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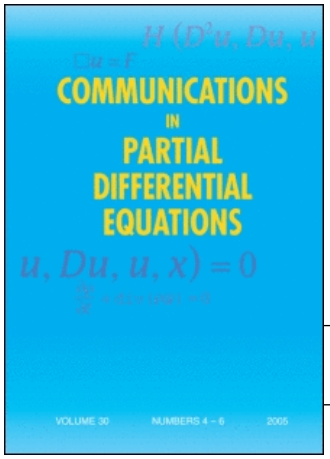

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Sobolev Inequalities for Weighted Gradients

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Sobolev Inequalities for Weighted Gradients

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We study symmetry, existence, and uniqueness properties of extremal functions for the weighted Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{2Q}{Q-2}}\,dx\,dy\right)^{\frac{Q-2}{Q}} \le c_{m,k,\alpha}\int_{\mathbb{R}^n}\left\{|\nabla_x u|^2 + (\alpha+1)^2|x|^{2\alpha}|\nabla_y u|^2\right\}dx\,dy,$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$ *with* $m, k \ge 1$ *and* $n = m + k$, $\alpha > 0$, *and* $Q = m + k(\alpha+1)$.

Keywords Grushin gradient; Hyperbolic symmetry; Rearrangements; Sobolev inequalities.

Mathematics Subject Classification 46E35.

1. Introduction

Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^k$, and $n = m + k$, with $m, k \ge 1$, introduce the weight function $w(x) = (\alpha+1)|x|^\alpha$, where $\alpha > 0$ is a positive real number, and consider the "homogeneous dimension" $Q = m + k(\alpha+1)$. We study the weighted Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{2Q}{Q-2}}\,dx\,dy\right)^{\frac{Q-2}{Q}} \le c_{m,k,\alpha}\int_{\mathbb{R}^n}\left\{|\nabla_x u|^2 + w^2|\nabla_y u|^2\right\}dx\,dy \tag{1.1}$$

for functions u in an appropriate Sobolev space. Here, $\nabla_x u$ and $\nabla_y u$ denote the gradients of u in the variables x and y, respectively. We are interested in symmetry, existence, and uniqueness properties of extremal functions for (1.1), i.e., functions achieving the best constant $c_{m,k,\alpha} > 0$. This article continues the research initiated in Monti and Morbidelli (2006), where Morbidelli and the author studied symmetry and uniqueness properties of non-negative, entire solutions to the critical semilinear

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equation

$$\Delta_x u + w^2 \Delta_y u = -u^{\frac{Q+2}{Q-2}} \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

This is the Euler equation associated with (1.1). When $\alpha = 0$, we have the Yamabe equation and positive, entire solutions are extremal functions for the Sobolev inequality (1.1) with $Q = n$ and $w = 1$. For $\alpha > 0$, the inequality is known as Sobolev inequality for the Grushin operator and it can be proven by general techniques in doubling metric spaces with Poincaré inequality. A general reference to the topic is Hajłasz and Koskela (2000) (see also Franchi et al., 1994 and Garofalo and Nhieu, 1996).

For $\alpha = 0$, extremal functions are radially symmetric and decreasing about some point in $\mathbb{R}^n$. This is no longer true for $\alpha > 0$, because inequality (1.1) is not invariant for the full group of Euclidean translations and orthogonal transformations. Let us introduce the following notion of symmetry. We say that a non-negative function $u : \mathbb{R}^n \to [0, +\infty)$ belongs to the class $\mathcal{S}$ if:

i) there is a function $v : \mathbb{R}^m \to [0, +\infty)$ such that

$$u(x, y) = |x|^{\frac{2-Q}{2}} v\left(\frac{(1 - |x|^{\alpha+1})^2 + |y|^2}{4|x|^{\alpha+1}} \frac{x}{|x|}\right), \quad x \neq 0; \tag{1.3}$$

ii) the function $t \mapsto t^{\frac{Q-2}{2}} u(t\zeta, 0)$ is nondecreasing on $[0, 1]$ for all $\zeta \in \mathbb{S}^{m-1}$, i.e., $\zeta \in \mathbb{R}^m$ and $|\zeta| = 1$.

In Monti and Morbidelli (2006), any positive function $u \in C^2(\mathbb{R}^n)$ solving equation (1.2) is proved to be of the form (1.3), up to translation in the variable y and scaling $u_\lambda(x, y) = \lambda^{\frac{Q-2}{2}} u(\lambda x, \lambda^{\alpha+1} y)$ for some positive number λ. The proof relies on the Kelvin transform for Grushin operators and on the moving spheres method. In this work, we take a different point of view and we show by a rearrangement argument that the family of competing functions for inequality (1.1) with sharp constant can be reduced to the class $\mathcal{S}$.

Let $D^1(\mathbb{R}^n)$ be the Sobolev space obtained as completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{D^1(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \{|\nabla_x u|^2 + w^2 |\nabla_y u|^2\} dx\, dy\right)^{1/2}.$$

The space $D^1(\mathbb{R}^n)$ is the natural framework for inequality (1.1). It can be equivalently defined as the space of functions $u \in L^{\frac{2Q}{Q-2}}(\mathbb{R}^n)$ with weak derivatives satisfying $\|u\|_{D^1(\mathbb{R}^n)} < +\infty$. Any function in $C_0^\infty(\mathbb{R}^n)$ can be approximated uniformly and in $D^1(\mathbb{R}^n)$ by compactly supported, piecewise affine functions u satisfying the additional condition

$$|\nabla_y u| \neq 0 \quad \text{on the support of } u. \tag{1.4}$$

We denote by $\mathcal{A}$ the family of all such functions. The set $\mathcal{A}$ is dense in $D^1(\mathbb{R}^n)$ and it is sufficient to work in this class.

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Theorem 1. *Let $m, k \in \mathbb{N}$, $\alpha \geq 0$, and $Q > 2$. For any non-negative function $u \in \mathscr{A}$, there exists a non-negative function $u^* \in D^1(\mathbb{R}^n)$ in the class $\mathscr{S}$ such that*

$$\int_{\mathbb{R}^n} \{|\nabla_x u^*|^2 + w^2|\nabla_y u^*|^2\} dx\, dy \leq \int_{\mathbb{R}^n} \{|\nabla_x u|^2 + w^2|\nabla_y u|^2\} dx\, dy. \tag{1.5}$$

Moreover,

$$\int_{\mathbb{R}^n} u^{*\frac{2Q}{Q-2}} dx\, dy = \int_{\mathbb{R}^n} u^{\frac{2Q}{Q-2}} dx\, dy. \tag{1.6}$$

The proof of Theorem 1 relies on a rearrangement argument in the hyperbolic space. We decompose $\mathbb{R}^n$ as $\mathbb{S}^{m-1} \times \mathbb{H}^{k+1}$, where $\mathbb{S}^{m-1}$ is the unit Euclidean sphere in $\mathbb{R}^m$ and $\mathbb{H}^{k+1}$ is the $(k+1)$-dimensional hyperbolic space. After a suitable functional change, functions are rearranged in each slice $\mathbb{H}^{k+1}$ with respect to hyperbolic volume and distance. The rearranged function is radially symmetric in each slice $\mathbb{H}^{k+1}$ about some fixed origin. Inverting the functional change, we get a rearranged function in the class $\mathscr{S}$. The procedure is introduced in Section 2, and Theorem 1 is proven in Section 3. Beckner (2001) recognized the connection between Grushin operators and hyperbolic symmetry: In the case of the plane ($m = k = 1$ and also $\alpha = 1$) he was able to determine extremal functions, controlling inequality (1.1) by the sharp Sobolev inequality on the sphere.

As functions in the class $\mathscr{S}$ are determined by their values in the "disk" $\{(x, 0) \in \mathbb{R}^m \times \mathbb{R}^k : |x| \leq 1\}$, inequality (1.1) is related to a Sobolev–Poincaré inequality in the unit ball $B = \{x \in \mathbb{R}^m\, |x| < 1\}$. Let us introduce the weights

$$p(x) = (1 - |x|^{2(\alpha+1)})^k, \quad q(x) = (Q-m)(Q-2)(1 - |x|^{2(\alpha+1)})^{k-1}|x|^{2\alpha}. \tag{1.7}$$

Both p and q are radial and non-negative on B. We omit reference to the relevant parameters m, k, α.

Theorem 2. *Let $m, k \in \mathbb{N}$, $\alpha \geq 0$, $Q > 2$, and p, q as in* (1.7). *Any non-negative function $v \in C^1(\overline{B})$ satisfies the inequality*

$$\left(\int_B v^{\frac{2Q}{Q-2}} p\, dx\right)^{\frac{Q-2}{Q}} \leq c'_{m,k,\alpha} \int_B \{|\nabla_x v|^2 p + v^2 q\} dx \tag{1.8}$$

with the sharp constant $c'_{m,k,\alpha} = (\omega_k 2^{-k})^{\frac{2}{Q}} c_{m,k,\alpha}$. Here, ω_k is the area of the unit sphere in $\mathbb{R}^{k+1}$ and $c_{m,k,\alpha}$ is the sharp constant in (1.1).

The vanishing of p and q at the boundary ∂B is, even for $\alpha = 0$, the main difference between inequality (1.8) and the standard Sobolev–Poincaré inequality in the ball. Its proof relies on the fact that a function in $C^1(\overline{B})$ can be extended to a function in $\mathbb{R}^n$ of the form (1.3) satisfying the Sobolev inequality (1.1). The reduction procedure from $\mathbb{R}^n$ to B, which produces the weights p and q, is performed in Theorem 3.2.

The Euler equation for (1.8) is

$$\mathrm{div}_x(p\nabla_x v) - qv = -pv^{\frac{Q+2}{Q-2}}, \quad |x| < 1.$$

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This equation, which can also be obtained from (1.2) assuming u of the form (1.3), was studied in Monti and Morbidelli (2006). We believe that the positive solution v of this equation with mixed boundary condition

$$\frac{\partial v}{\partial v} + \left(\frac{Q}{2} - 1\right)v = 0, \quad |x| = 1,$$

is unique. Equivalently, we believe that extremal functions for inequality (1.8) are radial in the variable x. We are able to prove this conjecture for $m = 1$. In this case, the inequality is one-dimensional

$$\left(\int_{-1}^{+1} v^{\frac{2Q}{Q-2}} p\, dx\right)^{\frac{Q-2}{Q}} \leq c'_{1,k,\alpha} \int_{-1}^{+1} \left\{v'^2 p + v^2 q\right\} dx, \tag{1.9}$$

where now $Q = 1 + k(\alpha + 1)$. We prove that (1.9) is optimized by even functions and we establish an existence result of extremals. As a corollary we have the following theorem.

Theorem 3. *Let $m = 1$, $k \in \mathbb{N}$, and $\alpha > 0$. There exist extremal functions in $D^1(\mathbb{R}^n)$ for inequality* (1.1)*, they are even in x and belong to the class $\mathcal{S}$* (*up to translation in y and scaling*).

Existence of extremals can be proven elementarly, because the loss of compactness for (1.1) is removed in the class $\mathcal{S}$ (see Theorem 4.2). The symmetry property $u(-x, y) = u(x, y)$ is the main point. The proof is based on a rearrangement argument for (1.9) combined with a uniqueness result for the corresponding Euler equation, which is proven in Monti and Morbidelli (2006) (see Theorem 4.5). The premise for Theorem 3 is a complicate functional change discussed at the beginning of Section 4 (see Proposition 4.1).

In the plane $m = k = 1$, we have uniqueness of extremals. In this case, a positive extremal of (1.9) solves the problem (see Lemma 4.4)

$$\begin{cases} (pv')' - qv = -pv^{\frac{Q+2}{Q-2}} & \text{in } (-1, 1) \\ 2v'(x) + (Q-2)xv(x) = 0 & x = \pm 1. \end{cases} \tag{1.10}$$

Here $Q = \alpha + 2$. By Theorem 6 in Monti and Morbidelli (2006), the positive solution to problem (1.10) is unique. In the case $\alpha = 1$, the solution can be expressed in closed form and it is

$$v(x) = \frac{1}{(1 + x^2)^{1/2}}. \tag{1.11}$$

Then we have the following theorem.

Theorem 4. *Let $m = k = 1$ and $\alpha > 0$. Extremal functions for inequality* (1.1) *are unique, up to translation in y, scaling and multiplication by a nonzero constant. In particular, for $\alpha = 1$ the extremal is*

$$u(x, y) = \frac{1}{((1 + x^2)^2 + y^2)^{1/4}}. \tag{1.12}$$

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Beckner's function (1.12) is obtained from (1.11) and (1.3). In the case of the plane, the sharp Sobolev inequality for the L^1 norm of the Grushin gradient was established in Monti and Morbidelli (2004) as a corollary of a sharp isoperimetric inequality. Grushin isoperimetric sets and level sets of extremal functions (1.12) are different.

If radiality in x is assumed, existence, symmetry, and uniqueness results can be obtained also for $m > 1$. Precisely, let $\mathscr{R}$ be the family of functions $u : \mathbb{R}^m \times \mathbb{R}^k \to [0, +\infty)$ such that $u(x, y) = v(|x|, y)$ for some $v : [0, +\infty) \times \mathbb{R}^k \to [0, +\infty)$ and define $c^{\mathscr{R}}_{m,k,\alpha} > 0$ by

$$\frac{1}{c^{\mathscr{R}}_{m,k,\alpha}} = \inf\left\{ \int_{\mathbb{R}^n} \left\{|\nabla_x u|^2 + w^2|\nabla_y u|^2\right\} dx\, dy : u \in \mathscr{R} \cap D^1(\mathbb{R}^n), \int_{\mathbb{R}^n} |u|^{\frac{2Q}{Q-2}} dx\, dy = 1\right\}.$$

By Theorem 1 and Lions's concentration-compactness principle (Lions, 1985), the constant $c^{\mathscr{R}}_{m,k,\alpha} > 0$ is achieved by functions belonging to the class $\mathscr{S}$ (up to translation in y and scaling). In the case $\alpha = 1$, this result is linked to the sharp Sobolev inequalities in groups of Heisenberg type proven by Garofalo and Vassilev (2001).

A short overview of the article is in order. In Section 2, we introduce the main notion of rearrangement and we describe its basic properties. In Section 3, we prove Theorems 1 and 2. Section 4 is devoted to the proof of Theorem 3.

2. Hyperbolic Type Rearrangement

Let $u : \mathbb{R}^n \to [0, +\infty)$ be a non-negative function and associate with it the new function $U : \mathbb{R}^n \to [0 + \infty)$ of the variables $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^k$

$$U(\xi, \eta) = |\xi|^{\frac{Q-2}{2(\alpha+1)}} u\big(|\xi|^{-\frac{\alpha}{\alpha+1}} \xi, \eta\big). \tag{2.1}$$

If u is of the form (1.3) and U is given by (2.1), then

$$U(\xi, \eta) = \Phi\left(\frac{(1 - |\xi|)^2 + |\eta|^2}{4|\xi|} \frac{\xi}{|\xi|}\right). \tag{2.2}$$

For any fixed direction $\xi/|\xi| \in \mathbb{S}^{m-1}$, a function U of the form (2.2) is radially symmetric for the hyperbolic metric in the $(k + 1)$-dimensional halfspace "spanned" by $\xi/|\xi|$ and η.

Let $\mathbb{H}^{k+1} = \mathbb{R}^+ \times \mathbb{R}^k$ be the $(k + 1)$-dimensional hyperbolic space with metric $r^{-2}(dr^2 + |d\eta|^2)$, $r > 0$, and $\eta \in \mathbb{R}^k$. The Riemannian volume μ and the Riemannian hypersurface measure ν are respectively

$$\mu = \frac{1}{r^{k+1}} \mathscr{L}^{k+1} \quad \text{and} \quad \nu = \frac{1}{r^k} \mathscr{H}^k,$$

where $\mathscr{L}^{k+1}$ is the Lebesgue measure in $\mathbb{R}^{k+1}$ and $\mathscr{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^{k+1}$. If $A \subset \mathbb{H}^{k+1}$ is an open set and B is a metric ball in $\mathbb{H}^{k+1}$, then the hyperbolic isoperimetric inequality states that

$$\mu(B) = \mu(A) \Rightarrow \nu(\partial B) \leq \nu(\partial A), \tag{2.3}$$

with equality only if A is a ball (see Burago and Zalgaller, 1988, §10.2).

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A ball B_s centered at $(1,0) \in \mathbb{H}^{k+1}$ having radius $s = \log\left(\frac{1+\varrho}{1-\varrho}\right)$, $\varrho \in (0,1)$, is the set of points $(r,\eta) \in \mathbb{H}^{k+1}$ satisfying

$$\frac{(1-r)^2 + |\eta|^2}{4r} < \frac{\varrho^2}{1-\varrho^2}. \tag{2.4}$$

The measure of B_s is

$$\mu(B_s) = \omega_k 2^{k+1} \int_0^{\varrho} \frac{r^k}{(1-r^2)^{k+1}} dr, \tag{2.5}$$

where ω_k is the surface measure of the unit Euclidean sphere in $\mathbb{R}^{k+1}$.

Formulae (2.4) and (2.5) can be easily proven in the ball-model of the hyperbolic space. Endow $\mathbb{B}^{k+1} = \{z \in \mathbb{R}^{k+1} : |z| < 1\}$ with the metric $4(1-|z|^2)^{-2}|dz|^2$ and with the Riemannian volume $2^{k+1}(1-|z|^2)^{-k-1}dz$. The Möbius map $S : \mathbb{H}^{k+1} \to \mathbb{B}^{k+1}$

$$S(r,\eta) = \left(\frac{1-r^2-|\eta|^2}{(1+r)^2+|\eta|^2}, \frac{-2\eta}{(1+r)^2+|\eta|^2}\right). \tag{2.6}$$

is an isometry and it maps the ball (2.4) onto the ball $\{|z| < \varrho\}$. Formula (2.5) easily follows.

Now we construct a function $u^* \in \mathcal{S}$ starting from a non-negative function $u \in C_0(\mathbb{R}^n)$. Let U be the function defined in (2.1) and for any $\zeta \in \mathbb{S}^{m-1}$ let $U_\zeta(r,\eta) = U(r\zeta,\eta)$. Analogously, for any $t \geq 0$ we have the superlevel sets $\Lambda_t = \{U > t\} \subset \mathbb{R}^n$ and $\Lambda_{\zeta,t} = \{U_\zeta > t\} \subset \mathbb{H}^{k+1}$. For fixed ζ, the distribution function $\varphi_\zeta(t) = \mu(\Lambda_{\zeta,t})$ is decreasing in t and $\varphi_\zeta(t) = \varphi_\zeta(t+0)$. Its "inverse" $\psi_\zeta : [0, \mu(\Lambda_{\zeta,0})) \to [0,+\infty)$ is defined by $\psi_\zeta(s) = \inf\{t > 0 : \varphi_\zeta(t) \leq s\}$. The rearranged function U_ζ^* is by definition radially symmetric about the point $(1,0) \in \mathbb{H}^{k+1}$ and with level sets $\Lambda^*_{\zeta,t}$ satisfying the condition

$$\mu(\Lambda^*_{\zeta,t}) = \mu(\Lambda_{\zeta,t}), \quad t > 0. \tag{2.7}$$

Precisely, we let

$$U_\zeta^*(r,\eta) = \psi_\zeta\big(\mu(B_{d(r,\eta)})\big), \tag{2.8}$$

where

$$d(r,\eta) = \log\frac{1+\varrho}{1-\varrho}, \quad \text{with } \varrho^2 = \frac{(1-r)^2+|\eta|^2}{(1+r)^2+|\eta|^2}, \tag{2.9}$$

is the hyperbolic distance of (r,η) from $(1,0)$. By Cavalieri principle, for any real number $p > 0$ condition (2.7), implies the identity

$$\int_{\mathbb{H}^{k+1}} U_\zeta^{*p} d\mu = \int_0^{+\infty} \mu(\Lambda^*_{\zeta,t^{1/p}}) dt = \int_0^{+\infty} \mu(\Lambda_{\zeta,t^{1/p}}) dt = \int_{\mathbb{H}^{k+1}} U_\zeta{}^p d\mu. \tag{2.10}$$

Letting $U^*(r\zeta,\eta) = U_\zeta^*(r,\eta)$, we can finally define u^* by

$$U^*(\xi,\eta) = |\xi|^{\frac{Q-2}{2(\alpha+1)}} u^*\big(|\xi|^{-\frac{\alpha}{\alpha+1}}\xi, \eta\big), \tag{2.11}$$

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as in (2.1). We call u^* the rearrangement of u. The function u^* is always in the class $\mathcal{S}$, but it does not necessarily have compact support even if u itself has compact support.

Now we discuss the properties of u^*, assuming u in $\mathcal{A}$, the family of piecewise affine functions with compact support satisfying (1.4). In this case, the level sets $\partial\Lambda_{\zeta,t} = \{U_\zeta = t\}$ are negligible and $(\zeta, t) \mapsto \varphi_\zeta(t)$ is therefore continuous. Moreover, $t \mapsto \varphi_\zeta(t)$ is strictly decreasing. We shall equivalently write $\varphi(\zeta, t) = \varphi_\zeta(t) = \varphi_t(\zeta)$, according to the relevant variable.

Throughout the article, we use the following notation. $\nabla_\xi U$ and $\nabla_\eta U$ denote the gradients of a function U in $\mathbb{R}^n$ in the variables ξ and η, respectively, $\nabla_R U = \langle \nabla_\xi U, \xi/|\xi| \rangle$ is the radial part of $\nabla_\xi U$ and $\nabla_T U = \nabla_\xi U - \nabla_R U \xi/|\xi|$ is the tangential part of $\nabla_\xi U$. We also denote by $\nabla_H U = |\xi|(\nabla_R U, \nabla_\eta U)$ the hyperbolic component of ∇U. Finally, if $W : \mathbb{S}^{m-1} \to \mathbb{R}$ is differentiable at $\zeta \in \mathbb{S}^{m-1}$ and $\mathrm{v} \in T_\zeta \mathbb{S}^{m-1}$ is a tangent vector, we define the tangential derivative in the direction v

$$\nabla_{\mathrm{v}} W(\zeta) = \lim_{\varepsilon \to 0} \frac{W(\gamma(\varepsilon)) - W(\zeta)}{\varepsilon},$$

where $\gamma : (-\delta, \delta) \to \mathbb{S}^{m-1}$ is a smooth curve such that $\gamma(0) = \zeta$ and $\dot{\gamma}(0) = \mathrm{v}$.

The next propositions describe some properties of the rearrangement u^* and of the distribution function $\mu(\Lambda_{\zeta,t})$. The proof is a standard application of coarea and area formulas and it is sketched in the Appendix at the end of the article.

Proposition 2.1. *Let $u \in \mathcal{A}$ be non-negative, U as in* (2.1) *and $\Lambda_{\zeta,t} = \{U_\zeta > t\}$. The function $\varphi(\zeta, t) = \mu(\Lambda_{\zeta,t})$ is differentiable at almost every $(\zeta, t) \in \mathbb{S}^{m-1} \times \mathbb{R}^+$ and moreover*

$$\frac{\partial \varphi_\zeta(t)}{\partial t} = -\int_{\partial\Lambda_{\zeta,t}} \frac{1}{|\nabla_H U_\zeta|} dv, \tag{2.12}$$

and, for $\mathrm{v} \in T_\zeta \mathbb{S}^{m-1}$ *(with $m \geq 2$),*

$$\nabla_{\mathrm{v}} \varphi_t(\zeta) = \int_{\partial\Lambda_{\zeta,t}} \frac{\nabla_{\mathrm{v}} U_\zeta}{|\nabla_H U_\zeta|} dv, \tag{2.13}$$

where $\nabla_{\mathrm{v}} U_\zeta$ denotes the tangential derivative of U_ζ in the variable ζ and direction v.

Proposition 2.2. *If $u \in \mathcal{A}$ is non-negative, then the rearrangement u^* is continuous in $\mathbb{R}^n$ minus the set $\{x = 0\}$ and locally Lipschitz continuous in $\mathbb{R}^n$ minus the set $\{x = 0\} \cup \{|x| = 1, y = 0\}$.*

In the next proposition we show that the function u^* is actually continuous on the whole $\mathbb{R}^n$. The proof also provides the profile of extremals for (1.1) on the subspace $x = 0$.

Proposition 2.3. *If $u \in \mathcal{A}$ is a non-negative function, then u^* can be continuously extended to $\mathbb{R}^n$. Moreover,*

$$u^*(0, y) = C(m, k, \alpha) \left(\int_{\Omega_0} s^{-k(\alpha+1)-1} ds\, dy \right)^{\frac{Q-2}{2k(\alpha+1)}} (1 + |y|^2)^{\frac{2-Q}{2(\alpha+1)}}, \tag{2.14}$$

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where $\Omega_0 = \{(s, y) \in \mathbb{R}^+ \times \mathbb{R}^k : s^{\frac{Q-2}{2}} u(0, y) > 1\}$ *and* $C(m, k, \alpha) > 0$ *depends on* m, k, α.

Proof. We claim that for any $\varepsilon > 0$ there is a $\delta > 0$ such that for $|x| \le \delta$ and $y \in \mathbb{R}^k$ we have $|u^*(x, y) - u^*(0, y)| \le \varepsilon$ with $u^*(0, y)$ given by (2.14). If $(|x|^\alpha x, y) \in \partial\Lambda_t^* = \partial\{U^* > t\}$ for some $t > 0$, then

$$u^*(x, y) = |x|^{\frac{2-Q}{2}} U^*(|x|^\alpha x, y) = |x|^{\frac{2-Q}{2}} t. \tag{2.15}$$

For $\zeta \in \mathbb{S}^{m-1}$ let $\Lambda_{\zeta,t} = \{U_\zeta > t\}$. As in (2.7) we have $\mu(\Lambda_{\zeta,t}) = \mu(\Lambda^*_{\zeta,t})$, where $\Lambda^*_{\zeta,t}$ is the ball B_s centered at $(1, 0)$ with radius $s = \log \frac{1-\varrho}{1+\varrho}$, which, by (2.5), is determined by

$$\begin{aligned}\omega_k 2^{k+1} \int_0^\varrho \frac{r^k}{(1-r^2)^{k+1}} dr = \mu(\Lambda_{\zeta,t}) &= \int_{U_\zeta > t} r^{-k-1} dr\, d\eta \\ &= (\alpha+1) t^{-\frac{2k(\alpha+1)}{Q-2}} \int_{\Omega_{\zeta,t}} s^{-k(\alpha+1)-1} ds\, dy = t^{-\frac{2k(\alpha+1)}{Q-2}} \lambda(\zeta, t),\end{aligned} \tag{2.16}$$

where $\Omega_{\zeta,t} = \{(s, y) \in \mathbb{R}^+ \times \mathbb{R}^k : s^{\frac{Q-2}{2}} u(t^{\frac{2}{Q-2}} s\zeta, y) > 1\}$ and $\lambda(\zeta, t)$ is defined by the last equality. In order to get the second equality in (2.16), we used (2.1) and we performed the change of variable $rt^{\frac{2(\alpha+1)}{Q-2}} = s^{\alpha+1}$. Notice that if $|x| \to 0$ then $t \to 0$ and that

$$\lim_{t \to 0} \lambda(\zeta, t) = (\alpha+1) \int_{\Omega_0} s^{-k(\alpha+1)-1} ds\, dy. \tag{2.17}$$

The convergence in (2.17) is uniform in ζ. In order to prove this fact, we note that

$$\lim_{s_0 \to +\infty} \int_{\Omega_{\zeta,t} \cap \{s > s_0\}} s^{-k(\alpha+1)-1} ds\, dy = 0$$

uniformly in ζ. Moreover, u is uniformly continuous and then for any $\sigma > 0$ and $s_0 > 0$ there exists $t_0 = t_0(\sigma, s_0) > 0$ such that

$$s^{\frac{Q-2}{2}} u(0, y) > 1 + \sigma \Rightarrow s^{\frac{Q-2}{2}} u\big(t^{\frac{2}{Q-2}} s\zeta, y\big) > 1 \Rightarrow s^{\frac{Q-2}{2}} u(0, y) > 1 - \sigma$$

for all $s \in (0, s_0)$, $t \in (0, t_0)$, and $\zeta \in \mathbb{S}^{m-1}$. The uniform convergence easily follows.

Now, from (2.15) and (2.16) we get

$$u^*(x, y) = \left(\frac{\omega_k 2^{k+1} |x|^{k(\alpha+1)}}{\lambda(x/|x|, t)} \int_0^\varrho \frac{r^k}{(1-r^2)^{k+1}} dr \right)^{\frac{2-Q}{2k(\alpha+1)}},$$

where t and ϱ are related through (2.16) and moreover, by (2.9) with $(|x|^\alpha x, y) \in \partial\Lambda_t^*$,

$$\varrho^2 = \frac{(1 - |x|^{\alpha+1})^2 + |y|^2}{(1 + |x|^{\alpha+1})^2 + |y|^2}.$$

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By Hôpital's rule

$$\lim_{|x|\to 0} |x|^{k(\alpha+1)} \int_0^{\varrho} \frac{r^k}{(1-r^2)^{k+1}} dr = \lim_{|x|\to 0} -\frac{|x|^{k(\alpha+1)+1}\varrho^k \varrho'}{k(\alpha+1)(1-\varrho^2)^{k+1}},$$

where ϱ' stands for the derivative of ϱ with respect to $|x|$. Using

$$2\varrho\varrho' = (\varrho^2)' = \frac{-4(\alpha+1)|x|^{\alpha}(1+|y|^2-|x|^{2(\alpha+1)})}{\left((1+|x|^{\alpha+1})^2+|y|^2\right)^2},$$

we find

$$-\frac{|x|^{k(\alpha+1)+1}\varrho^k\varrho'}{k(\alpha+1)(1-\varrho^2)^{k+1}} = \frac{2}{4^k k}\left(1+|y|^2-|x|^{2(\alpha+1)}\right)\left((1+|x|^{\alpha+1})^2+|y|^2\right)^{k-1},$$

which converges to $\frac{2}{4^k k}\left(1+|y|^2\right)^k$ as $|x|\to 0$, uniformly for $y \in \mathbb{R}^k$. The claim follows. □

3. Hyperbolic Symmetry for Competing Functions

Theorem 3.1. *Let $u \in \mathscr{A}$ be a non-negative function and let u^* be its rearrangement. Then $u^* \in D^1(\mathbb{R}^n)$ and relations* (1.5) *and* (1.6) *hold.*

Proof. The statement $u^* \in D^1(\mathbb{R}^n)$ follows from Propositions 2.2 and 2.3, as soon as we prove (1.5) and (1.6).

Let U be the function related to u as in (2.1). By a short computation, we find

$$\begin{aligned} |\nabla_x u|^2 &= |x|^{-Q}\Big\{|x|^{2(\alpha+1)}((\alpha+1)^2|\nabla_R U|^2+|\nabla_T U|^2) \\ &\qquad + \left(1-\frac{Q}{2}\right)^2 U^2 + (\alpha+1)(2-Q)|x|^{\alpha+1}U\nabla_R U\Big\}, \qquad (3.1) \\ |\nabla_y u|^2 &= |x|^{2-Q}|\nabla_\eta U|^2. \end{aligned}$$

Here, $\nabla_x u$ and $\nabla_y u$ are evaluated at (x, y), while U, $\nabla_R U$, $\nabla_T U$, and $\nabla_\eta U$ are evaluated at $(\xi, \eta) = (|x|^{\alpha}x, y)$.

For any $t > 0$ let $\Lambda_t = \{U > t\}$ and $\Omega_t = \Psi(\Lambda_t)$, where

$$\Psi(\xi, \eta) = (|\xi|^{-\frac{\alpha}{\alpha+1}}\xi, \eta). \qquad (3.2)$$

Using (3.1), we get

$$\begin{aligned} E_t &:= \int_{\Omega_t} \{|\nabla_x u|^2 + w^2|\nabla_y u|^2\} dx\, dy \\ &= \int_{\Omega_t} \Big\{(\alpha+1)^2|x|^{2(\alpha+1)}\left(|\nabla_R U|^2+|\nabla_\eta U|^2\right) + |x|^{2(\alpha+1)}|\nabla_T U|^2 \\ &\qquad + (\alpha+1)(2-Q)|x|^{\alpha+1}U\nabla_R U + \left(1-\frac{Q}{2}\right)^2 U^2\Big\}\frac{dx\, dy}{|x|^Q} \\ &= \int_{\Lambda_t} \left\{\delta|\nabla_H U|^2 + \frac{|\xi|^2}{\delta}|\nabla_T U|^2 + \frac{2\gamma|\xi|}{k}U\nabla_R U + \beta U^2\right\}\frac{d\xi\, d\eta}{|\xi|^n}, \qquad (3.3) \end{aligned}$$

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where we set

$$\beta = \frac{(2-Q)^2}{4(\alpha+1)}, \quad \gamma = \frac{k}{2}(2-Q), \quad \delta = \alpha + 1. \tag{3.4}$$

We got the last equality in (3.3) by the change of variable $(x, y) = \Psi(\xi, \eta)$, which has Jacobian determinant $|\det J\Psi| = (a+1)^{-1}|\xi|^{-\frac{m\alpha}{\alpha+1}}$.

Let $U_\zeta(r, \eta) = U(r\zeta, \eta)$, with $\zeta \in \mathbb{S}^{m-1}$ and $r > 0$, and $\Lambda_{\zeta,t} = \{U_\zeta > t\} \subset \mathbb{H}^{k+1}$. By Fubini–Tonelli theorem, we split the last integral in (3.3) into a double integral in $\mathbb{R}^k$ and $\mathbb{R}^m$. Then, introducing polar coordinates in $\mathbb{R}^m$, we obtain

$$E_t = \int_{\mathbb{S}^{m-1}} \int_{\Lambda_{\zeta,t}} \left\{ \delta |\nabla_H U_\zeta|^2 + \frac{r^2}{\delta} |\nabla_T U_\zeta|^2 + \frac{2\gamma}{k} r U_\zeta \nabla_R U_\zeta + \beta U_\zeta^2 \right\} \frac{dr\, d\eta}{r^{k+1}} d\mathcal{H}^{m-1}, \tag{3.5}$$

where $\nabla_T U_\zeta(r, \eta) = \nabla_T U(r\zeta, \eta)$ and $\nabla_R U_\zeta(r, \eta) = \nabla_R U(r\zeta, \eta) = \frac{\partial}{\partial r} U_\zeta(r, \eta)$. Note in the inner integral the hyperbolic measure $d\mu = \frac{dr\, d\eta}{r^{k+1}}$.

We have to estimate four terms. Let U_ζ^* be the rearrangement of U_ζ as in (2.8) and let $\Lambda_{\zeta,t}^* = \{U_\zeta^* > t\}$. We claim that for all $\zeta \in \mathbb{S}^{m-1}$ and for all $t > 0$, we have

$$\int_{\Lambda_{\zeta,t}^*} U_\zeta^{*2} d\mu = \int_{\Lambda_{\zeta,t}} U_\zeta^2 d\mu, \tag{3.6}$$

$$\int_{\Lambda_{\zeta,t}^*} r U_\zeta^* \frac{\partial U_\zeta^*}{\partial r} d\mu = \int_{\Lambda_{\zeta,t}} r U_\zeta \frac{\partial U_\zeta}{\partial r} d\mu, \tag{3.7}$$

$$\int_{\Lambda_{\zeta,t}^*} r^2 |\nabla_T U_\zeta^*|^2 d\mu \le \int_{\Lambda_{\zeta,t}} r^2 |\nabla_T U_\zeta|^2 d\mu, \tag{3.8}$$

$$\int_{\Lambda_{\zeta,t}^*} |\nabla_H U_\zeta^*|^2 d\mu \le \int_{\Lambda_{\zeta,t}} |\nabla_H U_\zeta|^2 d\mu. \tag{3.9}$$

Relations (3.6)–(3.9) imply

$$E_t \ge \int_{\Omega_t^*} \left\{ |\nabla_x u^*|^2 + w^2 |\nabla_y u^*|^2 \right\} dx\, dy,$$

where u^* is the function defined in (2.11) and $\Omega_t^* = \Psi(\Lambda_t^*)$. Our claim (1.5) then follows by monotone convergence letting $t \to 0$.

The proof of identity (3.6) is as in line (2.10). We prove (3.7). This is the part of the proof where we need to work with superlevel sets Λ_t with $t > 0$. For any $\eta \in \mathbb{R}^k$ let $\Lambda_{\zeta,\eta,t} = \{r > 0 : (r, \eta) \in \Lambda_{\zeta,t}\}$. Then

$$\int_{\Lambda_{\zeta,t}} r U_\zeta \frac{\partial U_\zeta}{\partial r} d\mu = \frac{1}{2} \int_{\Lambda_{\zeta,t}} \frac{\partial}{\partial r}(U_\zeta^2) \frac{dr\, d\eta}{r^k} = \frac{1}{2} \int_{\mathbb{R}^k} \int_{\Lambda_{\zeta,\eta,t}} \frac{\partial}{\partial r}(U_\zeta^2) \frac{dr}{r^k} d\eta.$$

The open set $\Lambda_{\zeta,\eta,t} \subset \mathbb{R}^+$ is an at most countable disjoint union of open intervals

$$\Lambda_{\zeta,\eta,t} = \bigcup_{h=1}^{+\infty} (a_h, b_h), \quad 0 < a_h \le b_h \le a_{h+1}.$$

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(The numbers a_h, b_h depend on ζ, η, t.) Then, integrating by parts, we obtain

$$\begin{aligned}\int_{\Lambda_{\zeta,\eta,t}} \frac{\partial}{\partial r}(U_\zeta^2)\frac{dr}{r^k} &= \sum_{h=1}^{+\infty}\int_{a_h}^{b_h}\frac{\partial}{\partial r}(U_\zeta^2)\frac{dr}{r^k}\\ &= \sum_{h=1}^{+\infty}\left(\frac{U_\zeta(b_h,\eta)^2}{b_h^k}-\frac{U_\zeta(a_h,\eta)^2}{a_h^k}+k\int_{a_h}^{b_h}U_\zeta^2\frac{dr}{r^{k+1}}\right)\\ &= k\int_{\Lambda_{\zeta,\eta,t}}U_\zeta^2\frac{dr}{r^{k+1}}+t^2\sum_{h=1}^{+\infty}\left(\frac{1}{b_h^k}-\frac{1}{a_h^k}\right)\\ &= k\int_{\Lambda_{\zeta,\eta,t}}(U_\zeta^2-t^2)\frac{dr}{r^{k+1}}.\end{aligned}$$

Hence, integrating in η and using (2.7) and (3.6), we get

$$\int_{\Lambda_{\zeta,t}}\frac{\partial}{\partial r}(U_\zeta^2)d\mu = k\int_{\Lambda_{\zeta,t}}(U_\zeta^2-t^2)d\mu = k\int_{\Lambda^*_{\zeta,t}}(U_\zeta^{*2}-t^2)d\mu = \int_{\Lambda^*_{\zeta,t}}\frac{\partial}{\partial r}(U_\zeta^{*2})d\mu. \tag{3.10}$$

This concludes the proof of (3.7).

We prove (3.8) (we need this part of the argument only if $m \geq 2$). First of all, notice that Proposition 2.1 applies also to U_ζ^*. From (2.12), we have for a.e. $t > 0$

$$\int_{\partial\Lambda_{\zeta,t}}\frac{1}{|\nabla_H U_\zeta|}dv = \int_{\partial\Lambda^*_{\zeta,t}}\frac{1}{|\nabla_H U_\zeta^*|}dv. \tag{3.11}$$

From (2.13), we have for a.e. $\zeta \in \mathbb{S}^{m-1}$ and for every tangential direction $\mathrm{v} \in T_\zeta\mathbb{S}^{m-1}$

$$\int_{\partial\Lambda_{\zeta,t}}\frac{\nabla_{\mathrm{v}}U_\zeta}{|\nabla_H U_\zeta|}dv = \int_{\partial\Lambda^*_{\zeta,t}}\frac{\nabla_{\mathrm{v}}U_\zeta^*}{|\nabla_H U_\zeta^*|}dv. \tag{3.12}$$

By the coarea formula, as in (A.1), we have

$$\int_{\Lambda_{\zeta,t}} r^2|\nabla_T U_\zeta|^2 d\mu = \int_t^{+\infty}\int_{\partial\Lambda_{\zeta,s}}\frac{r^2|\nabla_T U_\zeta|^2}{|\nabla_H U_\zeta|}dv\,ds. \tag{3.13}$$

Let $\mathrm{v}_1, \ldots, \mathrm{v}_{m-1} \in T_\zeta\mathbb{S}^{m-1}$ be an orthonormal basis. Then

$$r^2|\nabla_T U_\zeta|^2 = \sum_{i=1}^{m-1}|\nabla_{\mathrm{v}_i}U_\zeta|^2, \tag{3.14}$$

where $\nabla_{\mathrm{v}_i}U_\zeta$ is the tangential derivative of $\zeta \mapsto U_\zeta$ in the direction v_i.

By Hölder inequality, (3.11) and (3.12) we have

$$\begin{aligned}\int_{\partial\Lambda_{\zeta,s}}\frac{|\nabla_{\mathrm{v}_i}U_\zeta|^2}{|\nabla_H U_\zeta|}dv &\geq \left(\int_{\partial\Lambda_{\zeta,s}}\frac{\nabla_{\mathrm{v}_i}U_\zeta}{|\nabla_H U_\zeta|}dv\right)^2\left(\int_{\partial\Lambda_{\zeta,s}}\frac{1}{|\nabla_H U_\zeta|}dv\right)^{-1}\\ &= \left(\int_{\partial\Lambda^*_{\zeta,s}}\frac{\nabla_{\mathrm{v}_i}U_\zeta^*}{|\nabla_H U_\zeta^*|}dv\right)^2\left(\int_{\partial\Lambda^*_{\zeta,s}}\frac{1}{|\nabla_H U_\zeta^*|}dv\right)^{-1} = \frac{|\nabla_{\mathrm{v}_i}U_\zeta^*|^2}{|\nabla_H U_\zeta^*|}v(\partial\Lambda^*_{\zeta,s}).\end{aligned}$$

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The last equality follows from the fact that $\nabla_{\mathrm{v}_i} U_\zeta^*$ and $|\nabla_H U_\zeta^*|$ are constant on $\partial\Lambda_{\zeta,s}^*$. Indeed, the length of the gradient of a radial function in $\mathbb{H}^{k+1}$ is still radial. Precisely, let $V : \mathbb{H}^{k+1} \to \mathbb{R}$ be a function of the form $V(r,\eta) = \psi(d(r,\eta))$ where d stands for the hyperbolic distance from some fixed point and ψ is a differentiable function. Then

$$|\nabla_H V(r,\eta)| = |\psi'(d(r,\eta))\nabla_H d(r,\eta)| = |\psi'(d(r,\eta))|, \tag{3.15}$$

because $|\nabla_H d(r,\eta)| = 1$, and thus $|\nabla_H V(r,\eta)|$ is radial.

Now (3.8) is a consequence of (3.13), (3.14), and

$$\int_{\partial\Lambda_{\zeta,s}^*} \frac{|\nabla_{\mathrm{v}_i} U_\zeta^*|^2}{|\nabla_H U_\zeta^*|}\, dv \le \int_{\partial\Lambda_{\zeta,s}} \frac{|\nabla_{\mathrm{v}_i} U_\zeta|^2}{|\nabla_H U_\zeta|}\, dv.$$

In order to prove (3.9), note that by the coarea formula, analogously to (A.1), we have

$$\int_{\Lambda_{\zeta,t}} |\nabla_H U_\zeta|^2 = \int_t^{+\infty} \int_{\partial\Lambda_{\zeta,s}} |\nabla_H U_\zeta| dv\, ds, \tag{3.16}$$

and by Hölder inequality

$$v(\partial\Lambda_{\zeta,s}) \le \left(\int_{\partial\Lambda_{\zeta,s}} |\nabla_H U_\zeta| dv\right)^{1/2} \left(\int_{\partial\Lambda_{\zeta,s}} \frac{1}{|\nabla_H U_\zeta|}\, dv\right)^{1/2}. \tag{3.17}$$

For $\mu(\Lambda_{\zeta,s}^*) = \mu(\Lambda_{\zeta,s})$, the isoperimetric inequality (2.3) implies $v(\partial\Lambda_{\zeta,s}^*) \le v(\partial\Lambda_{\zeta,s})$. Thus, from (3.17) and (3.11) it follows that

$$\int_{\partial\Lambda_{\zeta,s}} |\nabla_H U_\zeta| dv \ge v(\partial\Lambda_{\zeta,s}^*)^2 \left(\int_{\partial\Lambda_{\zeta,s}^*} \frac{1}{|\nabla_H U_\zeta^*|}\, dv\right)^{-1} = \int_{\partial\Lambda_{\zeta,s}^*} |\nabla_H U_\zeta^*| dv.$$

In the last equality we used again the fact that, by (3.15), $|\nabla_H U_\zeta^*|$ is constant on $\partial\Lambda_{\zeta,s}^*$. Now (3.9) follows from (3.16) and the proof of (1.5) is concluded.

The proof of (1.6) follows from

$$\int_{\mathbb{R}^n} u^{\frac{2Q}{Q-2}}\, dx\, dy = \frac{1}{\alpha+1} \int_{\mathbb{S}^{m-1}} \int_{\mathbb{H}^{k+1}} U_\zeta^{\frac{2Q}{Q-2}}\, d\mu\, d\mathcal{H}^{m-1},$$

and from (2.10). □

We recall that $B = \{x \in \mathbb{R}^m : |x| < 1\}$. By Theorem 3.1, competing functions in the class $\mathcal{S}$ optimize inequality (1.1). Theorem 2 is a consequence of (1.1) and of the following theorem.

Theorem 3.2. *Let $m, k \in \mathbb{N}$, $\alpha \ge 0$, $Q = m + k(\alpha+1) > 2$. For any non-negative function $v \in C^1(\overline{B})$, the function u of the form* (1.3) *and such that $u(x,0) = v(x)$, $|x| \le 1$, satisfies the identities*

$$\int_{\mathbb{R}^n} \left\{|\nabla_x u|^2 + w^2 |\nabla_y u|^2\right\} dx\, dy = \frac{\omega_k}{2^k} \int_B \left\{|\nabla_x v|^2 p + v^2 q\right\} dx, \tag{3.18}$$

$$\int_{\mathbb{R}^n} u^{\frac{2Q}{Q-2}}\, dx\, dy = \frac{\omega_k}{2^k} \int_B v^{\frac{2Q}{Q-2}} p\, dx. \tag{3.19}$$

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Here, p and q are the weights (1.7), *and ω_k stands for the surface measure of the unit Euclidean sphere in $\mathbb{R}^{k+1}$.*

Proof. Let U be the function associated with u as in (2.1), $U_\zeta(r,\eta) = U(r\zeta,\eta)$, $\Lambda_t = \{U > t\}$, $\Lambda_{\zeta,t} = \{U_\zeta > t\} \subset \mathbb{H}^{k+1}$, $\Omega_t = \Psi(\Lambda_t)$, with $t > 0$ and $\zeta \in \mathbb{S}^{m-1}$. Here, Ψ is the mapping introduced in (3.2). In order to prove (3.18), we begin with the identity

$$E_t := \int_{\Omega_t} \left\{|\nabla_x u|^2 + w^2|\nabla_y u|^2\right\} dx\, dy = \int_{\mathbb{S}^{m-1}} E_{\zeta,t}\, d\mathcal{H}^{m-1}, \tag{3.20}$$

where

$$E_{\zeta,t} = \int_{\Lambda_{\zeta,t}} \left\{\delta|\nabla_H U_\zeta|^2 + \frac{r^2}{\delta}|\nabla_T U_\zeta|^2 + \frac{2\gamma}{k} r U_\zeta \nabla_R U_\zeta + \beta U_\zeta^2\right\} d\mu.$$

Identity (3.20) is proven in Theorem 3.1, formulae (3.1)–(3.5). The constants β, γ, δ are defined in (3.4). Using the identity

$$\int_{\Lambda_{\zeta,t}} r U_\zeta \nabla_R U_\zeta\, d\mu = \frac{k}{2}\int_{\Lambda_{\zeta,t}} (U_\zeta^2 - t^2)\, d\mu,$$

which is proven in (3.10), we also have

$$E_{\zeta,t} = \int_{\Lambda_{\zeta,t}} \left\{\delta|\nabla_H U_\zeta|^2 + \frac{r^2}{\delta}|\nabla_T U_\zeta|^2 + (\beta+\gamma)U_\zeta^2 - \gamma t^2\right\} d\mu. \tag{3.21}$$

The function U_ζ is radial in the hyperbolic metric about the point $(1,0) \in \mathbb{H}^{k+1}$. In order to exploit this symmetry, it is useful to work with the ball model of the hyperbolic space. Let $\mathbb{B}^{k+1} = \{z \in \mathbb{R}^{k+1} : |z| < 1\}$ and let $S : \mathbb{H}^{k+1} \to \mathbb{B}^{k+1}$ be the isometry defined in (2.6). Notice that $S^{-1} = S$ and then in the coordinates $z = (z_0, z') \in \mathbb{R}^+ \times \mathbb{R}^k$ the components (S_1, S_2) of S^{-1} are

$$S_1(z) = \frac{1 - |z|^2}{(1+z_0)^2 + |z'|^2}, \quad S_2(z) = \frac{-2z'}{(1+z_0)^2 + |z'|^2}.$$

We need the function

$$\sigma(r) = \frac{1-r}{1+r}, \quad \text{with } \sigma'(r) = -\frac{2}{(1+r)^2}, \quad r \neq -1. \tag{3.22}$$

Note that $S(r,0) = (\sigma(r), 0)$.

The function $V_\zeta : \mathbb{B}^{k+1} \to \mathbb{R}$, $V_\zeta(z) = U_\zeta(S(z))$, is radial in the variable z. Moreover, the hyperbolic gradient transforms as

$$|\nabla_H U_\zeta(S(z))|^2 = \frac{(1-|z|^2)^2}{4}|\nabla_z V_\zeta(z)|^2. \tag{3.23}$$

We transform the tangential component $r^2|\nabla_T U_\zeta|^2$. Fix a point $(r,\eta) \in \mathbb{H}^{k+1}$ and let $r_0 \in (0,1]$ be the solution of the equation

$$\frac{(1-r_0)^2}{4r_0} = \frac{(1-r)^2 + |\eta|^2}{4r}. \tag{3.24}$$

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For U_ζ is radial about $(1,0)$, we have $r^2|\nabla_T U_\zeta(r,\eta)|^2 = r_0^2|\nabla_T U_\zeta(r_0,0)|^2$. Letting $z = S(r,\eta)$, by (3.24) we have $|z| = \sigma(r_0)$, which is equivalent to $r_0 = \sigma(|z|)$. Then we find $U_\zeta(r_0,0) = V_\zeta(S(r_0,0)) = V_\zeta(\sigma(r_0),0)$, and hence

$$r^2|\nabla_T U_\zeta(r,\eta)|^2 = r_0^2|\nabla_T V_\zeta(\sigma(r_0),0)|^2 = \sigma(|z|)^2|\nabla_T V_\zeta(|z|,0)|^2. \tag{3.25}$$

The Möbius map S transforms the measure μ into the measure $\frac{2^{k+1}dz}{(1-|z|^2)^{k+1}}$ and the set $\Lambda_{\zeta,t}$ onto the set $Z_{\zeta,t} = \{V_\zeta > t\}$. Taking into account (3.23) and (3.25), identity (3.21) becomes

$$E_{\zeta,t} = \int_{Z_{\zeta,t}} \left\{ \delta \frac{(1-|z|^2)^2}{4} |\nabla_z V_\zeta|^2 + \frac{\sigma(|z|)^2}{\delta} |\nabla_T V_\zeta|^2 + (\beta+\gamma)V_\zeta^2 - \gamma t^2 \right\} \frac{2^{k+1}dz}{(1-|z|^2)^{k+1}}. \tag{3.26}$$

From $V_\zeta(z) = V_\zeta(|z|,0) = U_\zeta(S(|z|,0)) = U_\zeta(\sigma(|z|),0))$ and (3.22), we find

$$\frac{(1-|z|^2)^2}{4} |\nabla_z V_\zeta(z)|^2 = \sigma(|z|)^2 |\nabla_R U_\zeta(\sigma(|z|),0)|^2.$$

Thus, by polar coordinates we obtain from (3.26)

$$E_{\zeta,t} = 2^{k+1} \int_0^{r_{\zeta,t}} \left\{ \sigma(r)^2 \left(\delta|\nabla_R U_\zeta|^2 + \frac{1}{\delta}|\nabla_T U_\zeta|^2 \right) + (\beta+\gamma)U_\zeta^2 - \gamma t^2 \right\} \frac{\omega_k r^k}{(1-r^2)^{k+1}} dr,$$

where $r_{\zeta,t} \in (0,1)$ is a number such that $U_\zeta(\sigma(r_{\zeta,t}),0) = t$ and functions inside the integral are evaluated at $(\sigma(r),0)$. Now the change of variable $\varrho = \sigma(r)$ with $d\varrho = \sigma'(r)dr$ furnishes

$$E_{\zeta,t} = \frac{\omega_k}{2^k} \int_{\varrho_{\zeta,t}}^1 \left\{ \varrho^2 \left(\delta|\nabla_R U_\zeta|^2 + \frac{1}{\delta}|\nabla_T U_\zeta|^2 \right) + (\beta+\gamma)U_\zeta^2 - \gamma t^2 \right\} \frac{(1-\varrho^2)^k}{\varrho^{k+1}} d\varrho, \tag{3.27}$$

where $\varrho_{\zeta,t} = \sigma(r_{\zeta,t})$ and functions inside the integral are evaluated at $(\varrho,0)$.

In the next step we write (3.27) for the function v. By (3.1) with $v(x) = u(x,0)$ (recall also the constants in (3.4)) we have

$$\frac{1}{\delta}\varrho^{\frac{\varrho}{\alpha+1}} |\nabla_x v\big(\varrho^{\frac{1}{\alpha+1}}\zeta\big)|^2 = \varrho^2 \left(\delta|\nabla_R U_\zeta|^2 + \frac{1}{\delta}|\nabla_T U_\zeta|^2 \right) + \beta U_\zeta^2 + \frac{2\gamma}{k}\varrho U_\zeta \nabla_R U_\zeta, \tag{3.28}$$

where functions in the right-hand side are evaluated at $(\varrho,0)$. Identity (3.27) then becomes

$$E_{\zeta,t} = \frac{\omega_k}{2^k} \int_{\varrho_{\zeta,t}}^1 \left\{ \frac{1}{\delta}\varrho^{\frac{\varrho}{\alpha+1}} |\nabla_x v(\varrho^{\frac{1}{\alpha+1}}\zeta)|^2 + \gamma\big(U_\zeta^2 - t^2\big) - \frac{2\gamma}{k}\varrho U_\zeta \nabla_R U_\zeta \right\} \frac{(1-\varrho^2)^k}{\varrho^{k+1}} d\varrho, \tag{3.29}$$

with U_ζ and $\nabla_R U_\zeta$ evaluated at $(\varrho,0)$. Let us consider

$$\begin{aligned} -\frac{2\gamma}{k} \int_{\varrho_{\zeta,t}}^1 U_\zeta \nabla_R U_\zeta \frac{(1-\varrho^2)^k}{\varrho^k} d\varrho &= -\frac{\gamma}{k} \int_{\varrho_{\zeta,t}}^1 \nabla_R (U_\zeta^2 - t^2) \frac{(1-\varrho^2)^k}{\varrho^k} d\varrho \\ &= \frac{\gamma}{k} \int_{\varrho_{\zeta,t}}^1 (U_\zeta^2 - t^2) f(\varrho) d\varrho, \end{aligned} \tag{3.30}$$

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with

$$f(\varrho) = \frac{d}{d\varrho}\frac{(1-\varrho^2)^k}{\varrho^k} = -2k\frac{(1-\varrho^2)^{k-1}}{\varrho^{k-1}} - k\frac{(1-\varrho^2)^k}{\varrho^{k+1}} = f_1(\varrho) + f_2(\varrho). \tag{3.31}$$

Integration by parts in (3.30) produces no boundary contribution because $U_\zeta(\varrho_{\zeta,t}, 0) = t$. The term $\gamma(U_\zeta^2 - t^2)$ in (3.29) is cancelled by the part of (3.30) corresponding to f_2 in (3.31).

We are therefore left with

$$E_{\zeta,t} = \frac{\omega_k}{2^k}\int_{\varrho_{\zeta,t}}^1 \left\{\frac{1}{\delta}|\nabla_x v\big(\varrho^{\frac{1}{\alpha+1}}\zeta\big)|^2\varrho^{\frac{m}{\alpha+1}-1}(1-\varrho^2)^k - 2\gamma v\big(\varrho^{\frac{1}{\alpha+1}}\zeta\big)^2\varrho^{\frac{m-2}{\alpha+1}+1}(1-\varrho^2)^{k-1}\right\}d\varrho$$
$$+ \frac{\gamma\omega_k}{2^{k-1}}R(\zeta, t), \tag{3.32}$$

where

$$R(\zeta, t) = t^2\int_{\varrho_{\zeta,t}}^1 \varrho^{1-k}(1-\varrho^2)^{k-1}d\varrho.$$

We claim that $R(\zeta, t)$ converges to zero as $t \to 0$, uniformly in $\zeta \in \mathbb{S}^{m-1}$. For $U_\zeta(\varrho_{\zeta,t}, 0) = t$ and $U(\varrho\zeta, 0) = \varrho^{\frac{Q-2}{2(\alpha+1)}}u(\varrho^{\frac{1}{\alpha+1}}\zeta, 0)$, by (2.1), the claim follows from the boundedness of u and from

$$\lim_{s\to 0} s^{\frac{Q-2}{\alpha+1}}\int_s^1 \varrho^{1-k}(1-\varrho^2)^{k-1}d\varrho = 0.$$

Letting t tend to zero in (3.32) and performing the change of variable $\varrho = r^{\alpha+1}$ we get (recall (3.4))

$$E_{\zeta,0} := \lim_{t\to 0} E_{\zeta,t} = \frac{\omega_k}{2^k}\int_0^1 \big\{|\nabla_x v(r\zeta)|^2 p(r) + v(r\zeta)^2 q(r)\big\}r^{m-1}dr.$$

Here, consistently with (1.7), we let $p(r) = (1 - r^{2(\alpha+1)})^k$ and $q(r) = (Q-m)(Q-2)(1 - r^{2(\alpha+1)})^{k-1}r^{2\alpha}$. Integrating $E_{\zeta,0}$ over $\mathbb{S}^{m-1}$ and using the coarea formula (polar coordinates), we complete the proof of (3.18).

The proof of (3.19) is much easier and it is left to the reader. □

4. Existence and Even Symmetry of Extremals

In this section we prove Theorem 3. First, we transform inequality (1.8) into a Sobolev inequality of the Riemannian type. Then, in Theorem 4.2, we prove existence of extremals for inequality (1.9) and hence for inequality (1.1), in the case $m = 1$. Finally, in Theorem 4.5, we show that extremals are even in the variable x. Note that outside the class $\mathcal{S}$ there can be no extremals for (1.1), because, by Theorem 4 in Monti and Morbidelli (2006), any positive solution to equation (1.2) must be in $\mathcal{S}$, up to scaling and translation in y. The important technical point in this section is Proposition 4.1, which provides the functional transformation suitable to proving existence and symmetry of extremals. The coefficient functions and the constants appearing in this proposition will play a fundamental role.

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We introduce in $M = \{\xi \in \mathbb{R}^m : |\xi| < \tau\}$ a Riemannian structure with "axial symmetry". The number $\tau > 1$ will be fixed later. For $v \in C^1(M)$ let

$$|\nabla_M v|^2 = |\nabla_R v|^2 + \frac{1}{\kappa^2}|\nabla_T v|^2, \tag{4.1}$$

where $\nabla_R v$ and $\nabla_T v$ are the radial and tangential components of the Euclidean gradient of v and κ is a positive function. Let σ be the measure on M

$$\sigma = \kappa^{m-1}\mathcal{L}^m, \tag{4.2}$$

where $\mathcal{L}^m$ is Lebesgue measure in M.

Now consider the unit ball $B = \{x \in \mathbb{R}^m : |x| < 1\}$. Given $u \in C^1(\overline{B})$, define the function $v \in C^1(M)$ by letting

$$u(x) = \varphi(|x|)\, v\left(\psi(|x|)\frac{x}{|x|}\right), \tag{4.3}$$

where $\varphi, \psi : [0, 1) \to [0, +\infty)$ are the functions

$$\varphi(t) = (1 - t^{2(\alpha+1)})^{-\frac{Q-2}{2(\alpha+1)}} \quad \text{and} \quad \psi(t) = \int_0^t (1 - s^{2(\alpha+1)})^{-\frac{1}{\alpha+1}} ds. \tag{4.4}$$

The function ψ can be continuously extended to $[0, 1]$ and it is strictly increasing on this interval. Now we fix $\tau := \psi(1) > 1$ and we fix the function κ by letting

$$h = |x|\frac{\psi'}{\psi} \quad \text{and} \quad \kappa(\xi) = h(\psi^{-1}(|\xi|)), \ \ |\xi| < \tau. \tag{4.5}$$

We are ready to transform the left- and right-hand side of inequality (1.8). We draw the reader's attention to the function G and to the constant ϑ in line (4.7) below. The qualitative behaviour of G and the sign of ϑ will play an important role.

Proposition 4.1. *Let $m, k \geq 1$ and $\alpha > 0$. Given a non-negative function $u \in C^1(\overline{B})$, let v be defined as in* (4.3). *Then we have the identities*

$$\int_B \{|\nabla_x u|^2 p + u^2 q\} dx = \varepsilon_k \int_{\partial B} u^2 d\mathcal{H}^{m-1} + \int_M \{|\nabla_M v|^2 + F v^2\} d\sigma, \tag{4.6}$$

where $\varepsilon_k = Q - 2$ if $k = 1$ and $\varepsilon_k = 0$ otherwise, and $F(\xi) = G(\psi^{-1}(|\xi|))$ with

$$G(|x|) = \frac{\vartheta |x|^{2\alpha}}{(1 - |x|^{2(\alpha+1)})^{\frac{2\alpha}{\alpha+1}}}, \quad \vartheta = (Q-2)(k(\alpha+1) - 2\alpha - m); \tag{4.7}$$

and

$$\int_B u^{\frac{2Q}{Q-2}} p dx = \int_M v^{\frac{2Q}{Q-2}} d\sigma. \tag{4.8}$$

Proof. If u and v are related as in (4.3), then by a short computation we find

$$|\nabla_x u|^2 = \varphi^2 \psi'^2 |\nabla_R v|^2 + \frac{1}{|x|^2}\varphi^2\psi^2|\nabla_T v|^2 + 2\varphi\varphi'\psi' v \nabla_R v + \varphi'^2 v^2, \tag{4.9}$$

with v, $\nabla_R v$, $\nabla_T v$ evaluated at $\psi(|x|)x/|x|$.

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We explain how φ, ψ, and h can be determined. In order that the first two terms in the right-hand sides of (4.9) have the form (4.1), we require

$$\frac{1}{|x|^2}\varphi^2\psi^2 = \frac{1}{h^2}\varphi^2\psi'^2. \tag{4.10}$$

Expression (4.9) is multiplied by p, added to u^2q and finally integrated over the unit ball B. In the integral, we have to perform the change of variable $\xi = \psi(|x|)x/|x| = \Phi(x)$, which has the Jacobian determinant

$$|\det J\Phi| = \frac{\psi^{m-1}\psi'}{|x|^{m-1}}. \tag{4.11}$$

In order to get the measure (4.2) after this change of variable, we also require

$$\frac{\varphi^2\psi'^2 p|x|^{m-1}}{\psi^{m-1}\psi'} = h^{m-1}. \tag{4.12}$$

Analogously, the condition producing the measure (4.2) after the change of variable $\xi = \Phi(x)$ in (4.8) is

$$\frac{\varphi^{2^*} p|x|^{m-1}}{\psi^{m-1}\psi'} = h^{m-1}. \tag{4.13}$$

Here and in the following, we let $2^* = \frac{2Q}{Q-2}$.

Now, (4.10), (4.12), and (4.13) determine φ, ψ, and h. Indeed, comparing (4.12) and (4.13) we find

$$\varphi = \psi'^{\frac{2}{2^*-2}}. \tag{4.14}$$

From (4.10) we get

$$h = |x|\frac{\psi'}{\psi}. \tag{4.15}$$

Note that, with such an h, the Jacobian determinant (4.11) is

$$|\det J\Phi| = \psi'^m h^{1-m}. \tag{4.16}$$

Replacing (4.15) and (4.14) into (4.12), we determine

$$\psi'(|x|) = p^{-(2-m+4/(2^*-2))^{-1}} = p^{-\frac{1}{k(\alpha+1)}} = \left(1-|x|^{2(\alpha+1)}\right)^{-\frac{1}{\alpha+1}}. \tag{4.17}$$

With $\psi(0) = 0$, ψ is also determined. Note that

$$\tau = \psi(1) = \int_0^1 \left(1-t^{2(\alpha+1)}\right)^{-\frac{1}{\alpha+1}} dt \tag{4.18}$$

is finite for any $\alpha > 0$. From (4.14), we have

$$\varphi(|x|) = p^{-\frac{Q-2}{2k(\alpha+1)}} = \left(1-|x|^{2(\alpha+1)}\right)^{-\frac{Q-2}{2(\alpha+1)}}. \tag{4.19}$$

Finally, h is determined by (4.15). It is not difficult to check that $h(0) = 1$.

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By (4.9), (4.10), (4.12), and (4.15) we obtain

$$\begin{aligned} E &:= \int_B \left\{|\nabla_x u|^2 p + u^2 q\right\} dx \\ &= \int_B \left\{\left(|\nabla_R v|^2 + h^{-2}|\nabla_T v|^2\right)\psi'^m + p\varphi\varphi'\nabla_R(v^2) + v^2\left(\varphi'^2 p + \varphi^2 q\right)\right\} dx, \end{aligned} \tag{4.20}$$

where $\nabla_R(v^2) = 2\psi' v \nabla_R v$ and $v, \nabla_R v$ are evaluated at $\psi(|x|)x/|x|$. We are going to integrate by parts the term with $\nabla_R(v^2)$

$$\begin{aligned} \int_B p\varphi\varphi'\nabla_R(v^2)dx &= \int_{\mathbb{S}^{m-1}} \int_0^1 p\varphi\varphi' t^{m-1}\nabla_R\left(v^2\right)dt\, d\mathcal{H}^{m-1} \\ &= \int_{\mathbb{S}^{m-1}} L(x)d\mathcal{H}^{m-1} - \int_B \left((p\varphi\varphi')' + (m-1)\frac{p\varphi\varphi'}{|x|}\right)v^2 dx, \end{aligned} \tag{4.21}$$

where

$$L(x) = \lim_{t\to 1} p(t)\varphi(t)\varphi'(t)v(\psi(t)x)^2, \quad |x| = 1. \tag{4.22}$$

Letting $\lambda(t) = (1 - t^{2(\alpha+1)})$, $t = |x|$, we have

$$p = \lambda^k, \quad q = (Q-m)(Q-2)t^{2\alpha}\lambda^{k-1}, \quad \varphi = \lambda^b, \quad \text{with } b = -\frac{Q-2}{2(\alpha+1)}. \tag{4.23}$$

Hence

$$\lambda' = -2(\alpha+1)t^{2\alpha+1}, \quad p' = k\lambda^{k-1}\lambda', \quad \varphi' = b\lambda^{b-1}\lambda', \tag{4.24}$$

and

$$\varphi\varphi' p = b\lambda^{2b+k-1}\lambda', \quad (\varphi\varphi' p)' = b\lambda^{2b+k-2}\left((2b+k-1)\lambda'^2 + \lambda\lambda''\right). \tag{4.25}$$

Using (4.3) and (4.23)–(4.24), we can compute $L(x)$ in (4.22) for $|x| = 1$

$$L(x) = (Q-2)\lim_{t\to 1}\lambda(t)^{k-1}u(tx)^2 = \begin{cases} (Q-2)u(x)^2 & \text{for } k = 1, \\ 0 & \text{for } k > 1. \end{cases} \tag{4.26}$$

Taking into account (4.21), the coefficient of v^2 in (4.20) is

$$P := \varphi'^2 p + \varphi^2 q - (p\varphi\varphi')' - (m-1)p\varphi\varphi'/t.$$

Here and in the following, $t = |x|$. By (4.23)–(4.25) we find

$$\begin{aligned} P &= b\lambda z^{k+2b-2}\left\{(1-b-k)\lambda'^2 - \left(2(\alpha+1)(Q-m)t^{2\alpha} + \lambda'' + (m-1)\lambda'/t\right)\lambda\right\} \\ &= 2(\alpha+1)bt^{2\alpha}\lambda^{k+2b-2}\left\{2(\alpha+1)(1-b-k)t^{2(\alpha+1)} + (2\alpha+m-k(\alpha+1))(1-t^{2(\alpha+1)})\right\}. \end{aligned}$$

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By the definition of b in line (4.23), it can be checked that $2(\alpha+1)(1-b-k) = 2\alpha + m - k(\alpha+1)$ and we have

$$P = \frac{\vartheta |x|^{2\alpha}}{\left(1-|x|^{2(\alpha+1)}\right)^{\frac{m+2\alpha}{\alpha+1}}}, \quad \text{with } \vartheta = (Q-2)(k(\alpha+1)-2\alpha-m). \tag{4.27}$$

The function

$$G = \frac{P}{\psi'^m} = \frac{\vartheta |x|^{2\alpha}}{(1-|x|^{2(\alpha+1)})^{\frac{2\alpha}{\alpha+1}}}$$

is the one in (4.7). Now, by (4.21) and (4.26), (4.20) becomes

$$E = \varepsilon_k \int_{|x|=1} u^2 d\mathscr{H}^{m-1} + \int_B \left\{ \left|\nabla_R v\right|^2 + h^{-2}\left|\nabla_T v\right|^2 + Gv^2 \right\} \psi'^m dx,$$

and after the change of variable $\xi = \Phi(x)$ with Jacobian determinant (4.16), and with κ and F as in the statement of the proposition, we get (4.6).

Finally, (4.8) easily follows from (4.3), (4.13), and (4.11). □

We begin the proof of Theorem 3. Now let $m = 1$. Our first task is to prove existence of extremal functions for inequality (1.9). We introduce the following norm of a function $u \in C^1([-1,1])$

$$\|u\| = \left(\int_{-1}^{+1} \left\{ |u'|^2 p + |u|^2 q \right\} dx \right)^{1/2},$$

where p and q are the functions (1.7) with $m = 1$. Clearly, the norm $\|\cdot\|$ origins from a scalar product. Denote by H^1 the Hilbert space obtained completing $C^1([-1,1])$ in the norm $\|\cdot\|$. Vectors in this Sobolev space are measurable functions u with measurable distributional derivative u' satisfying $\|u\| < +\infty$.

Consider the following minimum problem

$$\frac{1}{c'_{1,k,\alpha}} = \inf\left\{ \|u\|^2 : u \in H^1, \int_{-1}^{+1} |u|^{\frac{2Q}{Q-2}} p\, dx = 1 \right\}. \tag{4.28}$$

Theorem 4.2. *The infimum* (4.28) *is attained in* H^1.

Proof. Let $(u_h)_{h\in\mathbb{N}}$ be a minimizing sequence for (4.28)

$$\lim_{h\to\infty} \|u_h\|^2 = \frac{1}{c'_{1,k,\alpha}} \quad \text{and} \quad \int_{-1}^{+1} |u_h|^{\frac{2Q}{Q-2}} p\, dx = 1, \quad h \in \mathbb{N}. \tag{4.29}$$

We can assume without loss of generality that $u_h \in C^1([-1,1])$ and $u_h(\pm 1) > 0$. Indeed, we can work with non-negative functions $u_h \geq 0$. Moreover, if $u_h(1) = 0$ or $u_h(-1) = 0$ we replace u_h with $au_h + b$, where $b > 0$ is close to zero and $a \in (0,1)$ is close to 1 as we wish and such that the integral constraint is maintained. The error in the norm can be made arbitrarily small. By weak compactness, there is a

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subsequence—which is still denoted by u_h—weakly converging to a function $u \in H^1$. The norm is lower semicontinuous for the weak convergence

$$\|u\| \leq \liminf_{h\to\infty} \|u_h\|. \tag{4.30}$$

Possibly extracting a new subsequence, we can also assume that $u_h(x) \to u(x)$ for almost every $x \in (-1, 1)$. If u satisfies the integral constraint

$$\int_{-1}^{+1} u^{\frac{2Q}{Q-2}} p\, dx = 1, \tag{4.31}$$

i.e., there is no concentration of measure, then from (4.29)–(4.31) it follows that u is a minimizer for (4.28).

In order to prove (4.31), consider for any $h \in \mathbb{N}$ the function $v_h \in C^1(-\tau, \tau)$ associated with u_h as in (4.3). Note that v_h is continuous on $[-\tau, \tau]$ and $v_h(\pm\tau) = 0$. The number $\tau > 1$ is defined in (4.18). By Proposition 4.1, we have the identities

$$\int_{-1}^{+1} u_h^{\frac{2Q}{Q-2}} p\, dx = \int_{-\tau}^{+\tau} v_h^{\frac{2Q}{Q-2}} d\xi, \tag{4.32}$$

and

$$\int_{-1}^{+1} \{{u_h'}^2 p + u_h^2 q\} dx = \varepsilon_k \{u_h(-1)^2 + u_h(1)^2\} + \int_{-\tau}^{+\tau} \{{v_h'}^2 + F v_h^2\} d\xi, \tag{4.33}$$

where $\varepsilon_k = Q - 2$ if $k = 1$ and $\varepsilon_k = 0$ otherwise. F is the function defined in the statement of Proposition 4.1. Examining the constant ϑ in (4.7), we notice that with $m = 1$ we have

$$F \leq 0 \text{ for } k = 1, \quad F \geq 0 \text{ for } k \geq 2.$$

We must discuss separately the cases $k = 1$ and $k \geq 2$.

We study the case $k = 1$. By continuity, for any $h \in \mathbb{N}$ there is a point $\xi_h \in [-\tau, \tau]$ such that $v_h(\xi_h) = \max_{[-\tau,\tau]} v_h := M_h$. Let z_h be the piecewise affine function such that $z_h(\pm\tau) = 0$ and $z_h(\xi_h) = M_h$. Define $\bar{v}_h = \max\{v_h, z_h\}$ and let $\bar{u}_h$ be associated with $\bar{v}_h$ as in (4.3). Note that, with $m = k = 1$, the functions in (4.17) and (4.19) are respectively $\psi'(x) = (1 - x^{2(\alpha+1)})^{-\frac{1}{\alpha+1}}$ and $\varphi(x) = (1 - x^{2(\alpha+1)})^{-\frac{\alpha}{2(\alpha+1)}}$, $x \in (0, 1)$. Then, by Hôpital's rule

$$\lim_{x\to 1^-} \frac{v_h(\psi(x))}{\psi(x) - \tau} = \lim_{x\to 1^-} \frac{\varphi(x)^{-1} u_h(x)}{\psi(x) - \tau} = -\alpha u_h(1) \lim_{x\to 1^-} \left(1 - x^{2(\alpha+1)}\right)^{-\frac{\alpha}{2(\alpha+1)}} = -\infty,$$

because $u_h(1) > 0$. It follows that $v_h \geq z_h$ in some left neighborhood of τ and this proves that $\bar{u}_h(1) = u_h(1)$. The same holds at $x = -1$.

For the L^2 norm of $\bar{v}_h'$ is less than or equal to the L^2 norm of v_h', we have

$$\int_{-\tau}^{+\tau} \{|\bar{v}_h'|^2 + F\bar{v}_h^2\} d\xi \leq \int_{-\tau}^{+\tau} \{|v_h'|^2 + F v_h^2\} d\xi.$$

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Indeed, $F \leq 0$ and $\bar{v}_h \geq v_h$. Analogously,

$$\int_{-\tau}^{+\tau} \bar{v}_h^{\frac{2Q}{Q-2}} d\xi \geq \int_{-\tau}^{+\tau} v_h^{\frac{2Q}{Q-2}} d\xi.$$

Then, possibly renormalizing each $\bar{u}_h$ by a positive factor smaller than 1 (in order that the integral constraint be satisfied), the sequence $(\bar{u}_h)_{h\in\mathbb{N}}$ is a new minimizing sequence for (4.28) and we can without loss of generality assume $u_h = \bar{u}_h$, i.e., $v_h \geq z_h$. Thus, letting $2^* = \frac{2Q}{Q-2}$, we have

$$1 = \left(\int_{-\tau}^{+\tau} v_h^{2^*} d\xi\right)^{1/2^*} \geq \left(\int_{-\tau}^{+\tau} z_h^{2^*} d\xi\right)^{1/2^*} \geq 2^{\frac{1}{2^*}-1}\tau^{\frac{1}{2^*}} M_h,$$

and then $\|v_h\|_\infty$ is uniformly bounded. Now, (4.31) easily follows from (4.32) by dominated convergence.

In the case $k \geq 2$, we have $\varepsilon_k = 0$ and $F \geq 0$ in (4.33), with left-hand side uniformly bounded in $h \in \mathbb{N}$. It follows that the sequence $(v_h)_{h\in\mathbb{N}}$ is bounded in $H_0^1(-\tau, \tau)$, the usual Sobolev space of functions vanishing at the boundary and with derivative in L^2. Then $\sup_{h\in\mathbb{N}} \|v_h\|_\infty < +\infty$ and the proof of (4.31) is concluded as in the case $k = 1$. □

The extension of a function $u \in H^1$ to a function on $\mathbb{R}^n$, $n = 1 + k$, of the form (1.3) produces a function which is in $D^1(\mathbb{R}^n)$, by Theorem 3.2. In view of the minimization problem, this function can also be assumed to satisfy condition ii) in the definition of the class $\mathcal{S}$, by Theorem 3.1. Then, without loss of generality, competing functions $u \in H^1$ for problem (4.28) can also be assumed to satisfy the condition

$$t \mapsto t^{\frac{Q-2}{2}} u(\pm t) \quad \text{is non decreasing for } t \in [0, 1). \tag{4.34}$$

To a minimum $u \in H^1$ for problem (4.28) there corresponds an extremal $u \in D^1(\mathbb{R}^n)$ for inequality (1.1) solving in weak sense equation (1.2). By abuse of notation, we denote the two functions with the same symbol u. By a result due to Serrin, weak solutions to (1.2) are locally bounded. For our purposes, the boundedness near $|x| = 1$ is enough. We state the next Lemma for general m.

Lemma 4.3. *Let $m, k \in \mathbb{N}$, $\alpha > 0$, $Q = m + k(\alpha + 1)$ and $n = m + k$. A non-negative weak solution $u \in D^1(\mathbb{R}^n)$ of equation* (1.2) *is in $L^\infty_{loc}(\Omega)$, with $\Omega = \mathbb{R}^n \backslash \{x = 0\}$.*

Proof. Letting $V = -u^{\frac{4}{Q-2}}$, the function u solves the equation

$$\Delta_x u + w^2 \Delta_y u = Vu.$$

If we show that for some positive $\varepsilon \in (0, 1]$

$$V \in L^{\frac{n}{2-\varepsilon}}_{loc}(\Omega), \tag{4.35}$$

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then the claim follows from Theorems 1 and 2 in Serrin (1964). If $n \geq 3$, we have $u \in L^{\frac{2n}{n-2}}_{loc}(\Omega)$, by the standard Sobolev–Poincaré inequality, and (4.35) follows from

$$\frac{4n}{(Q-2)(2-\varepsilon)} \leq \frac{2n}{n-2},$$

which is satisfied for some positive ε, because $Q > n$ for $\alpha > 0$.

If $n = 2$, i.e., $m = k = 1$, we have $u \in L^{\frac{2Q}{Q-2}}(\mathbb{R}^2)$, by the Sobolev inequality (1.1), and (4.35) follows from

$$\frac{8}{(Q-2)(2-\varepsilon)} \leq \frac{2Q}{Q-2},$$

which is satisfied for some positive ε, because $Q > 2$ for $\alpha > 0$. □

The next Lemma describes the regularity and the boundary behaviour of minimizers of (4.28).

Lemma 4.4. *Let $m = 1$, $k \in \mathbb{N}$ and $\alpha > 0$. A non-negative function $u \in H^1$ satisfying* (4.34) *and solving* (4.28) *is in $C^2(-1,1) \cap C^1([-1,1])$ and moreover $2u'(x) + (Q-2)xu(x) = 0$ for $x = \pm 1$.*

Proof. For it is $u \in H^1$, u is continuous in $(-1,1)$ with weak derivative in $L^2_{loc}(-1,1)$. By Lemma 4.3, we also have $u \in L^\infty(-1,1)$, and then by (4.34) u can be continuously extended to $[-1,1]$. By minimality, u solves in a weak sense the Euler equation for problem (4.28), and precisely (up to a multiplicative constant which can be assumed to be equal to 1 upon changing the value of the integral constraint)

$$(pu')' - qu = -pu^{\frac{Q+2}{Q-2}} \quad \text{in } (-1,1). \tag{4.36}$$

Then $u'' \in L^2_{loc}(-1,1)$ and consequently $u \in C^1(-1,1)$ and thus equation (4.36) implies $u \in C^2(-1,1)$. Higher regularity of u depends on $\alpha > 0$. The function $(pu')'$ and hence the function pu' can be continuously extended to $[-1,1]$. Il must be

$$L := \lim_{x \to 1} p(x)u'(x) = 0.$$

If it were $L > 0$, then $pu' > L/2$ in some left neighborhood of 1 and thus $u(1) = +\infty$, that is not possible. The case $L < 0$ can be excluded in the same way. Integrating (4.36) we obtain

$$u'(x) = \frac{1}{p(x)} \int_x^1 \left\{ pu^{\frac{Q+2}{Q-2}} - qu \right\} dt, \tag{4.37}$$

and by Hôpital's rule there exists

$$u'(1) = \lim_{x \to 1} u'(x) = \lim_{x \to 1} \frac{p(x)u(x)^{\frac{Q+2}{Q-2}} - q(x)u(x)}{p'(x)} = -\frac{Q-2}{2} u(1). \tag{4.38}$$

Analogously, $u'(-1) = \frac{Q-2}{2} u(-1)$. □

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We prove the even symmetry for extremals of (1.1) when $m = 1$. This finishes the proof of Theorem 3. The proof is different according to the case: $k = 1$ or $k \geq 2$. If $k \geq 2$, then the function F introduced in Proposition 4.1 satisfies $F \geq 0$. This makes possible a rearrangement argument. If $k = 1$, then $F \leq 0$, and we can use a uniqueness result for problem (1.10).

Theorem 4.5. *Let $m = 1$, $k \in \mathbb{N}$ and $\alpha > 0$. Any non-negative function $u \in H^1$ solving* (4.28) *is even.*

Proof. We begin with the case $k \geq 2$. We claim that for any non-negative function $u \in C^1([-1, 1])$ there exists an even function $u^* \geq 0$ such that

$$\int_{-1}^{+1} u^{*\frac{2Q}{Q-2}} p dx = \int_{-1}^{+1} u^{\frac{2Q}{Q-2}} p\, dx, \tag{4.39}$$

$$\int_{-1}^{+1} \left\{ |u^{*\prime}|^2 p + u^{*2} q \right\} dx \leq \int_{-1}^{+1} \left\{ |u'|^2 p + u^2 q \right\} dx. \tag{4.40}$$

Moreover, if $u \neq u^*$ the second inequality is strict.

Let v be the function associated with u as in (4.3). Identity (4.6) with $m = 1$ and $k \geq 2$ reads

$$\int_{-1}^{+1} \left\{ u'^2 p + u^2 q \right\} dx = \int_{-\tau}^{+\tau} \left\{ v'^2 + F v^2 \right\} d\xi,$$

where F is the function introduced in Proposition 4.1. Note that F is non-negative, because if $m = 1$ and $k \geq 2$ it is $\vartheta > 0$ in (4.7). Moreover, F is even and strictly increasing on $[0, \tau)$.

Given a measurable set $I \subset (-\tau, \tau)$, let $I^* = (-|I|/2, |I|/2)$ be the interval centered at the origin with length equal to $|I|$, the Lebesgue measure of I. Then

$$\int_I F d\xi = \int_{I \cap I^*} F d\xi + \int_{I \setminus I^*} F d\xi \leq \int_{I \cap I^*} F d\xi + \int_{I^* \setminus I} F d\xi = \int_{I^*} F d\xi, \tag{4.41}$$

because $|I^* \setminus I| = |I \setminus I^*|$ and F on $I^* \setminus I$ is less or equal than F on $I \setminus I^*$. Moreover, inequality in (4.41) is strict as soon as $|I \setminus I^*| > 0$, because of the strict monotonicity of F.

Denote by v^* the symmetric decreasing rearrangement of v. The function v^* is defined by the condition on level sets

$$\{v^* > t\} = \{v > t\}^*, \quad t > 0. \tag{4.42}$$

Note that, by (4.41) and (4.42), we have

$$\int_{-\tau}^{+\tau} F v^{*2} d\xi = \int_0^{+\infty} \int_{\{v^* > t^{1/2}\}} F d\xi\, dt \leq \int_0^{+\infty} \int_{\{v > t^{1/2}\}} F\, d\xi\, dt = \int_{-\tau}^{+\tau} F v^2 d\xi \tag{4.43}$$

and the inequality is strict if $v \neq v^*$. Moreover, the following well-known facts hold

$$\int_{-\tau}^{+\tau} |v^{*\prime}|^2 d\xi \leq \int_{-\tau}^{+\tau} |v'|^2 d\xi, \quad \int_{-\tau}^{+\tau} v^{*\frac{2Q}{Q-2}} d\xi = \int_{-\tau}^{+\tau} v^{\frac{2Q}{Q-2}} d\xi. \tag{4.44}$$

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Let u^* be defined by v^* as in (4.3). Then our claims (4.39)–(4.40) follow from (4.43) and (4.44), as soon as the integration by parts in (4.21) involving v^* and u^* produces no boundary contribution in (4.22), and precisely we need

$$\lim_{x\to\pm 1} p\varphi\varphi' v^*(\psi) = 0, \tag{4.45}$$

where φ and ψ are the functions in (4.4) with $m = 1$. This holds for v (this is (4.26) with $k \geq 2$). By the rearrangement property (4.42), for any $x \in (0, 1)$ there is an $\bar{x} \in (-1, 1)$ such that $|\bar{x}| \in [|x|, 1)$ and $v^*(\psi(x)) \leq v(\psi(\bar{x}))$. For the function $|p\varphi\varphi'|$ is even and increasing on $(0, 1)$, we have $|p\varphi\varphi' v^*(\psi)(x)| \leq |p\varphi\varphi' v^*(\psi)(\bar{x})|$ and the claim (4.45) is proven.

If $k = 1$, we argue in the following way. By Lemma 4.4, a minimizer u of problem (4.28) is in $C^2(-1, 1) \cap C^1([-1, 1])$ and solves the problem

$$\begin{cases} (pu')' - qu + pu^{\frac{Q+2}{Q-2}} = 0 & \text{in } (-1, 1) \\ u > 0 & \text{in } [-1, 1] \\ 2u'(x) + (Q-2)xu(x) = 0 & x = \pm 1. \end{cases} \tag{4.46}$$

By Theorem 6 in Monti and Morbidelli (2006), the solution $u \in C^1([-1, 1]) \cap C^2(-1, 1)$ of problem (4.46) is unique and thus even. The proof of the theorem is complete. □

Appendix

Proof of Proposition 2.1. We begin with (2.12). By the coarea formula (see Federer, 1969, 3.2.12)

$$\mu(\Lambda_{\zeta,t}) = \int_{\Lambda_{\zeta,t}} \frac{1}{r^{k+1}} dr\, d\eta = \int_t^\infty \int_{\partial\Lambda_{\zeta,s}} \frac{1}{|\nabla_{r\eta} U_\zeta|} \frac{d\mathcal{H}^k}{r^{k+1}} ds = \int_t^\infty \int_{\partial\Lambda_{\zeta,s}} \frac{1}{|\nabla_H U_\zeta|} dv\, ds. \tag{A.1}$$

Here, $\nabla_{r\eta} U_\zeta$ denotes the gradient of U_ζ in the variables r and η. Indeed, note that $r|\nabla_{r\eta} U_\zeta| = |\nabla_H U_\zeta|$ and $r^{-k}\mathcal{H}^k = v$. Since $u \in \mathcal{A}$ satisfies (1.4), $|\nabla_H U_\zeta|$ is bounded from below by a positive constant and from (A.1) it follows that $t \mapsto \varphi_\zeta(t)$ is locally Lipschitz for $t > 0$. Upon differentiating (A.1), we get (2.12).

We prove (2.13). Without loss of generality, assume $m = 2$, $\zeta = (1, 0)$ and $\mathrm{v} = (0, 1)$. Let $\zeta_\vartheta = (\cos\vartheta, \sin\vartheta)$, $\vartheta \in [0, 2\pi)$, and write—with abuse of notation—$\Lambda_\vartheta = \Lambda_{\zeta_\vartheta,t}$. Then we have

$$\nabla_{\mathrm{v}} \varphi_t(\zeta) = \lim_{\vartheta\to 0} \frac{\mu(\Lambda_\vartheta) - \mu(\Lambda_0)}{\vartheta}.$$

Let

$$\Omega_\vartheta = \{(r\zeta_s, \eta) \in \mathbb{R}^n : 0 < s < \vartheta, (r, \eta) \in \Lambda_s\},$$
$$S_\vartheta = \{(r\zeta_s, \eta) \in \mathbb{R}^n : 0 < s < \vartheta, (r, \eta) \in \partial\Lambda_s\},$$

and denote by N the exterior unit normal to $\partial\Omega_\vartheta$. The vector field $Y(\xi, \eta) = |\xi|^{-k-2}(-\xi_2, \xi_1, 0)$ is divergence free and $\langle Y, N\rangle = |\xi|^{-k-1}$ on the part of $\partial\Omega_\vartheta$

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corresponding to Λ_ϑ and $\langle Y, N\rangle = -|\xi|^{-k-1}$ on the part of $\partial\Omega_\vartheta$ corresponding to Λ_0. The divergence theorem therefore implies

$$\mu(\Lambda_\vartheta) - \mu(\Lambda_0) = -\int_{S_\vartheta} \langle Y, N\rangle d\mathcal{H}^{k+1},$$

where $\mathcal{H}^{k+1}$ is the $(k+1)$-dimensional Hausdorff measure in $\mathbb{R}^{k+2}$. From this formula we see that $\vartheta \mapsto \mu(\Lambda_\vartheta)$ is Lipschitz. Then, for general $m > 1$, $\zeta \mapsto \varphi_t(\zeta)$ is Lipschitz on $\mathbb{S}^{m-1}$ for any $t > 0$. By Rademacher theorem, φ is differentiable almost everywhere on $\mathbb{S}^{m-1} \times \mathbb{R}^+$.

We claim that

$$\lim_{\vartheta\to 0} \frac{1}{\vartheta} \int_{S_\vartheta} \langle Y, N\rangle d\mathcal{H}^{k+1} = -\int_{\partial\Lambda_{\zeta,t}} \frac{\nabla_{\mathrm{v}} U_\zeta}{|\nabla_H U_\zeta|} dv. \tag{A.2}$$

First of all, note that on S_ϑ we have $N = -\nabla U/|\nabla U|$ and thus

$$\langle Y, N\rangle = -\frac{\langle(-\xi_2, \xi_1), \nabla_\xi U\rangle}{|\xi|^{k+2}|\nabla U|}.$$

It is enough to prove (A.2) in a local form. Consider a $(k+1)$-dimensional surface S_ϑ parameterized by a Lipschitz function $f\colon [0, \vartheta] \times [0, 1]^k \to \mathbb{R}^{k+2}$ of the form $f = (r\cos s, r\sin s, \eta)$ where $r = r(s, \eta)$ is Lipschitz. Let T be the k-dimensional surface parameterized by $g(\eta) = f(0, \eta)$, $\eta \in [0, 1]^k$. Applying the area formula to the integral on S_ϑ (see Federer, 1969, 3.2.3), taking the limit in ϑ, and finally applying the area formula to the integral over T below, we get

$$\lim_{\vartheta\to 0} \frac{1}{\vartheta} \int_{S_\vartheta} \langle Y, N\rangle d\mathcal{H}^{k+1} = -\int_T \frac{\nabla_{\mathrm{v}} U_\zeta}{|\nabla U|} \left(\frac{\det(Jf^T Jf)}{\det(Jg^T Jg)}\right)^{1/2} |\xi|^{-k-2} d\mathcal{H}^k,$$

where Jf and Jg are the differential matrices of f and g, and Jf^T stands for the transposed matrix. Here, Jf is evaluated on T.

A short computation furnishes

$$\det\left(Jf^T Jf\right) = \left|\frac{\partial r}{\partial s}\right|^2 + r^2\left(1 + \left|\frac{\partial r}{\partial \eta}\right|^2\right), \quad \det\left(Jg^T Jg\right) = 1 + \left|\frac{\partial r}{\partial \eta}\right|^2,$$

and moreover, differentiating the identity $U(r\cos s, r\sin s, \eta) = 0$, we find

$$\left|\frac{\partial r}{\partial \eta}\right|^2 = \frac{|\nabla_\eta U|^2}{|\nabla_R U|^2}, \quad \left|\frac{\partial r}{\partial s}\right|^2 = r^2 \frac{|\nabla_T U|^2}{|\nabla_R U|^2}.$$

Thus

$$\left(\frac{\det\left(Jf^T Jf\right)}{\det\left(Jg^T Jg\right)}\right)^{1/2} = |\xi|^2 \frac{|\nabla U|}{|\nabla_H U|}$$

and the claim (A.2) follows. □

Proof of Proposition 2.2. Without loss of generality, we assume $\mu(\Lambda_{\zeta,0}) = +\infty$ for all $\zeta \in \mathbb{S}^{m-1}$. The distribution function $\varphi(\zeta, t) = \mu(\Lambda_{\zeta,t})$ is continuous and strictly

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decreasing in t. The inverse function $\psi_\zeta : [0, +\infty) \to (0, \max U_\zeta]$ is defined by the identity

$$\psi(\zeta, \varphi(\zeta, t)) = t.$$

We write $\psi_\zeta(s) = \psi(\zeta, s)$. Note that from (A.1) it follows that for any $\varepsilon > 0$ there is a constant $C > 0$ such that for $\varepsilon < t_1 < t_2 < \max U_\zeta - \varepsilon$

$$\varphi_\zeta(t_1) - \varphi_\zeta(t_2) \geq C(t_2 - t_1),$$

and thus also ψ_ζ is locally Lipschitz for $s \in (0, +\infty)$.

Now, from the Lipschitz continuity of φ, it easily follows that $\psi : \mathbb{S}^{m-1} \times \mathbb{R}^+ \to \mathbb{R}$ is locally Lipschitz continuous. The function $(r, \zeta, \eta) \mapsto U_\zeta^*(r, \eta) = \psi_\zeta\big(\mu(B_{d(r,\eta)})\big)$ is continuous for $r > 0$ and is locally Lipschitz continuous for $r > 0$ and $(r, \eta) \neq (1, 0)$. The statement of Proposition 2.2 immediately follows from (2.11). □

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