Dato $K \subset W^{1,p}(\Omega)$ si tratta di mostrare che K è totalmente limitato in $L^q(\Omega)$. Per il Teorema di immersione di Sobolev, dati $1 \leq q < p^*$ esiste una costante $C > 0$ tale che

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}.$$

Non entriamo nel merito di questo Teorema. Notiamo solo che occorre la Lipschitzianità della frontiera. In particolare, segue che K è limitato anche in $L^q(\Omega)$. Proviamo la convergenza uniforme

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{u \in K} \|u_\varepsilon - u\|_{L^q(\Omega)} = 0.$$

Consideriamo il caso $q = 1$. Sia $\delta > 0$ e poniamo $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$. Allora:

$$\int_{\Omega} |u_\varepsilon(x) - u(x)| dx = \int_{\Omega_\delta} |u_\varepsilon(x) - u(x)| dx + \int_{\Omega \setminus \Omega_\delta} |u_\varepsilon(x) - u(x)| dx = I_1 + I_2.$$

Stimiamo I_2 con la disuguaglianza di Hölder:

$$\begin{aligned} \int_{\Omega \setminus \Omega_\delta} |u(x)| dx &\leq |\Omega \setminus \Omega_\delta|^{\frac{1}{p'}} \|u\|_p, \\ \int_{\Omega \setminus \Omega_\delta} |u_\varepsilon(x)| dx &\leq |\Omega \setminus \Omega_\delta|^{\frac{1}{p'}} \|u_\varepsilon\|_p \leq |\Omega \setminus \Omega_\delta|^{\frac{1}{p'}} \|u\|_p \end{aligned}$$

E poiché $|\Omega \setminus \Omega_\delta| \rightarrow 0$ per $\delta \rightarrow 0$ (questo segue dalla Lipschitzianità della frontiera; non proviamo questo fatto), dalla limitatezza delle norme in K , per ogni $\eta > 0$ esiste $\delta > 0$ tale che

$$\sup_{u \in K} \int_{\Omega \setminus \Omega_\delta} |u_\varepsilon(x) - u(x)| dx \leq \frac{\eta}{2}.$$

Sia tale δ fissato e stimiamo I_1 (possiamo supporre che φ abbia supporto nella palla unitaria)

$$\begin{aligned} \int_{\Omega_\delta} |u_\varepsilon(x) - u(x)| dx &= \int_{\Omega_\delta} \left| \int_{B_1(0)} \varphi(w) (u(x - \varepsilon w) - u(x)) dw \right| dx \\ &\leq \int_{B_1(0)} \varphi(w) \int_{\Omega_\delta} |(u(x - \varepsilon w) - u(x))| dx dw. \end{aligned}$$

Ora si scrive

$$|u(x - \varepsilon w) - u(x)| = \left| \int_0^1 \frac{d}{dt} u(x - t\varepsilon w) dt \right| \leq \varepsilon |w| \int_0^1 |Du(x - t\varepsilon w)| dt,$$

e quindi, per $\varepsilon < \delta$

$$\begin{aligned} \int_{\Omega_\delta} |u_\varepsilon(x) - u(x)| dx &\leq \varepsilon \int_{B_1(0)} |w| \varphi(w) \int_{\Omega_\delta} \int_0^1 |Du(x - t\varepsilon w)| dt dx dw \\ &\leq \varepsilon \int_{B_1(0)} \varphi(w) \int_0^1 \int_{\Omega_\delta} |Du(x - t\varepsilon w)| dx dt dw \leq \varepsilon \int_{B_1(0)} \varphi(w) \int_{\Omega} |Du(x)| dx dw \end{aligned}$$

e infine

$$\int_{\Omega_\delta} |u_\varepsilon(x) - u(x)| dx \leq \varepsilon |\Omega|^{\frac{1}{p'}} \|u\|_{W^{1,p}(\Omega)}.$$

Per la limitatezza in K di $W^{1,p}(\Omega)$, si può fissare ε in modo che

$$\sup_{u \in K} \int_{\Omega_\delta} |u_\varepsilon(x) - u(x)| dx \leq \frac{\eta}{2}.$$

In conclusione, per ogni $\eta > 0$ esiste $\varepsilon > 0$ tale che

$$\sup_{u \in K} \int_{\Omega} |u_\varepsilon(x) - u(x)| dx \leq \eta,$$

e la convergenza uniforme nel caso $q = 1$ è dimostrata.

Il caso $1 < q < p^*$ si riduce al caso $q = 1$ con una disuguaglianza di Hölder. \square

5. Compattezza nella topologia debole

Sia $(X, \|\cdot\|)$ uno spazio normato e indichiamo con X^* il suo spazio duale. Introduciamo su X la topologia meno fine che rende continui tutti i funzionali $T \in X^*$. Fissati un punto $x \in X$, un numero $\varepsilon > 0$ e un insieme finito $\mathcal{I} \subset X^*$, definiamo l'intorno del punto x

$$V(x, \varepsilon, \mathcal{I}) = \{y \in X : |Tx - Ty| < \varepsilon, \text{ per ogni } T \in \mathcal{I}\}.$$

Una sottobase per la *topologia debole* di X è costituita dalla famiglia

$$\mathcal{B} = \{V(x, \varepsilon, \mathcal{I}) : x \in X, \varepsilon > 0, \mathcal{I} \subset X^* \text{ finito}\}.$$

Le intersezioni finite di insiemi di \mathcal{B} formano una base per la topologia debole.

Sia X^{**} il duale di X e definiamo $J : X \rightarrow X^{**}$ ponendo per ogni $x^* \in X^*$

$$\langle J(x), x^* \rangle = \langle x^*, x \rangle.$$

J è lineare e iniettivo. Inoltre $\|J(x)\|_{X^{**}} = \|x\|$. Lo spazio X si dice riflessivo se J è anche suriettivo.

I seguenti teoremi legano la riflessività dello spazio al compattezza debole degli insiemi limitati. Cfr. Dunford-Schwartz, *Linear Operators*, Part I, Cap. V, 6.

THEOREM 9.5.1 (Kakutani). *Uno spazio di Banach X è riflessivo se e solo se la palla chiusa $\{x \in X : \|x\| \leq 1\}$ è compatta nella topologia debole.*

THEOREM 9.5.2 (Eberlein-Shmulyan). *Uno spazio di Banach X è riflessivo se e solo se da ogni successione limitata si può estrarre una sottosuccessione debolmente convergente.*

Ricordiamo che una successione $(x_n)_{n \in \mathbb{N}}$ in X converge debolmente ad $x \in X$ se e solo se

$$\lim_{n \rightarrow \infty} T(x_n) = T(x) \quad \text{per ogni } T \in X^*.$$

EXAMPLE 9.5.3.

- 1) Gli spazi di Hilbert sono riflessivi.
- 2) Lo spazio $L^p(\mathbb{R}^n)$ è riflessivo se $p > 1$.
- 3) Lo spazio $L^1(\mathbb{R}^n)$ non è riflessivo.

DEFINITION 9.5.4. Un insieme $K \subset L^1(\mathbb{R}^n)$ si dice equi-integrabile se:

- i) Per ogni $\varepsilon > 0$ esiste un insieme misurabile $A \subset \mathbb{R}^n$ tale che $\mathcal{L}^n(A) < +\infty$ e

$$\sup_{u \in K} \int_{\mathbb{R}^n \setminus A} |u(x)| dx \leq \varepsilon;$$

- ii) Per ogni $\varepsilon > 0$ esiste $\delta > 0$ tale che per ogni insieme misurabile $A \subset \mathbb{R}^n$ si verificare

$$\mathcal{L}^n(A) \leq \delta \quad \Rightarrow \quad \sup_{u \in K} \int_A |u(x)| dx \leq \varepsilon.$$

THEOREM 9.5.5 (Dunford-Pettis). *Sia $K \subset L^1(\mathbb{R}^n)$ in insieme limitato. Sono equivalenti le seguenti affermazioni:*

- i) K è equi-integrabile;
- ii) Da ogni successione in K si può estrarre una sottosuccessione debolmente convergente in $L^1(\mathbb{R}^n)$.

Per una dimostrazione del Teorema: Ambrosio-Fusco-Pallara, *Functions of Bounded Variations and Free Discontinuity Problems*, Oxford, p. 18.

6. Compattezza nella topologia debole*

Sia X^* il duale di X . La topologia debole* di X^* è la topologia meno fine che rende continue tutte le applicazioni $J(x) \in X^{**}$. Una sottobase di intorni della topologia debole* di X^* è data da

$$V(T, \mathcal{I}, \varepsilon) = \{S \in X^* : |Tx - Sx| < \varepsilon \text{ per ogni } x \in \mathcal{I}\},$$

dove $T \in X^*$, $\mathcal{I} \subset X$ è un insieme finito, ed $\varepsilon > 0$.

THEOREM 9.6.1 (Alaoglu). *Sia $(X, \|\cdot\|)$ uno spazio normato. L'insieme $K = \{T \in X^* : \|T\| \leq 1\}$ è compatto nella topologia debole* di X^* .*

DIM. Proviamo che K è un sottoinsieme chiuso di un opportuno spazio topologico compatto Y . Per ogni $x \in X$ sia $\mathcal{I}(x) = [-\|x\|, \|x\|]$, e definiamo

$$Y = \prod_{x \in X} \mathcal{I}(x) = \{\varphi : X \rightarrow \mathbb{R} : \varphi(x) \in \mathcal{I}(x)\}.$$

Su Y si introduce la topologia prodotto, che è la topologia meno fine che rende continue tutte le proiezioni $P_x : Y \rightarrow \mathcal{I}(x)$ definite da $P_x(\varphi) = \varphi(x)$. Poiché ogni $\mathcal{I}(x)$ è compatto, per il teorema di Tychonov Y con la topologia prodotto è uno spazio topologico compatto. Chiaramente $K \subset Y$.

Una base di intorni aperti di Y nella topologia prodotto è data da

$$\begin{aligned} V_Y(\varphi, I, \varepsilon) &= \{\psi \in Y : |P_x(\varphi) - P_x(\psi)| < \varepsilon \text{ per } x \in I\} \\ &= \{\psi \in Y : |\varphi(x) - \psi(x)| < \varepsilon \text{ per } x \in I\}, \end{aligned}$$

dove $\varphi \in Y$, $I \subset X$ è un insieme finito, ed $\varepsilon > 0$. Dunque $V_Y(T_0, I, \varepsilon) \cap K = V(T_0, I, \varepsilon) \cap K$. Ovvero, la topologia prodotto di Y ristretta ad K coincide con la topologia debole* di X^* ristretta ad K .

Se K è un sottoinsieme chiuso di Y , allora K è compatto. Sia $\varphi \in \overline{K}^Y$ e mostriamo che $\varphi \in K$. Se $I \subset X$ è finito ed $\varepsilon > 0$, allora

$$K \cap V_Y(\varphi, I, \varepsilon) \neq \emptyset$$

e dunque è possibile trovare $T \in X^*$ con $\|T\| \leq 1$ tale che $|\varphi(x) - Tx| < \varepsilon$ per ogni $x \in I$.

Proviamo, ad esempio, che φ è additivo. Fissati $x, y \in X$ sia $I = \{x, y, x + y\}$. Per $\varepsilon > 0$ arbitrario esiste $T \in X^*$ tale che

$$|\varphi(x) - Tx| < \varepsilon, \quad |\varphi(y) - Ty| < \varepsilon, \quad |\varphi(x + y) - T(x + y)| < \varepsilon.$$

Dunque

$$|\varphi(x) + \varphi(y) - \varphi(x + y)| \leq |\varphi(x) - Tx| + |\varphi(y) - Ty| + |\varphi(x + y) - T(x + y)| \leq 3\varepsilon,$$

e in definitiva, dalla arbitrarietà di $\varepsilon > 0$ si ricava $\varphi(x) + \varphi(y) = \varphi(x + y)$. La omogeneità di φ si dimostra in modo analogo. Questo termina la dimostrazione. \square

CHAPTER 10

Exercises

EXERCISE 1. Compute the general solution of the following differential equations

$$\text{i) } y' = \frac{y \cos x}{1 + \sin x} + \sin x; \quad \text{ii) } y' = \frac{3}{x}y + x^2 + 1, \quad x > 0.$$

EXERCISE 2. Compute the solutions to the Cauchy Problems

$$\text{i) } \begin{cases} y' = \frac{y}{1 + e^x} + e^{-x} \\ y(0) = 0. \end{cases} \quad \text{iii) } \begin{cases} y' = y^2 \log(x + 3) \\ y(-2) = -\frac{1}{2}. \end{cases}$$

Answers: i) $y(x) = (1 + e^{\frac{1}{2}x^2+x})^{-1}$; ii) $y = (x - (x + 3) \log(x + 3))^{-1}$.

EXERCISE 3. Compute the general solution of the differential equation

$$y' = y - \frac{x^2}{y}.$$

EXERCISE 4. Compute the solution of the following Cauchy Problem

$$\begin{cases} y' = y(y - 1)(x + 1), & x \in \mathbb{R}, \\ y(0) = \frac{1}{2}. \end{cases}$$

Answer: $y(x) = (1 + e^{\frac{1}{2}x^2+x})^{-1}$, $x \in \mathbb{R}$.

EXERCISE 5. Consider the differential equation

$$(1 - \cos y)y' = x \sin x \sin y.$$

- i) Determine all constant solutions;
- ii) Compute (in implicit form) the general solution (i.e., the family of all solutions depending on a real parameter);
- iii) Compute explicitly the solution satisfying the initial condition $y(0) = \frac{5}{2}\pi$.

Answers: i) $y = k\pi$ with $k \in \mathbb{Z}$. ii) $\cos y = 2e^{x \cos x - \sin x - C} - 1$ with $C \in \mathbb{R}$. iii) $y(x) = 2\pi + \arccos(e^{x \cos x - \sin x} - 1)$.

EXERCISE 6. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that $f(0) = 0$, $f(x) > 0$ if $x \neq 0$, and

$$\int_0^1 \frac{dx}{f(x)} = +\infty.$$

Prove that $y = 0$ is the unique solution to the Cauchy Problem

$$\begin{cases} y' = f(y) \\ y(0) = 0. \end{cases}$$

EXERCISE 7. Compute the solution to the Cauchy Problem

$$\begin{cases} y' = \sin(x + y + 3) \\ y(0) = -3. \end{cases}$$

EXERCISE 8. Compute the solution to the Cauchy Problem

$$\begin{cases} y' = \frac{x^2 + y^2}{xy}, & x > 1 \\ y(\sqrt{2}) = \sqrt{2 \log 2}. \end{cases}$$

EXERCISE 9. Compute the solution to the Cauchy Problem

$$\begin{cases} y' = \frac{\pi \cos(xy)}{x^2} \\ y(1) = \pi. \end{cases}$$

EXERCISE 10. Compute the general solution of the second order, linear differential equation

$$y'' + 4y = x^2 e^{2x}, \quad x \in \mathbb{R}.$$

EXERCISE 11. Compute the solution to the Cauchy Problem

$$\begin{cases} y'' - 2y' + y = x(1 + e^x), & x \in \mathbb{R}, \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

EXERCISE 12. Let $y_1, y_2 \in C^2(\mathbb{R})$ be solutions to the differential equation

$$ay'' + by' + cy = 0, \quad \text{in } \mathbb{R},$$

where a, b, c are continuous functions in \mathbb{R} with $a(x) \neq 0$ for all $x \in \mathbb{R}$, and let $w \in C^1(\mathbb{R})$ be the function

$$w(x) = \det \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Prove that there exists a constant $C \in \mathbb{R}$ such that

$$w(x) = C \exp\left(-\int_0^x \frac{b(t)}{a(t)} dt\right), \quad x \in \mathbb{R}.$$

EXERCISE 13. Compute the solution to the Cauchy Problem

$$\begin{cases} y'' + y = \frac{1}{\cos x}, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ y(0) = 0 \\ y'(0) = 0. \end{cases}$$

Hint: Variation of constants. Solution: $y = \cos x \log(\cos x) + x \sin x$.

EXERCISE 14. Compute the solution of the Cauchy Problem

$$\begin{cases} x^2 y'' + 3xy' - 3y = 0, & x \in \mathbb{R}^+ \\ y(1) = 0 \\ y'(1) = 4. \end{cases}$$

EXERCISE 15. Show that the solution y of the following Cauchy problem is an even and convex function, and then compute it:

$$\begin{cases} \frac{y^2 y''}{(1 + y'^2)^{3/2}} = 1 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

EXERCISE 16. Let (X, d) be a compact metric space and let $T : X \rightarrow X$ be a mapping such that $d(T(x), T(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then there exists a unique $x \in X$ such that $x = T(x)$.

EXERCISE 17. Show that there exists a unique solution $y \in C^1(\mathbb{R})$ of the problem

$$\begin{cases} (y'(x))^2 = 1 + \sqrt{|y(x)|}, & x \in \mathbb{R}, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

EXERCISE 18. Consider the Cauchy Problem

$$\begin{cases} y'(x) = \sqrt{x} + \sqrt{|y(x)|}, & x \geq 0 \\ y(0) = 0. \end{cases}$$

- i) Prove that any solution y satisfies $y(x) \geq \frac{2}{3}x^{3/2}$ for $x \geq 0$.
- ii) Using the contraction principle, prove that there exists a unique local solution of the problem.
- iii) Show that the local solution can be continued to a global solution on $[0, +\infty)$.
- iv) Prove that

$$\lim_{x \rightarrow +\infty} \frac{y(x)}{x^2} = \frac{1}{4}, \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{y(x)}{x^{3/2}} = \frac{2}{3}.$$

EXERCISE 19. Consider the Cauchy Problem

$$\begin{cases} y' = \sqrt{y^2 + x^2 + 1} \\ y(0) = 0. \end{cases}$$

- i) Prove that the problem has a unique local solution;
- ii) Prove that the solution is even, i.e., $y(-x) = -y(x)$ for all x ;
- iii) Prove that the solution is convex for $x \geq 0$;
- iv) Prove that the solution is defined for all $x \in \mathbb{R}$;
- v) Show that $y(x) \geq \sinh(x)$ for all $x \geq 0$.

EXERCISE 20. Prove that there exists a unique solution $y \in C^1(\mathbb{R})$ of the Cauchy Problem

$$\begin{cases} y' = \sin\left(\frac{x}{y}\right) \\ y(0) = 1. \end{cases}$$

Show the y is even and that

$$\lim_{x \rightarrow +\infty} y(x) = +\infty.$$

EXERCISE 21. Let $f \in C^1(\mathbb{R}^2)$ be a bounded function. Show that any solution of the differential equation $y'(x) = f(x, y(x))$ can be uniquely continued to a global solution (i.e., to a solution defined for all $x \in \mathbb{R}$).

EXERCISE 22. Let y be the maximal solution of the Cauchy Problem

$$\begin{cases} y' = (y^2 - 1)(y^2 + x^2) \\ y(0) = y_0, \end{cases}$$

where $y_0 \in \mathbb{R}$ is a real parameter.

- i) Prove that the problem has a unique maximal solution;
- ii) Show that for $y_0 = 0$ it is $y(x) = -x^3/3 + O(x^5)$ as $x \rightarrow 0$;
- iii) Prove that for $|y_0| < 1$ the solution is globally defined on \mathbb{R} ;
- iv) Show that for $y_0 > 1$ the solution is not globally defined on \mathbb{R} .

EXERCISE 23. Prove that the solution of the Cauchy Problem

$$\begin{cases} y' = -(x+1)y^2 + x \\ y(-1) = 1 \end{cases}$$

is globally defined on \mathbb{R} .

EXERCISE 24. Consider the Cauchy Problem

$$\begin{cases} y' = \frac{1}{y} - \frac{1}{x} \\ y(1) = 1. \end{cases}$$

- i) Show that the problem has a unique global solution on the interval $(0, +\infty)$.
- ii) Draw a qualitative graph of the solution.

EXERCISE 25. For any real number $y_0 \in \mathbb{R}$ let y be the (maximal) solution to the Cauchy Problem

$$\begin{cases} y' = x \sin y \\ y(0) = y_0. \end{cases}$$

- i) Show that the problem has a unique solution which is globally defined in \mathbb{R} ;
- ii) Prove that the following limit exists

$$\lambda = \lim_{x \rightarrow +\infty} y(x),$$

and compute λ as a function of y_0 .

- iii) Show that for any $n \in \mathbb{N}$ it is $y(x) = \lambda + o(1/x^n)$ as $x \rightarrow +\infty$, i.e.,

$$\lim_{x \rightarrow +\infty} x^n (y(x) - \lambda) = 0.$$

EXERCISE 26. Given a real number $y_0 \neq 0$, consider the Cauchy Problem

$$\begin{cases} y' = \frac{1}{x^2 + y^2} \\ y(0) = y_0. \end{cases}$$

Prove that:

- i) the solutions are globally defined on \mathbb{R} ;
- ii) the limits $L^- = \lim_{x \rightarrow -\infty} y(x)$ and $L^+ = \lim_{x \rightarrow +\infty} y(x)$ exist and are finite;
- iii) Estimate L^+ and L^- with respect to y_0 .

EXERCISE 27. Let $f \in C^1(\mathbb{R})$ be a function such that $f(0) = 0$ and $|f'(x)| \leq f(x)$ for all $x \in \mathbb{R}$. Show that $f = 0$.

EXERCISE 28. Let $g \in C([0, +\infty))$ be a function such that

$$\int_0^{+\infty} |g(x)| dx < +\infty,$$

and let $y \in C^2([0, +\infty))$ be the solution to the Cauchy Problem

$$\begin{cases} y'' + y = gy, & x \geq 0, \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

Prove that there exists a constant $M > 0$ such that $|y(x)| \leq M$ for all $x \geq 0$.

EXERCISE 29. Let $f \in C(\mathbb{R})$ be a continuous function such that $tf(t) \geq 0$ for all $t \in \mathbb{R}$. Show that the Cauchy Problem

$$\begin{cases} y'' + e^{-x}f(y) = 0 \\ y(0) = y'(0) = 0 \end{cases}$$

has the unique solution $y = 0$.

Hint: multiply by $e^x y'$ and Gronwall Lemma.

EXERCISE 30. Let $F \in C^1([0, +\infty))$ be a function such that $F(0) > 0$ and $F'(x) \geq 0$ for all $x \geq 0$. Show that any solution $y \in C^2([0, +\infty))$ to the differential equation

$$y'' + F(x)y = 0, \quad x \geq 0,$$

is bounded. Hint: multiply by y' , integrate, Gronwall Lemma.

EXERCISE 31. Let $y \in C^2(\mathbb{R})$ be the solution to the Cauchy Problem

$$\begin{cases} y'' + 4y^3 = 0, & x \in \mathbb{R}, \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$$

Prove that the solution is in fact defined for all $x \in \mathbb{R}$ and show that $|y(x)| \leq 1$ for all $x \in \mathbb{R}$.

EXERCISE 32. Show that there exist periodic solutions $y \in C^1(\mathbb{R})$ to the differential equation

$$y'(x) = 2 \sin y(x) + \sin x, \quad x \in \mathbb{R}.$$

EXERCISE 33. Consider the differential equation

$$x^3 y' - 2y + 2x = 0.$$

Prove that:

- i) Any solution $y \in C^1(\mathbb{R} \setminus \{0\})$ can be continued to a function in $C^1(\mathbb{R})$;
 ii) There is no analytic solution to the equation in any neighborhood of $x = 0$.

EXERCISE 34. Let $\alpha \geq 0$ be a real number and consider the Cauchy Problem

$$\begin{cases} y'' + \frac{\alpha}{x}y' + y^3 = 0, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

- i) Prove that the problem has a unique solution $y \in C^2([0, T])$ for some $T > 0$.
 ii) Show that the solution is global on $[0, +\infty)$;
 iii) For $\alpha = 0$ prove that the solution y is periodic and that

$$\max_{x \in [0, +\infty)} y(x) = 1, \quad \min_{x \in [0, +\infty)} y(x) = -1.$$

Hint: Multiply by x^α and contraction principle.

EXERCISE 35. Let $a, b \in C(\mathbb{R})$ be continuous functions and let $y_1 \in C^2(\mathbb{R})$, $y_1(x) \neq 0$ for all $x \in \mathbb{R}$, be a solution to the differential equation

$$y'' + ay' + by = 0.$$

Determine a solution $y_2 \in C^2(\mathbb{R})$ which is linearly independent from y_1 .

EXERCISE 36. Let $f \in C^1(\mathbb{R})$ be a function such that $f(1) = 0$ and let $0 < x_0 < 1$ be a real number. Prove that the Cauchy Problem

$$\begin{cases} x' = xf(x^2 + y^2) - y \\ y' = yf(x^2 + y^2) + x \\ x(0) = x_0 \\ y(0) = 0 \end{cases}$$

has a (unique) solution which is bounded and globally defined on \mathbb{R} .

EXERCISE 37. Let A be the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Compute the general solution of the differential equation $y' = Ay$.

EXERCISE 38. For $0 < T \leq +\infty$ consider the vector space $X = \{b \in C([0, T]; \mathbb{R}^n) : b \text{ is bounded}\}$, endowed with the norm

$$\|b\|_\infty = \sup_{x \in [0, T]} |b(x)|.$$

Let $A \in M_n(\mathbb{R})$ and $y_0 \in \mathbb{R}^n$ and consider the Cauchy Problem

$$\begin{cases} y' = Ay + b & \text{in } [0, T) \\ y(0) = y_0. \end{cases}$$

Define $T : X \rightarrow C^1([0, T]; \mathbb{R}^n)$ on letting $F(b) = y$ if $y \in C^1([0, T]; \mathbb{R}^n)$ is the solution of the Cauchy problem with datum b .

- i) Prove that for all $0 < T < +\infty$ there exists a constant $0 < C_T < +\infty$ such that

$$\|F(b_1) - F(b_2)\|_\infty \leq C_T \|b_1 - b_2\|_\infty \quad (*)$$

for all $b_1, b_2 \in X$;

- ii) Prove that the continuity estimate (*) holds also in the case $T = +\infty$, provided that all eigenvalues of $A + A^t$ are strictly positive.

EXERCISE 39. Let $f \in C^2(\mathbb{R}^n)$ be a function such that:

- a) The sets $\{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ are compact for all $\lambda \in \mathbb{R}$.
 b) $\nabla f(x) = 0$ if and only if $x = 0$.

Consider the Cauchy Problem

$$\begin{cases} \dot{\gamma}(t) = -\nabla f(\gamma(t)), & t \geq 0, \\ \gamma(0) = x_0, \end{cases}$$

where $x_0 \in \mathbb{R}^n$. Prove that:

- i) The problem has a unique solution $\gamma_{x_0} \in C^2([0, +\infty))$;
 ii) $\lim_{t \rightarrow +\infty} \gamma_{x_0}(t) = 0$;
 iii) In the case $f(x) = |x|^2/2$, compute the flow $\Phi : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\Phi(t, x_0) = \gamma_{x_0}(t)$.

Hint: ii) Show that $0 \in \mathbb{R}^n$ is the unique minimum point of f ; then prove that $t \mapsto f(\gamma(t))$ is decreasing and converges to the minimum of f , as $t \rightarrow +\infty$.

EXERCISE 40. Let $Y = C([0, 1]; \mathbb{R})$ be the space of continuous functions on $[0, 1]$ with the norm $\|y\| = \max_{x \in [0, 1]} |y(x)|$.

- i) Prove that the set

$$K = \left\{ y \in Y : y(0) = 0, y \in C^1([0, 1]), \int_0^1 |y'(x)|^2 dx \leq 1 \right\}$$

is precompact in Y , i.e., it is equibounded and equicontinuous.

- ii) Prove that the set

$$K = \left\{ y \in Y : y(0) = 0, y \in C^1([0, 1]), \int_0^1 |y'(x)| dx \leq 1 \right\}$$

is not precompact in Y , i.e., there is a sequence in K which has no subsequence converging in the norm $\|\cdot\|$.

EXERCISE 41. Let $K \in C([0, 1] \times \mathbb{R})$ be a continuous function such that

$$\sup_{x \in [0, 1], y \in \mathbb{R}} |K(x, y)| < +\infty.$$

Prove that the integral equation

$$y(x) = \int_0^1 K(x, y(t)) dt$$

has at least one solution $y \in C([0, 1])$.

EXERCISE 42. Show that the boundary value problem

$$\begin{cases} y'' = -y^2, & -\alpha < x < \alpha, \\ y(-\alpha) = y(\alpha) = 0 \end{cases}$$

has exactly two solutions $y \in C^2([-\alpha, \alpha])$ for any real number $\alpha > 0$.

EXERCISE 43. For $\varepsilon > 0$ consider the boundary value problem

$$\begin{cases} -\varepsilon y'' + y'^2 - 1 = 0 & \text{on } [-1, 1], \\ y(1) = y(-1) = 0 \end{cases}$$

- i) Show that if the problem has a solution $y \in C^2([-1, 1])$ then it is unique. In particular y is even, i.e., $y(x) = y(-x)$.
- ii) Show that a solution y to the problem satisfies $|y'(x)| < 1$ for all $x \in [-1, 1]$;
- iii) Compute the solution $y = y_\varepsilon \in C^2([-1, 1])$ of the problem;
- iv) Compute the limit $z(x) = \lim_{\varepsilon \downarrow 0} y_\varepsilon(x)$.

Comment: $z'(x)^2 = 1$ for all $x \in [-1, 1] \setminus \{0\}$.

Hints: i) If y_1, y_2 are solutions, then $y_1 - y_2$ has a maximum (and/or a minimum) point. At this point we have $y_1' = y_2'$. ii) If $|y'(x)| = 1$ for some x , then $|y'(x)| = 1$ for all x ; iii) Separation of variables. It could be useful to use $y'(0) = 0$.

EXERCISE 44. Let $f \in C([0, 1])$ be a continuous function and $\lambda \in \mathbb{R}$ a real number. Consider the boundary value problem

$$\begin{cases} -y'' + \lambda y' = f & \text{on } [0, 1] \\ y(0) = y(1) = 0. \end{cases}$$

- i) Prove that the problem has a unique solution $y \in C^2([0, 1])$.
- ii) For fixed f , let y_λ be the solution depending on $\lambda \in \mathbb{R}$. Show that $\lambda \mapsto y_\lambda$ is continuous from \mathbb{R} to $Y = C^2([0, 1])$, where Y is normed as in (8.1.3).
- iii) For fixed $\lambda \in \mathbb{R}$, let y_f be the solution depending on the function f . Show that $f \mapsto y_f$ is continuous from $C([0, 1])$ to $C^2([0, 1])$.
- iv) Show that $y \leq g$ implies $y_f \leq y_g$.

EXERCISE 45. Let $\varphi \in C^1(\mathbb{R})$ be a given function. Compute the solution $u \in C^1(\mathbb{R}^2)$ to the Cauchy Problem

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u, & \text{in } \mathbb{R}^2 \\ u(x, 0) = \varphi(x), & x \in \mathbb{R}. \end{cases}$$

Solution: $u(x, y) = \varphi(x - y)e^y$.

EXERCISE 46. Let $\varphi \in C^1(\mathbb{R})$ be a given function. Compute the solution $u \in C^1(\mathbb{R}^2)$ to the Cauchy Problem

$$\begin{cases} \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = u, & \text{in } \mathbb{R}^2, \\ u(0, y) = \varphi(y), & y \in \mathbb{R}. \end{cases}$$

Solution: $u(x, y) = \varphi(ye^x)e^x$.

EXERCISE 47. Let $\varphi \in C^1(\mathbb{R})$ be a given function. Compute the solution $u \in C^1(\mathbb{R} \times \mathbb{R}^+)$ to the Cauchy Problem

$$\begin{cases} x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u, & y > 0 \\ u(x, 1) = \varphi(x), & x \in \mathbb{R}. \end{cases}$$

Solution: $u(x, y) = y^2 \varphi(x/y)$.

EXERCISE 48. Let $n \geq 3$ and $\lambda > 0$. Prove that the function

$$\varphi(r) = \left(\frac{\lambda}{\lambda^2 + r^2} \right)^{\frac{n-2}{2}}$$

is the unique solution of the Cauchy Problem

$$\begin{cases} \varphi''(r) + \frac{n-1}{r} \varphi'(r) = -n(n-2) \varphi(r)^{\frac{n+2}{n-2}}, & r > 0, \\ \varphi(0) = \lambda^{\frac{2-n}{2}} \\ \varphi'(0) = 0. \end{cases}$$

Hint: To prove uniqueness, use the Contraction Principle on the interval $r \in [0, \varepsilon]$ for some $\varepsilon > 0$.

EXERCISE 49. Let $\varphi \in C^2(0, +\infty)$ with $\varphi \geq 0$ be a solution to the equation

$$\varphi''(r) + \frac{n-1}{r} \varphi'(r) = -n(n-2) \varphi(r)^{\frac{n+2}{n-2}}, \quad r > 0, \quad (*)$$

and let $\psi \in C^2(\mathbb{R})$ be the function defined by $\varphi(r) = r^{\frac{2-n}{2}} \psi(-\log r)$. This is called Fowler-Emden substitution.

i) Prove that there exists a constant $C \in \mathbb{R}$ such that

$$\psi'(t)^2 = (n-2)^2 \left(\frac{1}{4} \psi(t)^2 - \psi(t)^{\frac{2n}{n-2}} \right) + C, \quad t \in \mathbb{R}. \quad (**)$$

- ii) Assuming φ bounded near $r = 0$, prove that it must be $C = 0$.
 iii) Let $\varphi \in C^2([0, +\infty))$, $\varphi \geq 0$, be a solution of (*) and let $\vartheta \in C^2(0, +\infty)$ be the function defined by $\varphi(r) = (r\vartheta(r))^{\frac{2-n}{2}}$. Using (**) with $C = 0$, prove that ϑ solves the Euler equation

$$r^2 \vartheta'' + r \vartheta' - \vartheta = 0,$$

and then determine φ .

EXERCISE 50. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence of functions in $C([0, 1]; \mathbb{R})$ and let $y \in C([0, 1]; \mathbb{R})$. Assume that:

- i) There exists a dense set $A \subset [0, 1]$ such that $y_n(x) \rightarrow y(x)$ as $n \rightarrow \infty$ for all $x \in A$;
 ii) Any subsequence of $(y_n)_{n \in \mathbb{N}}$ has a subsequence which converges uniformly.

Prove that $y_n \rightarrow y$ uniformly as $n \rightarrow +\infty$.

EXERCISE 51. Consider the differential equations $y' = \sqrt{|y|}$ and $Y'_\varepsilon = \sqrt[4]{Y_\varepsilon^2 + \varepsilon^2}$, for $\varepsilon > 0$, along with the initial data $y(0) = Y_\varepsilon(0) = 0$. Let y be the strictly increasing solution of the Cauchy Problem. Prove or disprove the following statement: there exists $\varepsilon_0 > 0$ such that $Y_\varepsilon(x) < y(x)$ for all $x \in (0, 1]$ and for all $0 < \varepsilon \leq \varepsilon_0$.

EXERCISE 52. For $\alpha > 0$ and $\lambda \in \mathbb{R}$, consider the differential problem

$$\begin{cases} y' = \frac{y \sin y}{1 + x^\alpha}, & x > 0, \\ \lim_{x \rightarrow +\infty} y(x) = \lambda. \end{cases}$$

For given α and λ , study existence and uniqueness of solutions $y \in C^1(0, +\infty)$.

EXERCISE 53. Let $f, g \in C^1(\mathbb{R})$ with $f, g > 0$ and fg monotone. Consider a nontrivial solution y to the differential equation $fy'' + f'y' + gy = 0$. Show that if $x_1 < x_2$ are two points of relative extremum of y then the following holds:

$$|y(x_1)| \leq |y(x_2)| \quad \text{if } fg \text{ is decreasing,}$$

$$|y(x_1)| \geq |y(x_2)| \quad \text{if } fg \text{ is increasing.}$$

Hint: Compute the first derivative of the function $v = y^2 + \frac{(fy')^2}{fg}$.

EXERCISE 54. Let $I = [0, \infty)$ and consider functions $f \in C^1(I)$ and $g \in C(I)$ with $f, g > 0$ and

$$\int_0^\infty \frac{1}{f(x)} dx = \int_0^\infty g(x) dx = \infty.$$

Show that every solution $y : I \rightarrow \mathbb{R}$ to the differential equation $fy'' + f'y' + gy = 0$ has infinitely many zero points.

Hint: Equivalent linear system. Polar coordinates.

EXERCISE 55. Study existence of periodic and global solutions to the equation

$$y'' = \alpha^2 - y^2,$$

where α is a real parameter.

EXERCISE 56. Let $T > 0$ and consider the problem

$$\begin{cases} y'' + e^y = 0 & \text{on } [0, T] \\ y(0) = y(T) = 0. \end{cases} \quad (*)$$

- i) Show that any solution has a maximum y_0 at $t = T/2$;
- ii) Prove that the function

$$\varphi(\xi) = \frac{1}{\sqrt{2}} \int_0^\xi \frac{dy}{\sqrt{e^\xi - e^y}}$$

satisfies $\varphi(y_0) = T/2$;

- iii) Prove that there exists $\delta > 0$ such that φ is increasing on $(0, \delta)$ and decreasing on $(\delta, +\infty)$;

- iv) Show that there exists $T_0 > 0$ such that the problem (*) has: 1) one solution for $T = T_0$; 2) two solutions for $T < T_0$; 3) no solution for $T > T_0$.

EXERCISE 57. Consider the Cauchy Problem

$$\begin{cases} y' = \sqrt{y^2 + x^2 + 1} \\ y(0) = 0. \end{cases}$$

- i) Prove that the problem has a unique local solution $y \in C^1(-\delta, \delta)$ for some $0 < \delta \leq +\infty$;
 ii) Prove that the solution is even, i.e., $y(-x) = -y(x)$ for all $x \in (-\delta, \delta)$;
 iii) Prove that the solution is convex for $x \geq 0$;
 iv) Prove that the solution is defined for all $x \in \mathbb{R}$;
 v) Show that $y(x) \geq \sinh(x)$ for all $x \geq 0$.

EXERCISE 58. Compute in implicit form the (maximal) solution $y \in C^\infty(a, b)$, $-\infty \leq a < 1 < b \leq +\infty$, of the Cauchy Problem

$$\begin{cases} y' = \frac{y-x}{y+x}, \\ y(1) = 0, \end{cases}$$

and draw a qualitative graph of y . Compute b and show that $a > -\frac{1}{2}e^{-\pi/2}$.