ISOPERIMETRIC PROBLEM AND MINIMAL SURFACES IN THE HEISENBERG GROUP

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Contents

Preface

This text is an extended version of the lecture notes of the course *Isoperimetric problem and minimal surfaces in the Heisenberg group* given at the *ERC-School on*

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1. Introduction to the Heisenberg group $\mathbb{H}^n$

1.1. **Algebraic structure.** The $2n+1$-dimensional Heisenberg group is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \in \mathbb{N}$, endowed with the group product

$$(z,t)\cdot(\zeta,\tau) = \big(z+\zeta, t+\tau+2\,\mathrm{Im}\langle z,\bar\zeta\rangle\big), \tag{1.1}$$

where $t,\tau \in \mathbb{R}$, $z,\zeta \in \mathbb{C}^n$ and $\langle z,\bar\zeta\rangle = z_1\bar\zeta_1 + \ldots + z_n\bar\zeta_n$. The Heisenberg group is a noncommutative Lie group. The identity element is $0 = (0,0) \in \mathbb{H}^n$. The inverse element of (z,t) is $(-z,-t)$. The center of the group is $Z = \{(z,t) \in \mathbb{H}^n : z = 0\}$. We denote elements of $\mathbb{H}^n$ by $p = (z,t) \in \mathbb{C}^n \times \mathbb{R}$.

The *left translation* by $p \in \mathbb{H}^n$ is the mapping $L_p : \mathbb{H}^n \to \mathbb{H}^n$

$$L_p(q) = p\cdot q.$$

Left translations are linear mappings in $\mathbb{H}^n = \mathbb{R}^{2n+1}$. For any $\lambda > 0$, the mapping $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t), \tag{1.2}$$

is called *dilation*. Dilations are linear mappings and form a 1-parameter group $(\delta_\lambda)_{\lambda>0}$ of automorphisms of $\mathbb{H}^n$.

We denote by $|E|$ the Lebesgue measure of a Lebesgue measurable set $E \subset \mathbb{H}^n = \mathbb{R}^{2n+1}$. The differential dL_p of any left translation is an upper triangular matrix with 1 along the principal diagonal. It follows that $\det dL_p = 1$ on $\mathbb{H}^n$ for any $p \in \mathbb{H}^n$ and, as a consequence,

$$|L_pE| = |E|, \quad \text{for any } p \in \mathbb{H}^n \text{ and for any } E \subset \mathbb{H}^n.$$

Lebesgue measure is the Haar measure of the Heisenberg group. Moreover, we have $\det \delta_\lambda = \lambda^Q$, where the integer

$$Q = 2n+2 \tag{1.3}$$

is called *homogeneous dimension* of $\mathbb{H}^n$. As a consequence, we have

$$|\delta_\lambda E| = \lambda^Q |E|.$$

We introduce the Lie algebra of left invariant vector fields of $\mathbb{H}^n$. A C^∞ vector field X in $\mathbb{H}^n$ is left invariant if for any function $f \in C^\infty(\mathbb{H}^n)$ and for any $p \in \mathbb{H}^n$ there holds

$$X(f \circ L_p) = (Xf) \circ L_p.$$

Equivalently, X is left invariant if $X(p) = dL_p X(0)$, where dL_p is the differential of the left translation by p. Left invariant vector fields with the bracket form a nilpotent Lie algebra $\mathfrak{h}_n$, called Heisenberg Lie algebra. The algebra $\mathfrak{h}_n$ is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad \text{and} \quad T = \frac{\partial}{\partial t}, \tag{1.4}$$

with $j = 1, \dots, n$. In other words, any left invariant vector field is a linear combination with real coefficients of the vector fields (1.4). We are using the notation $p = (z,t)$ and $z = x + iy$ with $x, y, \in \mathbb{R}^n$. The vector fields (1.4) are determined by the relations

$$\begin{aligned} X_j(p) &= dL_p X_j(0) = dL_p \frac{\partial}{\partial x_j}, \\ Y_j(p) &= dL_p Y_j(0) = dL_p \frac{\partial}{\partial y_j}, \\ T(p) &= dL_p T(0) = dL_p \frac{\partial}{\partial t}. \end{aligned}$$

The distribution of $2n$-dimensional planes H_p spanned by the vector fields X_j and Y_j, $j = 1, \dots, n$, is called *horizontal distribution*:

$$H_p = \operatorname{span}\big\{X_j(p), Y_j(p) \, : \, j = 1, \dots, n\big\}. \tag{1.5}$$

The horizontal distribution is nonintegrable. In fact, for any $j = 1, \dots, n$ there holds

$$[X_j, Y_j] = -4T \neq 0. \tag{1.6}$$

All other commutators vanish. The horizontal distribution is bracket generating of step 2.

When $n = 1$, we write $X = X_1$ and $Y = Y_1$.

1.2. **Metric structure.** We introduce the Carnot-Carathéodory metric of $\mathbb{H}^n$ and we describe the geodesics of this metric. In $\mathbb{H}^1$, these curves are important in the structure of H-minimal surfaces and surfaces with constant H-curvature.

A Lipschitz curve $\gamma : [0,1] \to \mathbb{H}^n$ is horizontal if $\dot\gamma(t) \in H_{\gamma(t)}$ for a.e. $t \in [0,1]$. Equivalently, γ is horizontal if there exist functions $h_j \in L^\infty([0,1])$, $j = 1, \dots, 2n$, such that

$$\dot\gamma = \sum_{j=1}^n h_j X_j(\gamma) + h_{n+j} Y_j(\gamma), \quad \text{a.e. on } [0,1]. \tag{1.7}$$

The coefficients h_j are unique, and by the structure of the vector fields X_j and Y_j they satisfy $h_j = \dot\gamma_j$, where $\gamma = (\gamma_1, \dots, \gamma_{2n+1})$ are the coordinates of γ given by the identification $\mathbb{H}^n = \mathbb{R}^{2n+1}$. We call the Lipschitz curve $\kappa : [0,1] \to \mathbb{R}^{2n}$, $\kappa = (\gamma_1, \dots, \gamma_{2n})$, *horizontal projection* of γ.

The vertical component of γ is determined by the horizontality condition (1.7). Namely, we have

$$\dot\gamma_{2n+1} = 2\sum_{j=1}^n h_j \gamma_{n+j} - h_{n+j}\gamma_j = 2\sum_{j=1}^n \dot\kappa_j \kappa_{n+j} - \dot\kappa_{n+j}\kappa_j,$$

and, by integrating, we obtain for any $t \in [0,1]$

$$\gamma_{2n+1}(t) = \gamma_{2n+1}(0) + 2\sum_{j=1}^n \int_0^t (\dot\kappa_j \kappa_{n+j} - \dot\kappa_{n+j}\kappa_j) ds. \tag{1.8}$$

If κ is a given Lipschitz curve in $\mathbb{R}^{2n}$, the curve γ with $(\gamma_1, \dots, \gamma_{2n}) = \kappa$ and γ_{2n+1} as in (1.8) is called a *horizontal lift* of κ and we write $\gamma = \mathrm{Lift}(\kappa)$. The horizontal lift is unique modulo the initial value $\gamma_{2n+1}(0)$.

Now we define the Carnot-Carathéodory metric of $\mathbb{H}^n$. For any pair of points $p, q \in \mathbb{H}^n$, there exists a horizontal curve $\gamma : [0,1] \to \mathbb{H}^n$ such that $\gamma(0) = p$ and $\gamma(1) = q$. This follows from the nonintegrability condition (1.6) and it can be checked via a direct computation. The basic observation is that for any $t \in \mathbb{R}$

$$\exp(-tY_j)\exp(-tX_j)\exp(tY_j)\exp(tX_j)(0,0) = (0, -4t^2),$$

where $\exp(tV)(p)$ is the flow of the vector field V at time t starting from p.

We fix on the horizontal distribution H_p the positive quadratic form $g(p;\cdot)$ making the vector fields $X_1, \dots, X_n, Y_1, \dots, X_n$ orthonormal at every point $p \in \mathbb{H}^n$. Since the vector fields are left invariant, the quadratic form is left invariant. We use the quadratic form g to define the length of a horizontal curve $\gamma : [0,1] \to \mathbb{H}^n$ with horizontal projection κ:

$$L(\gamma) = \int_0^1 g(\gamma; \dot\gamma)^{1/2} dt = \int_0^1 |\dot\kappa| dt,$$

where $|\dot\kappa|$ is the Euclidean norm in $\mathbb{R}^{2n}$ of $\dot\kappa$. For any couple of points $p, q \in \mathbb{H}^n$, we define

$$d(p,q) = \inf\Big\{ L(\gamma) : \gamma : [0,1] \to \mathbb{H}^n \text{ is horizontal, } \gamma(0) = p \text{ and } \gamma(1) = q \Big\}. \tag{1.9}$$

We already observed that the above set is nonempty for any $p, q \in \mathbb{H}^n$, and thus $0 \le d(p,q) < \infty$.

The function $d : \mathbb{H}^n \times \mathbb{H}^n \to [0, \infty)$ is a distance on $\mathbb{H}^n$, called *Carnot-Carathéodory distance*. It can be proved that for any compact set $K \subset \mathbb{H}^n = \mathbb{R}^{2n+1}$ there exists a constant $0 < C_K < \infty$ such that

$$d(p,q) \ge C_K |p - q| \tag{1.10}$$

for all $p, q \in K$, where $|p-q|$ is the Euclidean distance between the points. In particular, we have $d(p,q) \neq 0$ if $p \neq q$. The distance d is left invariant and 1-homogeneous. Namely, for any $p, q, w \in \mathbb{H}^n$ and $\lambda > 0$ there holds:

i) $d(w \cdot p, w \cdot q) = d(p,q)$;
ii) $d(\delta_\lambda(p), \delta_\lambda(q)) = d(p,q)$.

Statement i) follows from the fact that $L(w \cdot \gamma) = L(\gamma)$ for any horizontal curve γ and for any $w \in \mathbb{H}^n$, because the quadratic form g is left invariant. Analogously, ii) follows from $L(\delta_\lambda(\gamma)) = \lambda L(\gamma)$, that is a consequence of the identities

$$X_j(f \circ \delta_\lambda) = \lambda (X_j f) \circ \delta_\lambda, \quad Y_j(f \circ \delta_\lambda) = \lambda (Y_j f) \circ \delta_\lambda,$$

holding for any $f \in C^\infty(\mathbb{H}^n)$ and $\lambda > 0$.

For any $p \in \mathbb{H}^n$ and $r > 0$, we define the Carnot-Carathéodory ball

$$B_r(p) = \big\{ q \in \mathbb{H}^n \, : \, d(p,q) < r \big\}.$$

We also let $B_r = B_r(0)$. The size of Carnot-Carathéodory balls can be described by means of anisotropic homogeneous norms. For any $p = (z,t) \in \mathbb{H}^n$ let

$$\|p\|_\infty = \max\{|z|, |t|^{1/2}\}. \tag{1.11}$$

The "box norm" $\|\cdot\|_\infty$ has the following properties:

i) $\|\delta_\lambda(p)\|_\infty = \lambda \|p\|_\infty$, for all $p \in \mathbb{H}^n$ and $\lambda > 0$;
ii) $\|p \cdot q\|_\infty \leq \|p\|_\infty + \|q\|_\infty$, for all $p, q \in \mathbb{H}^n$.

By ii), the function $\varrho : \mathbb{H}^n \times \mathbb{H}^n \to [0, \infty)$,

$$\varrho(p,q) = \|p^{-1} \cdot q\|_\infty, \tag{1.12}$$

satisfies the triangle inequality and is a distance on $\mathbb{H}^n$. By an elementary argument based on continuity, compactness, and homogeneity, there exists an absolute constant $C > 0$ such that

$$C^{-1} d(p,q) \leq \varrho(p,q) \leq C d(p,q)$$

for all $p, q \in \mathbb{H}^n$. The distance functions d and ϱ are equivalent.

All the previous observations are still valid when the "box norm" $\|\cdot\|_\infty$ is replaced with the Koranyi norm $\|p\| = \big(|z|^4 + t^2\big)^{1/4}$.

The metric space $(\mathbb{H}^n, d)$ is complete and locally compact. By the definition of d, it is also a length space. Then, a standard application of Ascoli-Arzelà theorem shows that it is a geodesic space, namely for all $p, q \in \mathbb{H}^n$ there exists a horizontal curve $\gamma : [0,1] \to \mathbb{H}^n$ such that $\gamma(0) = q$, $\gamma(1) = p$, and $L(\gamma) = d(p,q)$. The curve γ is called *geodesic* or *length minimizing curve* joining q to p.

We classify geodesics in $\mathbb{H}^1$ starting from the initial point 0. Let $\Phi : [0, 2\pi] \times [-2\pi, 2\pi] \to \mathbb{H}^1$ be the mapping

$$\Phi(\psi, \varphi) = \Big(\frac{\mathrm{e}^{i\psi}(\mathrm{e}^{i\varphi} - 1)}{\varphi}, 2\frac{\varphi - \sin\varphi}{\varphi^2}\Big). \tag{1.13}$$

When $\varphi = 0$, the formula is determined by analytic continuation and we have $\Phi(\psi, 0) = (i\mathrm{e}^{i\psi}, 0)$. The set $S = \Phi([0, 2\pi] \times [-2\pi, 2\pi]) \subset \mathbb{H}^1$ is homeomorphic to a 2-dimensional sphere. It is a C^∞ surface at points $(z, t) \in S$ such that $z \neq 0$, i.e., $(z, t) = \Phi(\psi, \varphi)$ with $|\varphi| \neq 2\pi$. The antipodal points $(0, \pm 1/\pi) \in S$ are obtained for $\varphi = \pm 2\pi$ and are Lipschitz points. We will show that S is the unitary Carnot-Carathéodory sphere of $\mathbb{H}^1$ centered at 0, $S = \partial B_1(0)$.

Theorem 1.1. For any $\psi \in [0, 2\pi]$ and $\varphi \in [-2\pi, 2\pi]$, the curve $\gamma_{\psi,\varphi} : [0, 1] \to \mathbb{H}^1$

$$\gamma_{\psi,\varphi}(s) = \Big(\frac{\mathrm{e}^{i\psi}(\mathrm{e}^{i\varphi s} - 1)}{\varphi}, 2\frac{\varphi s - \sin\varphi s}{\varphi^2}\Big), \quad s \in [0, 1], \tag{1.14}$$

is length minimizing. When $|\varphi| < 2\pi$, $\gamma_{\psi,\varphi}$ is the unique length minimizing curve from 0 to $\Phi(\psi, \varphi)$. When $\varphi = \pm 2\pi$, for every $\psi \in [0, 2\pi]$ the curve $\gamma_{\psi,\varphi}$ is length minimizing from 0 to $(0, \pm 1/\pi)$.

Proof. Let $(z_0, t_0) \in \mathbb{H}^1$ be any point and introduce the family of admissible curves

$$\mathscr{A} = \Big\{\kappa \in \mathrm{Lip}([0, 1]; \mathbb{R}^2) : \kappa(0) = 0, \kappa(1) = z_0\Big\}.$$

The end-point mapping relative to the third coordinate $\mathrm{End} : \mathscr{A} \to \mathbb{R}$ is

$$\mathrm{End}(\kappa) = 2\int_0^1 \big(\kappa_2\dot\kappa_1 - \kappa_1\dot\kappa_2\big)ds = 2\int_0^1 \mathrm{Im}(\kappa\dot{\bar\kappa})ds,$$

where $\kappa\dot{\bar\kappa}$ is a complex product.

The geodesic γ joining 0 to (z_0, t_0) is the horizontal lift of the curve κ in the plane that solves the problem

$$\min\Big\{\int_0^1 |\dot\kappa|ds \,:\, \kappa \in \mathscr{A} \text{ and } \mathrm{End}(\kappa) = t_0\Big\}. \tag{1.15}$$

Let κ be a minimizer for problem (1.15). We compute the first variation of the length functional at the curve κ with constraint $\mathrm{End}(\kappa) = t_0$. For $\tau \in \mathbb{R}$ and $\vartheta \in C_c^\infty\big((0, 1); \mathbb{R}^2\big)$ the curve $\kappa^\tau = \kappa + \tau\vartheta$ satisfies

$$\begin{aligned}
\frac{d}{d\tau}\mathrm{End}(\kappa^\tau)\Big|_{\tau=0} &= 2\frac{d}{d\tau}\int_0^1 \big((\kappa_2 + \tau\vartheta_2)(\dot\kappa_1 + \tau\dot\vartheta_1) - (\kappa_1 + \tau\vartheta_1)(\dot\kappa_2 + \tau\vartheta_2)\big)ds\Big|_{\tau=0} \\
&= 2\int_0^1 \big(\vartheta_2\dot\kappa_1 + \kappa_2\dot\vartheta_1 - \kappa_1\vartheta_2 - \vartheta_1\dot\kappa_2\big)ds \\
&= 4\int_0^1 \big(\vartheta_2\dot\kappa_1 - \vartheta_1\dot\kappa_2\big)ds.
\end{aligned}$$

We have $|\dot\kappa| \neq 0$ a.e., and thus there exists $\vartheta \in C_c^\infty\big((0,1);\mathbb{R}^2\big)$ such that

$$\frac{d}{d\tau}\mathrm{End}(\kappa^\tau)\Big|_{\tau=0} \neq 0. \tag{1.16}$$

Fix a function ϑ satisfying (1.16) and let $\eta \in C_c^\infty\big((0,1);\mathbb{R}^2\big)$ be an arbitrary vector valued function. The curve $\kappa + \tau\vartheta + \varepsilon\eta$ belongs to $\mathscr{A}$. Define the function in the plane $E : \mathbb{R}^2 \to \mathbb{R}$

$$E(\varepsilon, \tau) = \mathrm{End}(\kappa + \tau\vartheta + \varepsilon\eta).$$

This function is C^1-smooth and $H := \partial E(0,0)/\partial\tau \neq 0$, by (1.16). By the implicit function theorem, there exist $\varepsilon_0 > 0$ and a function $\tau \in C^1(-\varepsilon_0, \varepsilon_0)$ such that $E(\varepsilon, \tau(\varepsilon)) = E(0,0) = t_0$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Moreover, we have

$$\begin{aligned} \tau'(0) &= -\Big(\frac{\partial E(0,0)}{\partial\tau}\Big)^{-1}\Big(\frac{\partial E(0,0)}{\partial\varepsilon}\Big) \\ &= -\frac{1}{H}\int_0^1 (\eta_2\dot\kappa_1 - \eta_1\dot\kappa_2)ds = -\frac{1}{H}\int_0^1 \langle \dot\kappa^\perp, \eta\rangle ds, \end{aligned} \tag{1.17}$$

where $\dot\kappa^\perp = (-\dot\kappa_2, \dot\kappa_1)$, or equivalently, in the complex notation $\dot\kappa^\perp = i\dot\kappa$.

Since κ is a solution to the minimum problem (1.15) and $\kappa + \tau(\varepsilon)\vartheta + \varepsilon\eta \in \mathscr{A}$ with $\mathrm{End}(\kappa + \tau(\varepsilon)\vartheta + \varepsilon\eta) = t_0$, then we have

$$1 = \int_0^1 |\dot\kappa| ds \leq \int_0^1 |\dot\kappa + \tau(\varepsilon)\dot\vartheta + \varepsilon\dot\eta| ds = L(\varepsilon),$$

and thus $L'(0) = 0$. We can without loss of generality assume that κ is parameterized by arc-length, i.e., $|\dot\kappa| = 1$. The equation $L'(0) = 0$ gives (we also use (1.17))

$$\begin{aligned} 0 &= \tau'(0)\int_0^1 \langle \dot\kappa, \dot\vartheta\rangle ds + \int_0^1 \langle \dot\kappa, \dot\eta\rangle ds \\ &= \varphi\int_0^1 \langle \dot\kappa^\perp, \eta\rangle ds + \int_0^1 \langle \dot\kappa, \dot\eta\rangle ds, \end{aligned}$$

where $\varphi \in \mathbb{R}$ is the constant

$$\varphi = -\frac{1}{H}\int_0^1 \langle \dot\kappa, \dot\vartheta\rangle ds.$$

Eventually, for any test function $\eta \in C_c^\infty\big((0,1);\mathbb{R}^2\big)$ we have

$$\int_0^1 \big\{\langle \dot\kappa, \dot\eta\rangle + \varphi\langle \dot\kappa^\perp, \eta\rangle\big\} ds = 0,$$

and a standard argument implies that κ is in $C^\infty([0,1];\mathbb{R}^2)$ and it solves the differential equation $\ddot\kappa = \varphi\dot\kappa^\perp = i\varphi\dot\kappa$. Then we have $\dot\kappa(s) = i\mathrm{e}^{i\psi}\mathrm{e}^{i\varphi s}$, $s \in \mathbb{R}$, for some $\psi \in [0, 2\pi]$. Integrating with $\kappa(0) = 0$, we find

$$\kappa(s) = \frac{\mathrm{e}^{i\psi}(\mathrm{e}^{i\varphi s} - 1)}{\varphi}, \quad s \in \mathbb{R}.$$

The vertical coordinate of the horizontal lift γ of κ is

$$\gamma_3(s) = 2\int_0^s \mathrm{Im}(\kappa(\sigma)\overline{\dot\kappa(\sigma)})d\sigma = 2\frac{\varphi s - \sin\varphi s}{\varphi^2},$$

and thus for any $\psi \in [0,\pi]$ and $\varphi \in \mathbb{R}$ we get the curve

$$\gamma_{\psi,\varphi}(s) = \Big(\frac{\mathrm{e}^{i\psi}(\mathrm{e}^{i\varphi s}-1)}{\varphi}, 2\frac{\varphi s - \sin\varphi s}{\varphi^2}\Big), \quad s \in \mathbb{R}. \tag{1.18}$$

When $\varphi = 0$, γ reduces to the line $\gamma(s) = (i\mathrm{e}^{i\psi}s, 0)$.

The curve $\gamma_{\psi,\varphi}$ is length minimizing on the interval $0 \le s \le 2\pi/|\varphi|$ and, after $s = 2\pi/|\varphi|$, it ceases to be length minimizing. We prove this claim in the case $\varphi = 2\pi$ by a geometric argument. For $s = 1$ we have

$$\gamma_{\psi,2\pi}(1) = \big(0, 1/\pi\big).$$

At the point $(0, 1/\pi) \in \mathbb{C}\times\mathbb{R}$, the surface $S = \Phi([0,2\pi]\times[-2\pi,2\pi])$ introduced in (1.13) has a conical point directed downwards. By this, we mean that near $(0,1/\pi)$ the surfaces S stays above the cone $t = 1/\pi + \delta|z|$ for some $\delta > 0$. Then for any $\varepsilon > 0$ small enough there exist $0 < \lambda < 1$ and $(\bar\psi, \bar\varphi) \in [0,2\pi]\times[-2\pi,2\pi]$ such that $\gamma_{\psi,2\pi}(1+\varepsilon) = \delta_\lambda\Phi(\bar\psi,\bar\varphi)$. Since $d(\Phi(\bar\psi,\bar\varphi),0) \le 1$ (a posteriori we have equality, here), we deduce that

$$d(\gamma_{\psi,2\pi}(1+\varepsilon),0) = \lambda d(\Phi(\bar\psi,\bar\varphi),0) \le \lambda < 1.$$

Since the length of $\gamma_{\psi,2\pi}$ on the interval $[0,1+\varepsilon]$ is $1+\varepsilon$, we see that the curve is not length minimizing.

For any point $(z_0,t_0) \in \mathbb{H}^1$ with $z_0 \neq 0$, the system of equations

$$\frac{\mathrm{e}^{i\psi}(\mathrm{e}^{i\varphi s}-1)}{\varphi} = z_0, \quad 2\frac{\varphi s - \sin\varphi s}{\varphi^2} = t_0, \tag{1.19}$$

has unique solutions $s \ge 0$, $\psi \in [0,2\pi)$, and $\varphi \in \mathbb{R}$ subject to the constraint $s|\varphi| < 2\pi$ (we omit details). Thus $\gamma_{\psi,\varphi}$ is the unique length minimizing curve from 0 to (z_0,t_0) and $s = d((z_0,t_0),0)$.

□

Remark 1.2. The Heisenberg isoperimetric problem is related to the classical Dido problem, that asks to bound a region of the half plane with a curve with minimal length, where the boundary of the half plane (the coast) is a free length.

Let $\gamma : [0,1] \to \mathbb{H}^1$ be a horizontal curve such that $\gamma(0) = 0$ and let $\kappa : [0,1] \to \mathbb{R}^2$ be its horizontal projection. By formula (1.8), the third coordinate of γ at time $t \in [0,1]$ is

$$\gamma_3(t) = 2\int_0^t \big(\kappa_2\dot\kappa_1 - \kappa_1\dot\kappa_2\big)ds = 2\int_{\kappa|_{[0,t]}} ydx - xdy.$$

Let $E_t \subset \mathbb{R}^2$ be the region of the plane bounded by the curve κ restricted to $[0,t]$ and by the line segment joining $\kappa(t)$ to 0. Assume that the concatenation of κ and of the line segment bounds E_t counterclockwise. Then by Stokes' theorem we have

$$\gamma_3(t) = -4 \int_{E_t} dx \wedge dy = -4|E_t|.$$

If the orientation is clockwise, $-4|E_t|$ is replaced by $4|E_t|$. If the orientation is different in subregions of E_t, there are area cancellations.

So the minimum problem (1.15) consists in finding the shortest curve in the plane enclosing an amount of area given by the t_0 coordinate of the final point (z_0, t_0). In the Heisenberg isoperimetric problem, the point z_0 is also fixed, differently from Dido problem.

2. Heisenberg perimeter and other equivalent measures

2.1. ***H*-perimeter.** We introduce the notion of H-perimeter for a set $E \subset \mathbb{H}^n$. We preliminarily need the definition of H-divergence of a vector valued function $\varphi \in C^1(\mathbb{H}^n; \mathbb{R}^{2n})$.

Let V be a smooth vector field in $\mathbb{H}^n = \mathbb{R}^{2n+1}$. We may express V using both the basis X_j, Y_j, T and the standard basis of vector fields of $\mathbb{R}^{2n+1}$:

$$\begin{aligned} V &= \sum_{j=1}^n \big(\varphi_j X_j + \varphi_{n+j} Y_j\big) + \varphi_{2n+1} T \\ &= \sum_{j=1}^n \Big\{ \varphi_j \frac{\partial}{\partial x_j} + \varphi_{n+j} \frac{\partial}{\partial y_j} + \big(2y_j \varphi_j - 2x_j \varphi_{n+j}\big) \frac{\partial}{\partial t} \Big\} + \varphi_{2n+1} \frac{\partial}{\partial t}, \end{aligned} \tag{2.1}$$

where $\varphi_j, \varphi_{n+j}, \varphi_{2n+1} \in C^\infty(\mathbb{H}^n)$ are smooth functions. The standard divergence of V is

$$\begin{aligned} \operatorname{div} V &= \sum_{j=1}^n \Big\{ \frac{\partial \varphi_j}{\partial x_j} + \frac{\partial \varphi_{n+j}}{\partial y_j} + \Big(2y_j \frac{\partial \varphi_j}{\partial t} - 2x_j \frac{\partial \varphi_{n+j}}{\partial t}\Big) \Big\} + \frac{\partial \varphi_{2n+1}}{\partial t} \\ &= \sum_{j=1}^n \big(X_j \varphi_j + Y_j \varphi_{n+j}\big) + T\varphi_{2n+1}. \end{aligned} \tag{2.2}$$

The vector field V is said to be horizontal if $V(p) \in H_p$ for all $p \in \mathbb{H}^n$. Namely, a vector field V as in (2.1) is horizontal when $\varphi_{2n+1} = 0$. These observations suggest the following definition.

Let $A \subset \mathbb{H}^n$ be an open set. We define the horizontal divergence of a vector valued mapping $\varphi \in C^1(A; \mathbb{R}^{2n})$ as

$$\operatorname{div}_H \varphi = \sum_{j=1}^n \big(X_j \varphi_j + Y_j \varphi_{n+j}\big). \tag{2.3}$$

By (2.2), $\mathrm{div}_H\varphi = \mathrm{div}\, V$ is the standard divergence of the horizontal vector field V with coordinates $\varphi = (\varphi_1, \dots, \varphi_{2n})$ in the basis $X_1, \dots, X_n, Y_1, \dots, Y_n$. If $\|\cdot\|$ is the norm on H_p that makes $X_1, \dots, X_n, Y_1, \dots, Y_n$ orthonormal, then we have

$$\|V(p)\| = |\varphi(p)|,$$

where $|\cdot|$ is the standard norm on $\mathbb{R}^{2n}$.

The following definition is the starting point of the fundamental paper [27] (see also [33]).

Definition 2.1 (H-perimeter)**.** The H-perimeter in an open set $A \subset \mathbb{H}^n$ of a Lebesgue measurable set $E \subset \mathbb{H}^n$ is

$$P(E; A) = \sup\left\{ \int_E \mathrm{div}_H\varphi\, dzdt \,:\, \varphi \in C_c^1(A; \mathbb{R}^{2n}),\ \|\varphi\|_\infty \leq 1 \right\}. \tag{2.4}$$

Above, we let

$$\|\varphi\|_\infty = \sup_{p\in A} |\varphi(p)|.$$

If $P(E; A) < \infty$, we say that E has finite H-perimeter in A. If $P(E; A') < \infty$ for any open set $A' \subset\subset A$, we say that E has locally finite H-perimeter in A.

H-perimeter has the following invariance properties.

Proposition 2.2. Let $E \subset \mathbb{H}^n$ be a set with finite H-perimeter in an open set $A \subset \mathbb{H}^n$. Then for any $p \in \mathbb{H}^n$ and for any $\lambda > 0$ we have:

i) $P(L_pE; L_pA) = P(E; A)$;
ii) $P(\delta_\lambda E; \delta_\lambda A) = \lambda^{Q-1} P(E; A)$.

Proof. Statement i) follows from the fact that the vector fields X_j and Y_j are left invariant, and thus

$$(\mathrm{div}_H\varphi) \circ L_p = \mathrm{div}_H(\varphi \circ L_p).$$

We prove ii) in the case $A = \mathbb{H}^n$. First notice that for any $\varphi \in C_c^1(\mathbb{H}^n; \mathbb{R}^{2n})$ we have

$$\mathrm{div}_H(\varphi \circ \delta_\lambda) = \lambda(\mathrm{div}_H\varphi) \circ \delta_\lambda,$$

and thus

$$\int_{\delta_\lambda E} \mathrm{div}_H\varphi\, dzdt = \lambda^Q \int_E (\mathrm{div}_H\varphi) \circ \delta_\lambda\, dzdt = \lambda^{Q-1} \int_E \mathrm{div}_H(\varphi \circ \delta_\lambda)\, dzdt.$$

The claim easily follows.

□

Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter in an open set $A \subset \mathbb{H}^n$. The linear functional $T : C_c^1(A; \mathbb{R}^{2n}) \to \mathbb{R}$

$$T(\varphi) = \int_E \mathrm{div}_H\varphi(z, t)\, dzdt$$

is locally bounded in $C_c(A;\mathbb{R}^{2n})$. Namely, for any open set $A' \subset\subset A$ we have

$$T(\varphi) \le \|\varphi\|_\infty P(E;A') \tag{2.5}$$

for all $\varphi \in C^1_c(A';\mathbb{R}^{2n})$. By density, T can be extended to a bounded linear operator on $C_c(A';\mathbb{R}^{2n})$ satisfying the same bound (2.5). Thus, by Riesz' representation theorem we deduce the following proposition.

Proposition 2.3. Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter in the open set $A \subset \mathbb{H}^n$. There exist a positive Radon measure μ_E on A and a μ_E-measurable function $\nu_E : A \to \mathbb{R}^{2n}$ such that:

1) $|\nu_E| = 1$ μ_E-a.e. on A.
2) The following generalized Gauss-Green formula holds
$$\int_E \mathrm{div}_H \varphi \, dzdt = -\int_A \langle \varphi, \nu_E\rangle d\mu_E \tag{2.6}$$
for all $\varphi \in C^1_c(A;\mathbb{R}^{2n})$.

Above, $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{R}^{2n}$.

Definition 2.4 (Horizontal normal)**.** The measure μ_E is called *H-perimeter measure* and the function ν_E is called *measure theoretic inner horizontal normal* of E.

We shall refer to ν_E simply as to the *horizontal normal*. In Section 3, we describe geometrically ν_E in the smooth case (see (3.3)). In Proposition 2.10 below, we shall see that the vertical hyperplane in $\mathbb{H}^n$ orthogonal to $\nu_E(p)$ is the "tangent plane" to ∂E at points of the reduced boundary.

Remark 2.5. By Proposition 2.3, the open sets mapping $A' \mapsto P(E;A')$, with $A' \subset\subset A$ open, extends to the Radon measure μ_E. We show that for any open set $A' \subset\subset A$ we have $\mu_E(A') = P(E;A')$.

The inequality $P(E;A') \le \mu_E(A')$ follows from the sup-definition (2.4) of H-perimeter. The opposite inequality can be proved by a standard approximation argument. By Lusin's theorem, for any $\varepsilon > 0$ there exists a compact set $K \subset A'$ such that $\mu_E(A' \setminus K) < \varepsilon$ and $\nu_E : K \to \mathbb{R}^{2n}$ is continuous. By Titze's theorem, there exists $\psi \in C_c(A';\mathbb{R}^{2n})$ such that $\psi = \nu_E$ on K and $\|\psi\|_\infty \le 1$. Finally, by mollification there exists $\varphi \in C^\infty_c(A';\mathbb{R}^{2n})$ such that $\|\varphi - \psi\|_\infty < \varepsilon$ and $\|\varphi\|_\infty \le 1$. Then we have

$$P(E;A') \ge \int_E \mathrm{div}_H \varphi dzdt = -\int_{A'} \langle \varphi, \nu_E\rangle d\mu_E \ge (1-\varepsilon)\mu_E(A') - 2\varepsilon,$$

and the claim follows.

In the sequel, we need a metric structure on $\mathbb{H}^n$. For most purposes, the Carnot-Carathéodory metric would be fine. In some cases, however, as in the characterization

(2.13) of H-perimeter by means of spherical Hausdorff measures, the structure of Carnot-Carathéodory balls is less manageable. For this reason, we closely follow [27] and we use the metric ϱ introduced in (1.12) via the "box-norm" $\|\cdot\|_\infty$ in (1.11). We denote the open ball in ϱ centered at $p \in \mathbb{H}^n$ and with radius $r > 0$ in the following way

$$U(p,r) = \big\{q \in \mathbb{H}^n : \|p^{-1}\cdot q\|_\infty < r\big\}. \tag{2.7}$$

We also let $U_r(p) = U(p,r)$ and $U_r = U_r(0)$.

Definition 2.6 (Measure theoretic boundary)**.** The *measure theoretic boundary* of a measurable set $E \subset \mathbb{H}^n$ is the set

$$\partial E = \big\{p \in \mathbb{H}^n : |E \cap U_r(p)| > 0 \text{ and } |U_r(p) \setminus E| > 0 \text{ for all } r > 0\big\}.$$

The measure theoretic boundary is a subset of the topological boundary. The definition does not depend on the specific balls $U_r(p)$. We may also consider the set of points with density $1/2$:

$$E_{1/2} = \Big\{p \in \mathbb{H}^n : \lim_{r\to 0} \frac{|E \cap U_r(p)|}{|U_r(p)|} = \frac{1}{2}\Big\}.$$

We clearly have $E_{1/2} \subset \partial E$. The definition of $E_{1/2}$ is sensitive to the choice of the metric.

The perimeter measure μ_E is concentrated in a subset of $E_{1/2}$ called reduced boundary. The following definition is introduced and studied in [27].

Definition 2.7 (Reduced boundary)**.** The *reduced boundary* of a set $E \subset \mathbb{H}^n$ with locally finite H-perimeter is the set $\partial^* E$ of all points $p \in \mathbb{H}^n$ such that the following three conditions hold:

(1) $\mu_E(U_r(p)) > 0$ for all $r > 0$.

(2) We have
$$\lim_{r\to 0} \fint_{U_r(p)} \nu_E \, d\mu_E = \nu_E(p).$$

(3) There holds $|\nu_E(p)| = 1$.

As usual $\fint$, stands for the averaged integral. The definition of reduced boundary is sensitive to the metric. It also depends on the representative of ν_E.

The proof of the Euclidean model of Proposition 2.8 below relies upon Lebesgue-Besicovitch differentiation theorem for Radon measures in $\mathbb{R}^n$. In $\mathbb{H}^n$ with metrics equivalent to the Carnot-Carathéodory distance, however, Besicovitch's covering theorem fails (see [36] and [65]). This problem is bypassed in [27] using an asymptotic doubling property established, in a general context, in [1].

Proposition 2.8. Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter. Then the perimeter measure μ_E is concentrated on $\partial^* E$. Namely, we have $\mu_E(\mathbb{H}^n \setminus \partial^* E) = 0$.

Proof. By [1], Theorem 4.3, there exists a constant $\tau(n) > 0$ such that for μ_E-a.e. $p \in \mathbb{H}^n$ there holds

$$\tau(n) \le \liminf_{r\to 0} \frac{\mu_E(U_r(p))}{r^{Q-1}} \le \limsup_{r\to 0} \frac{\mu_E(U_r(p))}{r^{Q-1}} < \infty.$$

As a consequence, we have the following asymptotic doubling formula

$$\limsup_{r\to 0} \frac{\mu_E(U_{2r}(p))}{\mu_E(U_r(p))} < \infty, \tag{2.8}$$

for μ_E-a.e. $p \in \mathbb{H}^n$. Thus, by Theorems 2.8.17 and 2.9.8 in [25], for any function $f \in L^1_{\mathrm{loc}}(\mathbb{H}^n; \mu_E)$ there holds

$$\lim_{r\to 0} \fint_{U_r(p)} f \, d\mu_E = f(p)$$

for μ_E-a.e. $p \in \mathbb{H}^n$.

Assume that $p \in \mathbb{H}^n \setminus \partial^* E$. There are three possibilities:

1) We have $\mu_E(U_r(p)) = 0$ for some $r > 0$. The set of points with this property has null μ_E measure.
2) We have
$$\lim_{r\to 0} \fint_{U_r(p)} \nu_E \, d\mu_E \neq \nu_E(p).$$
By the above argument with $f = \nu_E$, the set of such points has null μ_E measure.
3) We have $|\nu_E(p)| \neq 1$. By Proposition 2.3, the set of such points has null μ_E measure.

This ends the proof. □

Definition 2.9 (Vertical plane)**.** For any $\nu \in \mathbb{R}^{2n}$ with $|\nu| = 1$, we call the set

$$H_\nu = \big\{ (z,t) \in \mathbb{H}^n : \langle \nu, z \rangle \ge 0, \, t \in \mathbb{R} \big\}$$

the *vertical half-space* through $0 \in \mathbb{H}^n$ with inner normal ν. The boundary of H_ν, the set

$$\partial H_\nu = \big\{ (z,t) \in \mathbb{H}^n : \langle \nu, z \rangle = 0, \, t \in \mathbb{R} \big\},$$

is called *vertical plane* orthogonal to ν passing through $0 \in \mathbb{H}^n$.

At points $p \in \partial^* E$, the set E blows up to the vertical half space H_ν with $\nu = \nu_E(p)$. In this sense, the boundary of H_ν is the anisotropic tangent space of $\partial^* E$ at p. The problem of the characterization of blow-ups in Carnot groups is still open. In general, it is known that in the blow-up of blow-ups there are vertical hyperplanes (see [3]). Hereafter, we let $E_\lambda = \delta_\lambda E$ for $\lambda > 0$.

Theorem 2.10 (Blow-up)**.** Let $E \subset \mathbb{H}^n$ be a set with finite H-perimeter, assume that $0 \in \partial^* E$ and let $\nu = \nu_E(0)$. Then we have

$$\lim_{\lambda \to \infty} \chi_{E_\lambda} = \chi_{H_\nu}, \tag{2.9}$$

where the limit is in $L^1_{\mathrm{loc}}(\mathbb{H}^n)$. Moreover, for a.e. $r > 0$ we have

$$\lim_{\lambda \to \infty} P(E_\lambda; U_r) = P(H_\nu; U_r) = c_n r^{Q-1}, \tag{2.10}$$

where $c_n = P(H_\nu; U_1) > 0$ is an absolute constant.

Proof. Let $\varphi \in C^1_c(\mathbb{H}^n; \mathbb{R}^{2n})$ be a test vector valued function. For a.e. $r > 0$, we have the following integration by parts formula

$$\int_{E \cap U_r} \mathrm{div}_H \varphi \, dzdt = -\int_{U_r} \langle \varphi, \nu_E \rangle d\mu_E - \int_{\partial U_r \cap E} \langle \varphi, \nu_{U_r} \rangle d\mu_{U_r}. \tag{2.11}$$

This formula can be proved in the following way. Let $(f_j)_{j \in \mathbb{N}}$ be a sequence of functions $f_j \in C^\infty(\mathbb{H}^n)$ such that $f_j \to \chi_E$, as $j \to \infty$, in $L^1_{\mathrm{loc}}(\mathbb{H}^n)$ and $\nabla_H f_j dzdt \rightharpoonup \nu_E d\mu_E$ in the weak sense of Radon measures. We are denoting by

$$\nabla_H f = (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$$

the horizontal gradient of a function f.

The set U_r supports the standard divergence theorem and therefore we have

$$\int_{U_r} f_j \mathrm{div}_H \varphi \, dzdt = -\int_{U_r} \langle \varphi, \nabla_H f_j \rangle dzdt - \int_{\partial U_r} f_j \langle \varphi, \nu_{U_r} \rangle d\mu_{U_r}. \tag{2.12}$$

We can assume that, for a.e. $r > 0$, $f_j \to \chi_E$ in $L^1(\partial U_r)$ and $\mu_E(\partial U_r) = 0$. Letting $j \to \infty$ in (2.12) we obtain (2.11).

Let $\varphi \in C^1_c(\mathbb{H}^n; \mathbb{R}^{2n})$ be such that $\varphi(z,t) = \nu_E(0)$ for all $(z,t) \in U_r$. From (2.11), we have

$$0 = -\int_{U_r} \langle \nu_E(0), \nu_E \rangle d\mu_E - \int_{\partial U_r \cap E} \langle \nu_E(0), \nu_{U_r} \rangle d\mu_{U_r}.$$

Using $|\nu_E(0)| = |\nu_{U_r}| = 1$ a.e. and Proposition 2.2, we have

$$\int_{U_r} \langle \nu_E(0), \nu_E \rangle d\mu_E = -\int_{\partial U_r \cap E} \langle \nu_E(0), \nu_{U_r} \rangle d\mu_{U_r} \le P(U_r; \mathbb{H}^n) = r^{Q-1} P(U_1; \mathbb{H}^n).$$

Since $0 \in \partial^* E$, there holds

$$\int_{U_r} \langle \nu_E(0), \nu_E \rangle d\mu_E = (1 + o(1)) P(E; U_r),$$

where $o(1) \to 0$ as $r \to 0$. Using these estimates, we conclude that for any $\lambda \ge 1$ we have

$$P(E_\lambda; U_r) = \lambda^{Q-1} P(E; U_{r/\lambda}) \le 2P(U_1; \mathbb{H}^n) r^{Q-1}.$$

The family of sets $(E_\lambda)_{\lambda > 1}$ has locally uniformly bounded perimeter. By the compactness theorem for BV_H functions (see [33]), there exists a set $F \subset \mathbb{H}^n$ with locally finite perimeter and a sequence $\lambda_j \to \infty$ such that $E_{\lambda_j} \to F$ in the $L^1_{\rm loc}(\mathbb{H}^n)$ convergence of characteristic functions. From the Gauss-Green formula (2.6), it follows that

$$\nu_{E_{\lambda_j}}\mu_{E_{\lambda_j}} \rightharpoonup \nu_F\mu_F, \quad \text{as } j\to\infty,$$

in the sense of the weak convergence of Radon measures.

Starting from the identity

$$\fint_{U_r} \nu_{E_{\lambda_j}} d\mu_{E_{\lambda_j}} = \fint_{U_{r/\lambda_j}} \nu_E d\mu_E,$$

using $0\in\partial^* E$, and choosing $r>0$ such that $\mu_F(\partial U_r)=0$ – this holds for a.e. $r>0$, – letting $j\to\infty$ we find

$$\fint_{U_r} \langle \nu_F, \nu_E(0)\rangle d\mu_F = 1.$$

This implies that $\nu_F = \nu_E(0)$ μ_F-a.e. in $\mathbb{H}^n$, because $r>0$ is otherwise arbitrary. By the characterization of sets with constant horizontal normal (see Remark 5.7 below), we have $F = H_\nu$ with $\nu = \nu(0)$. We are omitting the proof that $0\in\partial F$. The limit $F=H_\nu$ is thus independent of the sequence $(\lambda_j)_{j\in\mathbb{N}}$ and this observation concludes the proof of (2.9).

We prove (2.10). From

$$\fint_{U_r} \langle \nu, \nu_{E_\lambda}\rangle d\mu_{E_\lambda} = \fint_{U_{r/\lambda}} \langle \nu, \nu_E\rangle d\mu_E = 1+o(1), \qquad \text{as} \qquad \lambda\to\infty,$$

we deduce that

$$P(E_\lambda; U_r) = (1+o(1))\int_{U_r} \langle \nu, \nu_{E_\lambda}\rangle d\mu_{E_\lambda}.$$

Letting $\lambda\to\infty$, using the weak convergence $\nu_{E_\lambda} d\mu_{E_\lambda} \to \nu_F d\mu_F$ and choosing $r>0$ with $\mu_F(\partial U_r)=0$, we get the claim. □

2.2. **Equivalent notions for H-perimeter.** In this section, we describe some characterizations of H-perimeter related to the metric structure of $\mathbb{H}^n$.

2.2.1. *Hausdorff measures.* The Heisenberg perimeter has a representation in terms of spherical Hausdorff measures. We use the metric ϱ in (1.12). The diameter of a set $K\subset\mathbb{H}^n$ is

$$\operatorname{diam} K = \sup_{p,q\in K} \varrho(p,q).$$

If U_r is a ball in the distance ϱ with radius r, then we have $\operatorname{diam} U_r = 2r$. Let $E\subset\mathbb{H}^n$ be a set. For any $s\ge 0$ and $\delta>0$ define the premeasures

$$\mathscr{H}^{s,\delta}_\varrho(E) = \inf\Big\{ \sum_{i\in\mathbb{N}} (\operatorname{diam} K_i)^s : E\subset \bigcup_{i\in\mathbb{N}} K_i,\ K_i\subset\mathbb{H}^n,\ \operatorname{diam} K_i<\delta\Big\},$$

$$\mathscr{S}^{s,\delta}_\varrho(E) = \inf\Big\{ \sum_{i\in\mathbb{N}} (\operatorname{diam} U_i)^s : E\subset \bigcup_{i\in\mathbb{N}} U_i, U_i\ \varrho\text{-balls in } \mathbb{H}^n, \operatorname{diam} U_i<\delta\Big\},$$

Letting $\delta \to 0$, we define

$$\mathscr{H}^s_\varrho(E) = \sup_{\delta>0} \mathscr{H}^{s,\delta}_\varrho(E) = \lim_{\delta\to 0} \mathscr{H}^{s,\delta}_\varrho(E),$$
$$\mathscr{S}^s_\varrho(E) = \sup_{\delta>0} \mathscr{S}^{s,\delta}_\varrho(E) = \lim_{\delta\to 0} \mathscr{S}^{s,\delta}_\varrho(E).$$

By Carathèodory's construction, $E \mapsto \mathscr{H}^s_\varrho(E)$ and $E \mapsto \mathscr{S}^s_\varrho(E)$ are Borel measures in $\mathbb{H}^n$. The measure $\mathscr{H}^s_\varrho$ is called s-dimensional Hausdorff measure. The measure $\mathscr{S}^s_\varrho$ is called s-dimensional spherical Hausdorff measure. These measures are equivalent, in the sense that for any $E \subset \mathbb{H}^n$ there holds

$$\mathscr{H}^s_\varrho(E) \le \mathscr{S}^s_\varrho(E) \le 2^s \mathscr{H}^s_\varrho(E).$$

The measures $\mathscr{H}^Q_\varrho(E)$ and $\mathscr{S}^Q_\varrho$ are Haar measures in $\mathbb{H}^n$ and therefore they coincide with the Lebesgue measure, up to a multiplicative constant factor. The natural dimension to measure hypersurfaces, as the boundary of smooth sets, is $s = Q - 1$.

The following theorem is proved in [27], Theorem 7.1 part (iii). The proof relies on Federer's differentiation theorems, Theorem 2.10.17 and Theorem 2.10.19 part (3) of [25]. Extensions of this result are based on general differentiation theorems for measures, see [41]. Formula (2.14) for the geometric constant c_n in (2.13) depends on the shape (convexity and symmetries) of the metric unit ball U_1, [42].

Theorem 2.11 (Franchi-Serapioni-Serra Cassano)**.** For any set $E \subset \mathbb{H}^n$ with locally finite H-perimeter we have

$$\mu_E = c_n \mathscr{S}^{Q-1}_\varrho \llcorner \partial^* E, \tag{2.13}$$

where μ_E is the perimeter measure of E, $\mathscr{S}^{Q-1}_\varrho \llcorner \partial^* E$ is the restriction of $\mathscr{S}^{Q-1}_\varrho$ to the reduced boundary $\partial^* E$, and the constant $c_n > 0$ is given by

$$c_n = P(H_\nu; U_1). \tag{2.14}$$

Remark 2.12. It is not known whether in (2.13) the spherical measure $\mathscr{S}^{Q-1}_\varrho$ can be replaced by the Hausdorff measure $\mathscr{H}^{Q-1}_\varrho$, even when $\partial^* E$ is a smooth set. In $\mathbb{R}^n$ with the standard perimeter, the identity $\mathscr{S}^{n-1} \llcorner \partial^* E = \mathscr{H}^{n-1} \llcorner \partial^* E$ follows from Besicovitch's covering theorem, that fails to hold in the Heisenberg group, see [36] and [65].

2.2.2. *Minkowski content and H-perimeter.* In the description of H-perimeter in terms of Minkowski content, the correct choice of the metric is the Carnot-Carathéodory distance d on $\mathbb{H}^n$.

The Carnot-Carathéodory distance from a closed set $K \subset \mathbb{H}^n$ is the function

$$\mathrm{dist}_K(p) = \min_{q\in K} d(p,q), \qquad p \in \mathbb{H}^n.$$

For $r > 0$, the r-tubular neighborhood of K is the set

$$I_r(K) = \big\{ p \in \mathbb{H}^n : \mathrm{dist}_K(p) < r \big\}.$$

The *upper* and *lower Minkowski content* of K in an open set $A \subset \mathbb{H}^n$ are, respectively,

$$\mathscr{M}^+(K;A) = \limsup_{r\to 0} \frac{|I_r(K)\cap A|}{2r},$$
$$\mathscr{M}^-(K;A) = \liminf_{r\to 0} \frac{|I_r(K)\cap A|}{2r}.$$

Above, $|\cdot|$ stands for Lebesgue measure. If $\mathscr{M}^+(K;A) = \mathscr{M}^-(K;A)$, the common value is called *Minkowski content* of K in A and it is denoted by $\mathscr{M}(K;A)$.

Below, $\mathscr{H}^{2n}$ is the standard $2n$-dimensional Hausdorff measure in $\mathbb{H}^n = \mathbb{R}^{2n+1}$.

Theorem 2.13 (Monti-Serra Cassano)**.** Let $A \subset \mathbb{H}^n$ be an open set and let $E \subset \mathbb{H}^n$ be a bounded set with C^2 boundary such that $\mathscr{H}^{2n}(\partial E \cap \partial A) = 0$. Then we have

$$P(E;A) = \mathscr{M}(\partial E;A). \tag{2.15}$$

This result is proved in [54], in a general framework. It is an open problem to prove formula (2.15) for sets E with less regular boundary. The tools used in the proof in [54] are the eikonal equation for the Carnot-Carathéodory distance and the coarea formula. Assume $A = \mathbb{H}^n$. We have

$$|I_r(\partial E)| = \int_{I_r(\partial E)} |\nabla_H \mathrm{dist}_{\partial E}(z,t)| dzdt,$$

because $|\nabla_H \mathrm{dist}_K(z,t)| = 1$ a.e. in $\mathbb{H}^n$. By the coarea formula in the sub-Riemannian setting, we have

$$\int_{I_r(\partial E)} |\nabla_H \mathrm{dist}_{\partial E}(z,t)| dzt = \int_0^r P(I_s(\partial E);\mathbb{H}^n) ds.$$

We refer the reader to [54] and [40] for a discussion on coarea formulas. Now formula (2.15) follows proving that

$$\lim_{r\to 0} \frac{1}{2r} \int_0^r P(I_s(\partial E);\mathbb{H}^n) ds = P(E;\mathbb{H}^n).$$

The regularity of ∂E is used at this final step: the Riemannian approximation of the distance function from ∂E is of class C^2, if ∂E is of class C^2.

2.2.3. *Integral differential quotients.* H-perimeter can be also expressed as the limit of certain integral differential quotients.

Let $k_n > 0$ be the following geometric constant

$$k_n = \fint_{B_1} |\langle \nu, z\rangle| dzdt,$$

where $B_1 \subset \mathbb{H}^n$ is the unitary Carnot-Carathéodory ball centered at the origin. By the rotational symmetry of B_1, the definition of k_n is independent of the unit vector $\nu \in \mathbb{R}^{2n}$, $|\nu| = 1$. The following theorem is proved in [62].

Theorem 2.14. A Borel set $E \subset \mathbb{H}^n$ with finite measure has finite H-perimeter in $\mathbb{H}^n$ if and only if

$$\liminf_{r\downarrow 0} \frac{1}{r} \int_{\mathbb{H}^n} \fint_{B_r(q)} |\chi_E(p) - \chi_E(q)| \, dp \, dq < \infty.$$

Moreover, if E has also finite Euclidean perimeter then

$$\lim_{r\downarrow 0} \frac{1}{r} \int_{\mathbb{H}^n} \fint_{B_r(q)} |\chi_E(p) - \chi_E(q)| \, dp \, dq = k_n P(E; \mathbb{H}^n). \tag{2.16}$$

For the proof, we refer to [62], where the result is proved in the setting of BV_H functions. It is an open question whether the identity (2.16) holds dropping the assumption "if E has also finite Euclidean perimeter".

The characterization of H-perimeter in Theorem 2.14 is useful in the theory of rearrangements in the Heisenberg group proposed in [49].

2.3. **Rectifiability of the reduced boundary.** The reduced boundary of sets with finite H-perimeter needs not be rectifiable in the standard sense. However, it is rectifiable in an intrinsic sense that we are going to explain. The main reference is the paper [27]. A systematic treatment of these topics in the setting of stratified groups can be found in [39].

We need first the notion of C^1_H-regular function.

Definition 2.15 (C^1_H-function)**.** Let $A \subset \mathbb{H}^n$ be an open set. A function $f : A \to \mathbb{R}$ is of class $C^1_H(A)$ if:

1) $f \in C(A)$;
2) the derivatives $X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f$ in the sense of distributions are (represented by) continuous functions in A.

The horizontal gradient of a function $f \in C^1_H(A)$ is the vector valued mapping $\nabla_H f \in C(A; \mathbb{R}^{2n})$, $\nabla_H f = \big(X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f\big)$.

For C^1_H-regular functions there is an implicit function theorem (Theorem 6.5 in [27]) that can be used to represent the zero set $\{f = 0\}$ as an "intrinsic Lipschitz graph" (see Section 3.1.4).

Definition 2.16 (H-regular hypersurface)**.** A set $S \subset \mathbb{H}^n$ is an H-regular hypersurface if for all $p \in S$ there exists $r > 0$ and a function $f \in C^1_H(B_r(p))$ such that:

1) $S \cap B_r(p) = \big\{q \in B_r(p) : f(q) = 0\big\}$;
2) $|\nabla_H f(p)| \neq 0$.

If $S \subset \mathbb{H}^n$ is a hypersurface of class C^1 in the standard sense, then for any $p \in S$ there exist $r > 0$ and a function $f \in C^1(B_r(p))$ such that $S \cap B_r(p) = \big\{q \in B_r(p) : f(q) = 0\big\}$ and $|\nabla f(p)| \neq 0$. However, the set S needs not be an H-regular hypersurface because it may happen that $|\nabla_H f(p)| = 0$ at some (many) points $p \in S$. On the other hand, the following theorem, proved in [35] Theorem 3.1, shows that, in general, H-regular hypersurfaces are not rectifiable.

Theorem 2.17 (Kirchheim-Serra Cassano)**.** There exists an H-regular surface $S \subset \mathbb{H}^1$ such that

$$\mathscr{H}^{(5-\varepsilon)/2}(S) > 0 \quad \text{for all } \varepsilon \in (0,1).$$

In particular, the set S is not 2-rectifiable.

Above, $\mathscr{H}^s$ is the standard s-dimensional Hausdorff measure in $\mathbb{R}^3$. The set S constructed in [35] has Euclidean Hausdorff dimension 5/2. Any H-regular surface $S \subset \mathbb{H}^1$ can be locally parameterized by a 1/2-Hölder continuous map $\Phi : \mathbb{R}^2 \to \Phi(\mathbb{R}^2) = S \subset \mathbb{H}^1$, i.e., $d(\Phi(u), \Phi(v)) \leq C|u-v|^{1/2}$ for $u, v \in \mathbb{R}^2$, where $C > 0$ is a constant and d is the Carnot-Carathéodory distance, see Theorem 4.1 in [35].

Definition 2.18. A set $\Gamma \subset \mathbb{H}^n$ is $\mathscr{S}_\varrho^{Q-1}$-rectifiable if there exists a sequence of H-regular hypersurfaces $(S_j)_{j\in\mathbb{N}}$ in $\mathbb{H}^n$ such that

$$\mathscr{S}_\varrho^{Q-1}\Big(\Gamma \setminus \bigcup_{j\in\mathbb{N}} S_j\Big) = 0.$$

This definition is generalized in [43], where the authors study the notion of a s-rectifiable set in $\mathbb{H}^n$ for any integer $1 \leq s \leq Q-1$. The definition of s-rectifiability has a different nature according to whether $s \leq n$ or $s \geq n+1$. Definition 2.18 is relevant because the reduced boundary of sets with finite H-perimeter is rectifiable precisely in this sense. The following theorem is the main result of [27].

Theorem 2.19. Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter. Then the reduced boundary $\partial^* E$ is $\mathscr{S}_\varrho^{Q-1}$-rectifiable.

The proof of Theorem 2.19 goes as follows, for details see Theorem 7.1 in [27]. By Lusin's theorem there are compact sets $K_j \subset \partial^* E$, $j \in \mathbb{N}$, and a set $N \subset \partial^* E$ such that:

i) $\mu_E(N) = 0$;
ii) $\nu_E : K_j \to \mathbb{R}^{2n}$ is continuous, for each $j \in \mathbb{N}$;
iii) $\partial^* E = N \cup \bigcup_{j\in\mathbb{N}} K_j$.

By a Whitney extension theorem (Theorem 6.8 in [27]), it is possible to construct functions $f_j \in C_H^1(\mathbb{H}^n)$ such that $\nabla_H f_j = \nu_E$ and $f_j = 0$ on K_j. Then the sets $S_j = \{f_j = 0\}$ are H-regular hypersurfaces near K_j and $K_j \subset S_j$.

3. Area formulas, first variation and H-minimal surfaces

3.1. **Area formulas.** In this section, we derive some area formulas for H-perimeter of sets with regular boundary. In particular, we study sets with Euclidean Lipschitz boundary and sets with "intrinsic Lipschitz boundary".

3.1.1. *Sets with Lipschitz boundary.* Let $E \subset \mathbb{H}^n$ be a set with Lipschitz boundary and denote by N the Euclidean outer unit normal to ∂E. This vector is defined at $\mathscr{H}^{2n}$-a.e. point of ∂E. Here and hereafter, $\mathscr{H}^{2n}$ denotes the standard $2n$-dimensional Hausdorff measure of $\mathbb{R}^{2n+1}$. Using the projections of $X_1, \dots, X_n, Y_1, \dots, Y_n$ along the normal N, we can define the $2n$-dimensional vector field $N_H : \partial E \to \mathbb{R}^{2n}$

$$N_H = \big(\langle X_1, N\rangle, \dots, \langle X_n, N\rangle, \langle Y_1, N\rangle \dots, \langle Y_n, N\rangle\big), \tag{3.1}$$

where the vector fields X_j, Y_j and N are identified with elements of $\mathbb{R}^{2n+1}$ and $\langle\cdot,\cdot\rangle$ is the standard scalar product.

Proposition 3.1. Let $E \subset \mathbb{H}^n$ be a set with Lipschitz boundary. Then the H-perimeter of E in an open set $A \subset \mathbb{H}^n$ is

$$P(E; A) = \int_{\partial E \cap A} |N_H| d\mathscr{H}^{2n}, \tag{3.2}$$

where N is the Euclidean (outer) unit normal to ∂E and $|N_H|$ is the Euclidean norm of N_H.

Proof. For any $\varphi \in C^1_c(A; \mathbb{R}^{2n})$ let $V = \sum_{j=1}^n \varphi_j X_j + \varphi_{n+j} Y_j$ be the horizontal vector field with coordinates φ. By the standard divergence theorem and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_E \mathrm{div}_H \varphi \, dzdt &= \int_E \mathrm{div} V \, dzdt = \int_{\partial E} \langle V, N\rangle d\mathscr{H}^{2n} \\ &= \int_{\partial E} \sum_{j=1}^n \varphi_j \langle X_j, N\rangle + \varphi_{n+j}\langle Y_j, N\rangle d\mathscr{H}^{2n} \le \int_{\partial E} \sum_{j=1}^n |\varphi||N_H| d\mathscr{H}^{2n},\end{aligned}$$

and taking the supremum with $\|\varphi\|_\infty \le 1$ it follows that $P(E; A) \le \displaystyle\int_{\partial E \cap A} |N_H| d\mathscr{H}^{2n}$.

The opposite inequality can be obtained by approximation. By Lusin's theorem, for any $\varepsilon > 0$ there exists a compact set $K \subset \partial E \cap A$ such that

$$\int_{(\partial E \setminus K)\cap A} |N_H| d\mathscr{H}^{2n} < \varepsilon,$$

and $N_H : K \to \mathbb{R}^{2n}$ is continuous and nonzero. By Tietze's theorem, there exists $\psi \in C_c(A; \mathbb{R}^{2n})$ such that $\|\psi\|_\infty \le 1$ and $\psi = N_H/|N_H|$ on K. By mollification there exists $\varphi \in C^1_c(A;\mathbb{R}^{2n})$ such that $\|\varphi\|_\infty \le 1$ and $\|\psi - \varphi\|_\infty < \varepsilon$. For such a test function φ we have

$$\int_E \mathrm{div}_H \varphi \, dzdt \ge (1-\varepsilon)\int_{\partial E\cap A} |N_H| d\mathscr{H}^{2n} - 2\varepsilon.$$

This ends the proof. $\square$

3.1.2. *Formulas for the horizontal inner normal.* Let $E \subset \mathbb{H}^n$ be a set with Lipschitz boundary and let $\varphi \in C^1_c(\mathbb{H}^n;\mathbb{R}^{2n})$. From the Gauss-Green formula (2.6) and from the standard divergence theorem, we have

$$\int_{\partial E} \langle \varphi, N_H\rangle d\mathscr{H}^{2n} = \int_E \mathrm{div}_H \varphi \, dzdt = -\int_{\mathbb{H}^n} \langle \varphi, \nu_E\rangle d\mu_E.$$

It follows that the perimeter measure has the following representation

$$\mu_E = |N_H| \mathscr{H}^{2n} \llcorner \partial E,$$

and the measure theoretic inner normal is

$$\nu_E = -\frac{N_H}{|N_H|} \quad \mu_E\text{-a.e. on } \partial E. \tag{3.3}$$

Next, we express ν_E in terms of a defining function for the boundary. Assume that ∂E is a C^1-surface and $f \in C^1(A)$ is a defining function for ∂E, i.e., $\partial E \cap A = \{p \in A : f(p) = 0\}$ with $|\nabla f| \neq 0$ and $f < 0$ inside E. Then the outer Euclidean normal to ∂E is

$$N = \frac{\nabla f}{|\nabla f|} \quad \text{on } \partial E \cap A,$$

and therefore the vector N_H introduced in (3.1) is

$$N_H = \frac{\nabla_H f}{|\nabla f|} \quad \text{on } \partial E \cap A.$$

From (3.3), we conclude that the horizontal inner normal is given by

$$\nu_E = -\frac{\nabla_H f}{|\nabla_H f|} \quad \text{on } \partial E \cap A,\ |\nabla_H f| \neq 0. \tag{3.4}$$

Let N_E be the horizontal vector with coordinates ν_E in the basis $X_1, \dots, X_n, Y_1, \dots, Y_n$. The vector N_E can be recovered in the following way. Fix on $\mathbb{H}^n$ the Riemannian metric making $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ orthonormal. The Riemannian exterior normal to the surface $\{f = 0\}$ is the vector

$$N_R = \frac{\nabla_R f}{|\nabla_R f|_R}.$$

where $\nabla_R f = \sum_{j=1}^n (X_j f)X_j + (Y_j f)Y_j + (Tf)T$ is the gradient of f and $|\nabla_R f|_R$ is its Riemannian length. Let $\pi_R : T_p\mathbb{H}^n \to H_p$ be the orthogonal projection onto the horizontal plane. Then the vector N_E is precisely

$$N_E = \frac{\pi_R(N_R)}{|\pi_R(N_R)|_R}.$$

3.1.3. *Area formula for t-graphs.* We specialize formula (3.2) to the case of t-graphs. Let $D \subset \mathbb{R}^{2n} = \mathbb{C}^n$ be an open set and let $f : D \to \mathbb{R}$ be a function. The set $E_f = \big\{(z,t) \in \mathbb{H}^n : t > f(z),\, z \in D\big\}$ is called t-epigraph of f. The set $\mathrm{gr}(f) = \big\{(z,t) \in \mathbb{H}^n : t = f(z),\, z \in D\big\}$ is called t-graph of f.

Proposition 3.2 (Area formula for t-graphs)**.** Let $D \subset \mathbb{R}^{2n}$ be an open set and let $f : D \to \mathbb{R}$ be a Lipschitz function. Then we have

$$P(E_f; D \times \mathbb{R}) = \int_D |\nabla f(z) + 2z^\perp| dz, \tag{3.5}$$

where $z^\perp = (x,y)^\perp = (-y,x)$.

Proof. The outer normal to $\partial E_f \cap (D \times \mathbb{R}) = \mathrm{gr}(f)$ is $N = (\nabla f, -1)/\sqrt{1 + |\nabla f|^2}$, and so, for any $j = 1, \dots, n$, we have

$$\langle N, X_j \rangle = \frac{\partial_{x_j} f - 2y_j}{\sqrt{1 + |\nabla f|^2}}, \quad \langle N, Y_j \rangle = \frac{\partial_{y_j} f + 2y_j}{\sqrt{1 + |\nabla f|^2}},$$

and thus

$$|N_H| = \frac{|\nabla f + 2z^\perp|}{\sqrt{1 + |\nabla f|^2}}.$$

By formula (3.2) and by the standard area formula for graphs, we obtain

$$P(E_f; D \times \mathbb{R}) = \int_{\mathrm{gr}(f)} |N_H| d\mathscr{H}^{2n} = \int_D |\nabla f(z) + 2z^\perp| dz$$

□

The area formula (3.5) is the starting point of many investigations on H-minimal surfaces. Epigraphs of the form $E_f = \{t > f(z)\}$ are systematically studied in [71]. In particular, in Theorem 3.2 of [71] the authors compute the relaxed functional in $L^1(D)$ of the area functional $\mathscr{A} : C^1(D) \to [0, \infty]$

$$\mathscr{A}(f) = \int_D |\nabla f(z) + 2z^\perp| dz.$$

They also prove existence of minimizers with a trace constraint when D is a bounded open set with Lipschitz boundary (Theorem 1.4) and they show that minimizers are locally bounded (Theorem 1.5). The Lipschitz regularity of minimizers under the bounded slope condition is proved in [63].

3.1.4. *Area formula for intrinsic graphs.* Let $S \subset \mathbb{H}^n$ be a C^1_H-regular hypersurface. Then we have $S = \{f = 0\}$ with $f \in C^1_H$ satisfying $|\nabla_H f| \neq 0$. Up to a change of coordinates, we can assume that locally we have $X_1 f > 0$. Then each integral line of X_1 meets S in one single point: S is a graph along X_1. These considerations lead to the following definitions.

The line flow of the vector field X_1 starting from the point $(z,t) \in \mathbb{H}^n$ is

$$\exp(sX_1)(z,t) = (z + s\mathrm{e}_1, t + 2y_1 s), \quad s \in \mathbb{R},$$

where $\mathrm{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{2n}$ and $z = (x, y) \in \mathbb{C}^n = \mathbb{R}^{2n}$, with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

We fix a domain of initial points. The most natural choice is to consider the vertical hyperplane $W = \big\{(z,t) \in \mathbb{H}^n : x_1 = 0\big\}$, that is identified with $\mathbb{R}^{2n}$ with the coordinates $w = (x_2, \ldots, x_n, y_1, \ldots, y_n, t)$.

Definition 3.3 (Intrinsic epigraph and graph)**.** Let $D \subset W$ be a set and let $\varphi : D \to \mathbb{R}$ be a function. The set

$$E_\varphi = \big\{ \exp(sX_1)(w) \in \mathbb{H}^n : s > \varphi(w),\, w \in D \big\}$$

is called *intrinsic epigraph* of φ along X_1. The set

$$\mathrm{gr}(\varphi) = \big\{ \exp(\varphi(w)X_1)(w) \in \mathbb{H}^n : w \in D \big\} \tag{3.6}$$

is called *intrinsic graph* of φ along X_1.

In Definition 3.8, there is an equivalent point of view on intrinsic graphs.

We are going to introduce a nonlinear gradient for functions $\varphi : D \to \mathbb{R}$. First, let us introduce the Burgers' operator : $\mathrm{Lip}_{\mathrm{loc}}(D) \to L^\infty_{\mathrm{loc}}(D)$

$$\varphi = \frac{\partial \varphi}{\partial y_1} - 4\varphi \frac{\partial \varphi}{\partial t}. \tag{3.7}$$

Next, notice that the vector fields $X_2, \ldots, X_n, Y_2, \ldots, Y_n$ can be naturally restricted to W.

Definition 3.4 (Intrinsic gradient)**.** The *intrinsic gradient* of a function $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(D)$ is the vector valued mapping $\nabla^\varphi \varphi \in L^\infty_{\mathrm{loc}}(D; \mathbb{R}^{2n-1})$

$$\nabla^\varphi \varphi = \big(X_2\varphi, \ldots, X_n\varphi, \varphi, Y_2\varphi, \ldots, Y_n\varphi\big).$$

When $n = 1$, the definition reduces to $\nabla^\varphi \varphi = \varphi$.

With abuse of notation, we define the cylinder over $D \subset W$ along X_1 as the set

$$D \cdot \mathbb{R} = \big\{ \exp(sX_1)(w) \in \mathbb{H}^n : w \in D \text{ and } s \in \mathbb{R} \big\}.$$

When $D \subset W$ is open, the cylinder $D \cdot \mathbb{R}$ is an open set in $\mathbb{H}^n$. The general version of the following proposition is presented in Theorem 3.9.

Proposition 3.5. Let $D \subset W$ be an open set and let $\varphi : D \to \mathbb{R}$ be a Lipschitz function. Then the H-perimeter of the intrinsic epigraph E_φ in the cylinder $D \cdot \mathbb{R}$ is

$$P(E_\varphi; D \cdot \mathbb{R}) = \int_D \sqrt{1 + |\nabla^\varphi \varphi|^2} dw, \tag{3.8}$$

where dw is the Lebesgue measure in $\mathbb{R}^{2n}$.

Proof. We prove the claim in the case $n = 1$. The intrinsic graph mapping $\Phi : D \to \mathbb{H}^1$ is $\Phi(y, t) = \exp(\varphi(y, t)X)(0, y, t) = (\varphi, y, t + 2y\varphi)$, and thus

$$\Phi_y \wedge \Phi_t = \begin{vmatrix} \mathrm{e}_1 & \mathrm{e}_2 & \mathrm{e}_3 \\ \varphi_y & 1 & 2\varphi + 2\varphi_y \\ \varphi_t & 0 & 1 + 2y\varphi_t \end{vmatrix} = \big(1 + 2y\varphi_t\big)\mathrm{e}_1 + \big(2\varphi\varphi_t - \varphi_y\big)\mathrm{e}_2 - \varphi_t \mathrm{e}_3.$$

The Euclidean outer normal to the intrinsic graph $\partial E_\varphi \cap (D \cdot \mathbb{R})$ is the vector $N = -\Phi_y \wedge \Phi_t / |\Phi_y \wedge \Phi_t|$ and thus

$$\langle N, X\rangle = \frac{-1}{|\Phi_y \wedge \Phi_t|} \quad \text{and} \quad \langle N, Y\rangle = \frac{\varphi_y - 4\varphi\varphi_t}{|\Phi_y \wedge \Phi_t|} = \frac{\varphi}{|\Phi_y \wedge \Phi_t|}.$$

From formula (3.2) and from the standard area formula for graphs, we obtain

$$\begin{aligned} P(E_\varphi; D \cdot \mathbb{R}) &= \int_{\partial E_\varphi \cap D \cdot \mathbb{R}} |N_H| d\mathscr{H}^2 \\ &= \int_D \sqrt{\frac{1}{|\Phi_y \wedge \Phi_t|^2} + \frac{(\varphi)^2}{|\Phi_y \wedge \Phi_t|^2}} |\Phi_y \wedge \Phi_t| dy dt \\ &= \int_D \sqrt{1 + |\nabla^\varphi \varphi|^2} dy dt. \end{aligned}$$

□

The area formula (3.8) was originally proved for boundaries that are C^1_H-regular hypersurfaces (see [27] Theorem 6.5 part (vi) and [4] Proposition 2.22). It was later generalized to *intrinsic Lipschitz* graphs.

Definition 3.6. Let $D \subset W = \mathbb{R}^{2n}$ be an open set and let $\varphi \in C(D)$ be a continuous function.

i) We say that $\mathscr{B}\varphi$ exists in the sense of distributions and is represented by a locally bounded function, $\varphi \in L^\infty_{\mathrm{loc}}(D)$, if there exists a function $\psi \in L^\infty_{\mathrm{loc}}(D)$ such that for all $\vartheta \in C^1_c(D)$ there holds

$$\int_D \vartheta\, \psi\, dw = -\int_D \Big\{ \varphi \frac{\partial \vartheta}{\partial y_1} - 2\varphi^2 \frac{\partial \vartheta}{\partial t} \Big\} dw.$$

ii) We say that the intrinsic gradient $\nabla^\varphi \varphi \in L^\infty_{\mathrm{loc}}(D; \mathbb{R}^{2n-1})$ exists in the sense of distributions if $X_1\varphi, \ldots, X_n\varphi, \varphi, Y_2\varphi, \ldots, Y_n\varphi$ are represented by locally bounded functions in D.

We introduce intrinsic Lipschitz graphs along any direction. Theorem 3.9 below relates such graphs to the boundedness of the intrinsic gradient $\nabla^\varphi\varphi$.

Let $\nu \in \mathbb{R}^{2n}$, $|\nu| = 1$, be a unit vector that is identified with $(\nu, 0) \in \mathbb{H}^n$. For any $p \in \mathbb{H}^n$, we let $\nu(p) = \langle p, \nu\rangle\nu \in \mathbb{H}^n$ and we define $\nu^\perp(p) \in \partial H_\nu \subset \mathbb{H}^n$ as the unique point such that

$$p = \nu^\perp(p) \cdot \nu(p). \tag{3.9}$$

Recall that $\|\cdot\|_\infty$ is the box-norm introduced in (1.11).

Definition 3.7 (Intrinsic cones)**.** i) The (open) cone with vertex $0 \in \mathbb{H}^n$, axis $\nu \in \mathbb{R}^{2n}$, $|\nu| = 1$, and aperture $\alpha \in (0, \infty]$ is the set

$$C(0, \nu, \alpha) = \big\{p \in \mathbb{H}^n : \|\nu^\perp(p)\|_\infty < \alpha\|\nu(p)\|_\infty\big\}. \tag{3.10}$$

ii) The cone with vertex $p \in \mathbb{H}^n$, axis $\nu \in \mathbb{R}^{2n}$, and aperture $\alpha \in (0, \infty]$ is the set $C(p, \nu, \alpha) = p \cdot C(0, \nu, \alpha)$.

Definition 3.8 (Intrinsic Lipschitz graphs)**.** Let $D \subset \partial H_\nu$ be a set and let $\varphi : D \to \mathbb{R}$ be a function.

i) The *intrinsic graph* of φ is the set

$$\mathrm{gr}(\varphi) = \big\{p \cdot \varphi(p)\nu \in \mathbb{H}^n : p \in D\big\}. \tag{3.11}$$

ii) The function φ is *L-intrinsic Lipschitz* if there exists $L \geq 0$ such that for any $p \in \mathrm{gr}(\varphi)$ there holds

$$\mathrm{gr}(\varphi) \cap C(p, \nu, 1/L) = \emptyset. \tag{3.12}$$

When $\nu = \mathrm{e}_1$, the definition in (3.11) reduces to the definition in (3.6). Namely, let $\varphi : D \to \mathbb{R}$ be a function with $D \subset W = \{x_1 = 0\}$. For any $w \in D$, we have the identity

$$\exp(\varphi(w)X_1)(w) = w \cdot (\varphi(w)\mathrm{e}_1),$$

where $\varphi(w)\mathrm{e}_1 = (\varphi(w), 0 \ldots, 0) \in \mathbb{H}^n$. Then the intrinsic graph of φ is the set

$$\mathrm{gr}(\varphi) = \big\{w \cdot (\varphi(w)\mathrm{e}_1) \in \mathbb{H}^n \,:\, w \in D\big\}.$$

The notion of intrinsic Lipschitz function of Definition 3.8 is introduced in [30]. The cones (3.10) are relevant in the theory of H-convex sets [5]. The following theorem is the final result of many contributions.

Theorem 3.9. Let $\nu = \mathrm{e}_1$, $D \subset \partial H_\nu$ be an open set, and $\varphi : D \to \mathbb{R}$ be a continuous function. The following statements are equivalent:

A) We have $\nabla^\varphi\varphi \in L^\infty_{\mathrm{loc}}(D; \mathbb{R}^{2n-1})$.

B) For any $D' \subset\subset D$, the function $\varphi : D' \to \mathbb{R}$ is intrinsic Lipschitz.

Moreover, if A) or B) holds then the intrinsic epigraph $E_\varphi \subset \mathbb{H}^n$ has locally finite H-perimeter in the cylinder $D\cdot\mathbb{R}$, the inner horizontal normal to ∂E_φ is

$$\nu_{E_\varphi}(w\cdot\varphi(w)) = \Big(\frac{1}{\sqrt{1+|\nabla^\varphi\varphi(w)|^2}}, \frac{-\nabla^\varphi\varphi(w)}{\sqrt{1+|\nabla^\varphi\varphi(w)|^2}}\Big), \quad \text{for } \mathscr{L}^{2n}\text{-a.e. } w \text{ on } D, \tag{3.13}$$

and, for any $D'\subset D$, we have

$$P(E_\varphi; D'\cdot\mathbb{R}) = \int_{D'}\sqrt{1+|\nabla^\varphi\varphi|^2}dw = c_n\mathscr{S}^{Q-1}_\varrho(\mathrm{gr}(\varphi)\cap D'\cdot\mathbb{R}). \tag{3.14}$$

The equivalence between A) and B) is a deep result that is proved in [7], Theorem 1.1. Formula (3.13) for the normal and the area formula (3.14) are proved in [16] Corollary 4.2 and Corollary 4.3, respectively. A related result can be found in [56], where it is proved that if $E\subset\mathbb{H}^n$ is a set with finite H-perimeter having controlled normal ν_E, say $\langle\nu_E, \mathrm{e}_1\rangle \geq k > 0$ μ_E-a.e., then the reduced boundary $\partial^* E$ is an intrinsic Lipschitz graph along X_1.

3.2. **First variation and H-minimal surfaces.** In this section, we deduce the minimal surface equation for H-minimal surfaces in the special but important case of t-graphs. We show that H-minimal surfaces in $\mathbb{H}^1$ are ruled surfaces. These facts have been observed by several authors.

In Section 3.2.2, we review some results established in [12] and [14] about the characteristic set of surfaces in $\mathbb{H}^1$ with "controlled curvature", see Theorem 3.15 below.

3.2.1. *First variation of the area for t-graphs.* Let $D\subset\mathbb{R}^{2n}$ be an open set and let $f\in C^2(D)$ be a function. Assume that the t-epigraph of f, the set

$$E = \big\{(z,t)\in\mathbb{H}^n : t > f(z),\, z\in D\big\},$$

is H-perimeter minimizing in the cylinder $A = D\times\mathbb{R}$. This means that if $F\subset\mathbb{H}^n$ is a set such that $E\Delta F \subset\subset A$ then $P(E;A)\leq P(F;A)$. Here and in the following, $E\Delta F = E\setminus F\cup F\setminus E$ denotes the symmetric difference of sets.

Let $\Sigma(f) = \{z\in D : \nabla f(z) + 2z^\perp = 0\}$ be the *characteristic set of f*. At points $p = (z, f(z))\in\partial E$ with $z\in\Sigma(f)$ we have $T_p\partial E = H_p$, the horizontal plane and the tangent plane to ∂E at p coincide. These points are called *characteristic points* of the surface $S=\partial E$. The set of characteristic points of S is denoted by $\Sigma(S)$.

By the area formula (3.5), we have

$$P(E;A) = \int_D |\nabla f(z) + 2z^\perp|dz = \int_{D\setminus\Sigma(f)}|\nabla f + 2z^\perp|dz.$$

By the minimality of E, for any $\varepsilon \in \mathbb{R}$ and $\varphi \in C_c^\infty(D)$ we have

$$\begin{aligned}\int_{D\setminus\Sigma(f)} |\nabla f + 2z^\perp| dz &\leq \int_D |\nabla f + \varepsilon\nabla\varphi + 2z^\perp| dz \\ &= \int_{D\setminus\Sigma(f)} |\nabla f + \varepsilon\nabla\varphi + 2z^\perp| dz + |\varepsilon| \int_{\Sigma(f)} |\nabla\varphi| dz = \psi(\varepsilon).\end{aligned}$$

If $f \in C^2$ then $\Sigma(f)$ is (contained in) a C^1 hypersurface of D, see Section 3.2.2, and therefore $|\Sigma(f)| = 0$. If we only have $f \in C^1$, this is no longer true. When $|\Sigma(f)| = 0$, the function ψ is differentiable at $\varepsilon = 0$ and the minimality of E implies $\psi'(0) = 0$. We deduce that for any test function φ we have

$$\int_{D\setminus\Sigma(f)} \frac{\langle \nabla f + 2z^\perp, \nabla\varphi\rangle}{|\nabla f + 2z^\perp|} dz = 0.$$

If $\varphi \in C_c^1(D \setminus \Sigma(f))$, we can integrate by parts with no boundary contribution obtaining

$$\int_{D\setminus\Sigma(f)} \operatorname{div}\Big(\frac{\nabla f + 2z^\perp}{|\nabla f + 2z^\perp|}\Big)\varphi dz = 0. \tag{3.15}$$

When the support of φ intersects $\Sigma(f)$, there is a contribution to the first variation due to the characteristic set, see Theorem 3.17. From (3.15), we deduce that the function f satisfies the following partial differential equation

$$\operatorname{div}\Big(\frac{\nabla f + 2z^\perp}{|\nabla f + 2z^\perp|}\Big) = 0 \quad \text{in } D \setminus \Sigma(f). \tag{3.16}$$

This is the *H-minimal surface* equation for f, in the case of t-graphs. It is a degenerate elliptic equation. A solution $f \in C^2(D)$ to (3.16) is calibrated and the epigraph of f is H-perimeter minimizing over the cylinder $D \setminus \Sigma(f) \times \mathbb{R}$

Definition 3.10 (H-curvature and H-minimal graphs)**.** For any $f \in C^2(D)$ and $z \in D \setminus \Sigma(f)$, the number

$$H(z) = \operatorname{div}\Big(\frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|}\Big),$$

is called *H-curvature* of the graph of f at the point $(z, f(z))$. If $H = 0$ we say that $\mathrm{gr}(f)$ is an H-minimal graph (surface).

We specialize the analysis to the dimension $n = 1$, where the minimal surface equation (3.16) has a clear geometric meaning. If $n = 1$, then $\partial E \cap (D \times \mathbb{R}) = \mathrm{gr}(f)$ is a 2-dimensional surface.

At noncharacteristic points $p = (z, f(z)) \in \partial E$ with $z \in D \setminus \Sigma(f)$, we have $\dim(T_p\partial E \cap H_p) = 1$. A section of $T_p\partial E \cap H_p$ is the vector field

$$V = \frac{1}{|\nabla f + 2z^\perp|}\big(-(f_y + 2x)X + (f_x - 2y)Y\big).$$

Let $\gamma : (-\delta, \delta) \to \mathbb{H}^1$, $\delta > 0$, be the curve such that $\gamma(0) = p \in \partial E$ and $\dot\gamma = V(\gamma)$. The curve γ is horizontal because V is horizontal. Moreover, we have $\gamma(t) \in \partial E$ for all $t \in (-\delta, \delta)$ because V is tangent to ∂E.

Consider the vector fields in $D \setminus \Sigma(f)$

$$N_f(z) = \frac{\nabla f + 2z^\perp}{|\nabla f + 2z^\perp|} \quad \text{and} \quad N_f(z)^\perp = \frac{\big(-f_y - 2x, f_x - 2y\big)}{|\nabla f + 2z^\perp|}.$$

The vector field $N_f^\perp$ is the projection of V onto the xy-plane. The horizontal projection of γ, the curve $\kappa = (\gamma_1, \gamma_2)$, satisfies $\kappa(0) = z_0$ and solves the differential equation $\dot\kappa = N_f^\perp(\kappa)$. Then the vector N_f is a normal vector to the curve κ.

Viceversa, let κ be the solution of $\dot\kappa = N_f^\perp(\kappa)$ and $\kappa(0) = z_0$ and let γ be the horizontal lift of κ with $\gamma(0) = p = (z_0, f(z_0)) \in \partial E$. Then γ solves $\dot\gamma = V(\gamma)$ and is contained in ∂E.

We summarize these observations in the following proposition.

Proposition 3.11. Let $S = \mathrm{gr}(f) \subset \mathbb{H}^1$ be the graph of a function $f \in C^1(D)$. Then:

1) The horizontal projection κ of a horizontal curve γ contained in $S \setminus \Sigma(S)$ solves $\dot\kappa = N_f^\perp(\kappa)$.
2) The horizontal lift γ of a curve κ solving $\dot\kappa = N_f^\perp(\kappa)$ in $D \setminus \Sigma(f)$ is contained in S, if γ starts from S.

Now it is straightforward to prove the following result.

Theorem 3.12 (Structure of H-minimal surfaces)**.** Let $D \subset \mathbb{C}$ be an open set and let $f \in C^2(D)$ be a function such that $\mathrm{gr}(f)$ is an H-minimal surface. Then for any $z_0 \in D \setminus \Sigma(f)$ there exists a horizontal line segment contained in $\mathrm{gr}(f)$ and passing through $(z_0, f(z_0))$.

Proof. Let γ be the horizontal curve passing through $p = (z_0, f(z_0))$ and contained in $\mathrm{gr}(f)$. The horizontal projection κ solves $\dot\kappa = N_f(\kappa)^\perp$. The minimal surface equation (3.16) reads

$$\operatorname{div} N_f(z) = 0 \quad \text{in } D \setminus \Sigma(f),$$

where N_f is a unit normal vector field of κ. Thus κ is a curve with curvature 0 and thus it is a line segment. Its horizontal lift is also a line segment. $\square$

Remark 3.13. If $H : D \setminus \Sigma(f) \to \mathbb{R}$ is the H-curvature of the graph of f, then the partial differential equation

$$\operatorname{div}\Big(\frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|}\Big) = H(z), \quad \text{in } D \setminus \Sigma(f) \subset \mathbb{C},$$

implies that an integral curve κ of the vector field $N_f^\perp$ has curvature $H(\kappa)$. When H is a nonzero constant, κ is a circle. This is relevant in the Heisenberg isoperimetric problem.

Equation (3.16) can be given a meaning along integral curves of $N_f^\perp$ without assuming the full C^2 regularity of f, see [13]. See also Section 4.3 for the problem of integrating the H-curvature equation for a convex function f.

3.2.2. *Characteristic points.* Let $D \subset \mathbb{C}^{2n}$ be an open set and let $f \in C^2(D)$. Consider the mapping $\Phi : D \to \mathbb{R}^{2n}$

$$\Phi(z) = \nabla f(z) + 2z^\perp, \quad z \in D.$$

The point $z = x + iy \in \Sigma(f)$ is characteristic if and only if $\Phi(z) = 0$, namely,

$$\begin{cases} \Phi_1(z) = \nabla_x f(z) - 2y = 0 \\ \Phi_2(z) = \nabla_y f(z) + 2x = 0. \end{cases}$$

If $z_0 \in \Sigma(f)$ is a point such that $\det(J\Phi(z_0)) \neq 0$ then Φ is a local C^1 diffeomorphism at z_0 and thus z_0 is an isolated point of $\Sigma(f)$.

In general, for any $z_0 \in \Sigma(f)$ there exists $\varepsilon > 0$ such that $\Sigma(f) \cap \{|z - z_0| < \varepsilon\}$ is *contained* in the graph of a C^1 function. For instance, in the case $n = 1$ we have

$$|\partial_y \Phi_1(z)| + |\partial_x \Phi_2(z)| = |f_{xy}(z) - 2| + |f_{xy}(z) + 2| \neq 0,$$

and the claim follows from the implicit function theorem. We used the C^2 regularity of f to have equality of mixed derivatives $f_{xy} = f_{yx}$.

When f is less than C^2-regular, the characteristic set $\Sigma(f)$ may be large.

Theorem 3.14 (Balogh)**.** Let $D = (0,1) \times (0,1) \subset \mathbb{R}^2$ be the square. For any $\varepsilon > 0$ there exists a function $f \in \bigcap_{0<\alpha<1} C^{1,\alpha}(D)$ such that $|\Sigma(f)| > 1 - \varepsilon$.

This theorem is proved in [6] by the following construction. Given a continuous mapping $F : D \to \mathbb{R}^2$ one has to find a function such that $\nabla f = F$ on a large subset of D. The construction starts from a Cantor type subset of D with large measure. The function f is defined in a recursive way starting from suitable means of F in the subsquares of D generating the Cantor set.

The following theorem, proved in [14], shows that if H-curvature is suitably bounded near characteristic points then $\Sigma(f)$ consists, for $n = 1$, either of isolated points or, locally, of C^1 graphs over intervals. Generalizations to the case $f \in C^1(D)$, with some further technical assumptions, are given in [15]. For surfaces of class C^2 the curvature H needs not be integrable for the standard area element near the characteristic set, see [22].

Theorem 3.15 (Cheng-Hwang-Malchiodi-Yang)**.** Let $D \subset \mathbb{C}$ be an open set, $f \in C^2(D)$ and $z_0 \in \Sigma(f)$. Assume that:

1) $\det(J\Phi(z_0)) = 0$.
2) For any $z \in D \setminus \Sigma(f)$ we have
$$\mathrm{div}\Big(\frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|}\Big) = H(z), \tag{3.17}$$
where $H : D \setminus \Sigma(f) \to \mathbb{R}$ is a continuous function such that
$$|H(z)| \leq \frac{C}{|z - z_0|}, \quad z \in D \setminus \Sigma(f) \tag{3.18}$$
for some constant $C > 0$.

Then there exists $\varepsilon > 0$ such that $\Sigma(f) \cap \{|z - z_0| < \varepsilon\}$ is the graph of a C^1 function defined over an open interval.

Proof. Since $\det(J\Phi(z_0)) = 0$ then the Jacobian matrix $J\Phi(z_0)$ has rank at most 1. On the other hand, the antidiagonal of $J\Phi(z_0)$ never vanishes and thus the rank is precisely 1. Up to the sign, there exists a unique unit vector $w \in \mathbb{R}^2$, $|w| = 1$, that is orthogonal to the range of the transposed Jacobian matrix $J\Phi(z_0)^*$.

For $u \in \mathbb{R}^2$, we define the function $\Phi_u : D \to \mathbb{R}$, $\Phi_u = \langle \Phi, u\rangle = u_1(f_x - 2y) + u_2(f_y + 2x)$. If $u \notin \mathrm{Ker}(J\Phi(z_0)^*)$ then
$$\nabla\Phi_u(z_0) = J\Phi(z_0)^* u \neq 0,$$
and thus the equation $\Phi_u = 0$ defines a C^1 curve $\kappa_u : (-s_0, s_0) \to \mathbb{R}^2$, for some $s_0 > 0$, such that $\kappa_u(0) = z_0$ and $\Phi_u(\kappa_u) = 0$. The image of this curve is a graph over an interval. We can assume that $|\dot\kappa_u| = 1$. Differentiating $\Phi_u(\kappa_u) = 0$ we obtain $\langle\nabla\Phi_u(\kappa_u), \dot\kappa_u\rangle = 0$, and therefore at $s = 0$ we have
$$\langle J\Phi(z_0)^* u, \dot\kappa_u(0)\rangle = 0.$$
Then, up to the sign we have $\dot\kappa_u(0) = w$. The derivative $\dot\kappa_u(0)$ is independent of $u \notin \mathrm{Ker}(J\Phi(z_0)^*)$.

For some small $\varepsilon > 0$, we have $\Sigma(f) \cap \{|z - z_0| < \varepsilon\} \subset \{\kappa_u(s) \in \mathbb{R}^2 : |s| < s_0\} \cap \{|z - z_0| < \varepsilon\}$. We claim that the inclusion is an identity of sets. By contradiction assume that for any $\delta > 0$ there are $0 \leq s_1 < s_2 \leq \delta$ such that $\kappa_u(s) \notin \Sigma(f)$ for $s_1 < s < s_2$, and $\kappa_u(s_1), \kappa_u(s_2) \in \Sigma(f)$. Without loss of generality, we assume that $s_1 = 0$ and $s_2 = \delta$, where $\delta > 0$ is as small as we wish.

The defining equation $\langle\Phi(\kappa_u), u\rangle = \Phi_u(\kappa_u) = 0$ implies that, for $0 < s < \delta$, the vector
$$N_f(\kappa_u(s)) = \frac{\Phi(\kappa_u(s))}{|\Phi(\kappa_u(s))|} = \pm u^\perp \tag{3.19}$$
is constant, either $+u^\perp$ or $-u^\perp$, where $u^\perp = (-u_2, u_1)$.

There exists a unit vector $v \in \mathbb{R}^2$ such that $v \notin \mathrm{Ker}(J\Phi(z_0)^*)$,

$$\langle u - v, w\rangle \neq 0 \quad \text{and} \quad \langle u + v, w\rangle \neq 0. \tag{3.20}$$

The equation $\Phi_v = 0$ defines a C^1 curve $\kappa_v : (-\bar{s}_0, \bar{s}_0) \to \mathbb{R}^2$ such that $\kappa_v(0) = z_0$, $\dot{\kappa}_v(0) = w$, $|\dot{\kappa}_v| = 1$ and $\Phi_v(\kappa_v) = 0$. There is a number $\bar{\delta} > 0$ such that $\kappa_v(\bar{\delta}) = \kappa_u(\delta)$ and $\kappa_v(s) \notin \Sigma(f)$ for $0 < s < \bar{\delta}$. As above, the equation $\langle \Phi(\kappa_v), v\rangle = \Phi_v(\kappa_v) = 0$ implies that, for $0 < s < \bar{\delta}$, the vector $N_f(\kappa_v(s)) = \pm v^\perp$ is constant.

Let $A \subset \mathbb{R}^2$ be the region enclosed by the curves κ_u restricted to $[0, \delta]$ and κ_v restricted to $[0, \bar{\delta}]$. Integrating the equation (3.17) over A, using the divergence theorem and (3.18), we obtain

$$\int_{\partial A} \langle N_f, N\rangle d\mathscr{H}^1 = \int_A \operatorname{div} N_f(z)\, dz = \int_A H(z) dz \leq C \int_A \frac{1}{|z - z_0|} dz, \tag{3.21}$$

where N is the exterior normal to ∂A. Namely, along κ_u we have $N = \dot{\kappa}_u^\perp$ and along κ_v we have $N = -\dot{\kappa}_v^\perp$, or viceversa.

Using (3.19), we can compute the integral

$$\begin{aligned}
\int_{\kappa_u([0,\delta])} \langle N_f, N\rangle d\mathscr{H}^1 &= \int_0^\delta \langle N_f(\kappa_u(s)), \dot{\kappa}_u^\perp(s)\rangle ds \\
&= \langle \pm u^\perp, \kappa_u(\delta)^\perp - z_0^\perp\rangle \\
&= \langle \pm u, \kappa_u(\delta) - z_0\rangle,
\end{aligned}$$

where $\kappa_u(\delta) - z_0 = \delta w + o(\delta)$ as $\delta \to 0$. Analogously, using $\kappa_v(\bar{\delta}) = \kappa_u(\delta)$ we obtain

$$\int_{\kappa_v([0,\bar{\delta}])} \langle N_f, N\rangle d\mathscr{H}^1 = -\int_0^{\bar{\delta}} \langle N_f(\kappa_v(s)), \dot{\kappa}_v^\perp(s)\rangle ds = -\langle \pm v, \kappa_u(\delta) - z_0\rangle,$$

and, therefore, by (3.20) we have for $\delta > 0$ small

$$\Big| \int_{\partial A} \langle N_f, N\rangle d\mathscr{H}^1 \Big| \geq \big| \langle u \pm v, \delta w + o(\delta)\big| \geq \frac{\delta}{2} |\langle u \pm v, w\rangle|. \tag{3.22}$$

Fix a parameter $\varepsilon > 0$. For $\delta > 0$ small, we have the inclusion $A \subset \big\{ z_0 + rw\mathrm{e}^{i\vartheta} \in \mathbb{C} : 0 \leq r \leq \delta,\, |\vartheta| \leq \varepsilon \big\}$. Using polar coordinates centered at z_0, we find

$$\int_A \frac{1}{|z - z_0|} dz \leq 2\varepsilon\delta, \tag{3.23}$$

and, from (3.21)-(3.22)-(3.23), we obtain $\frac{\delta}{2}|\langle u \pm v, w\rangle| \leq 2\varepsilon\delta C$, that is a contradiction if we choose $\varepsilon > 0$ such that $4\varepsilon C < |\langle u \pm v, w\rangle|$.

□

Let $D \subset \mathbb{C}$ be an open set, $f \in C^2(D)$, and assume that $\Sigma(f)$ is a C^1 curve disconnecting D. Then we have the partition

$$D = D^+ \cup D^- \cup \Sigma(f)$$

where $D^+, D^- \subset D$ are disjoint open sets. In [14], Proposition 3.5, it is shown that the vector N_f extends to $\Sigma(f)$ from D^+ and from D^-, separately.

Theorem 3.16. In the above setting, for any $z_0 \in \Sigma(f)$ the following limits do exist

$$\begin{aligned} N_f(z_0)^+ &= \lim_{\substack{z\to z_0\\ z\in D^+}} N_f(z),\\ N_f(z_0)^- &= \lim_{\substack{z\to z_0\\ z\in D^-}} N_f(z), \end{aligned}$$

and moreover $N_f(z_0)^+ = -N_f(z_0)^-$.

Proof. Without loss of generality, we assume that $z_0 = 0$. We have either $f_{xy}(0) - 2 \neq 0$ or $f_{xy}(0) + 2 \neq 0$. Assume that $f_{xy}(0) - 2 > 0$. Then $f_x - 2y = 0$ is a defining equation for $\Sigma(f)$ near 0 and $\Sigma(f) = \big\{(x, \varphi(x)) \in \mathbb{R}^2 : |x| < \delta\big\}$, where $\varphi \in C^1(-\delta, \delta)$ is such that $\varphi(0) = 0$, and

$$\begin{aligned} D^+ &= \big\{(x,y) \in D : y > \varphi(x)\big\} = \big\{z \in D : f_x(z) - 2y > 0\big\},\\ D^- &= \big\{(x,y) \in D : y < \varphi(x)\big\} = \big\{z \in D : f_x(z) - 2y < 0\big\}. \end{aligned}$$

By Cauchy theorem, for any $x \in (-\delta, \delta)$ and for any $y > \varphi(x)$ there exists $\bar\varphi(x) \in (\varphi(x), y)$ such that

$$\frac{f_y(x,y) + 2x}{f_x(x,y) - 2y} = \frac{f_{yy}(x, \bar\varphi(x))}{f_{xy}(x, \bar\varphi(x)) - 2}.$$

When $x \to 0$ and $y \to 0$ we also have $\varphi(\bar x) \to 0$. Then we have

$$\lim_{\substack{z\to 0\\ z\in D^+}} \frac{f_y(z) + 2x}{f_x(z) - 2y} = \frac{f_{yy}(0)}{f_{xy}(0) - 2} = b.$$

Using $f_x(z) - 2y > 0$ on D^+, it follows that

$$N_f(0)^+ = \lim_{\substack{z\to 0\\ z\in D^+}} N_f(z) = \lim_{\substack{z\to 0\\ z\in D^+}} \frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|} = \frac{(1,b)}{\sqrt{1+b^2}}.$$

An analogous computation using $f_x(z) - 2y < 0$ on D^- shows that

$$N_f(0)^- = \lim_{\substack{z\to 0\\ z\in D^-}} N_f(z) = \lim_{\substack{z\to 0\\ z\in D^-}} \frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|} = -\frac{(1,b)}{\sqrt{1+b^2}}.$$

□

For H-minimal graphs, the vectors N_f^+ and N_f^- are tangent to the C^1 curve $\Sigma(f)$. The following theorem and Theorem 3.16 fail when we have only $f \in C^{1,1}$, see Section 5.2.2.

Theorem 3.17. In the above setting, assume that the epigraph of $f \in C^2(D)$ is H-perimeter minimizing in the cylinder $D \times \mathbb{R}$. Then we have

$$\langle N_f^+, N\rangle = \langle N_f^-, N\rangle = 0 \quad \text{on } \Sigma(f),$$

where N is the normal to the C^1 curve $\Sigma(f)$.

Proof. Let $\varphi \in C^1_c(D)$ be a test function and consider the function

$$\psi(\varepsilon) = \int_D |\nabla f + \varepsilon\nabla\varphi + 2z^\perp| dz, \quad \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

If the epigraph of f is H-perimeter minimizing then

$$0 = \psi'(0) = \int_D \frac{\langle \nabla f + 2z^\perp, \nabla\varphi\rangle}{|\nabla f + 2z^\perp|} dz.$$

By $|\Sigma(f)| = 0$ and by (3.16), this is equivalent to

$$\int_{D^+} \operatorname{div}\Big(\varphi \frac{\nabla f + 2z^\perp}{|\nabla f + 2z^\perp|}\Big) dz + \int_{D^-} \operatorname{div}\Big(\varphi \frac{\nabla f + 2z^\perp}{|\nabla f + 2z^\perp|}\Big) dz = 0.$$

Denoting by N the exterior unit normal to D^+ along $\Sigma(f)$ and by N_f^+ and N_f^- the traces of N_f onto $\Sigma(f)$ from D^+ and D^-, the divergence theorem gives

$$0 = \int_{\Sigma(f)} \varphi\langle N, N_f^+\rangle d\mathscr{H}^1 - \int_{\Sigma(f)} \varphi\langle N, N_f^-\rangle d\mathscr{H}^1 = 2\int_{\Sigma(f)} \varphi\langle N, N_f^+\rangle d\mathscr{H}^1.$$

In fact, by Theorem 3.16 we have $N_f^- = -N_f^+$. Since φ is arbitrary, we conclude that $\langle N, N_f^+\rangle = 0$ on $\Sigma(f)$.

□

3.2.3. *First variation of the area functional for intrinsic graphs.* By (3.14), the H-perimeter of the intrinsic epigraph E_φ along X_1 of an intrinsic Lipschitz function $\varphi : D \to \mathbb{R}$, $D \subset \mathbb{C}^n$ open set, is

$$\mathscr{A}(\varphi) = P(E_\varphi; D\cdot\mathbb{R}) = \int_D \sqrt{1 + |\nabla^\varphi\varphi|^2} dw, \tag{3.24}$$

where $\nabla^\varphi\varphi$ is a distribution represented by $L^\infty(D;\mathbb{R}^{2n-1})$ functions. It is not clear how to compute the first variation of the area functional $\mathscr{A}$ within the class of intrinsic Lipschitz functions. In fact, this class is not a vector space because the Burgers' operator is nonlinear. Even for a smooth function $\psi \in C^\infty(D)$ we have

$$\mathscr{B}(\varphi+\psi) = \varphi_y + \psi_y - 4(\varphi+\psi)(\varphi_t+\psi_t) = \mathscr{B}\varphi + \mathscr{B}\psi - 4(\varphi\psi_t + \psi\varphi_t),$$

and the distributional derivative φ_t is not represented by an L^∞ function. So, if φ is only intrinsic Lipschitz it may happen that $P(E_{\varphi+\psi}; D\cdot\mathbb{R}) = \infty$ for any small perturbation $\psi \neq 0$. The reason of this phenomenon is that the variation of the intrinsic graph of φ along X_1 is not a contact deformation. On the other hand, if we had $\varphi_t \in L^\infty_{\mathrm{loc}}$ then the intrinsic graph would have the standard Lipschitz regularity.

Assuming the Lipschitz regularity for φ, the first variation for the area functional $\mathscr{A}$ in (3.24), namely the condition

$$\frac{d}{d\varepsilon}\mathscr{A}(\varphi+\varepsilon\psi) = 0 \quad \text{for any } \psi \in C^\infty_c(D),$$

leads to the following minimal surface equation for a minimizer φ in D:

$$\Big(\frac{\partial}{\partial y} - 4\varphi\frac{\partial}{\partial t}\Big)\frac{\mathscr{B}\varphi}{\sqrt{1+|\nabla^{\varphi}\varphi|^2}} + \sum_{j=2}^{n} X_j\Big(\frac{X_j\varphi}{\sqrt{1+|\nabla^{\varphi}\varphi|^2}}\Big) + Y_j\Big(\frac{Y_j\varphi}{\sqrt{1+|\nabla^{\varphi}\varphi|^2}}\Big) = 0. \tag{3.25}$$

This equation, but in a different system of coordinates, is the starting point of the papers [10] and [9], where the authors study the regularity of vanishing viscosity Lipschitz continuous solutions. When $n \geq 2$, vanishing viscosity solutions are C^∞-smooth. When $n = 1$, their intrinsic graph is foliated by horizontal lines.

3.3. **First variation along a contact flow.** In this section, we present a formula for computing the first variation of H-perimeter for any set with finite H-perimeter. This result can be extended to $\mathscr{S}_\varrho^{Q-1}$-rectifiable sets in the sense of Definition 2.18 and is a joint result with D. Vittone. We give the proof in the smooth case, the technical details for the general case will appear elsewhere. First and second order variation formulas are discussed also in [45], [18], and [31].

Let $A \subset \mathbb{H}^n$ be an open set. A diffeomorphism $\Psi : A \to \mathbb{H}^n$ is said to be a *contact map* if for any $p \in A$ the differential $\Psi_* : T_p\mathbb{H}^n \to T_{\Psi(p)}\mathbb{H}^n$ maps the horizontal space H_p into $H_{\Psi(p)}$:

$$\Psi_*(H_p) = H_{\Psi(p)}, \quad p \in A. \tag{3.26}$$

A one-parameter flow $(\Psi_s)_{s\in\mathbb{R}}$ of diffeomorphisms in $\mathbb{H}^n$ is a *contact flow* if each Ψ_s is a contact map. Contact flows are generated by contact vector fields.

A contact vector field in $\mathbb{H}^n$ is a vector field of the form

$$V_\psi = -4\psi T + \sum_{j=1}^{n} (Y_j\psi)X_j - (X_j\psi)Y_j, \tag{3.27}$$

where $\psi \in C^\infty(\mathbb{H}^n)$ is the *generating function* of the vector field (see [36]). For any compact set $K \subset \mathbb{H}^n$, there exist $\delta = \delta(\psi, K) > 0$ and a flow $\Psi : [-\delta,\delta] \times K \to \mathbb{H}^n$ defined by $\dot\Psi(s,p) = V_\psi(\Psi(s,p))$ and $\Psi(0,p) = p$ for any $s \in [-\delta,\delta]$ and $p \in K$. We call Ψ the flow generated by ψ. We also let $\Psi_s = \Psi(s,\cdot)$.

Related to the function ψ, we have, at any point $p \in \mathbb{H}^n$, the real quadratic form $\mathscr{Q}_\psi : H_p \to \mathbb{R}$

$$\mathscr{Q}_\psi\Big(\sum_{j=1}^{n} x_jX_j + y_jY_j\Big) = \sum_{i,j=1}^{n} x_ix_j\, X_jY_i\psi + x_jy_i\,(Y_iY_j\psi - X_jX_i\psi) - y_iy_j\, Y_jX_i\psi, \tag{3.28}$$

where $x_j, y_j \in \mathbb{R}$, and ψ with its derivatives are evaluated at p. In the sequel, we identify a vector $\nu = \nu(p) \in \mathbb{R}^{2n}$, $p \in \mathbb{H}^n$, with the horizontal vector $\sum_{j=1}^n \nu_jX_j(p) + \nu_{n+j}Y_j(p)$. The quadratic form $\mathscr{Q}_\psi(\nu)$ is defined accordingly.

Theorem 3.18. Let $A \subset \mathbb{H}^n$ be an open set and let $\Psi : [-\delta, \delta] \times A \to \mathbb{H}^n$, $\delta = \delta(\psi, A) > 0$, be the flow generated by $\psi \in C^\infty(\mathbb{H}^n)$. Then there exists a constant $C = C(\psi, A) > 0$ such that for any set $E \subset \mathbb{H}^n$ with finite H-perimeter in A we have

$$\left| P(\Psi_s(E); \Psi_s(A)) - P(E; A) + s \int_A \big\{ 4(n+1)T\psi + \mathscr{Q}_\psi(\nu_E) \big\} d\mu_E \right| \leq C\, P(E; A)\, s^2 \tag{3.29}$$

for any $s \in [-\delta, \delta]$.

Proof. We prove the theorem when $\partial E \cap A$ is a C^∞ smooth hypersurface. We deduce formula (3.29) from the Taylor expansion for the standard perimeter. Let $E_s = \Psi_s(E)$ and $A_s = \Psi_s(A)$. Then $\partial E_s \cap A_s = \Psi_s(\partial E \cap A)$ is a C^∞ smooth $2n$-dimensional hypersurface. By the area formula (3.2), we have

$$P(E; A) = \int_{\partial E \cap A} K\, d\mathscr{H}^{2n} \quad \text{and} \quad P(E_s; A_s) = \int_{\partial E_s \cap A_s} K_s\, d\mathscr{H}^{2n},$$

where $\mathscr{H}^{2n}$ is the standard $2n$-dimensional Hausdorff measure of $\mathbb{R}^{2n+1}$,

$$K = \Big(\sum_{j=1}^n \langle X_j, N\rangle^2 + \langle Y_j, N\rangle^2 \Big)^{1/2},$$
$$K_s = \Big(\sum_{j=1}^n \langle X_j, N_s\rangle^2 + \langle Y_j, N_s\rangle^2 \Big)^{1/2},$$

and N, N_s are the standard Euclidean unit normals to $\partial E \cap A$ and $\partial E_s \cap A_s$, respectively. We fix a coherent orientation.

By the standard Taylor formula for the area, we have

$$\int_{\partial E_s \cap A_s} K_s\, d\mathscr{H}^{2n} = \int_{\partial E \cap A} K_s \circ \Psi_s\, \mathscr{J}\Psi_s\, d\mathscr{H}^{2n}, \tag{3.30}$$

where $\mathscr{J}\Psi_s : \partial E \cap A \to \mathbb{R}$ is the Jacobian determinant of Ψ_s restricted to ∂E:

$$\mathscr{J}\Psi_s = \sqrt{\det \big[J\Psi_s\big|_{\partial E}^* \circ J\Psi_s\big|_{\partial E} \big]}. \tag{3.31}$$

This Jacobian determinant has the following first order Taylor expansion in s

$$\mathscr{J}\Psi_s = 1 + s\big(\operatorname{div} V_\psi - \langle (JV_\psi)N, N\rangle \big) + O(s^2) \quad \text{on } \partial E \cap A, \tag{3.32}$$

where $\operatorname{div} V_\psi$ is the standard divergence of the vector field V_ψ generating the flow and JV_ψ is the Jacobian matrix of V_ψ. Here, the vector field V_ψ is identified with the mapping given by the coefficients of V_ψ in the standard basis. The remainder $O(s^2)$ in (3.32) satisfies $|O(s^2)| \leq C_1 s^2$ for some constant $C_1 = C_1(\psi, A) > 0$.

We compute the derivative of the function $s \mapsto K_s \circ \Psi_s$. We start from the derivative of $s \mapsto M(s) = N_s(\Psi_s)$. Let us fix a frame $V_1, \dots, V_{2n}$ of orthonormal vector fields (in the standard scalar product) tangent to $\partial E \cap \Omega$. This frame does always exist locally.

As the vector fields $J\Psi_s V_1, \dots, J\Psi_s V_{2n}$ are tangent to $\partial E_s \cap \Omega_s$ we can differentiate the identities $\langle J\Psi_s V_i, M(s)\rangle = 0$, $i = 1, \dots, 2n$. We obtain

$$\langle JV_\psi(\Psi_s)V_i), M(s)\rangle + \langle J\Psi_s V_i, M'(s)\rangle = 0. \tag{3.33}$$

On the other hand, differentiating the identity $|N_s|^2 = 1$ we deduce that $\langle M'(s), N_s(\Psi_s)\rangle = 0$. Using (3.33), we deduce that at the point $s = 0$ we have

$$\begin{aligned} M'(0) &= \sum_{i=1}^{2n} \langle V_i, M'(0)\rangle V_i = -\sum_{i=1}^{2n} \langle (JV_\psi)V_i, N\rangle V_i \\ &= -\sum_{i=1}^{2n} \langle V_i, (JV_\psi)^* N\rangle V_i = \langle (JV_\psi)^* N, N\rangle N - (JV_\psi)^* N. \end{aligned} \tag{3.34}$$

Using the property of flows, we can repeat the computation for any s and we find the formula

$$M'(s) = \langle (JV_\psi)^* N_s, N_s\rangle N_s - (JV_\psi)^* N_s, \tag{3.35}$$

where the right-hand side is evaluated at Ψ_s.

Now let X be any smooth vector field in $\mathbb{H}^n$ and consider the function $F_X(s) = \langle X, N_s\rangle(\Psi_s)$. The derivative of F_X is

$$F'_X(s) = \langle (JX)V_\psi(\Psi_s), M(s)\rangle + \langle X(\Psi_s), M'(s)\rangle,$$

where JX is the Jacobian matrix of the mapping given by the coefficients of X. We may also use the notation $(JX)V_\psi = V_\psi X$, where V_ψ acts on the coefficients of X. Using (3.35), we obtain

$$\begin{aligned} F'_X(s) &= \langle (JX)V_\psi, N_s\rangle + \big\langle X, \langle (JV_\psi)^* N_s, N_s\rangle N_s - (JV_\psi)^* N_s \big\rangle \\ &= \langle [V_\psi, X], N_s\rangle + \langle (JV_\psi)N_s, N_s\rangle \langle X, N_s\rangle. \end{aligned} \tag{3.36}$$

The right-hand side is evaluated at Ψ_s.

As V_ψ is of the form (3.27), the commutators $[V_\psi, X_j]$ and $[V_\psi, Y_j]$ are horizontal vector fields, i.e., linear combinations of X_i and Y_i. From (3.36) it follows that F'_{X_j} and F'_{Y_j} are homogeneous functions of degree 1 with respect to $\langle X_i, N_s\rangle$ and $\langle Y_i, N_s\rangle$, $i = 1, \dots, n$.

As Ψ_s is a contact flow, by (3.26) we have $K(p) = 0$ if and only if $K_s(\Psi_s(p)) = 0$. Assuming that $K(p) \neq 0$, we can thus compute the derivative (in the sequel we omit reference to $p \in \partial E \cap A$)

$$\frac{dK_s \circ \Psi_s}{ds} = \frac{1}{K_s} \sum_{j=1}^{n} \langle X_j, N_s\rangle F'_{X_j}(s) + \langle Y_j, N_s\rangle F'_{Y_j}(s), \tag{3.37}$$

and using (3.36) we obtain the formula

$$\frac{dK_s\circ\Psi_s}{ds} = K_s(\langle (JV_\psi)N_s, N_s\rangle + \frac{1}{K_s}\sum_{j=1}^n \Big\langle \langle X_j, N_s\rangle [V_\psi, X_j] + \langle Y_j, N_s\rangle [V_\psi, Y_j], N_s\Big\rangle. \tag{3.38}$$

The right hand side is evaluated at Ψ_s and it is bounded by K_s. Namely, there exists a constant $C_2 = C_2(\psi, A)$ such that

$$\Big|\frac{dK_s\circ\Psi_s}{ds}\Big| \leq C_2 K_s. \tag{3.39}$$

Then we can interchange integral and derivative in s in the derivative of $P(E_s; A_s)$:

$$\frac{d}{ds}\int_{\partial E\cap A} K_s\circ\Psi_s\, \mathscr{J}\Psi_s\, d\mathscr{H}^{2n} = \int_{\partial E\cap A} \frac{d}{ds}\Big(K_s\circ\Psi_s\, \mathscr{J}\Psi_s\Big)\, d\mathscr{H}^{2n}.$$

A formula for the second derivative of $s\mapsto K_s\circ\Psi_s$ can be obtained starting from (3.37) and using (3.36). We do not compute this formula, here. It suffices to notice that also the second derivative is bounded by K_s, and namely:

$$\Big|\frac{d^2K_s\circ\Psi_s}{ds^2}\Big| \leq C_3 K_s \tag{3.40}$$

for some $C_3 = C_3(\psi, A) > 0$. This follows again from the formula (3.36). Thus we can differentiate twice in s inside the integral (3.30) defining $P(E_s; A_s)$.

From (3.32) and (3.38), we get the first order Taylor development

$$K_s\circ\Psi_s\, \mathscr{J}\Psi_s = K\Big\{1 + s\Big[\operatorname{div} V_\psi + \frac{1}{K^2}\sum_{j=1}^n \langle N_{X_j}[V_\psi, X_j] + N_{Y_j}[V_\psi, Y_j], N\rangle\Big] + O(s^2)\Big\}, \tag{3.41}$$

where we let $N_{X_j} = \langle X_j, N\rangle$ and $N_{Y_j} = \langle Y_j, N\rangle$, and $O(s^2)/s^2$ is bounded uniformly in N by some constant $C_4 = C_4(\psi, A) > 0$. Now, using the structure (3.27) of V_ψ, we get

$$\sum_{j=1}^n \langle N_{X_j}[V_\psi, X_j] + N_{Y_j}[V_\psi, Y_j], N\rangle = -\mathscr{Q}_\psi\Big(\sum_{j=1}^n N_{X_j}X_j + N_{Y_j}Y_j\Big), \tag{3.42}$$

and

$$\operatorname{div} V_\psi = -4T\psi + \sum_{j=1}^n X_jY_j\psi - Y_jX_j\psi = -4(n+1)T\psi. \tag{3.43}$$

Formula (3.29) follows from (3.30) along with (3.41)–(3.43). □

Remark 3.19. Let $\Gamma\subset\mathbb{H}^n$ be an $\mathscr{S}_\varrho^{Q-1}$-rectifiable set in $\mathbb{H}^n$ in the sense of Definition 2.18. Using the C^1_H-regular surfaces that cover Γ, a unit horizontal normal ν_Γ can be defined $\mathscr{S}_\varrho^{Q-1}$-a.e. on Γ. When Γ is bounded and with finite measure, formula (3.29) reads as follows:

$$\Big|\mathscr{S}_\varrho^{Q-1}(\Psi_s(\Gamma)) - \mathscr{S}_\varrho^{Q-1}(\Gamma) + s\int_\Gamma \big\{4(n+1)T\psi + \mathscr{Q}_\psi(\nu_\Gamma)\big\}d\mathscr{S}_\varrho^{Q-1}\Big| \leq C\mathscr{S}_\varrho^{Q-1}(\Gamma)\, s^2 \tag{3.44}$$

for any $s \in [-\delta, \delta]$, where $\psi \in C^\infty(\mathbb{H}^n)$ is a generating function and $\delta > 0$. The details of the proof of (3.44) will appear elsewhere.

If Γ is locally measure minimizing in an open set $A \subset \mathbb{H}^n$, from (3.44) we deduce the necessary condition

$$\int_\Gamma \big\{4(n+1)T\psi + \mathscr{Q}_\psi(\nu_\Gamma)\big\}d\mathscr{S}_\varrho^{Q-1} = 0$$

for any function $\psi \in C^\infty(A)$.

4. Isoperimetric problem

4.1. **Existence of isoperimetric sets and Pansu's conjecture.** For a measurable set $E \subset \mathbb{H}^n$ with positive and finite measure, the *isoperimetric quotient* is defined as

$$\mathrm{Isop}(E) = \frac{P(E;\mathbb{H}^n)}{|E|^{(Q-1)/Q}}.$$

The isoperimetric problem consists in minimizing the isoperimetric quotient among all admissible sets

$$C_{\mathrm{isop}} = \inf\big\{\mathrm{Isop}(E) \,:\, E \subset \mathbb{H}^n \text{ measurable set with } 0 < |E| < \infty\big\}. \tag{4.1}$$

A measurable set $E \subset \mathbb{H}^n$ with $0 < |E| < \infty$ realizing the infimum is called *isoperimetric set*. Isoperimetric sets are defined up to null sets.

If a set E is isoperimetric, then also the left translates $L_pE = p \cdot E$, $p \in \mathbb{H}^n$, are isoperimetric because perimeter and volume are left invariant. Also the dilated sets $\lambda E = \delta_\lambda E$ are isoperimetric, because the isoperimetric quotient is 0-homogeneous, $\mathrm{Isop}(\lambda E) = \mathrm{Isop}(E)$, for any $\lambda > 0$. It follows that the infimum C_{isop} in (4.1) is the infimum of perimeter for fixed volume

$$C_{\mathrm{isop}} = \inf\big\{P(E;\mathbb{H}^n) \,:\, E \subset \mathbb{H}^n \text{ measurable set with } |E| = 1\big\}. \tag{4.2}$$

Hence, isoperimetric sets are precisely the sets that have least Heisenberg perimeter for given volume.

The infimum in (4.1) is in fact positive, $C_{\mathrm{isop}} > 0$, and we have the isoperimetric inequality

$$P(E;\mathbb{H}^n) \geq C_{\mathrm{isop}}|E|^{\frac{Q-1}{Q}}, \tag{4.3}$$

holding for any measurable set E with finite measure. The constant C_{isop} is the largest constant making true the above inequality (i.e., the *sharp constant*). Isoperimetric sets are precisely the sets for which the inequality (4.3) is an equality.

Inequality (4.3) with a positive nonsharp constant can be obtained by several methods (see, for example, [58], [59], [26], and [33]). The functional analytic proof casts the isoperimetric inequality as a special case of Sobolev-Poincarè inequalities. Indeed, for any $1 \le p < Q$ there exists a constant $C_{n,p} > 0$ such that

$$C_{p,n}\Big(\int_{\mathbb{H}^n} |u|^{\frac{pQ}{Q-p}} dzdt\Big)^{\frac{Q-p}{pQ}} \le \Big(\int_{\mathbb{H}^n} |\nabla_H u|^p dzdt\Big)^{1/p} \tag{4.4}$$

for any $u \in C_c^1(\mathbb{H}^n)$. The inequality extends to appropriate Sobolev or BV spaces. The case $p = 1$ is the *geometric case* and reduces to the Heisenberg isoperimetric inequality (4.3). In fact, for the characteristic function of a set $u = \chi_E$ we have

$$\int_{\mathbb{H}^n} |\nabla_H u| = \sup\Big\{\int_{\mathbb{H}^n} \chi_E \mathrm{div}_H \varphi \, dzdt \,:\, \varphi \in C_c^1(A; \mathbb{R}^{2n}),\ \|\varphi\|_\infty \le 1\Big\} = P(E; \mathbb{H}^n).$$

Inequality (4.4) can be obtained starting from the potential estimate

$$|u(z,t)| \le C_n \int_{\mathbb{H}^n} \frac{|\nabla_H u(\zeta,\tau)|}{d((z,t),(\zeta,\tau))^{Q-1}} d\zeta d\tau = C_n I_{Q-1}(|\nabla_H u|)(z,t), \quad u \in C_c^1(\mathbb{H}^n),$$

and using the fact that the singular integral operator $I_{Q-1} : L^p(\mathbb{H}^n) \to L^q(\mathbb{H}^n)$ is bounded for $q = pQ/(Q-p)$ and $1 \le p < Q$.

The existence of isoperimetric sets is established in [38] and follows from a concentration-compactness argument. See also [32] for a proof of existence that avoids to use the concavity of the isoperimetric profile function.

Theorem 4.1 (Leonardi-Rigot)**.** Let $n \ge 1$. There exists a measurable set $E \subset \mathbb{H}^n$ with $|E| = 1$ realizing the minimum in (4.2).

Proof. We give a sketch of the proof. Let $(E_j)_{j\in\mathbb{N}}$ be a minimizing sequence of sets for (4.2):

1) $|E_j| = 1$ for all $j \in \mathbb{N}$;
2) $\lim\limits_{j\to\infty} P(E_j; \mathbb{H}^n) = C_{\mathrm{isop}}$.

The key step of the proof is a concentration argument. We claim that there exists an $R > 0$ such that (after a left translation, truncation, and dilation of each E_j) the sequence $(E_j)_{j\in\mathbb{N}}$ can be also assumed to lie in a bounded region. Namely, there exists $R > 0$ such that:

3) $E_j \subset Q_R = \big\{(z,t) \in \mathbb{H}^n : |x_i|, |y_i|, |t|^2 < R,\ i = 1,\dots,n\big\}$ for all $j \in \mathbb{N}$.

Then, by the compactness theorem for $BV_H(Q_R)$ functions (see [33]), there exists a subsequence, still denoted by $(E_j)_{j\in\mathbb{N}}$, that converges in $L^1(\mathbb{H}^n)$ to a set $E \subset \mathbb{H}^n$ such that:

i) $|E| = \lim\limits_{j\to\infty} |E_j| = 1$, by the $L^1(\mathbb{H}^n)$ convergence;
ii) $P(E; \mathbb{H}^n) \le \liminf\limits_{j\to\infty} P(E_j; \mathbb{H}^n) = C_{\mathrm{isop}}$, by the lower semicontinuity of perimeter.

So we have $P(E; \mathbb{H}^n) = C_{\mathrm{isop}}$ with $|E| = 1$, and E is therefore an isoperimetric set. This ends the proof, provided that we show 3). □

Claim 3) follows from the following lemma.

Lemma 4.2. Let $n \geq 1$. There exist constants $\varepsilon_0 > 0$, $C > 0$, and $R > 0$ such that for each $0 < \varepsilon < \varepsilon_0$ and for all sets $E \subset \mathbb{H}^n$ such that $|E| = 1$ and $P(E;\mathbb{H}^n) \leq (1+\varepsilon)C_{\mathrm{isop}}$ there exists a set $F \subset \mathbb{H}^n$ such that:

i) $|F| = 1$;
ii) $F \subset Q_R = \big\{(z,t) \in \mathbb{H}^n : |x_i|, |y_i|, |t|^2 < R,\ i = 1, \dots, n\big\}$;
iii) $P(F;\mathbb{H}^n) \leq \big(1 - C\varepsilon^{\frac{Q}{Q-1}}\big)^{-(Q-1)/Q} P(E;\mathbb{H}^n)$.

Proof. For $s \in \mathbb{R}$, let us define the following sets:

$$\Pi_s^- = \big\{(z,t) \in \mathbb{H}^n : x_1 < s\big\} \quad \text{and} \quad \Pi_s^+ = \big\{(z,t) \in \mathbb{H}^n : x_1 > s\big\}.$$

We also let $\Pi_s = \big\{(z,t) \in \mathbb{H}^n : x_1 = s\big\}$. Let $E \subset \mathbb{H}^n$ be a set with $|E| = 1$ and finite H-perimeter. We define the sets

$$E_s^- = E \cap \Pi_s^- \quad \text{and} \quad E_s^+ = E \cap \Pi_s^+.$$

By the Heisenberg isoperimetric inequality (4.3), we have

$$P(E_s^-;\mathbb{H}^n) \geq C_{\mathrm{isop}} |E_s^-|^{\frac{Q-1}{Q}}, \quad P(E_s^+;\mathbb{H}^n) \geq C_{\mathrm{isop}} |E_s^+|^{\frac{Q-1}{Q}}, \tag{4.5}$$

where

$$\begin{aligned} P(E_s^-;\mathbb{H}^n) &= P(E; H_s^-) + P(E_s^-;\Pi_s), \\ P(E_s^+;\mathbb{H}^n) &= P(E; H_s^+) + P(E_s^+;\Pi_s). \end{aligned} \tag{4.6}$$

The number $P(E_s^-;\Pi_s)$ is the standard $2n$-dimension measure of the trace of E_s^- onto Π_s. Analogously, the number $P(E_s^+;\Pi_s)$ is the standard $2n$-dimension measure of the trace of E_s^+ onto Π_s. The function $v(s) = |E_s^-|$ is continuous and increasing. Therefore it is differentiable almost everywhere. Hence, at differentiability points $s \in \mathbb{R}$ of v we have

$$v'(s) = P(E_s^-;\Pi_s) = P(E_s^+;\Pi_s).$$

We do not prove these claims, here. From (4.6) and (4.5), we obtain

$$\begin{aligned} P(E;\mathbb{H}^n) + 2v'(s) &\geq P(E;\Pi_s^-) + P(E;\Pi_s^+) + 2v'(s) \\ &= P(E;\Pi_s^-) + P(E;\Pi_s^+) + P(E_s^-;\Pi_s) + P(E_s^+;\Pi_s) \\ &= P(E_s^-;\mathbb{H}^n) + P(E_s^+;\mathbb{H}^n) \\ &\geq C_{\mathrm{isop}} \Big\{ |E_s^-|^{\frac{Q-1}{Q}} + |E_s^+|^{\frac{Q-1}{Q}} \Big\}. \end{aligned}$$

Using $P(E;\mathbb{H}^n) \leq C_{\mathrm{isop}}(1+\varepsilon)$ and $|E| = 1$, the inequality above implies

$$C_{\mathrm{isop}}(1+\varepsilon) + 2v'(s) \geq C_{\mathrm{isop}} \big\{ v(s)^{\frac{Q-1}{Q}} + (1 - v(s))^{\frac{Q-1}{Q}} \big\},$$

and letting $\psi(v) = v^{\frac{Q-1}{Q}} + (1-v)^{\frac{Q-1}{Q}} - 1$ for $v \in [0,1]$, we finally obtain

$$C_{\mathrm{isop}}\varepsilon + 2v'(s) \geq C_{\mathrm{isop}} \psi(v(s)). \tag{4.7}$$

The function ψ is strictly concave with $\psi(0) = \psi(1) = 0$. Then there exist $0 < v_- < v_+ < 1$ such that $\psi(v_-) = \psi(v_+) = 2\varepsilon$. By concavity, we have

$$\psi(v) \geq 2\varepsilon \quad \text{for all } v_- \leq v \leq v_+.$$

There exist numbers $s_- < s_+$ such that $v(s_-) = v_-$ and $v(s_+) = v_+$. Thus, from (4.7) we get

$$\begin{aligned} s_+ - s_- &\leq \int_{s_-}^{s_+} \frac{C_{\mathrm{isop}}\varepsilon + 2v'(s)}{C_{\mathrm{isop}}\psi(v(s))} ds \\ &\leq \frac{1}{2}(s_+ - s_-) + \int_{s_-}^{s_+} \frac{2v'(s)}{C_{\mathrm{isop}}\psi(v(s))} ds \\ &\leq \frac{1}{2}(s_+ - s_-) + \int_0^1 \frac{2}{C_{\mathrm{isop}}\psi(v)} dv. \end{aligned} \tag{4.8}$$

We obtain the bound

$$\frac{s_+ - s_-}{2} \leq \widehat{R} = \frac{2}{C_{\mathrm{isop}}} \int_0^1 \frac{1}{\psi(v)} dv < \infty.$$

The set $\widehat{E} = E \cap \{(z,t) \in \mathbb{H}^n : s_- < x_1 < s_+\}$ has volume

$$|\widehat{E}| = |E^-_{s_+}| - |E^-_{s_-}| = 1 - 2v_-.$$

We used the identity $v_+ = 1 - v_-$. The number $0 < v_- < 1/2$ satisfies $\psi(v_-) = 2\varepsilon$. There are constants $\varepsilon_0 > 0$ and $C > 0$ such that if $0 < \varepsilon < \varepsilon_0$ we have $v_- \leq C\varepsilon^{\frac{Q}{Q-1}}$. Let $\lambda > 0$ be such that $|\lambda\widehat{E}| = 1$. Then we have $1 = \lambda^Q|\widehat{E}| \geq \lambda^Q(1 - 2C\varepsilon^{\frac{Q}{Q-1}})$, and thus

$$\lambda \leq \Big(\frac{1}{1 - 2C\varepsilon^{\frac{Q}{Q-1}}}\Big)^{1/Q}.$$

A calibration argument shows that $P(\widehat{E}; \mathbb{H}^n) \leq P(E; \mathbb{H}^n)$. We do not prove this claim, here. So we get

$$P(\lambda\widehat{E}; \mathbb{H}^n) = \lambda^{Q-1} P(\widehat{E}; \mathbb{H}^n) \leq \Big(\frac{1}{1 - 2C\varepsilon^{\frac{Q}{Q-1}}}\Big)^{(Q-1)/Q} P(E; \mathbb{H}^n).$$

After a left translation, we may assume that

$$\lambda\widehat{E} \subset \big\{(z,t) \in \mathbb{H}^n : |x_1| < R\big\},$$

where we let $R = \lambda\widehat{R}$. Repeating the argument for each coordinate axis, we obtain the claim of the lemma. The argument in the t coordinate requires easy adaptations. $\square$

In 1983, Pansu conjectured a possible solution to the Heisenberg isoperimetric problem, see [59]. The conjecture can be formulated in the following way. Up to a null set, a left translation, and a dilation, the isoperimetric set in $\mathbb{H}^1$ is precisely the set

$$E_{\rm isop} = \Big\{ (z,t) \in \mathbb{H}^1 \,:\, |t| < \arccos|z| + |z|\sqrt{1-|z|^2},\ |z|<1 \Big\}. \tag{4.9}$$

Pansu did not give the formula for the conjectured isoperimetric set but he described how to construct it. Let us consider a geodesic $\gamma : [0,\pi] \to \mathbb{H}^1$ joining the point $\gamma(0) = 0$ to the point $\gamma(\pi) = (0,\pi) \in \mathbb{H}^1$. Using the formula (1.14) with $\vartheta = 0$ and $\varphi = 2$, we have the following formula for γ

$$\gamma(s) = \Big(\frac{\mathrm{e}^{2is}-1}{2}, s - \sin s \cos \Big).$$

The horizontal projection of γ, namely the curve $\kappa(s) = \frac{\mathrm{e}^{2is}-1}{2}$, is a circle with diameter 1. Letting $|z| = |\kappa(s)|$ we find $|z|^2 = 1-\cos^2 s$, and when $s \in [0,\pi/2]$ we get

$$s = \arccos\sqrt{1-|z|^2}.$$

We can thus define the profile function $\varphi : [0,1] \to \mathbb{R}$ by letting

$$\begin{aligned}
\varphi(|z|) &= s - \sin s \cos s - \frac{\pi}{2} \\
&= \arccos\sqrt{1-|z|^2} - |z|\sqrt{1-|z|^2} - \frac{\pi}{2} \\
&= -\arccos|z| - |z|\sqrt{1-|z|^2}.
\end{aligned}$$

The profile φ gives the radial value of the function whose graph is the bottom part of the boundary of the set $E_{\rm isop}$ in (4.9).

Pansu's conjecture is in $\mathbb{H}^1$. Of course, the formula defining $E_{\rm isop}$ in (4.9) makes sense in $\mathbb{H}^n$ for $n \geq 2$ and the conjecture can be naturally extended to any dimension.

Proposition 4.3. The set $E_{\rm isop} \subset \mathbb{H}^1$ has the following properties:

1) The boundary $\partial E_{\rm isop}$ is of class C^2 but not of class C^3.
2) The set $E_{\rm isop}$ is convex.
3) The set $E_{\rm isop}$ is axially symmetric.

Proof. 1) The boundary $\partial E_{\rm isop}$ is of class C^∞ away from the center of the group $Z = \{(0,t) \in \mathbb{H}^1 : t \in \mathbb{R}\}$. We claim that the function $\varphi : [0,1] \to \mathbb{R}$,

$$\varphi(r) = \arccos r + r\sqrt{1-r^2},$$

satisfies $\varphi'(0) = \varphi''(0) = 0$ but $\varphi'''(0) \neq 0$. This implies that $\partial E_{\rm isop}$ is of class C^2 but not of class C^3. In fact, we have

$$\varphi'(r) = \frac{-2r^2}{\sqrt{1-r^2}}, \quad \varphi''(r) = -2r\frac{2-r^2}{(1-r^2)^{3/2}},$$

and thus $\varphi'''(0) = -4 \neq 0$.

2) The set $E_{\rm isop}$ is convex because the function φ satisfies $\varphi'' \leq 0$ on $[0,1]$ and $\varphi'(0)=0$.

3) The set $E_{\rm isop}$ is axially symmetric:

$$(z,t) \in E_{\rm isop} \quad \Rightarrow \quad (\zeta,t) \in E_{\rm isop} \quad \text{for all } |\zeta| = |z|.$$

In fact, the profile function depends on $|z|$.

□

Pansu's conjecture is known to hold assuming some regularity, symmetry, or structure for the isoperimetric set. In the next sections, we describe the following recent results:

1) If $E \subset \mathbb{H}^1$ is isoperimetric and ∂E is of class C^2 then $E = E_{\rm isop}$, up to dilation and left translation. This result is not known when $n \geq 2$.
2) If $E \subset \mathbb{H}^1$ is isoperimetric and convex then $E = E_{\rm isop}$, up to dilation and left translation. This result is not known when $n \geq 2$.
3) Let $n \geq 1$. If $E \subset \mathbb{H}^n$ is isoperimetric and axially symmetric then $E = E_{\rm isop}$, up to a vertical translation and a dilation.
4) Let $n \geq 1$. If $E \subset \mathbb{H}^n$ is contained in a vertical cylinder and has a circular horizontal section, then $E = E_{\rm isop}$, up to dilation and left translation.

In general, Pansu's conjecture is still open.

4.2. **Isoperimetric sets of class C^2.** In this section, we show that isoperimetric sets in $\mathbb{H}^1$ of class C^2 are of the form (4.9). This result is due to [69] (Theorems 6.10 and 7.2) and relies upon two facts: the structure of the characteristic set of surfaces of class C^2; the geometric interpretation of the equation for surfaces with constant H-curvature. Both results are limited to $\mathbb{H}^1$.

Theorem 4.4 (Ritoré-Rosales)**.** Let $E \subset \mathbb{H}^1$ be a bounded isoperimetric set with boundary ∂E of class C^2. Then we have $E = E_{\rm isop}$, up to dilation and left translation.

Proof. Let $D \subset \mathbb{C}$ be an open set and let $f \in C^2(D)$ be a function such that

$$\mathrm{gr}(f) = \big\{ (z, f(z)) \in \mathbb{H}^1 : z \in D \big\} \subset \partial E.$$

We denote by $\Sigma(f) = \big\{ z \in D : \nabla f(z) + 2z^\perp = 0 \big\}$ the characteristic set of f. It may be $\Sigma(f) = \emptyset$. We always have $|\Sigma(f)| = 0$.

For $\varphi \in C_c^\infty(D \setminus \Sigma(f))$ and $\varepsilon \in \mathbb{R}$ small, consider the set $E_\varepsilon \subset \mathbb{H}^1$ that is obtained from E perturbing the piece of boundary of E given by the graph of f, through the function $f + \varepsilon\varphi$. Then, for small ε we have

$$\frac{P(E;\mathbb{H}^1)}{|E|^{3/4}} = \mathrm{Isop}(E) \leq \mathrm{Isop}(E_\varepsilon) = \frac{P(E_\varepsilon;\mathbb{H}^1)}{|E_\varepsilon|^{3/4}} = \frac{p(\varepsilon)}{v(\varepsilon)^{3/4}} = \psi(\varepsilon), \tag{4.10}$$

where $p(\varepsilon) = P(E_\varepsilon;\mathbb{H}^1)$ and $p(\varepsilon) = |E_\varepsilon|$. Using the area formula for H-perimeter (3.5) we find

$$p'(0) = \int_D \frac{\langle \nabla f + 2z^\perp, \nabla\varphi\rangle}{|\nabla f + 2z^\perp|} dz, \quad v'(0) = -\int_D \varphi(z)\, dz.$$

Here, we are assuming that the set E lies above the graph of f. Moreover, we have $\psi' = p'v^{-3/4} - \frac{3}{4}pv^{-7/4}v'$. From (4.10) we deduce that $\psi'(0) = 0$ and thus

$$\begin{aligned} 0 &= \frac{1}{|E|^{3/4}} \int_D \frac{\langle \nabla f + 2z^\perp, \nabla\varphi\rangle}{|\nabla f + 2z^\perp|} dz + \frac{3}{4}\frac{P(E;\mathbb{H}^1)}{|E|^{7/4}} \int_D \varphi\, dz \\ &= -\frac{1}{|E|^{3/4}} \int_D \varphi \operatorname{div}\Big(\frac{\nabla f + 2z^\perp}{|\nabla f + 2z^\perp|}\Big) dz + \frac{3}{4}\frac{P(E;\mathbb{H}^1)}{|E|^{7/4}} \int_D \varphi\, dz. \end{aligned}$$

Since $\varphi \in C_c^\infty(D \setminus \Sigma(f))$ is arbitrary, we deduce that the function f satisfies the partial differential equation

$$\operatorname{div}\Big(\frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|}\Big) = \frac{3}{4}\frac{P(E;\mathbb{H}^1)}{|E|} =: H, \quad z \in D \setminus \Sigma(f). \tag{4.11}$$

We conclude that for any $z \in D \setminus \Sigma(f)$ there exists an arc of circle κ_z with curvature H passing through z and such that $\gamma_z = \mathrm{Lift}(\kappa_z)$ is contained in $\mathrm{gr}(f) \subset \partial E$. See Remark 3.13.

Let $\Sigma(\partial E)$ be the characteristic set of ∂E. The above argument shows that for any $p \in \partial E \setminus \Sigma(\partial E)$ there exists a geodesic γ_p contained in $\partial E \setminus \Sigma(\partial E)$ and passing through p. There exists a maximal interval (a, b) such that we have $\gamma_p : (a, b) \to \partial E \setminus \Sigma(\partial E)$. Since E is bounded, γ_p can be extended to a and b with $\gamma(a), \gamma(b) \in \Sigma(\partial E)$.

In a neighborhood of the point $(z_0, t_0) = \gamma(a) \in \Sigma(f)$, the surface ∂E is a graph of the form $t = f(z)$ for some $f \in C^2(D)$ and $D \subset \mathbb{C}$ open set with $z_0 \in D$. This is because the tangent space to ∂E at this point coincides with the horizontal plane. Let (D, f) be the maximal pair such that $\mathrm{gr}(f) \subset \partial E$ with D open set containing z_0 and $f \in C^2(D)$.

By Theorem 3.15, there are two cases:

i) z_0 is an isolated point of $\Sigma(f)$;

ii) Near z_0, $\Sigma(f)$ is a C^1 curve κ_{z_0} passing through z_0.

In the case ii), let κ_{z_0} be the maximal C^1 curve contained in $\Sigma(f)$ and passing through z_0. The curve κ_{z_0} cannot reach the boundary ∂D because this would contradict the maximality of D. The curve κ_{z_0} cannot have limit points inside D that are singular, because of Theorem 3.15. Then κ_{z_0} must be a simple closed curve inside D. But this is not possible because the horizontal lift of κ_{z_0} grows in the t coordinate by an amount that equals 4 times the area of the region enclosed by the simple closed curve.

So we are left with the case $\Sigma(f) = \{z_0\}$ for some $z_0 \in D$. Through any point $z \in D \setminus \{z_0\}$ passes a circle with curvature H starting from z_0. Now the boundary of

E is determined in a neighborhood of $(z_0, f(z_0)) \in \partial E$. The regularity of ∂E forces D to be a circle centered at z_0 and E to be a left translation and dilatation of E_{isop}. □

4.3. **Convex isoperimetric sets.** We say that a set $E \subset \mathbb{H}^1$ is convex if it is convex for the standard linear structure of $\mathbb{H}^1 = \mathbb{R}^3$. Left translations and dilations preserve convexity. In [53], Pansu's conjecture is proved assuming the convexity of isoperimetric sets. Recall the $E_{\text{isop}} \subset \mathbb{H}^1$ is the set in (4.9).

Theorem 4.5 (Monti-Rickly)**.** Let $E \subset \mathbb{H}^1$ be a convex (open) isoperimetric set. Then, up to a left translation and a dilation we have $E = E_{\text{isop}}$.

Using the concentration argument of Theorem 4.1, it is possible to prove the existence of isoperimetric sets within the class of convex sets. However, it is not clear how to compute the first variation remaining inside this class of sets. Theorem 4.5 is not known when $n \geq 2$. It would be also interesting to prove the theorem assuming for isoperimetric sets only H-convexity (convexity along horizontal lines, see [5]) rather than standard convexity.

Here, we describe the technical steps of the proof of Theorem 4.5. For details, we refer the reader to [53]. Let $E \subset \mathbb{H}^1$ be a convex isoperimetric set. Then we have

$$E = \big\{(z,t) \in \mathbb{H}^1 \,:\, z \in D,\ f(z) < t < g(z), \big\}, \tag{4.12}$$

where $D \subset \mathbb{C} = \mathbb{R}^2$ is a bounded convex open set in the plane, and $-g, f : D \to \mathbb{R}$ are convex functions. In particular, f and g are locally Lipschitz continuous and their first derivatives are locally of bounded variation. The function f satisfies the following partial differential equation

$$\operatorname{div}\left(\frac{\nabla f + 2z^\perp}{|\nabla f + 2z^\perp|}\right) = \frac{3P(E;\mathbb{H}^1)}{4|E|} = H \quad \text{in } D. \tag{4.13}$$

Equation (4.13) can be deduced in the same way as in (4.11), with the difference that the equation is now verified only in the weak sense. As a matter of fact, the vector field

$$N_f(z) = \frac{\nabla f(z) + 2z^\perp}{|\nabla f(z) + 2z^\perp|} \quad z \in D,$$

is only in $L^\infty(D)$. However, we have $\nabla f(z) + 2z^\perp \in BV_{\text{loc}}(D)$.

The goal is to prove that integral curves of $N_f^\perp$ are circles with curvature H. The vector N_f will be the "normal vector" to the curve.

The first step of the proof of Theorem 4.5 is an improved regularity for solutions of (4.13): the candidate "normal vector" satisfies $N_f \in W^{1,1}_{\text{loc}}(D;\mathbb{R}^2)$, see [53].

The second step of the proof consists in the analysis of the flow of the vector field $v(z) = 2z - \nabla f^\perp(z)$. This vector field is orthogonal to N_f. Since f is convex, we have $v \in BV_{\text{loc}}(\text{int}(D);\mathbb{R}^2)$. Moreover, the distributional divergence of v is in L^∞, in fact

$\operatorname{div} v = 4$ in $\operatorname{int}(D)$. Thus, by Ambrosio's theory on the Cauchy Problem for vector fields of bounded variation [2], for any compact set $K \subset D$ there exist $r > 0$ and a (unique regular) Lagrangian flow $\Phi : K \times [-r, r] \to D$. In particular, for any $z \in K$, the curve $\gamma_z(s) = \Phi(z, s)$ is an integral curve of v passing through z at time $s = 0$.

The third step of the proof uses the fact that $v/|v|$ is in $W^{1,1}_{\mathrm{loc}}(D; \mathbb{R}^2)$ to show that (a suitable reparameterization of) the integral curve γ_z is twice differentiable in a weak sense. With this regularity, the distributional equation (4.13) can be given a formal meaning along the integral curve γ_z: it says that the curvature of γ_z is the constant H.

Theorem 4.6. Let $E \subset \mathbb{H}^1$ be a convex isoperimetric set with curvature $H > 0$ (the constant in (4.13)) and let $\Phi : K \times [-r, r] \to D$ be the flow introduced above. Then for a.e. $z \in K$ the curve $s \mapsto \Phi(z, s)$ is an arc of circle with radius $1/H$.

The shape of a convex isoperimetric set E can now be reconstructed starting from the structure of the characteristic set of ∂E. A point $(z, t) \in \partial E$ is characteristic if the horizontal plane at (z, t) is a supporting plane for E at (z, t). For convex sets, the characteristic set is the disjoint union of at most four compact disjoint horizontal segments, possibly points, see [53]. This property and Theorem 4.6 yield Theorem 4.5 as explained in the final part of the proof of Theorem 4.4.

4.4. **Axially symmetric solutions.** We denote by $\mathscr{S}$ the family of all measurable subsets $E \subset \mathbb{H}^n$ with $0 < |E| < \infty$ that are axially symmetric:

$$(z, t) \in E \quad \Rightarrow \quad (\zeta, t) \in E \quad \text{for all } |\zeta| = |z|.$$

The isoperimetric problem in the family $\mathscr{S}$ consists in proving existence and classifying all minimizers of the infimum problem

$$C^{\mathscr{S}}_{\mathrm{isop}} = \inf \big\{ \mathrm{Isop}(E) \, : \, E \in \mathscr{S} \big\}. \tag{4.14}$$

A set $E \in \mathscr{S}$ for which the infimum in (4.14) is attained is called an *axially symmetric isoperimetric set*. Clearly, we have $C^{\mathscr{S}}_{\mathrm{isop}} \geq C_{\mathrm{isop}}$. Even though we believe that $C^{\mathscr{S}}_{\mathrm{isop}} = C_{\mathrm{isop}}$, we are not able to prove this.

In the axially symmetric setting, Pansu's conjecture amounts to show that the solution to Problem (4.14) is the set

$$E_{\mathrm{isop}} = \Big\{ (z, t) \in \mathbb{H}^n \; : \; |t| < \arccos|z| + |z|\sqrt{1 - |z|^2}, \, |z| < 1 \Big\}. \tag{4.15}$$

for any dimension $n \geq 1$. This result is proved in [48] and, in this section, we present the scheme of the proof.

Theorem 4.7 (Monti)**.** The infimum $C^{\mathscr{S}}_{\mathrm{isop}} > 0$ is attained and any axially symmetric isoperimetric set coincides with the set E_{isop} in (4.15), up to a dilation, a vertical translation, and a Lebesgue negligible set.

By a rearrangement argument, Theorem 4.7 can be reduced to a one dimensional problem. The first step is the reduction to an isoperimetric problem in the half plane $\mathbb{R}^2_+ = \mathbb{R}^+ \times \mathbb{R}$.

Using spherical coordinates in $\mathbb{C}^n$, a measurable axially symmetric set $E \subset \mathbb{H}^n$ is generated by a measurable set $F \subset \mathbb{R}^2_+$ (and viceversa), and we have the following formula

$$P(E;\mathbb{H}^n) = \omega_{2n-1} Q(F;\mathbb{R}^2_+), \tag{4.16}$$

where $Q(\cdot;\mathbb{R}^2_+)$ is a weighted perimeter functional in the half-plane

$$Q(F;\mathbb{R}^2_+) = \sup\left\{ \int_F \left\{ \partial_r\big(r^{2n-1}\psi_1\big) + \partial_t\big(2r^{2n}\psi_2\big)\right\} drdt : \psi \in C^1_c(\mathbb{R}^2_+;\mathbb{R}^2), \|\psi\|_\infty \le 1 \right\}. \tag{4.17}$$

Above, $\omega_{2n-1} = \mathscr{H}^{2n-1}(\mathbb{S}^{2n-1})$ is the standard surface measure of the $(2n-1)$-dimensional unit sphere. For any axially symmetric set $E \subset \mathbb{H}^n$, the volume transforms according to the following rule

$$|E| = \omega_{2n-1} \int_F r^{2n-1} drdt = \omega_{2n-1} V(F), \tag{4.18}$$

where $V(\cdot)$ is a volume functional in the half-plane. From (4.17) and (4.18), the axially symmetric isoperimetric problem (4.14) transforms into the weighted isoperimetric problem in the half plane

$$C^{\mathscr{S}}_{\mathrm{isop}} = \omega_{2n-1}^{1/Q} \inf\left\{ \frac{Q(F;\mathbb{R}^2_+)}{V(F)^{\frac{Q-1}{Q}}} : F \subset \mathbb{R}^2_+ \text{ such that } 0 < V(F) < \infty \right\}. \tag{4.19}$$

The observation made in [48] is that the isoperimetric quotient for sets $F \subset \mathbb{R}^2_+$ is improved by a certain rearrangement of F in the variable r for fixed t that is tailored to the perimeter $Q(\cdot;\mathbb{R}^2_+)$. We measure the t-sections of F, the sets $F_t = \big\{ r > 0 : (r,t) \in F \big\}$, using the line density $\tau(r) = 2r^{2n}$. The function τ is the weight appearing in the definition of the functional $Q(\cdot;\mathbb{R}^2_+)$ in (4.17). We let

$$\Theta(r) = \int_0^r \tau(s)\, ds = \frac{2}{2n+1} r^{2n+1}, \tag{4.20}$$

and we say that a measurable set $F \subset \mathbb{R}^2_+$ is τ-rearrangeable if the function $f : \mathbb{R} \to [0,+\infty]$

$$f(t) = \int_{F_t} \tau(r)\, dr \tag{4.21}$$

is in $L^1_{\mathrm{loc}}(\mathbb{R})$. In this case, we call the set

$$F^\sharp = \big\{ (r,t) \in \mathbb{R}^2_+ : \Theta(r) < f(t) \big\} \tag{4.22}$$

the τ-rearrangement of F. The t-sections of $F^\sharp$ are intervals $(0, \Theta^{-1}(f(t)))$ with the same τ-measure as the t-sections F_t.

The following intermediate result is proved in [48].

Theorem 4.8. Let $F \subset \mathbb{R}^2_+$ be a τ-rearrangeable set. Then:

i) We have $Q(F^\sharp; \mathbb{R}^2_+) \le Q(F; \mathbb{R}^2_+)$, and in case of equality there holds $F = F^\sharp$, up to a negligible set.
ii) We have $V(F^\sharp) \ge V(F)$.

Using Theorem 4.8, it is easy to find a compact minimizing sequence, thus getting the existence of axially symmetric isoperimetric sets. Moreover, a set F minimizing (4.19) satisfies:

i) $F = F^\sharp$, up to a negligible set;
ii) the sections $F_r = \{t \in \mathbb{R} : (r,t) \in F\}$ are equivalent to intervals, for $\mathscr{L}^1$-a.e. $r \in \mathbb{R}^+$.

Now the boundary of ∂F inside $\mathbb{R}^2_+$ is a Lipschitz curve that can be computed by a standard variational argument. This curve is the profile of the isoperimetric set conjectured by Pansu and, as a matter of fact, it does not depend on the dimension n.

4.5. **Calibration argument.** In [67], Ritoré proved Pansu's conjecture within a special class of sets by a calibration argument. The sets have one circular horizontal section and are contained in a vertical cylinder, see also [19]. The argument works in any dimension. We let

$$B = \{(z,0) \in \mathbb{H}^n : |z| < 1\} \quad \text{and} \quad C = \{(z,t) \in \mathbb{H}^n : |z| < 1,\, t \in \mathbb{R}\}.$$

We identify $B = \{|z| < 1\} \subset \mathbb{C}^n$.

Theorem 4.9 (Ritoré)**.** Let $E \subset \mathbb{H}^n$, $n \ge 1$, be a bounded open set with finite H-perimeter such that:

i) $B \subset E \subset C$;
ii) $|E| = |E_{\mathrm{isop}}|$, where E_{isop} is the set in (4.15).

Then, we have $P(E_{\mathrm{isop}}; \mathbb{H}^n) \le P(E; \mathbb{H}^n)$.

Proof. Let $\varphi : \bar{B} \to \mathbb{R}$ be the profile function of E_{isop},

$$\varphi(z) = \arccos|z| + |z|\sqrt{1-|z|^2}, \quad |z| \le 1.$$

The function $f : \bar{C} \to \mathbb{R}$, $f(z,t) = |t| - \varphi(z)$, is a defining function for $\partial E_{\mathrm{isop}}$. Let us define the vector field $\psi : \bar{C} \setminus Z \to \mathbb{R}^{2n}$

$$\psi(z,t) = \frac{\nabla_H f(z,t)}{|\nabla_H f(z,t)|}, \quad 0 < |z| < 1, \quad t \ne 0.$$

The vector field ψ is not defined when $z = 0$ or $t = 0$; it can be extended to $|z| = 1$; it jumps at $t = 0$. In the set $\{0 < |z| < 1,\, t \ne 0\}$, ψ is of class C^∞ and there is a constant $H \neq 0$ such that

$$\mathrm{div}_H \psi(z,t) = H, \quad 0 < |z| < 1, \quad t \neq 0. \tag{4.23}$$

We consider the following sets:

$$E^+ = E \cap \{t > 0\}, \quad E^+_{\mathrm{isop}} = E_{\mathrm{isop}} \cap \{t > 0\}$$
$$E^- = E \cap \{t < 0\}, \quad E^-_{\mathrm{isop}} = E_{\mathrm{isop}} \cap \{t < 0\}.$$

By i), we have $E^+ \Delta E^+_{\mathrm{isop}} \subset C$ and moreover the boundary of $E^+ \Delta E^+_{\mathrm{isop}}$ does not intersect the base B of the cylinder. Let $F^+ = E^+_{\mathrm{isop}} \setminus E^+$ and $G^+ = E^+ \setminus E^+_{\mathrm{isop}}$. Then we have $F^+, G^+ \subset B \times \mathbb{R}^+$ and $E^+ \Delta E^+_{\mathrm{isop}} = F^+ \cup G^+$. Moreover, denoting by $N_H^{F^+}$ and $N_H^{G^+}$ the horizontal outer normals to ∂F^+ and ∂G^+, respectively:

a) $N_H^{F^+} = N_H^{E_{\mathrm{isop}}}$ a.e. on $\partial F^+ \cap \partial E_{\mathrm{isop}}$ and $N_H^{F^+} = -N_H^E$ a.e. on $\partial F^+ \cap \partial E$;

b) $N_H^{G^+} = -N_H^{E_{\mathrm{isop}}}$ a.e. on $\partial G^+ \cap \partial E_{\mathrm{isop}}$ and $N_H^{G^+} = N_H^E$ a.e. on $\partial G^+ \cap \partial E$.

Integrating (4.23) on F^+ we find

$$\begin{aligned} H|F^+| &= \int_{F^+} \mathrm{div}_H \psi(z,t)\, dzdt = \int_{\partial F^+} \langle N_H^{F^+}, \psi \rangle d\mu_{F^+} \\ &= \int_{\partial F^+} \langle N_H^{E_{\mathrm{isop}}}, \psi \rangle d\mu_{E_{\mathrm{isop}}} - \int_{\partial F^+} \langle N_H^E, \psi \rangle d\mu_E \\ &\geq P(E_{\mathrm{isop}}; \partial F^+) - P(E; \partial F^+), \end{aligned} \tag{4.24}$$

because $\langle N_H^{E_{\mathrm{isop}}}, \psi \rangle = 1$ on $\partial F^+ \cap \partial E_{\mathrm{isop}}$ and $\langle N_H^E, \psi \rangle \leq 1$ on $\partial F^+ \cap \partial E$.

In the same way, we find the inequality

$$\begin{aligned} H|G^+| &= \int_{G^+} \mathrm{div}_H \psi(z,t)\, dzdt = \int_{\partial G^+} \langle N_H^{G^+}, \psi \rangle d\mu_{G^+} \\ &= -\int_{\partial G^+} \langle N_H^{E_{\mathrm{isop}}}, \psi \rangle d\mu_{E_{\mathrm{isop}}} + \int_{\partial G^+} \langle N_H^E, \psi \rangle d\mu_E \\ &\leq -P(E_{\mathrm{isop}}; \partial G^+) + P(E; \partial G^+). \end{aligned} \tag{4.25}$$

From (4.24) and (4.25), we obtain

$$\begin{aligned} H(|F^+| - |G^+|) &\geq P(E_{\mathrm{isop}}; \partial F^+) - P(E; \partial F^+) + P(E_{\mathrm{isop}}; \partial G^+) - P(E; \partial G^+) \\ &= P(E_{\mathrm{isop}}; \{t > 0\}) - P(E; \{t > 0\}). \end{aligned} \tag{4.26}$$

Let $F^- = E^-_{\mathrm{isop}} \setminus E^-$ and $G^- = E^- \setminus E^-_{\mathrm{isop}}$. Then we have $F^-, G^- \subset B \times \mathbb{R}^-$ and $E^- \Delta E^-_{\mathrm{isop}} = F^- \cup G^-$. Computations analogous to the ones above show that

$$H(|F^-| - |G^-|) \geq P(E_{\mathrm{isop}}; \{t < 0\}) - P(E; \{t < 0\}). \tag{4.27}$$

Since $|E| = |E_{\rm isop}|$ we have $|F^+| + |F^-| = |G^+| + |G^-|$. Adding (4.26) and (4.27), we obtain

$$\begin{aligned} 0 &= H(|F^+| + |F^-| - |G^+| - |G^-|) \\ &\geq P(E_{\rm isop}; \{t \neq 0\}) - P(E; \{t \neq 0\}) \\ &= P(E_{\rm isop}; \mathbb{H}^n) - P(E; \mathbb{H}^n). \end{aligned}$$

This concludes the proof. □

Remark 4.10. In [67], Ritoré also discusses the equality case. Namely, in the setting of Theorem 4.9 and assuming that $\partial E \setminus Z$ is a C^1_H-regular surface, he shows that the equality $P(E; \mathbb{H}^n) = P(E_{\rm isop}; \mathbb{H}^n)$ implies $E = E_{\rm isop}$.

5. Regularity problem for H-perimeter minimizing sets

The regularity of H-perimeter minimizing boundaries is a challenging open problem. We list the main steps and the main technical difficulties.

1) *Lipschitz approximation.* The first step in the regularity theory of perimeter minimizing sets in $\mathbb{R}^n$ is a good approximation of minimizers. In De Giorgi's original approach, the approximation is made by convolution and the estimates are based on the monotonicity formula. In the Heisenberg group, the validity of a monotonicity formula is not clear, see [21]. A more flexible approach is the approximation of minimizing boundaries by Lipschitz graphs. This scheme works also in the Heisenberg group. An H-minimizing boundary is approximated in measure by an *intrinsic Lipschitz* graph. The estimate involves the notion of horizontal excess, see Theorem 5.9 and [50].

2) *Harmonic approximation.* The minimal set can be blown-up at a point of the reduced boundary by a quantity depending on excess. It can be shown that the corresponding approximating intrinsic Lipschitz functions converge to a limit function. This holds when $n \geq 2$ thanks to a Poincaré inequality valid on vertical hyperplanes, see [17]. We do not present the details, here. It is an open problem to prove that this limit function is harmonic for the natural (linear) sub-Laplacian of the vertical hyperplane.

3) *Decay estimate for excess.* Known estimates for sub-elliptic harmonic functions should give the decay estimate for excess

$$\mathrm{Exc}(E, B_{\alpha r}) \leq C\alpha^2 \mathrm{Exc}(E, B_r), \quad r > 0,$$

for some $0 < \alpha < 1$ and $C > 0$. By standard facts, this implies the Hölder continuity of the horizontal normal on the reduced boundary. In turn, the continuity of the normal implies that the reduced boundary is a C^1_H-regular surface in the sense of Definition 2.16, see [56], and thus it is locally the intrinsic graph of a continuous function φ having Hölder continuous *distributional* intrinsic gradient $\nabla^\varphi \varphi$, see Definition 3.4.

4) *Schauder-type regularity.* The function φ is a local minimizer of the area functional (see (3.14))

$$\mathscr{A}(\varphi) = \int_D \sqrt{1 + |\nabla^\varphi \varphi|^2} dw.$$

It is an open problem to deduce further regularity for φ, beyond the Hölder continuity of the distributional gradient ∇^φ. It is not even clear how to prove that φ solves the minimal surface equation (3.25).

This is the state of the art on the regularity of H-perimeter minimizing boundaries. In Section 5.3, we present the Lipschitz approximation of H-perimeter minimizing sets, Theorem 5.9, and also the so-called height estimate, giving a certain flatness of the boundary in the regime of small excess, see Theorem 5.10. The proofs are rather technical and are omitted.

In Section 5.2, we also study some examples of nonsmooth minimizers in $\mathbb{H}^1$, including sets with constant horizontal normal. No similar examples of nonsmooth minimizers are known in $\mathbb{H}^n$ with $n \geq 2$.

5.1. **Existence and density estimates.** We start from the definition of a local minimizer of H-perimeter.

Definition 5.1. A set $E \subset \mathbb{H}^n$ with locally finite H-perimeter in an open set $A \subset \mathbb{H}^n$ is *H-perimeter minimizing in A* if for all $p \in \mathbb{H}^n$ and $r > 0$ and for any $F \subset \mathbb{H}^n$ such that $E \Delta F \subset\subset B_r(p) \subset\subset A$ we have

$$P(E; B_r(p)) \leq P(F; B_r(p)). \tag{5.1}$$

The existence of local minimizers with some boundary condition easily follows by a compactness argument. Let $A \subset \mathbb{H}^n$ be a bounded open set and let $B \subset \mathbb{H}^n$ be a set such that $P(B; \mathbb{H}^n) < \infty$. Define the family of sets:

$$\mathscr{F}(A, B) = \big\{ F \subset \mathbb{H}^n \,:\, F \text{ has finite } H\text{-perimeter in } \mathbb{H}^n \text{ and } F \Delta B \subset \bar{A} \ \big\}.$$

Clearly, $\mathscr{F}(A, B) \neq \emptyset$ because $B \in \mathscr{F}(A, B)$. The set B determines a natural boundary condition.

Proposition 5.2. Let A and B be as above. Then there exists a set $E \in \mathscr{F}(A, B)$ such that

$$P(E; \mathbb{H}^n) \leq P(F; \mathbb{H}^n) \quad \text{ for all } F \in \mathscr{F}(A, B).$$

Proof. Define the infimum

$$m = \inf \big\{ P(F; \mathbb{H}^n) \,:\, F \in \mathscr{F}(A, B) \big\} \geq 0,$$

and let $(E_j)_{j \in \mathbb{N}}$ be a minimizing sequence of sets $E_j \in \mathscr{F}(A, B)$:

$$\lim_{j \to \infty} P(E_j; \mathbb{H}^n) = m.$$

Let $\Omega \subset \mathbb{H}^n$ be a bounded open set such that $\bar{A} \subset \Omega$ and supporting the compact embedding $BV_H(\Omega) \subset\subset L^1(\Omega)$. The C^2 regularity of the boundary $\partial\Omega$ is a sufficient condition for compactness (see [33] and [52]). Then we have:

i) $P(E_j; \mathbb{H}^n) \le m+1$ for all $j \in \mathbb{N}$ large enough;
ii) $|E_j \cap \Omega| \le |\Omega| < \infty$ for all $j \in \mathbb{N}$.

By compactness, there exists a subsequence, still denoted by $(E_j)_{j\in\mathbb{N}}$, and a measurable set $E \subset \mathbb{H}^n$ such that $\chi_{E_j} \to \chi_E$ in $L^1(\Omega)$. Since $\chi_{E_j} = \chi_B$ in $\mathbb{H}^n \setminus \bar{A}$, we can also assume that $\chi_E = \chi_B$ in $\mathbb{H}^n \setminus \bar{A}$, that is $E \in \mathscr{F}(A,B)$. In particular, we have $\chi_{E_j} \to \chi_E$ in $L^1(\mathbb{H}^n)$.

By the lower semicontinuity of perimeter for the L^1 convergence of sets, we obtain

$$P(E; \mathbb{H}^n) \le \liminf_{j\to\infty} P(E_j; \mathbb{H}^n) = m,$$

If now F is a set such that $E\Delta F \subset\subset B_r(p) \subset\subset A$, then $F \in \mathscr{F}(A,B)$ and $P(E; \mathbb{H}^n \setminus \bar{B}_r(p)) = P(F; \mathbb{H}^n \setminus \bar{B}_r(p))$. Therefore, we have

$$\begin{aligned} P(E; B_r(p)) &= P(E; \mathbb{H}^n) - P(E; \mathbb{H}^n \setminus \bar{B}_r(p)) \\ &\le P(F; \mathbb{H}^n) - P(F; \mathbb{H}^n \setminus \bar{B}_r(p)) = P(F; B_r(p)). \end{aligned}$$

□

As for the standard perimeter, sets that are H-perimeter minimizing admit lower and upper density estimates with geometric constants.

Lemma 5.3. If $E \subset \mathbb{H}^n$ is an H-perimeter minimizing set in a ball B_ϱ for some $\varrho > 0$, then we have

$$P(E; B_\varrho) \le c_1 \varrho^{Q-1}, \tag{5.2}$$

where $c_1 = P(B_1; \mathbb{H}^n)$.

Proof. Let $0 < s < r < \varrho$. Since the sets E and $E \setminus B_s$ agree inside $B_r \setminus \bar{B}_s$, we have

$$P(E; B_r \setminus \bar{B}_s) = P(E \setminus B_s; B_r \setminus \bar{B}_s) = P(E \setminus B_s; B_r) - P(E \setminus B_s; \bar{B}_s).$$

On the other hand, using $P(E \setminus B_s; B_s) = 0$ and (2.13) we obtain

$$\begin{aligned} P(E \setminus B_s; \bar{B}_s) = P(E \setminus B_s; \partial B_s) &= c_n \mathscr{S}_\varrho^{Q-1}(\partial^*(E \setminus B_s) \cap \partial B_s) \\ &\le c_n \mathscr{S}_\varrho^{Q-1}(\partial B_s) = P(B_s; \mathbb{H}^n) = c_1 s^{Q-1}. \end{aligned}$$

The formula $P(B_s; \mathbb{H}^n) = s^{Q-1} P(B_1; \mathbb{H}^n)$ follows by an elementary homogeneity argument. Then we obtain the inequality $P(E \setminus B_s; B_r) \le P(E; B_r \setminus \bar{B}_s) + c_1 s^{Q-1}$. Since E is H-perimeter minimizing in B_ϱ, by (5.1) we get

$$P(E; B_r) \le P(E \setminus B_s; B_r) \le P(E; B_r \setminus \bar{B}_s) + c_1 s^{Q-1}.$$

Letting $s \uparrow r$ and using $P(E; B_r) < \infty$, we obtain $P(E; B_r) \le c_1 r^{Q-1}$. Letting $r \uparrow \varrho$, we obtain (5.2). □

The density estimates from below are proved in [71], Proposition 2.14 (see also Theorem 2.4 therein).

Lemma 5.4. There exist constants $c_2, c_3 > 0$ depending on $n \geq 1$ such that for any set $E \subset \mathbb{H}^n$ that is H-perimeter minimizing in $B_{2\varrho}$, $\varrho > 0$, we have, for all $p \in \partial E \cap B_\varrho$ and for all $0 < r < \varrho$,

$$\min\big\{|E \cap B_r(p)|, |B_r(p) \setminus E|\big\} \geq c_2 r^Q, \tag{5.3}$$

and

$$P(E; B_r(p)) \geq c_3 r^{Q-1}. \tag{5.4}$$

For any set, the reduced boundary is a subset of the measure theoretic boundary, $\partial^* E \subset \partial E$, and moreover $\mu_E(\partial E \setminus \partial^* E) = 0$, see Proposition 2.8. For local minimizers the difference $\partial E \setminus \partial^* E$ is also small in terms of Hausdorff measures.

Lemma 5.5. For any set $E \subset \mathbb{H}^n$ that is H-perimeter minimizing in $\mathbb{H}^n$, we have

$$\mathscr{S}_\varrho^{Q-1}\big(\partial E \setminus \partial^* E\big) = 0. \tag{5.5}$$

Proof. Let $K = \partial E \setminus \partial^* E$, let A be an open set containing K, and fix $\delta > 0$. For any $p \in K$ there is an $0 < r_p < \delta/10$ such that $B_{5r_p}(p) \subset A$. Then $\{B_{r_p}(p) : p \in K\}$ is a covering of K and by the 5-covering lemma, there exists a sequence $p_i \in K$, $i \in \mathbb{N}$, such that the balls $B_i = B_{r_i}(p_i)$, with $r_i = r_{p_i}$, are pairwise disjoint and

$$K \subset \bigcup_{i \in \mathbb{N}} B_{5r_i}(p_i).$$

It follows that

$$\begin{aligned} \mathscr{S}_\varrho^{Q-1,\delta}(K \cap A) &\leq \sum_{i \in \mathbb{N}} \mathrm{diam}(B_{5r_i}(p_i))^{Q-1} = 10^{Q-1} \sum_{i \in \mathbb{N}} r_i^{Q-1} \\ &\leq 10^{Q-1} c_3^{-1} \sum_{i \in \mathbb{N}} P(E; B_{r_i}(p_i)) \leq 10^{Q-1} c_3^{-1} P(E; A). \end{aligned}$$

Since $\delta > 0$ is arbitrary, we deduce that $\mathscr{S}_\varrho^{Q-1}(K) \leq 10^{Q-1} c_3^{-1} P(E; A)$. As A is arbitrary and, by (2.13), $P(E; K) = 0$, we conclude that $\mathscr{S}_\varrho^{Q-1}(K) = 0$. □

5.2. **Examples of nonsmooth H-minimal surfaces.** The existence of nonsmooth H-minimal surfaces in $\mathbb{H}^1$ was already observed in [61]. Then this phenomenon was noticed by several authors, see [14], [66], [71], [55]. In the next examples, we prove perimeter minimality of certain H-minimal surfaces by a calibration argument, see [8], [55].

5.2.1. *A Lipschitz H-minimal surface.* In this example, we study a local minimizer of H-perimeter with boundary ∂E that is only Lipschitz-regular. The surface ∂E is, however, C^1_H-regular: whereas the standard normal jumps, the horizontal normal is continuous.

In the open half-space $A = \big\{(z,t) \in \mathbb{H}^1 : y = \mathrm{Im}(z) > 0\big\}$, consider the set

$$E = \big\{(z,t) \in A : x = \mathrm{Re}(z) < 0 \text{ and } t < 0\big\}.$$

The set E has locally finite H-perimeter in A and its boundary $S = \partial E \cap A$ is a Lipschitz surface consisting of two pieces of plane meeting at the singular line $L = \big\{(z,t) \in A : x = 0 \text{ and } t = 0\big\}$. The horizontal inner normal $\nu_E : S \to \mathbb{R}^2$ is the restriction to S of the mapping $\varphi : A \to \mathbb{R}^2$

$$\varphi(z,t) = \begin{cases} \dfrac{(-y,x)}{\sqrt{x^2+y^2}} & \text{if } x \le 0,\\ (-1,0) & \text{if } x \ge 0. \end{cases}$$

The function φ is continuous in A and thus S is an H-regular surface. In fact, φ is locally Lipschitz continuous in A.

We claim that E is a local minimizer of H-perimeter in A. Namely, we prove that for any bounded open set $\Omega \subset\subset A$ and for any $F \subset A$ such that $E\Delta F \subset\subset \Omega$ we have

$$P(E;\Omega) \le P(F;\Omega). \tag{5.6}$$

The proof is a calibration argument and the calibration is provided by the vector field V in A defined by

$$V(z,t) = \varphi_1(z,t)X + \varphi_2(z,t)Y,$$

where $\varphi = (\varphi_1, \varphi_2)$. Then, at points $(z,t) \in A$ where $x \le 0$ we have

$$\begin{aligned} \operatorname{div} V = \operatorname{div}_H \varphi &= X\Big(\frac{-y}{\sqrt{x^2+y^2}}\Big) + Y\Big(\frac{x}{\sqrt{x^2+y^2}}\Big) \\ &= \frac{\partial}{\partial x}\Big(\frac{-y}{\sqrt{x^2+y^2}}\Big) + \frac{\partial}{\partial y}\Big(\frac{x}{\sqrt{x^2+y^2}}\Big) = 0. \end{aligned}$$

Trivially, we have $\operatorname{div} V = 0$ where $x \ge 0$.

Without loss of generality, we assume that F is closed, that $\partial F \cap A$ is a smooth (say, Lipschitz) surface, and that $F \setminus E = \emptyset$ in such a way that $E\Delta F = E \setminus F = E \cap F'$, where $F' = \mathbb{H}^n \setminus F$.

Let N^E, N^F, and $N^{E\cap F'}$ denote the Euclidean outer unit normals to the boundary of $\partial E, \partial F$, and $\partial(E\cap F')$, respectively. By the divergence theorem, we have

$$\begin{aligned} 0 = \int_{E\cap F'} \operatorname{div} V \, dzdt &= \int_{\partial(E\cap F')} \langle V, N^{E\cap F'}\rangle d\mathscr{H}^2 \\ &= \int_{\partial E\cap F'} \langle V, N^E\rangle d\mathscr{H}^2 - \int_{\partial F\cap E} \langle V, N^F\rangle d\mathscr{H}^2. \end{aligned} \tag{5.7}$$

On ∂E, we have

$$\varphi = \frac{\big(\langle X, N^E\rangle, \langle Y, N^E\rangle\big)}{\sqrt{\langle X, N^E\rangle^2 + \langle Y, N^E\rangle^2}},$$

and thus

$$\langle V, N^E\rangle = \varphi_1\langle X, N^E\rangle + \varphi_2\langle Y, N^E\rangle = \sqrt{\langle X, N^E\rangle^2 + \langle Y, N^E\rangle^2}.$$

By the area formula (3.2), it follows that

$$\int_{\partial E\cap F'} \langle V, N^E\rangle d\mathscr{H}^2 = \int_{\partial E\cap F'} \sqrt{\langle X, N^E\rangle^2 + \langle Y, N^E\rangle^2} d\mathscr{H}^2 = P(E; F').$$

On the other hand, on ∂F we have $|\varphi| = 1$ and by the Cauchy-Schwarz inequality we obtain

$$\langle V, N^F\rangle = \varphi_1\langle X, N^F\rangle + \varphi_2\langle Y, N^F\rangle \leq \sqrt{\langle X, N^F\rangle^2 + \langle Y, N^F\rangle^2}.$$

So we deduce that

$$\int_{\partial F\cap E} \langle V, N^F\rangle d\mathscr{H}^2 \leq \int_{\partial F\cap E} \sqrt{\langle X, N^F\rangle^2 + \langle Y, N^F\rangle^2} d\mathscr{H}^2 = P(F; E).$$

So (5.7) implies $P(E; F') \leq P(F; E)$, and this is equivalent to (5.6).

5.2.2. *An H-minimal intrinsic graph with discontinuous normal.* In this example, we study an H-minimal intrinsic Lipschitz graph with discontinuous horizontal normal. This surface is a t-graph with standard $C^{1,1}$-regularity.

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be the function $\varphi(y, t) = \mathrm{sgn}(t)\sqrt{|t|}$. The intrinsic epigraph of φ in the sense of Definition 3.3 is the set

$$E = \big\{(s, y, t + 2ys) \in \mathbb{H}^1 \,:\, (y, t) \in \mathbb{R}^2,\ s > \varphi(y, t)\big\}.$$

The boundary of E is the intrinsic graph of φ:

$$\partial E = \big\{(\varphi(y, t), y, t + 2y\varphi(y, t)) \in \mathbb{H}^1 \,:\, (y, t) \in \mathbb{R}^2\big\}.$$

The intrinsic gradient of φ in the sense of Definition 3.4 reduces to the Burgers' component

$$\nabla^\varphi\varphi = \mathscr{B}\varphi = \varphi_y - 4\varphi\varphi_t = -2\mathrm{sgn}(t), \quad t \neq 0.$$

Then $\nabla^\varphi\varphi \in L^\infty(\mathbb{R}^2)$ and $\mathrm{gr}(\varphi)$ is an intrinsic Lipschitz graph, see Theorem 3.9. Moreover, by formula (3.13) the horizontal normal to ∂E is

$$\nu_E = \frac{(1, -\nabla^\varphi\varphi)}{\sqrt{1 + |\nabla^\varphi\varphi|^2}} = \frac{1}{\sqrt{5}}\big(1, 2\mathrm{sgn}(t)\big).$$

The normal can be extended in a constant way to $\mathbb{H}^1 \setminus \{x \neq 0\}$, when $x > 0$ and $x < 0$, separately.

Letting $x = \mathrm{sgn}(t)\sqrt{|t|}$, we realize that ∂E is the t-graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x|x| + 2xy$:

$$\partial E = \big\{\big(x, y, f(x, y)\big) \in \mathbb{H}^1 \,:\, (x, y) \in \mathbb{R}^2\big\}.$$

Clearly, we have $f \in C^{1,1}(\mathbb{R}^2)$.

We claim that E is a local minimizer for H-perimeter in $\mathbb{H}^1$. Namely, we prove that for any bounded open set $A \subset \mathbb{H}^1$ and for any measurable set $F \subset \mathbb{H}^1$ with locally finite H-perimeter and such that $E\Delta F \subset\subset A$ there holds

$$P(E; A) \le P(F; A). \tag{5.8}$$

Without loss of generality, we assume that $\partial F \cap A$ is a smooth surface. Let $G = E\Delta F$ and consider the subsets of G:

$$G^- = (E\Delta F) \cap \{x < 0\} \quad \text{and} \quad G^+ = (E\Delta F) \cap \{x > 0\}.$$

Let N^E, N^F, N^G be the Euclidean outer normals to $\partial E, \partial F$, and ∂G, respectively. To fix ideas, we assume that $F \setminus E = \emptyset$, so that we have

$$\begin{aligned} N^G &= N^E \quad \text{a.e. on } \partial E \cap \bar{G}, \\ N^G &= -N^F \quad \text{a.e. on } \partial F \cap \bar{G}. \end{aligned}$$

Define the horizontal vector field V in $\mathbb{H}^1$ by $V = \sqrt{5}(\nu_E^1 X + \nu_E^2 Y)$, where $\nu_E = (\nu_E^1, \nu_E^2)$ is the extended horizontal normal. Namely, we let

$$V = \begin{cases} X - 2Y & x < 0 \\ X + 2Y & x > 0. \end{cases}$$

The vector field V is not defined on the plane $x = 0$. When $x \neq 0$ we have $\mathrm{div}V = \mathrm{div}_H \nu_E = 0$. By the divergence theorem applied to G^- and G^+, we obtain

$$\begin{aligned} 0 &= \int_G \mathrm{div}V\, dzdt = \int_{G^-} \mathrm{div}V\, dzdt + \int_{G^+} \mathrm{div}V\, dzdt \\ &= \int_{\partial G^-} \langle V, N_{G^-}\rangle d\mathscr{H}^2 + \int_{\partial G^+} \langle V, N_{G^+}\rangle d\mathscr{H}^2. \end{aligned}$$

We denote by V^- and V^+ the traces of V onto $\{x = 0\}$, from the left and from the right. The integral on ∂G^- is

$$\begin{aligned} \int_{\partial G^-} \langle V, N_{G^-}\rangle d\mathscr{H}^2 &= \int_{\partial G \cap \{x<0\}} \langle V, N^G\rangle d\mathscr{H}^2 + \int_{G \cap \{x=0\}} \langle V^-, N^G\rangle d\mathscr{H}^2 \\ &= \int_{\partial E \cap \{x<0\}} \langle V, N^E\rangle d\mathscr{H}^2 - \int_{\partial F \cap \{x<0\}} \langle V, N^F\rangle d\mathscr{H}^2 \\ &\quad + \int_{G \cap \{x=0\}} \langle V^-, \mathrm{e}_1\rangle d\mathscr{H}^2. \end{aligned}$$

Using the identities and inequalities

$$\begin{aligned}
\langle V, N^E\rangle &= -|N^E_H| = -\sqrt{\langle N^E, X\rangle^2 + \langle N^E, Y\rangle^2} \quad \text{on } \partial E,\\
\langle V, N^F\rangle &\geq -|N^F_H| = -\sqrt{\langle N^F, X\rangle^2 + \langle N^F, Y\rangle^2} \quad \text{on } \partial F,
\end{aligned}$$

and $\langle V^-, \mathrm{e}_1\rangle = 1$, we conclude that

$$\begin{aligned}
\int_{\partial G^-} \langle V, N_{G^-}\rangle d\mathscr{H}^2 \leq &-\int_{\partial E\cap\{x<0\}} |N^E_H| d\mathscr{H}^2 + \int_{\partial F\cap\{x<0\}} |N^F_H| d\mathscr{H}^2\\
&+ \mathscr{H}^2(G\cap\{x=0\}).
\end{aligned}$$

A similar computation for G^+ yields

$$\begin{aligned}
\int_{\partial G^+} \langle V, N_{G^+}\rangle d\mathscr{H}^2 \leq &-\int_{\partial E\cap\{x>0\}} |N^E_H| d\mathscr{H}^2 + \int_{\partial F\cap\{x>0\}} |N^F_H| d\mathscr{H}^2\\
&- \mathscr{H}^2(G\cap\{x=0\}).
\end{aligned}$$

Adding the last two inequalities, the contribution from $G \cap \{x = 0\}$ cancels, and using the area formula (3.2), we finally obtain

$$P(E; \{x \neq 0\} \cap A) \leq P(F; \{x \neq 0\} \cap A) \leq P(F; A).$$

Since $P(E; \{x = 0\}) = 0$, this proves the claim (5.8).

5.2.3. *Sets with constant horizontal normal.* In the previous two examples, the calibration is provided by a suitable extension of the horizontal normal ν_E, extension that is divergence free. A special but interesting case of this situation is when the normal is in fact constant. In this section, we describe sets in $\mathbb{H}^1$ that have, locally, constant horizontal normal (see [50]).

For $r > 0$ and $p \in \mathbb{H}^1$, we let

$$\begin{aligned}
Q_r &= \big\{(x, y, t) \in \mathbb{H}^1 : |x| < r,\ |y| < r,\ |t| < r^2\big\},\\
Q_r(p) &= p \cdot Q_r.
\end{aligned} \tag{5.9}$$

For $r > 0$ and $(y_0, t_0) \in \mathbb{R}^2$, we also define

$$D_r(y_0, t_0) = \big\{(y, t) \in \mathbb{R}^2 : |y - y_0| < r,\ |t - t_0| < r^2\big\}, \tag{5.10}$$

and we let $D_r = D_r(0)$

Theorem 5.6. Let $E \subset \mathbb{H}^1$ be a set with finite $\mathbb{H}$-perimeter in Q_{4r}, $r > 0$, with $0 \in \partial E$. Assume that $\nu_E(p) = (1, 0) \in \mathbb{S}^1$ for μ_E-a.e. $p \in Q_{4r}$. Then there exists a function $g : D_r \to (-r/4, r/4)$ such that:

i) We have, up to a negligible set,

$$E \cap Q_r = \big\{(x, y, t) \in Q_r : x > g(y, t)\big\}.$$

ii) $g(0)=0$ and for all $(y,t),(y',t')\in D_r$

$$|g(y,t)-g(y',t')|\leq |y-y'|+\frac{1}{2r}|t-t'|. \tag{5.11}$$

iii) The graph of g consists of integral lines of the vector field Y.

Proof. For the sake of simplicity, we assume that E is open. For any $\alpha,\beta\in\mathbb{R}$ with $\alpha\geq 0$, let $Z=\alpha X+\beta Y$. Then, for any $\varphi\in C^1_c(Q_{4r})$ with $\varphi\geq 0$, by the Gauss-Green formula (2.6) we have

$$\int_E Z\varphi\, dzdt = -\alpha\int_{Q_{4r}}\varphi\, d\mu_E\leq 0,$$

that is $Z\chi_E\geq 0$ in the sense of distribution. It follows that

$$p\in E\cap Q_{4r}\quad\Rightarrow\quad \exp(sZ)(p)\in E, \tag{5.12}$$

for all $s>0$ such that $\exp(sZ)(p)\in Q_{4r}$.

For any point $q\in E\cap Q_{2r}$ consider the set $E_q=q^{-1}\cdot E$. The set E_q has constant measure theoretic normal $(1,0)\in\mathbb{S}^1$ in Q_{2r}. We can apply (5.12) to the set E_q starting first from the point $0\in E_q$ and then from a generic point $p=(0,y,0)\in E_q$ with $|y|<2r$. We deduce that

$$\big\{(x,y,t)\in Q_{2r}: x>0,\, |t|<4rx\big\}\subset E_q.$$

In other words, we have

$$q\in E\cap Q_{2r}\quad\Rightarrow\quad q\cdot\big\{(x,y,t)\in Q_{2r}: x>0,\, |t|<4rx\big\}\subset E. \tag{5.13}$$

From (5.13), it follows that $E\cap Q_{2r}\cap\{y=0\}$ is a planar set with the cone property, the cones having all axis parallel to the x-axis and aperture $4r$. We deduce that there exists a Lipschitz function $h:(-r^2,r^2)\to\mathbb{R}$ such that:

(a) $\big\{(x,t)\in\mathbb{R}^2:(x,0,t)\in E\big\}=\big\{(x,t)\in D_{2r}: x>h(t)\big\}$;

(b) $|h(t)-h(t')|\leq\dfrac{1}{4r}|t-t'|$ for all $t,t'\in(-r^2,r^2)$.

Since $0\in\partial E$, we infer that $h(0)=0$. From (5.13), we also deduce that ∂E consists of integral lines of Y in Q_{2r}. Then we have

$$\partial E\cap Q_{2r}=\big\{(h(\tau),\sigma,\tau-2\sigma h(\tau))\in\mathbb{H}^1:(\sigma,\tau)\in D_{2r}\big\}. \tag{5.14}$$

For any $(y,t)\in D_r$, the system of equations

$$\sigma=y,\quad \tau-2\sigma h(\tau)=t$$

has a unique solution $(\sigma,\tau)\in D_{2r}$. This is an easy consequence of the Banach fixed point theorem. We claim that the solution $\tau=\tau(y,t)$ of the equation $\tau-2yh(\tau)=t$

is Lipschitz continuous. Namely, by (b), we have for $(y,t),(y',t')\in D_r$

$$\begin{aligned}|\tau(y,t)-\tau(y',t')| &= |t-2yh(\tau(y,t))-t'+2y'h(\tau(y',t'))|\\ &\le |t-t'|+2|y||h(\tau(y,t))-h(\tau(y',t'))|+2|h(\tau(y',t'))||y-y'|\\ &\le |t-t'|+\frac{1}{2}|\tau(y,t)-\tau(y',t')|+\frac{1}{2r}|\tau(y',t')||y-y'|,\end{aligned}$$

and this implies

$$|\tau(y,t)-\tau(y',t')|\le 4r|y-y'|+2|t-t'|. \tag{5.15}$$

The function $g=h\circ\tau$ satisfies i), ii), and iii). In particular, (5.11) follows from (5.15), and $|g(y,t)|<r/4$ follows from (b). □

There are H-perimeter minimizing surfaces in $\mathbb{H}^1$ with a diffuse Lipschitz regularity. In fact, if $g : D_r \to (-r/4, r/4)$ is a function satisfying ii) and iii) of Theorem 5.6, then its x-graph is a Lipschitz surface that has, $\mathscr{H}^2$-a.e., constant horizontal normal. This vector can be used to show that the x-graph of g is locally minimizing H-perimeter.

Remark 5.7. If, in Theorem 5.6, the radius r can be taken arbitrarily large, then from (5.11) we deduce that the function g does not depend on t. Then from statement iii), we deduce that g does not depend on y, either. Thus E is a vertical half-space. This fact is used in Theorem 2.10.

When $n\ge 2$, the situation is different and easier because if the horizontal normal ν_E is constant in a small convex set then, inside this set, E is a vertical hyperplane orthogonal to the normal (see [27]).

5.3. **Lipschitz approximation and height estimate.** The notion of horizontal excess is natural:

Definition 5.8 (Horizontal excess)**.** Let $E\subset\mathbb{H}^n$ be a set with locally finite H-perimeter. The *horizontal excess* of E in a ball $B_r(p)$, where $p\in\mathbb{H}^n$ and $r>0$, is

$$\mathrm{Exc}(E,B_r(p)) = \min_{\substack{\nu\in\mathbb{R}^{2n}\\|\nu|=1}} \frac{1}{r^{Q-1}}\int_{B_r(p)} |\nu_E-\nu|^2 d\mu_E.$$

Intrinsic Lipschitz graphs are introduced in Definition 3.3, the notion of L-intrinsic Lipschitz function is introduced in Definition 3.8. The following theorem is proved in [50].

Theorem 5.9 (Monti)**.** Let $n\ge 1$ and let $L>0$ be a constant that is suitably large when $n=1$. There are constants $k>1$ and $c(L,n)>0$ with the following property. For any set $E\subset\mathbb{H}^n$ that is H-perimeter minimizing in B_{kr} with $0\in\partial E$ and $r>0$, there exist $\nu\in\mathbb{R}^{2n}$ with $|\nu|=1$ and an L-intrinsic Lipschitz function $\varphi : H_\nu\to\mathbb{R}$ such that

$$\mathscr{S}_\varrho^{Q-1}\big((\mathrm{gr}(\varphi)\Delta\partial E)\cap B_r\big)\le c(L,n)(kr)^{Q-1}\mathrm{Exc}(E,B_{kr}). \tag{5.16}$$

The following extension of the so-called "height estimate" to H-perimeter minimizing sets will be proved in the forthcoming paper [57].

Let $\nu = (1, 0 \dots, 0) \in \mathbb{R}^{2n}$ and let $W = \partial H_\nu \subset \mathbb{H}^n$ be the vertical hyperplane orthogonal to ν, i.e., $W = \{x_1 = 0\}$. For any $r > 0$ we let

$$D_r = \big\{ w \in W : \|w\|_\infty < r \big\},$$

and we define the truncated cylinder over D_r

$$C_r = D_r \cdot (-r, r) = \big\{ w \cdot (s\nu) \in \mathbb{H}^n : |s| < r \big\}.$$

The ν-directional excess of E inside the cylinder D_r is

$$\mathrm{Exc}(E, C_r, \nu) = \frac{1}{r^{Q-1}} \int_{C_r} |\nu_E - \nu|^2 d\mu_E.$$

Theorem 5.10 (Monti-Vittone)**.** Let $n \geq 2$. There exist constants $\varepsilon_0 > 0$, $c_0 > 0$, and $k > 0$ such that if $E \subset \mathbb{H}^n$ is an H-perimeter minimizing set in C_{kr} with

$$\mathrm{Exc}(E, C_{kr}, \nu) \leq \varepsilon_0,$$

then we have

$$\sup \{ x_1 = \mathrm{Re}(z_1) \in \mathbb{R} : (z, t) \in \partial E \cap C_r \} \leq c_0 r \mathrm{Exc}(E, C_{kr}, \nu)^{\frac{1}{2(Q-1)}}. \tag{5.17}$$

The proof follows the scheme of [70]. It relies on a nontrivial slicing technique and on a lower dimensional isoperimetric inequality. The estimate (5.17) does not hold when $n = 1$ because of the examples of Section 5.2.3, for which $\mathrm{Exc}(E, B_r) = 0$ but ∂E is not flat.

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