Degenerate Perturbations of a Two-Phase Transition Model

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Received May 15, 2002

We study the $\Gamma$–convergence as $\varepsilon \to 0^+$ of the family of degenerate functionals

$$Q_\varepsilon(u) = \varepsilon \int_\Omega \langle ADu, Du \rangle \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx,$$

where $A(x)$ is a symmetric, non negative $n \times n$ matrix on $\Omega$ (i.e. $\langle A(x)\xi, \xi \rangle \geq 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$) with regular entries and $W : \mathbb{R} \to [0, +\infty)$ is a double well potential having two isolated minimum points. Moreover, under suitable assumptions on the matrix $A$, we obtain a minimal interface criterion for the $\Gamma$–limit functional exploiting some tools of Analysis in Carnot-Carathéodory spaces. We extend some previous results obtained for the non degenerate perturbations $Q_\varepsilon$ in the classical gradient theory of phase transitions.

Keywords: Phase transitions, $\Gamma$-convergence, Carnot-Carathéodory spaces, minimal interface criterion

2000 Mathematics Subject Classification: 49J45, 49Q05, 49Q20

1. Introduction

In this paper we study the variational convergence for a family of anisotropic degenerate perturbations of a non convex functional which arises in the theory of two-phase transitions. Let us consider the family of functionals

$$Q_\varepsilon(u) = \varepsilon \int_\Omega q(x, Du) \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx, \quad \varepsilon > 0,$$

where $\Omega$ is a smooth, bounded open set of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}$, and $W : \mathbb{R} \to [0, +\infty)$ is a double-well potential that supports two phases of the model (i.e. $W$ has two isolated global minimum points). For the sake of simplicity we assume here $W(u) = u^2(1-u)^2$ but $W$ can be more general (see Section 3). The integral perturbation with integrand

*The authors were supported by GNAMPA, project “Analysis in metric spaces and subelliptic equations”, by MURST, Italy, and by University of Trento, Italy. The second author thanks the hospitality of the Department of Mathematics of Jyväskylä University, Finland, where part of the work was done.
function $q : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a term that penalizes the formation of interfaces in the model and it may degenerate in the sense that $q$ could vanish on big parts of $\Omega \times \mathbb{R}^n$.

Functionals of type (1) have arisen in a variety of applications as, for instance, in the study of stable configurations in the context of Van der Waals-Cahn-Hilliard theory of phase transitions (see [16], [33]). This model can be described by a fluid under isothermal conditions which is confined in a bounded container $\Omega$ and whose Gibbs free energy per unit volume is a prescribed non convex function $W$ of the density function $u$. The space of admissible smooth densities is the class

$$\mathcal{A} = \left\{ u : \Omega \to [0, 1] : u \in C^1(\Omega), \int_\Omega u \, dx = V \right\},$$

where $0 < V < |\Omega|$ is the given total mass of the fluid in $\Omega$.

In the classic isotropic model to every density $u$ one can associate the energy

$$\mathcal{E}_\varepsilon(u) = \varepsilon Q_\varepsilon(u)$$

where

$$q(x, \xi) = ||\xi||^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

and $\varepsilon > 0$ is a small parameter (see [33] for a physical motivation and also [1] for a simple nice introduction to the subject). The problem of determining the stable configurations is the study of the variational problem

$$\inf\left\{ \mathcal{E}_\varepsilon(u) : u \in \mathcal{A} \right\},$$

and the mathematical problem is then to study the asymptotic behavior as $\varepsilon \downarrow 0$ of the solutions $u_\varepsilon$ of these problems or equivalently, as the sets of the solutions agree, the ones of the rescaled problems

$$\inf\left\{ Q_\varepsilon(u) : u \in \mathcal{A} \right\}. \quad (P_\varepsilon)$$

A relevant variational convergence which turned out to be very useful to this goal is the $\Gamma-$convergence introduced by De Giorgi (see [20] and [19] for an introduction to this topic). More precisely, the functional $Q_\varepsilon : \mathcal{A} \to [0, +\infty]$ can be extended, with a slight abuse of notation, to a functional $Q_\varepsilon : L^1(\Omega) \to [0, +\infty]$ defined $+\infty$ outside $\mathcal{A}$, and now the variational problem is the existence and characterization of $Q = \Gamma(L^1(\Omega)) - \lim_{\varepsilon \to 0} Q_\varepsilon$.

In the isotropic scalar case, i.e. when $q$ is as in (2), the existence and characterization of the $\Gamma$-limit functional was first conjectured by De Giorgi and Franzoni ([20]). Then, this variational problem was studied in some particular situations by Gurtin ([33]), who also proposed several conjectures (see also [34]). Following a Gurtin’s conjecture and using previous $\Gamma-$convergence arguments contained in [42], Modica ([41]) proved that

$$Q(u) = \begin{cases} 
2\alpha|\partial E|(\Omega) & \text{if } u = \chi_E \in BV(\Omega), \ |E \cap \Omega| = V \\
+\infty & \text{otherwise,}
\end{cases} \quad (3)$$

where $|\partial E|(\Omega)$ is the perimeter of $E$ in $\Omega$, $BV(\Omega)$ is the set of functions with bounded variation in $\Omega$ (see [6]) and

$$\alpha = \int_0^1 \sqrt{W(s)} \, ds, \quad (4)$$
(see also [49]). Let us recall that by a well-known De Giorgi’s result

$$|\partial E|(\Omega) = \mathcal{H}^{n-1}(\partial^*E \cap \Omega)$$

where $\partial^*E \subset \partial E$ is the reduced boundary of $E$ and $\mathcal{H}^{n-1}$ is the $(n - 1)$—dimensional Hausdorff measure in $\mathbb{R}^n$ (see [6]).

Moreover, in [41] Modica also proved the existence of a sequence $(u_{\varepsilon,h})_{h \in \mathbb{N}}$ of solutions of the relaxed problems $(P_{\varepsilon,h})$ strongly converging in $L^1(\Omega)$ as $\varepsilon,h \downarrow 0$ to a function $u_0 = \chi_E$ solution of the geometric problem

$$\inf\{2\alpha \mathcal{H}^{n-1}(\partial^*E \cap \Omega) : \chi_E \in BV(\Omega), |E \cap \Omega| = V\}. \tag{5}$$

In particular, this result yields a “selection criterion” singling out a solution $u_0$ among the infinite collection of the ones of the imperturbated classical physical problem

$$\min \left\{ \int_{\Omega} W(u) \, dx : u \in L^1(\Omega), \int_{\Omega} u \, dx = V \right\} \tag{6}$$

(see [33] for a discussion of the physical meaning of this problem).

These results were generalized by Bouchitté ([14]) and Owen-Sternberg ([47]) to anisotropic functionals $Q$, allowing the function $q$ to be very general but always assuming at least a coercivity property which, in the case when $q$ is a positive quadratic form, i.e.

$$q(x,\xi) = \langle A(x)\xi, \xi \rangle \quad x \in \Omega \text{ and } \xi \in \mathbb{R}^n, \tag{7}$$

with $A(x)$ symmetric $n \times n$ matrix, amounts to the existence of a constant $\lambda_0 > 0$ such that

$$\langle A(x)\xi, \xi \rangle \geq \lambda_0||\xi||^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n. \tag{8}$$

Under this hypothesis Bouchitté proved in [14] that there exists a limit solution $u_0 = \chi_E$ which solves the following geometric problem

$$\inf \left\{ 2\alpha \int_{\Omega \cap \partial E} \langle A(x)\nu_E(x), \nu_E(x) \rangle^{1/2} d\mathcal{H}^{n-1} : \chi_E \in BV(\Omega), |E \cap \Omega| = V \right\} \tag{9}$$

where $\nu_E$ denotes the generalized outward normal to $E$ (see [6]) and $\alpha$ is the constant (4).

The isotropic vector valued-case, i.e. if $u : \Omega \rightarrow \mathbb{R}^p$ and $q : \Omega \times \mathbb{R}^{pm} \rightarrow [0, +\infty)$ is as in (2), was studied by Sternberg ([49]), by Kohn and Sternberg ([38]), by Baldo [9] and by Fonseca and Tartar ([22]). The anisotropic vector-valued case was also studied by Barroso and Fonseca ([10]) and recently by Ambrosio, Colli Franchi and Savaré when a degeneration in the potential $W$ is admitted too ([5]). Moreover, other variations of the functionals $Q_e$ in (1) have been studied by Alberti and Bellettini ([2] and [3]), Alberti, Bouchitté and Seppecher ([4]) and Fonseca and Mantegazza ([21]). Finally, Baldi and Franchi ([8]) informed us of a $\Gamma$–convergence result for the family of functionals $(Q_{\varepsilon})_\varepsilon$ in the special case when $q(x,\xi) = ||\xi||^2 \omega(x)^{1-2/n}$ and $\omega$ is a strong–$A_\infty$ weight on $\mathbb{R}^n$.

In this paper we obtain $\Gamma$–convergence results in the case when $q : \Omega \times \mathbb{R}^n \rightarrow [0, +\infty)$ is a non negative quadratic form, i.e. $q$ is as in (7) but the matrix $A(x)$ is only non negative definite on $\Omega$; in particular (8) may fail. More precisely, suppose that there exists a $m \times n$ matrix $C(x) = [c_{ij}(x)]$ with Lipschitz continuous entries on $\mathbb{R}^n$ such that

$$A(x) = C(x)^T C(x) \quad \text{for all } x \in \Omega, \tag{10}$$
where \( C(x)^T \) denotes the transposed matrix of \( C(x) \), define the \( A-variation \) in \( \Omega \) of a function \( f \in L^1(\Omega) \) as

\[
|Df|_{A}(\Omega) = \sup \left\{ \int_{\Omega} f \text{div}(C^T \psi) \, dx : \psi = (\psi_1, ..., \psi_m) \text{ is such that} \right. \\
\left. C^T \psi \in \text{Lip}_0(\Omega; \mathbb{R}^n), |\psi| \leq 1 \right\}.
\]

Note that \( |Df|_{A}(\Omega) \) does not depend on the particular factorization (10) (see (27), Proposition 2.1 and Remark 2.3). Finally define

\[
\text{BV}_A(\Omega) = \{ f \in L^1(\Omega) : |Df|_{A}(\Omega) < +\infty \}.
\]

In a natural way the \( A-perimeter measure \) in \( \Omega \) of a measurable set \( E \subset \mathbb{R}^n \) is

\[
|\partial E|_{A}(\Omega) = |D\chi_E|_{A}(\Omega). \quad (11)
\]

Now, let \( Q : L^1(\Omega) \rightarrow [0, +\infty] \) be the functional

\[
Q(u) = \begin{cases} 
2\alpha |\partial E|_{A}(\Omega) & \text{if } u = \chi_E \in \text{BV}_A(\Omega), |E \cap \Omega| = V \\
+\infty & \text{otherwise},
\end{cases} \quad (12)
\]

where \( \alpha \) is the constant (4).

Then, under assumption (10) we prove that

\[
Q = \Gamma(L^1(\Omega)) - \lim_{\varepsilon \downarrow 0} Q_{\varepsilon} \quad (13)
\]

for every bounded open set \( \Omega \subset \mathbb{R}^n \) with boundary of class \( C^2 \) (see Theorem 3.5 and Remark 3.6). The proof relies on some preliminary results that have been established in [43].

The result (13) shows that the definition of the \( A- \)perimeter measure \( |\partial E|_{A} \) is stable with respect to \( \Gamma- \)convergence. Indeed, only assumption (10), which is satisfied for instance by all matrices \( A(x) \) with smooth entries (see Lemma 2.2), is needed in order to assure the \( \Gamma- \)convergence result.

Under the weak assumption (10) only, the result (13) does not provide a significative selection criterion to single out preferred solutions among the ones of the limit geometric problem

\[
\inf \{ 2\alpha |\partial E|_{A}(\Omega) : E \subset \mathbb{R}^n, |E \cap \Omega| = V \}, \quad (14)
\]

because a minimizing sequence \( (u_{\varepsilon_h})_{h \in \mathbb{N}} \) of the problems \( (P_{\varepsilon_h}) \) need not be relatively compact in \( L^1(\Omega) \) if \( A \) vanishes on big parts of \( \Omega \).

On the other hand, we are able to prove a selection criterion providing a control to this lack of coerciveness by means of a \textit{Carnot-Carathéodory} (hereafter cc) \textit{distance} \( d \) induced by the matrix \( A \). This results also requires that the geometry of \( \Omega \) be smooth in the metric space \( (\mathbb{R}^n, d) \).

Namely, let \( X(x) = (X_1(x), ..., X_m(x)) \) be the family of Lipschitz continuous vector fields whose coefficients are the rows of the matrix \( C(x) \) in (10), i.e.

\[
X_j(x) = \sum_{i=1}^{n} c_{ji}(x) \partial_i, \quad x \in \mathbb{R}^n, j = 1, ..., m, \quad (15)
\]
and call $X-$subunit a Lipschitz continuous curve $\gamma : [0, T] \to \mathbb{R}^n$ such that

$$\langle \dot{\gamma}(t), \xi \rangle^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 \quad \text{for a.e. } t \in [0, T] \text{ and for all } \xi \in \mathbb{R}^n,$$

denoting $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$. The cc distance between the points $x, y \in \mathbb{R}^n$ is defined as

$$d(x, y) = \inf \{ T \geq 0 : \text{there exists an } X-\text{subunit curve } \gamma : [0, T] \to \mathbb{R}^n \text{ such that } \gamma(0) = x \text{ and } \gamma(T) = y \}.$$  

If the above set is empty put $d(x, y) = +\infty$. If $d$ is finite on $\mathbb{R}^n$ it turns out to be a metric and the metric space $(\mathbb{R}^n, d)$ is called cc space.

Under the hypotheses

(H1) $X$ is a family of Hörmander or Grushin’s type vector fields (see respectively Example 5.1 and Example 5.2 in Section 5) and

(H2) $\Omega$ is a bounded open set of class $C^2$ and a Boman domain in $(\mathbb{R}^n, d)$ (see Definition 5.4)

we prove that the relaxed problem of $(P_\varepsilon)$ has a solution $u_\varepsilon$ in the anisotropic Sobolev space $H^1_X(\Omega)$, the set of functions $f \in L^2(\Omega)$ such that $X_j f \in L^2(\Omega)$ ($j = 1, \ldots, m$) in distributional sense (see (69) and Theorem 4.3). Moreover, a sequence of solutions $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ is relatively compact in $L^1(\Omega)$, and using the $\Gamma-$convergence result (13) we show that, up to a subsequence, it strongly converges in $L^1(\Omega)$ to a solution $u_0 = \chi_E$ of problem (14) (see Theorem 5.8).

We stress that the degeneration makes things deeply different from the coercive case. Indeed, if the matrix $A(x)$ is not positive definite in $\Omega$ the domain of the functional $Q$ defined in (12) may be bigger than the domain of the one in (3). Moreover, Rellich-Kondrachov compactness theorems for anisotropic Sobolev spaces are critical and depend on the cc geometry of the domain $\Omega$.

Finally, a natural question is whether the geometric problem (14) can be translated in a minimum problem involving Hausdorff measures induced by the cc distance $d$. A representation of the perimeter measure $|\partial E|_A$ in terms of Hausdorff measures is in general not possible (see Section 5 Example 5.15 Remark 5.19), but in some special cases such a representation is available (see Section 5 Example 5.9).

We would like to notice that the use of cc metrics to control the lack of coerciveness of a quadratic form is well known in the literature, specially in applications in the setting of degenerate elliptic PDE’s (see, for instance, [25], [26], [23], [24], [17] and references therein). In this paper we show that such metrics can be useful also in the study of some functionals of Calculus of Variations.

We give a short abstract of the paper. In Section 2 we introduce our notation and some preliminary technical results. In Section 3 we prove the $\Gamma-$convergence results for the involved perturbated functionals and in Section 4 we study the asymptotic behavior of their minimizers and minima. Finally, in Section 5 we give some examples where our main results apply.
Acknowledgments. We are grateful to L. Ambrosio, B. Franchi and R. Serapioni for their constant support. We also thank G. Buttazzo for some useful discussions about Theorem 3.5 and L. Capogna for pointing out to us some references about Section 5.

2. Definitions and preliminary results

Let $X = (X_1, ..., X_m)$ be a family of locally Lipschitz continuous vector fields of the form (15). Let us denote the matrix of their coefficients

$$C(x) = [c_{ji}(x)]_{j=1,...,m\atop i=1,...,n},$$

(18)

and let $d_X \equiv d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ be the cc metric induced by $X$ according to (17).

The following $X-$connectivity assumption

(Xc) the metric $d$ is finite and the identity map $\text{Id} : (\mathbb{R}^n, d) \to (\mathbb{R}^n, | \cdot |)$ is a homeomorphism,

will be discussed in Section 5. In this section we shall introduce some functional spaces associated with vector fields and recall some properties of cc spaces.

We denote by $X_j^*$ the operator formally adjoint to $X_j$ in $L^2(\mathbb{R}^n)$, that is the operator which for all $\varphi, \psi \in C^\infty_0(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \varphi X_j \psi \, dx = \int_{\mathbb{R}^n} \psi X_j^* \varphi \, dx.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set. If $f \in C^1(\Omega)$ and $\varphi \in C^1(\Omega; \mathbb{R}^m)$, define the $X-$gradient and $X-$divergence

$$Xf := (X_1 f, ..., X_m f), \quad \text{div}_X(\varphi) := -\sum_{j=1}^m X_j^* \varphi_j.$$

If $1 \leq p \leq \infty$ we can introduce the anisotropic Sobolev space

$$H^{1,p}_X(\Omega) = \{ f \in L^p(\Omega) : \text{there exists } X_j f \in L^p(\Omega) \text{ for } j = 1, ..., m, \text{ in distributional sense} \}.$$

(19)

It is well known that $H^{1,p}_X(\Omega)$ endowed with the norm

$$\|u\|_{H^{1,p}_X(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{j=1}^m \|X_j u\|_{L^p(\Omega)}$$

is a Banach space. We shall write $H^1_\lambda(\Omega) := H^{1,2}_X(\Omega)$.

We introduce the functions with bounded variation with respect to the vector fields $X$. Let

$$F(\Omega; \mathbb{R}^m) := \{ \varphi \in C^1_0(\Omega; \mathbb{R}^m) : |\varphi(x)| \leq 1 \text{ for all } x \in \Omega \},$$

(20)

and if $f \in L^1(\Omega)$ define

$$\|Xf\|_{\Omega} := \sup_{\varphi \in F(\Omega; \mathbb{R}^m)} \int_{\Omega} f \text{div}_X(\varphi) \, dx < +\infty.$$

(21)
The space of the functions with bounded $X-$variation is
\[ \text{BV}_X(\Omega) := \{ f \in L^1(\Omega) : \|Xf\|(\Omega) < +\infty \}. \] (22)

A measurable set $E \subset \mathbb{R}^n$ is of locally finite $X-$perimeter (or an $X-$Caccioppoli set) if $\chi_E \in \text{BV}_X(U)$ for any open set $U \subset \mathbb{R}^n$, namely if
\[ |\partial E|_X(U) := \|X\chi_E\|(U) < +\infty. \] (23)

By means of Riesz representation Theorem one can prove that if $f \in \text{BV}_X(\Omega)$ then $|Xf|$ is a Radon measure on $\Omega$. Moreover, the total variation is lower semicontinuous with respect to the $L^1(\Omega)$ convergence, i.e. if $f, f_k \in L^1(\Omega)$, $k \in \mathbb{N}$, and $f_k \to f$ in $L^1(\Omega)$ then
\[ \liminf_{k \to \infty} |Xf_k|(\Omega) \geq |Xf|(\Omega). \] (24)

Finally, the $X-$perimeter has the following representation. If $E \subset \mathbb{R}^n$ is an $X-$Caccioppoli set with $C^1$ boundary then
\[ |\partial E|_X(\Omega) = \int_{\partial E \cap \Omega} |Cn| d\mathcal{H}^{n-1}, \] (25)

where $n(x)$ is the Euclidean normal to $\partial E$ at $x$ and $C(x)$ is the matrix in (18).

We now recall the definition of the space of functions with bounded variation with respect to a symmetric, non negative matrix, space introduced in [28] (see also [12] for some general motivations in the case when the matrix is is positive definite).

Let $A(x)$ be a symmetric, non negative $n \times n$ matrix defined for $x \in \Omega$. Let $V_x \subset \mathbb{R}^n$ be the range of $A(x)$, i.e. $V_x = \{ A(x)\xi : \xi \in \mathbb{R}^n \}$. Denote the linear map associated with $A(x)$ by $L_x : V_x \to V_x$, i.e. $L_x(\xi) = A(x)\xi$ for all $x \in \Omega$ and $\xi \in V_x$. The map $L_x$ is invertible and it can be easily checked that
\[ |v|_x := \langle v, L_x^{-1}v \rangle^{1/2}, \quad v \in V_x \]
is a norm on $V_x$. Let
\[ F_A(\Omega) := \{ \psi \in \text{Lip}_0(\Omega; \mathbb{R}^n) : \psi(x) \in V_x \text{ and } |\psi(x)|_x \leq 1 \text{ for all } x \in \Omega \}, \] (26)
and define
\[ |Df|_A(\Omega) := \sup_{\psi \in F_A(\Omega)} \int_\Omega f \text{div}(\psi) \, dx, \quad |\partial E|_A(\Omega) := |D\chi_E|_A(\Omega) \] (27)
and
\[ \text{BV}_A(\Omega) := \{ f \in L^1(\Omega) : |Df|_A(\Omega) < +\infty \}. \] (28)

An interesting relation between the spaces $\text{BV}_X(\Omega)$ and $\text{BV}_A(\Omega)$ is given by the following result (see [28, Proposition 2.1.7 and Remark 2.1.8]).

**Proposition 2.1.** If $A(x) = C(x)^T C(x)$ for all $x \in \Omega$ for some $m \times n-$matrix $C$ with locally Lipschitz continuous entries, then $\text{BV}_X(\Omega) = \text{BV}_A(\Omega)$, the total variations in (21) and (27) are equal, and moreover
\[ \|Xf\|(\Omega) = |Df|_A(\Omega) = \inf \left\{ \liminf_{h \to \infty} \int_\Omega \langle ADf_h, Df_h \rangle^{1/2} \, dx : (f_h)_{h \in \mathbb{N}} \subset C^1(\Omega), \right. \]
\[ \left. f_h \to f \text{ in } L^1(\Omega) \right\}. \] (29)
The factorization $A = C^T C$, the matrix $C$ having Lipschitz continuous entries, is not always possible. The following lemma gives a sufficient condition (see for instance [48, Theorem 5.2.3]).

**Lemma 2.2.** Let $A(x)$ be a symmetric, non negative $n \times n$ matrix with entries of class $C^2(\mathbb{R}^n)$ and assume there exists $\Lambda_0 > 0$ such that

$$\left| \langle \frac{\partial^2 A}{\partial x_i^2} (x) \xi, \xi \rangle \right| \leq \Lambda_0 |\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^n \text{ and } i = 1, \ldots, n. \quad (30)$$

Then there exists a symmetric $n \times n$ matrix $C(x)$ with Lipschitz continuous entries such that $A(x) = C(x)^T C(x)$ for all $x \in \mathbb{R}^n$.

**Remark 2.3.** If $A(x) = C(x)^T C(x)$ definition (27) can be equivalently given as

$$|Df|_A(\Omega) = \sup \left\{ \int_{\Omega} f \text{div}(C^T \psi) \, dx : \psi = (\psi_1, \ldots, \psi_m) \text{ is such that } C^T \psi \in \text{Lip}_0(\Omega; \mathbb{R}^n), |\psi| \leq 1 \right\}. \quad (32)$$

Moreover, if $A$ is positive definite on $\Omega$, i.e. there exists a constant $\lambda_0 > 0$ such that

$$\langle A(x)\xi, \xi \rangle \geq \lambda_0 |\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

then $BV_A(\Omega) = BV(\Omega)$ (see [28]). On the other hand, the inclusion $BV(\Omega) \subset BV_A(\Omega)$ always holds but it may be strict (see Remark 5.12).

We turn back to cc metrics and recall some results that will be needed. Consider a cc space $(\mathbb{R}^n, d)$. A function $f : (\mathbb{R}^n, d) \to \mathbb{R}$ is $L$-Lipschitz if

$$|f(x) - f(y)| \leq Ld(x, y) \quad (31)$$

for all $x, y \in \mathbb{R}^n$. In this case we shall write $f \in \text{Lip}(\mathbb{R}^n, d)$. The infimum of the constants $L$ such that (31) holds will be denoted by $\text{Lip}(f)$.

The following coarea formulas were proved in [28], [30], [43].

**Theorem 2.4.** Let $X_1, \ldots, X_m \in \text{Lip}(\mathbb{R}^n; \mathbb{R}^n)$. Then, if $f \in BV_X(\Omega)$

$$\|Xf\|_A(\Omega) = \int_{-\infty}^{\infty} |\partial E_t|_X(\Omega) \, dt, \quad (32)$$

where $E_t = \{ x \in \mathbb{R}^n : f(x) > t \}$.

Moreover, if $X = (X_1, \ldots, X_m)$ satisfies (Xc), then for every $f \in \text{Lip}(\mathbb{R}^n, d)$ and $u \in L^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} u |Xf| \, dx = \int_{-\infty}^{\infty} \left( \int_{\{f = t\}} u \, d\mu_t \right) \, dt, \quad (33)$$

where $\mu_t = |\partial E_t|_X$ is the perimeter measure of the level set $E_t$. 

The following result shows that, in view of those applications which are local in nature, we can always assume the vector fields to be bounded and globally Lipschitz on \( \mathbb{R}^n \). If \( x \in \mathbb{R}^n \) and \( r \geq 0 \) define the open Euclidean and cc ball respectively as

\[
B(x, r) = \{ y \in \mathbb{R}^n : |y - x| < r \} \quad \text{and} \quad B_{\text{cc}}(x, r) = \{ y \in \mathbb{R}^n : d(y, x) < r \}.
\]

If \( K \subset \mathbb{R}^n \) define its Euclidean and cc diameter respectively as

\[
diam(K) = \sup\{|x - y| : x, y \in K\} \quad \text{and} \quad diam_{\text{cc}}(K) = \sup\{d(x, y) : x, y \in K\}.
\]

**Proposition 2.5.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set such that \( \Omega \Subset \Omega_0 := B(x_0, r_0) \) with \( x_0 \in \Omega \) and \( r_0 > 0 \). Let \( X = (X_1, \ldots, X_m) \), \( X_j := \sum_{i=1}^n c_{ji} \partial_i \), be a family of vector fields on \( \Omega_0 \) such that

1. \( X_j \in \text{Lip}(\Omega_0; \mathbb{R}^n) \) for \( j = 1, \ldots, m \);
2. the cc metric \( d \) induced by \( X \) on \( \Omega_0 \) is finite and the map \( \text{Id} : (\Omega_0, d) \to (\Omega_0, | \cdot |) \) is a homeomorphism.

Then there exists a family \( \tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_{m+n}) \) of vector fields on \( \mathbb{R}^n \), \( \tilde{X}_j = \sum_{i=1}^n \tilde{c}_{ji} \partial_i \), and there exists \( L > 0 \) such that

1. \( |\tilde{X}_j(x)| := \left( \sum_{i=1}^n \tilde{c}_{ji}(x)^2 \right)^{1/2} \leq L \) for all \( x \in \mathbb{R}^n \) and \( j = 1, \ldots, m + n \);
2. \( |\tilde{X}_j(x) - \tilde{X}_j(y)| \leq L|x - y| \) for all \( x, y \in \mathbb{R}^n \) and \( j = 1, \ldots, m + n \);
3. \( \tilde{X}(x) = (X_1(x), \ldots, X_m(x), 0, \ldots, 0) \) for all \( x \in \Omega \);
4. hypothesis (\( \tilde{X}c \)) holds;
5. let \( M_0 := \sup_{x \in \Omega_0} |X(x)| \) and assume that

\[
diam(\Omega) < \frac{r_0}{2} \quad \text{and} \quad diam_{\text{cc}}(\Omega) < \frac{r_0}{2M_0}.
\]

Then \( d(x, y) = \tilde{d}(x, y) \) for all \( x, y \in \Omega \).

**Proof.** Fix \( 0 < s < t < 1 \) and define \( \Omega_1 := B(x_0, tr_0) \) and \( \Omega_2 := B(x_0, sr_0) \). We can choose \( s \in (0, 1) \) such that \( \Omega \Subset \Omega_2 \). By the Lipschitz extension theorem we can assume \( c_{ji} \in \text{Lip}(\mathbb{R}^n) \) and denote by \( \Lambda \) a Lipschitz constant for \( X_1, \ldots, X_m \). Define for \( j = 1, \ldots, m \) and \( i = 1, \ldots, n \)

\[
b_{ji}(x) := \max\{-M_0, \min\{M_0, c_{ji}(x)\}\}.
\]

Clearly, \( b_{ji} \in \text{Lip}(\mathbb{R}^n) \), \( |b_{ji}(x)| \leq M_0 \) for all \( x \in \mathbb{R}^n \), and \( b_{ji}(x) = c_{ji}(x) \) for all \( x \in \Omega_0, i = 1, \ldots, n, j = 1, \ldots, m \).

Let \( \varphi \in C^\infty(\mathbb{R}^n) \) be a function such that \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 0 \) on \( \Omega_2 \) and \( \varphi \equiv 1 \) on \( \mathbb{R}^n \setminus \Omega_1 \). Define

\[
\tilde{X}_j(x) = \sum_{i=1}^n b_{ji}(x) \partial_i \quad \text{if} \quad j = 1, \ldots, m \quad \text{and} \quad \tilde{X}_j(x) = \varphi(x) \partial_{j-m} \quad \text{if} \quad j = m + 1, \ldots, m + n.
\]

Let \( L_1 = \max\{1, M_0\} \) and \( L_2 = \max\{\Lambda, \max_{x \in \mathbb{R}^n} |D\varphi(x)|\} \). If we choose \( L = \max\{L_1, L_2\} \) then claims (1), (2) and (3) are verified.
It is easy to check that $\mathbb{R}^n$ is $\tilde{X}$-connected, i.e. that for any couple of points $x, y \in \mathbb{R}^n$ there exists an $\tilde{X}$-subunit curve connecting them. We prove that $(\mathbb{R}^n, \tilde{d})$ and $(\mathbb{R}^n, | \cdot |)$ are homeomorphic. First of all notice that for all $x, y \in \mathbb{R}^n$

$$|x - y| \leq L_1 \tilde{d}(x, y).$$

(34)

Indeed, if $\gamma : [0, T] \to \mathbb{R}^n$ is an $\tilde{X}$-subunit curve such that $\gamma(0) = x$ and $\gamma(T) = y$

$$|x - y| = |\gamma(0) - \gamma(T)| = \left| \int_0^T \dot{\gamma}(s) \ ds \right| \leq \int_0^T |\dot{\gamma}(s)| \ ds \leq L_1 T,$$

as the subunit condition implies

$$|\dot{\gamma}(s)| = \left| \sum_{j=1}^{m+n} h_j(s) \tilde{X}_j(\gamma(s)) \right| \leq L_1.$$

From (34) it follows that the map $\text{Id} : (\mathbb{R}^n, \tilde{d}) \to (\mathbb{R}^n, | \cdot |)$ is continuous. We prove that $\text{Id}^{-1} : (\mathbb{R}^n, | \cdot |) \to (\mathbb{R}^n, \tilde{d})$ is continuous, too. We show that if $|x_h - x| \to 0$ then $\tilde{d}(x_h, x) \to 0$. If $x \in \Omega_0$ we can assume $x_h \in \Omega_0$ for all $h \in \mathbb{N}$, and since $\tilde{d}(x_h, x) \leq d(x_h, x)$, the claim follows from hypothesis (ii). If $x \in \mathbb{R}^n \setminus \Omega_0$ we can assume $x_h \in \mathbb{R}^n \setminus \Omega_1$ for all $h \in \mathbb{N}$. And since $\tilde{d}(x_h, x) \leq |x_h - x|$ if $h$ is large enough, the claim follows.

We prove (5). Since every $X$-subunit curve is also $\tilde{X}$-subunit then $\tilde{d}(x, y) \leq d(x, y)$ for all $x, y \in \mathbb{R}^n$. Fix $s \in (0, 1)$ in such a way that

$$\text{diam}(\Omega) < \frac{sr_0}{2} \quad \text{and} \quad \text{diam}_C(\Omega) < \frac{sr_0}{2M_0},$$

and choose $0 < \varepsilon < (sr_0/(2M_0) - \text{diam}_C(\Omega))$. Let $x, y \in \Omega$. Every $\tilde{X}$-subunit curve $\gamma : [0, T] \to \mathbb{R}^n$ such that $\gamma(0) = x \text{ and } \gamma(T) = y$ with $T \leq d(x, y) + \varepsilon$ is $X$-subunit (with the same coefficients). Indeed

$$TM_0 \leq (\text{diam}_C(\Omega) + \varepsilon)M_0 < \frac{sr_0}{2},$$

and the argument in [37, Lemma 11.1] implies that $|\gamma(t) - x| < sr_0/2$ for all $t \in [0, T]$. Since $|x - x_0| \leq sr_0/2$ it follows that $|\gamma(t) - x_0| < sr_0$, that is $\gamma(t) \in \Omega_2$ for all $t \in [0, T]$. \qed

**Remark 2.6.** From (3) in Proposition 2.5 it follows that if $u \in H^1_\chi(\Omega)$ then $|Xu| = |\tilde{X}u|$ a.e., and hence $H^1_\chi(\Omega) = H^1_{\tilde{X}}(\Omega)$, $p \geq 1$. Analogously, $\|Xu\|(\Omega) = \|\tilde{X}u\|(\Omega)$ for all $u \in \text{BV}_X(\Omega)$ and thus $\text{BV}_X(\Omega) = \text{BV}_{\tilde{X}}(\Omega)$.

**Remark 2.7.** Assume that there exists $L > 0$ such that

$$|X_j(x)| = \left( \sum_{i=1}^{n} c_{ji}(x)^2 \right)^{1/2} \leq L$$

(35)

for all $x \in \mathbb{R}^n$ and $j = 1, ..., m$, and

$$|X_j(x) - X_j(y)| \leq L|x - y|$$

(36)
for all \( x, y \in \mathbb{R}^n \) and \( j = 1, \ldots, m \).

Let \( \sigma > 0 \) and consider the family of vector fields \( X_{\sigma, \eta} = (X_1^\eta, \ldots, X_m^\eta, \sigma \partial_1, \ldots, \sigma \partial_n) \) where

\[
X_j^\eta = \sum_{i=1}^{n} (c_{ji} * J_{\eta}) \partial_i, \quad j = 1, \ldots, m,
\]

and \( (J_{\eta})_{\eta > 0} \) is a family of mollifiers. We claim that

\[
\sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \leq \sigma^2 |\xi|^2 + \sum_{j=1}^{m} \langle X_j^\eta(x), \xi \rangle^2
\]

for all \( x \in \mathbb{R}^n \), for all \( \xi \in \mathbb{R}^n \) and for all \( 0 < \eta \leq \sigma^2/(2mL^2) \), where \( L > 0 \) is a constant such that (35) and (36) hold. Indeed

\[
\sigma^2 |\xi|^2 + \sum_{j=1}^{m} \langle X_j^\eta(x), \xi \rangle^2 = \sigma^2 |\xi|^2 + \sum_{j=1}^{m} \left( \langle X_j^\eta(x) - X_j(x), \xi \rangle - \langle X_j(x), \xi \rangle \right)^2
\]

\[
\geq \sigma^2 |\xi|^2 + \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 - 2|\xi|^2 \sum_{j=1}^{m} |X_j(x)| |X_j^\eta(x) - X_j(x)|
\]

\[
\geq (\sigma^2 - 2mL^2)|\xi|^2 + \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2.
\]

We used \( |X_j^\eta(x) - X_j(x)| \leq L\eta \).

Now let \( \eta_\sigma = \sigma^2/(4mL^2) \) and define

\[
X_{\sigma} = X_{\sigma, \eta_\sigma}.
\]

The coefficients of the vector fields \( X_{\sigma} \) are of class \( C^\infty \) and if \( d_\sigma \) is the cc metric induced by them then the cc space \( (\mathbb{R}^n, d_\sigma) \) is actually a complete Riemannian manifold.

### 3. The results of \( \Gamma \)-convergence

This section deals with the \( \Gamma \)-convergence results. For a comprehensive introduction to \( \Gamma \)-convergence we refer to [19]. We introduce the involved functionals.

Let \( W \in C^2(\mathbb{R}) \) be a function with two “wells” of equal depth

\[
W(0) = W(1) = 0, \quad W(s) > 0 \text{ if } s \neq 0, 1, \quad W''(0) > 0, \quad W''(1) > 0.
\]

Fix a bounded open set \( \Omega \subset \mathbb{R}^n \) and for \( \varepsilon > 0 \) define the functionals \( F_\varepsilon, F : L^1(\Omega) \to [0, +\infty] \)

\[
F_\varepsilon(u) = \left\{ \begin{array}{ll}
\int_{\Omega} \left( \varepsilon |Xu|^2 + \frac{1}{\varepsilon} W(u) \right) dx & \text{if } u \in H^1_X(\Omega) \\
+\infty & \text{if } u \in L^1(\Omega) \setminus H^1_X(\Omega),
\end{array} \right.
\]

and

\[
F(u) = \left\{ \begin{array}{ll}
2\alpha |\partial E|_X(\Omega) & \text{if } u = \chi_E \in BV_X(\Omega) \\
+\infty & \text{otherwise}
\end{array} \right.
\]
where $\alpha = \int_0^1 \sqrt{W(s)} \, ds$.

Let $0 < V < |\Omega|$, introduce the set of admissible functions

$$\mathcal{A}_V = \left\{ u \in L^1(\Omega) : \int_\Omega u \, dx = V, \ 0 \leq u \leq 1 \ \text{a.e. in} \ \Omega \right\},$$

(40)

and let $I_V$ be the indicator function of $\mathcal{A}_V$, i.e. the function which takes the value 0 on $\mathcal{A}_V$ and $+\infty$ outside. Finally, define

$$G_\varepsilon = F_\varepsilon + I_V \quad \text{and} \quad G = F + I_V.$$ 

(41)

Let $(\varepsilon_h)_{h \in \mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$ and let $G_h = G_{\varepsilon_h}$, $F_h = F_{\varepsilon_h}$.

**Theorem 3.1.** Suppose that $X_1, \ldots, X_m \in \text{Lip}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, let $W \in C^2(\mathbb{R})$ be as in (39) and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^2$ boundary. Then

$$G = \Gamma(L^1(\Omega)) \cdot \lim_{h \to \infty} G_h,$$

i.e. by definition

$$\forall u \in L^1(\Omega) \ \text{and} \ \forall (u_h) \subset L^1(\Omega) \ \text{if} \ u_h \to u \ \text{in} \ L^1(\Omega) \ \text{then} \ G(u) \leq \liminf_{h \to \infty} G_h(u_h),$$

(42)

$$\forall u \in L^1(\Omega) \ \exists (u_h) \subset L^1(\Omega) \ \text{such that} \ u_h \to u \ \text{in} \ L^1(\Omega) \ \text{and} \ G(u) \geq \limsup_{h \to \infty} G_h(u_h).$$

(43)

**Remark 3.2.** The $\Gamma$–convergence of the family $(F_\varepsilon)_{\varepsilon > 0}$ to $F$ (with $W(u) = u^2(1 - u)^2$ and without volume constraint) was proved in [43] assuming the regularity of the vector fields $X_1, \ldots, X_m$ and of $\Omega$ ($c_{ji} \in C^\infty(\mathbb{R}^n)$ and $\Omega$ with $C^\infty$ boundary), and finally assuming hypothesis (Xc) and an eikonal equation for the $cc$ metric $d$. Even under all these stronger regularity assumptions Theorem 3.1 is not implied by the results in [43] since the indicator function $I_V$ is not a continuous perturbation of $F_\varepsilon$ in the $L^1(\Omega)$ topology.

We begin with a refinement of the approximation theorem for BV$_X$ functions which is necessary in order to bypass the following technical difficulty. In the Euclidean setting one of the main tools in the approximation of a set of finite perimeter in $\Omega$ by means of sets with regular boundary in $\mathbb{R}^n$ (not only in $\Omega$) is the property of a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ to be extendible to a function $\tilde{u} \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $|D\tilde{u}|(\partial\Omega) = 0$, if $\Omega$ has Lipschitz boundary (see [41, Lemma 1] and [49, Lemma 1]). It is not known if such a property does hold for BV$_X(\Omega)$ functions. Nevertheless, we can prove the following result.

**Proposition 3.3.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^2$ boundary, and let $E \subset \Omega$ be a measurable set such that $|\partial E|_X(\Omega) < +\infty$ and $0 < |E| < |\Omega|$. Then there exists a sequence $(E_h)_{h \in \mathbb{N}}$ of open sets of $\mathbb{R}^n$ such that

(i) $E_h$ is bounded and $\partial E_h$ is of class $C^\infty$ for all $h \in \mathbb{N}$;

(ii) $E_h \to E$ in $L^1(\Omega)$;

(iii) $|\partial E_h|_X(\Omega) \to |\partial E|_X(\Omega)$;

(iv) $H^{n-1}(\partial E_h \cap \partial \Omega) = 0$ for all $h \in \mathbb{N}$;

(v) $|E_h \cap \Omega| = |E|$ for all $h \in \mathbb{N}$.

As a first step we prove the following Lemma.
Lemma 3.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $E \subset \Omega$ be a measurable set such that $|\partial E|_X(\Omega) < +\infty$ and $0 < |E| < |\Omega|$. Then there exists a sequence $(E_h)_{h \in \mathbb{N}}$ of open sets in $\mathbb{R}^n$ such that

(i) $E_h$ is bounded and $\partial E_h \cap \Omega$ is of class $C^\infty$ for all $h \in \mathbb{N}$;
(ii) $E_h \to E$ in $L^1(\Omega)$;
(iii) $|\partial E_h|_X(\Omega) \to |\partial E|_X(\Omega)$;
(iv) $|E_h \cap \Omega| = |E|$ for all $h \in \mathbb{N}$.

Proof. We first show that it is not restrictive to assume $\text{int}(E) \neq \emptyset$ and $\text{int}(\Omega \setminus E) \neq \emptyset$. Recall the definition of interior in measure of a set $F \subset \mathbb{R}^n$

$$
\text{int}_M(F) = \left\{ x \in \mathbb{R}^n : \text{there exists } \lim_{r \downarrow 0} \frac{|F \cap B(x, r)|}{|B(x, r)|} = 1 \right\}.
$$

Since $0 < |E| < |\Omega|$ from Lebesgue differentiation Theorem there exist $x_1 \in \text{int}_M(\Omega \setminus E)$ and $x_2 \in \text{int}_M(E)$. Let $r_0 := \min\{\text{dist}(x_1, \partial \Omega), \text{dist}(x_2, \partial \Omega), |x_1 - x_2|\}$, and if $0 \leq r_1, r_2 < r_0$ define

$$
\varphi(r_1, r_2) = |(E \cup B(x_1, r_1)) \setminus B(x_2, r_2)| - |E|.
$$

If $0 < r < r_0$ then

$$
\varphi(r, 0) = |E \cup B(x_1, r)| - |E| > 0,
\quad \varphi(0, r) = |E \setminus B(x_2, r)| - |E| < 0.
$$

Since $\varphi$ is continuous, for all $0 < r < r_0$ there exists $\alpha_r \in (0, 1)$ such that $\varphi(\alpha_r r, (1 - \alpha_r)r) = 0$. Define

$$
E_r = (E \cup B(x_1, \alpha_r r)) \setminus B(x_2, (1 - \alpha_r)r),
$$

and notice that $\text{int}(E_r) \neq \emptyset$, $\text{int}(\Omega \setminus E_r) \neq \emptyset$, $|E_r \Delta E| \leq 2\omega_n r^n$, $|E_r| = |E|$ and

$$
|\partial E_r|_X(\Omega) \leq |\partial E|_X(\Omega) + |\partial B(x_1, \alpha_r r)|_X(\mathbb{R}^n) + \left|\partial B(x_2, (1 - \alpha_r)r)|_X(\mathbb{R}^n) \right|
\leq |\partial E|_X(\Omega) + C r^{n-1}.
$$

These inequalities and the lower semicontinuity of the perimeter with respect to the convergence $E_r \to E$ in $L^1(\Omega)$ as $r \downarrow 0$ imply

$$
|\partial E|_X(\Omega) \leq \liminf_{r \downarrow 0} |\partial E_r|_X(\Omega) \leq \limsup_{r \downarrow 0} |\partial E_r|_X(\Omega) \leq |\partial E|_X(\Omega),
$$

and thus equalities hold and $|\partial E_r|_X(\Omega) \to |\partial E|_X(\Omega)$.

We now turn to the proof of the lemma. There exist $x_1 \in E$, $x_2 \in \Omega \setminus E$ and $r_0 > 0$ such that

$$
B_1 = B(x_1, r_0) \subset E, \quad B_2 = B(x_2, r_0) \subset \Omega \setminus E.
$$

Using the same notation as in [28, Theorem 2.2.2] write $u = \chi_E$ and let $\Omega_i = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \frac{1}{k + 1}\}$ for $i \in \mathbb{N}$. If $k$ is sufficiently large we can assume that $\Omega_k$ is such that $B_1 \cup B_2 \subset \Omega_k \subset \Omega$. There exists a sequence $(u_h)_{h \in \mathbb{N}} \subset C^\infty(\Omega)$ such that

$$
u_h \to u \text{ in } L^1(\Omega) \quad \text{and} \quad \lim_{h \to \infty} \int_{\Omega} |X u_h(x)| \, dx = |\partial E|_X(\Omega).
$$
Such functions may assumed to be of the form

$$u_h = \sum_{i=0}^{\infty} (u \varphi_i) \ast J_{\epsilon_i},$$

where \((J_{\epsilon})_{\epsilon>0}\) is a family of mollifiers, \(\epsilon_i\) depend on \(h\) and are small, and \((\varphi_i)_{i\in\mathbb{N}}\) is a partition of unity of \(\Omega\) subordinate to the covering \(\{\Omega_{i+1} \setminus \overline{\Omega}_{i-1} : i \in \mathbb{N}\}\) \((\Omega_{-1} = \emptyset)\). In particular \(\varphi_0 \equiv 1\) on \(\Omega_0\) and \(\varphi_i \equiv 0\) on \(\Omega_0\) if \(i \geq 1\). Moreover, we can choose \(\epsilon_i\) sufficiently small in order that \(\text{supp}((u \varphi_i) \ast J_{\epsilon_i}) \subset \Omega \setminus \overline{\Omega}_0\) for all \(i \geq 1\).

If \(h \in \mathbb{N}, \epsilon_0 < r_0/2\) and \(x \in B(x_1, r_0/2) \cup B(x_2, r_0/2)\) then

$$u_h(x) = \sum_{i=0}^{\infty} ((u \varphi_i) \ast J_{\epsilon_i})(x) = ((u \varphi_0) \ast J_{\epsilon_0})(x) = u(x). \quad (44)$$

For suitable sequences \((h_i)_{i \in \mathbb{N}}\) and \((t_i)_{i \in \mathbb{N}}\) the sets \(\widehat{E}_k = \{x \in \Omega : u_{\lambda_k}(x) > t_k\}\) are regular and verify

$$\widehat{E}_k \rightarrow E \text{ in } L^1(\Omega) \quad \text{and} \quad \lim_{k \rightarrow \infty} |\partial \widehat{E}_k|_X(\Omega) = |\partial E|_X(\Omega). \quad (45)$$

This can be proved exactly as in [43, Theorem 7.1].

The sets \(\widehat{E}_k\) can be modified in order that the volume constraint be satisfied. Let \(\lambda_k = |\widehat{E}_k| - |E|\) and define

$$E_k = \begin{cases} \widehat{E}_k \setminus B(x_1, r_k) & \text{if } \lambda_k > 0 \\ \widehat{E}_k & \text{if } \lambda_k = 0 \\ \widehat{E}_k \cup B(x_2, r_k) & \text{if } \lambda_k < 0, \end{cases}$$

where \(r_k > 0\) is such that \(|B(x_1, r_k)| = |B(x_2, r_k)| = |\lambda_k|\).

We show that \(|E_k \cap \Omega| = |E|\). Notice that

$$|\lambda_k| \leq |(\widehat{E}_k \Delta E) \cap \Omega| \rightarrow 0 \quad \text{as } k \rightarrow \infty, \quad (46)$$

and therefore \(\lim_{k \rightarrow \infty} r_k = 0\). For \(k\) sufficiently large we can assume \(r_k < r_0/2\). Moreover, by (44) \(B(x_1, r_0/2) \subset E_k\) and \(B(x_2, r_0/2) \subset \Omega \setminus E_k\), whence

$$|E_k| = |\widehat{E}_k| - |B(x_1, r_k)| = |E| \quad \text{if } \lambda_k > 0, \quad |E_k| = |\widehat{E}_k| + |B(x_2, r_k)| = |E| \quad \text{if } \lambda_k < 0.$$

This proves (iv). From (46) we also get (ii). Indeed

$$|(E_k \Delta E) \cap \Omega| \leq (\widehat{E}_k \Delta E_k) \cap \Omega| + |\widehat{E}_k \Delta E| \leq |\lambda_k| + |(\widehat{E}_k \Delta E) \cap \Omega| \rightarrow 0.$$

Finally notice that

$$|\partial E_k|_X(\Omega) = |\partial \widehat{E}_k|_X(\Omega) + \int_{\partial B(x_i, r_k)} |Cn| \, d\mathcal{H}^{n-1}$$

for \(i = 1\) or \(i = 2\), where \(n\) is the Euclidean normal to \(\partial B(x_i, r_k)\) and \(C\) is the matrix \(C(x) = [(c_{ij}(x))]\). From (45) and \(r_k \rightarrow 0\) we get (iii). \(\square\)
Proof of Proposition 3.3. By Lemma 3.4 we can assume without loss of generality that $E \subset \Omega$ is an open set such that $\partial E \cap \Omega$ is of class $C^\infty$. We shall divide the proof in two steps.

**Step 1.** Assume that $|\partial E|_X(\partial \Omega) = 0$. In this case

$$|\partial E|_X(\mathbb{R}^n) = |\partial E|_X(\Omega) + |\partial E|_X(\partial \Omega) + |\partial E|_X(\mathbb{R}^n \setminus \Omega)$$

$$= |\partial E|_X(\Omega) < +\infty.$$ 

Let $(J_\varepsilon)_{\varepsilon > 0}$ be a family of mollifiers, write $u = \chi_E$ and define $u_\varepsilon = u * J_\varepsilon$. From [28, Theorem 2.2.2] it follows that $u_\varepsilon \to u$ in $L^1(\mathbb{R}^n)$, $\lim_{\varepsilon \to 0} |\{ x \in \mathbb{R}^n : |u_\varepsilon(x) - u(x)| \geq \eta \}| = 0$ for any $\eta > 0$ and $\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} |Xu_\varepsilon(x)| \, dx = |\partial E|_X(\mathbb{R}^n)$. Moreover, since $|\partial E|_X(\partial \Omega) = 0$ we also have

$$\lim_{\varepsilon \to 0} \int_{\Omega} |Xu_\varepsilon| \, dx = |\partial E|_X(\Omega).$$

Thus we can proceed exactly as in [41, Lemma 1, proof] replacing the gradient $\nabla$ with $X$, the Euclidean perimeter $|\partial E|$ with $|\partial E|_X$ and taking into account the coarea formula (33).

**Step 2.** Assume that $|\partial E|_X(\partial \Omega) > 0$. We show that for all $\varepsilon > 0$ there exists an open set $E_\varepsilon \subset \mathbb{R}^n$ such that $|\partial E_\varepsilon|_X(\Omega) < \infty$, $E_\varepsilon \cap \Omega = |E|$, $|\partial E_\varepsilon|_X(\partial \Omega) = 0$ and

$$\lim_{\varepsilon \to 0} |(E_\varepsilon \Delta E) \cap \Omega| = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} |\partial E_\varepsilon|_X(\Omega) = |\partial E|_X(\Omega). \quad (47)$$

Because $E$ is open, $\partial E \cap \Omega$ is $C^\infty$ and $0 < |E| < |\Omega|$ there exist $x_1 \in E$, $x_2 \in \Omega \setminus E$ and $r_0 > 0$ such that $B_1 = B(x_1, r_0) \subset E$ and $B_2(x_2, r_0) \subset \Omega \setminus E$. We shall use the technique introduced in [43, Proposition 6.3, Step 3]. For $\varepsilon > 0$ fixed let $0 < t_0$, $\Omega_0$ and $\hat{E}$ be as in [43, (6.8)]. If $\Omega$ is of class $C^2$ then $\partial \hat{E} \cap \Omega$ is of class $C^1$. If $t_0$ and $r_0$ are small enough then

$$B_1 \subset \Omega_0 \cap E \quad \text{and} \quad B_2 \subset \Omega_0 \setminus E. \quad (48)$$

Let $\lambda_\varepsilon = |\hat{E} \cap \Omega| - |E|$ ($\hat{E}$ depends on $\varepsilon$) and define

$$E_\varepsilon = \begin{cases} \hat{E} \setminus B(x_1, r_\varepsilon) & \text{if } \lambda_\varepsilon > 0 \\ \hat{E} & \text{if } \lambda_\varepsilon = 0 \\ \hat{E} \cup B(x_2, r_\varepsilon) & \text{if } \lambda_\varepsilon < 0 \end{cases}$$

where $r_\varepsilon > 0$ is chosen in such a way that $|B(x_1, r_\varepsilon)| = |B(x_2, r_\varepsilon)| = |\lambda_\varepsilon|.$

Since $B_1 \subset \hat{E}$ and $B_2 \subset \Omega \setminus \hat{E}$, arguing as in the proof of Lemma 3.4 we get (ii), (iii) and (iv).

We finally prove that $|\partial E_\varepsilon|_X(\partial \Omega) = 0$. Since $\partial E_\varepsilon \cap \partial \Omega = (\partial \hat{E} \cup \partial B(x_i, r_\varepsilon)) \cap \partial \Omega$ for $i = 1$ or $i = 2$ with $\partial B(x_i, r_\varepsilon) \cap \partial \Omega = \emptyset$, from the definition of $E_\varepsilon$ we get $|\partial E_\varepsilon|_X(\partial \Omega) = |\partial \hat{E}|_X(\partial \Omega) = 0$, because of the definition of $\hat{E}$ (see [43, (6.10)]). □

**Proof of Theorem 3.1.** We divide the proof in two steps.

**Step 1.** Assume that $X_1, \ldots, X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $j = 1, \ldots, m$, and that the family $X = (X_1, \ldots, X_m)$ satisfies hypothesis (Xc) and let $d$ be the induced cc metric. We also assume the following eikonal equation:
Let $K \subset \mathbb{R}^n$ be a closed set. If $d_K(x) := \inf_{y \in K} d(x, y)$ then $X d_K(x) = (X_1 d_K(x), \ldots, X_m d_K(x)) \in \mathbb{R}^m$ exists and $|X d_K(x)| = 1$ for a.e. $x \in \mathbb{R}^n \setminus K$.

Under such hypotheses we shall prove the thesis.

The proof of the lower bound estimate (42) is verbatim contained in [43, Theorem 6.5, proof]. A few modifications will be needed in order to prove the upper bound estimate (43).

By Proposition 3.3 and by [42, Lemma IV] we can reduce to prove (43) for $u = \chi_E, E \subset \mathbb{R}^n$ bounded open set with $C^\infty$ boundary such that $|E \cap \Omega| = V$ and $H^{n-1}(\partial \Omega \cap \partial E) = 0$.

Define $\varrho : \mathbb{R}^n \to [0, +\infty)$

$$
\varrho(x) = \begin{cases} 
\min_{y \in \partial E} d(x, y) & x \in E \\
- \min_{y \in \partial E} d(x, y) & x \in \mathbb{R}^n \setminus E,
\end{cases}
$$

and write $\chi_0(t) = \chi_{(0, +\infty)}(t)$. Then $u(x) = \chi_0(\varrho(x))$ for all $x \in \mathbb{R}^n$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be the maximal solution of the Cauchy problem

$$
\begin{cases} 
\chi'(t) = \sqrt{W(\chi(t))} \\
\chi(0) = \frac{1}{2}.
\end{cases}
$$

It is easy to see that, as $W(0) = W(1) = 0$, $\chi$ is a strictly increasing $C^2$ function such that $\lim_{t \to +\infty} \chi(t) = 1$ and $\lim_{t \to -\infty} \chi(t) = 0$. Moreover, there exist $\bar{t} \in \mathbb{R}, c_1, c_2 > 0$ such that (see [49, (1.21)])

$$1 - \chi(t) \leq c_1 e^{-c_2 t}, \quad \text{for all } t \geq \bar{t}. \quad (49)$$

We follow the proof contained in [43] (see also [12]). Fix $\varepsilon > 0$ and write $t_\varepsilon = \vartheta \varepsilon \log 1/\varepsilon$ where $\vartheta \geq 3$ is a constant that will be determined later. Define the function $\Lambda_\varepsilon : \mathbb{R} \to \mathbb{R}$ in the following way

$$
\Lambda_\varepsilon(t) = \begin{cases} 
\chi(t) & \text{if } 0 \leq t < \frac{t_\varepsilon}{\vartheta} \\
p_\varepsilon(t) & \text{if } \frac{t_\varepsilon}{\vartheta} \leq t < \frac{2t_\varepsilon}{\vartheta} \\
1 & \text{if } t \geq \frac{2t_\varepsilon}{\vartheta} \\
1 - \Lambda_\varepsilon(-t) & \text{if } t < 0,
\end{cases}
$$

where $p_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the uniquely determined polynomial of degree 3 for which $\Lambda_\varepsilon \in C^{1,1}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{\pm t_\varepsilon/\varepsilon, \pm 2t_\varepsilon/\varepsilon\})$ (see [13] for the construction of $p_\varepsilon$) satisfying

$$
|p_\varepsilon - 1|_{L^\infty(t_\varepsilon/\varepsilon, 2t_\varepsilon/\varepsilon)} = O(\varepsilon^{2\vartheta - 1}) \quad \text{and} \quad \|p_\varepsilon'\|_{L^\infty(t_\varepsilon/\varepsilon, 2t_\varepsilon/\varepsilon)} = O(\varepsilon^{2\vartheta}). \quad (50)
$$

Now define $\chi_\varepsilon(t) = \Lambda_\varepsilon(t/\varepsilon)$ for $t \in \mathbb{R}$ and $v_\varepsilon(x) = \chi_\varepsilon(\varrho(x))$. It is easy to see that $v_\varepsilon \in H^{1,\infty}(\Omega)$ and $X v_\varepsilon(x) = \chi_\varepsilon(\varrho(x)) X \varrho(x)$ a.e. Then, from [43, Theorem 6.5]

$$
\lim_{\varepsilon \downarrow 0} \int_\Omega |v_\varepsilon - u| dx = 0, \quad (51)
$$

$$
\limsup_{\varepsilon \downarrow 0} F_\varepsilon(v_\varepsilon) \leq F(u) = G(u). \quad (52)
$$

The functions $v_\varepsilon$ will be now perturbated so as to satisfy the integral constraint without disturbing inequality (52). Let us begin to show that if $\delta_\varepsilon = \int_\Omega v_\varepsilon dx - V$, then $\delta_\varepsilon = O(\varepsilon)$
Consider now the family of functions \( \vartheta \) and the integral in the last expression is bounded because of (49). In conclusion, if we notice that by (49) it follows that (see also [49, Theorem 1]).

Because of (50), if \( \vartheta \geq 1 \) the second and fourth integrals are \( O(\varepsilon) \).

We estimate the first one. By hypothesis (Ek) \( |X\vartheta| = 1 \) a.e. on \( \mathbb{R}^n \) and using the coarea formula (32) we get for \( t \geq 0 \)

\[
V^+(t) := |\{ x \in \Omega : 0 < \vartheta(x) \leq t \}| = \int_0^t |\partial E_s| \chi(\Omega) \, ds,
\]

where \( E_s := \{ x \in \mathbb{R}^n : \vartheta(x) > s \} \). By the coarea formula (33) and integrating by parts

\[
\int_{\{ x \in \Omega : 0 < \vartheta(x) \leq t_v \}} (1 - \chi(\vartheta(x)/\varepsilon)) \, dx = \int_0^{t_v} (1 - \chi(s/\varepsilon)) |\partial E_s| \chi(\Omega) \, ds
\]

\[= V^+(t_v)(1 - \chi(\vartheta \log(1/\varepsilon))) + \frac{1}{\varepsilon} \int_0^{t_v} \chi'(s/\varepsilon)V^+(s) \, ds.\]

By [43, Theorem 5.1] (see also [7]) \( V^+(t) = Lt + t\delta^+(t) \), where \( L = |\partial E| \chi(\Omega) \) and \( \delta^+ : [0, +\infty) \to \mathbb{R} \) is a function such that

\[
\limsup_{\varepsilon \to 0} \sup_{s \in [0, t_v]} |\delta^+(s)| = 0.
\]

By (49) it follows that \( V^+(t_v)(1 - \chi(\vartheta \log(1/\varepsilon))) = O(\varepsilon) \) if \( \vartheta c_2 \geq 1 \). Moreover

\[
\left| \frac{1}{\varepsilon} \int_0^{t_v} \chi'(s/\varepsilon)V^+(s) \, ds \right| \leq \frac{1}{\varepsilon} \int_0^{t_v} \sqrt{W(\chi(s/\varepsilon))}V^+(s) \, ds
\]

\[\leq (L + \sup_{s \in [0, t_v]} |\delta^+(s)|) \frac{1}{\varepsilon} \int_0^{t_v} s \sqrt{W(\chi(s/\varepsilon))} \, ds
\]

\[\leq \varepsilon (L + \sup_{s \in [0, t_v]} |\delta^+(s)|) \int_0^{+\infty} s \sqrt{W(\chi(s))} \, ds,
\]

and the integral in the last expression is bounded because of (49). In conclusion, if we choose \( \vartheta \geq \max\{3, 1/c_2\} \) this ends the proof of \( \delta_\varepsilon = O(\varepsilon) \).

Consider now the family of functions \( u_\varepsilon = (1 + \eta_\varepsilon) v_\varepsilon \) with \( \eta_\varepsilon = -\delta_\varepsilon/\int_\Omega v_\varepsilon \, dx \). Of course, \( u_\varepsilon \in H^{1,\infty}_X(\Omega) \) and \( u_\varepsilon \in \mathcal{A}_V \) since \( 1 + \eta_\varepsilon > 0 \) and \( \int_\Omega u_\varepsilon \, dx = V \). If we show that

\[
\limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon), \tag{53}
\]

statement (43) will be proved.
Notice that
\[
G(u_\varepsilon) = \int_{\{x \in \Omega : \varepsilon|x| \leq 2_t\varepsilon\}} \left(\varepsilon(1 + \eta_\varepsilon)^2|Xv_\varepsilon|^2 + \frac{1}{\varepsilon}W(v_\varepsilon + \eta_\varepsilon v_\varepsilon)\right) dx \\
+ \frac{1}{\varepsilon}W(1 + \eta_\varepsilon)|\{x \in \Omega : \varepsilon(x) > 2_t\varepsilon\}|
\]
\[
\leq \varepsilon \int_{\Omega} |Xv_\varepsilon|^2 dx + \frac{\eta_\varepsilon(2 + \eta_\varepsilon)}{\varepsilon} \int_{\{x \in \Omega : \varepsilon(x) \leq 2_t\varepsilon\}} |\Lambda_\varepsilon'(\varepsilon)|^2 dx \\
+ \frac{1}{\varepsilon} \int_{\{x \in \Omega : \varepsilon(x) \leq 2_t\varepsilon\}} W(v_\varepsilon + \eta_\varepsilon v_\varepsilon) dx + \frac{1}{\varepsilon}W(1 + \eta_\varepsilon)|\{x \in \Omega : \varepsilon(x) > 2_t\varepsilon\}|.
\]
By (39) and by Taylor’s formula
\[
\frac{1}{\varepsilon}W(1 + \eta_\varepsilon)|\{x \in \Omega : \varepsilon(x) > 2_t\varepsilon\}| \leq \frac{|\Omega|}{2\varepsilon} W''(\xi_\varepsilon) \eta_\varepsilon^2
\]
for some \(\xi_\varepsilon \in (1 - \eta_\varepsilon, 1 + \eta_\varepsilon)\) and hence this term is \(O(\varepsilon)\). Moreover, since
\[
\int_{\{x \in \Omega : \varepsilon(x) \leq 2_t\varepsilon\}} |\Lambda_\varepsilon'(\varepsilon)|^2 dx \leq \sup |\chi'|^2|\{x \in \Omega : |\varepsilon(x)| \leq t\varepsilon\}|
\]
\[
+ \|p_\varepsilon\|_{L^\infty(t_\varepsilon/\varepsilon, 2t_\varepsilon/\varepsilon)}|\{x \in \Omega : t_\varepsilon < |\varepsilon(x)| \leq 2t_\varepsilon\}|,
\]
by (50) we get
\[
\lim_{\varepsilon \to 0} \frac{\eta_\varepsilon(2 + \eta_\varepsilon)}{\varepsilon} \int_{\{x \in \Omega : \varepsilon(x) \leq 2_t\varepsilon\}} |\Lambda_\varepsilon'(\varepsilon)|^2 dx = 0.
\]
In order to prove (53) it suffices to show that
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\{x \in \Omega : \varepsilon(x) < 2_t\varepsilon\}} (W(u_\varepsilon) - W(v_\varepsilon)) dx = 0.
\]
Indeed, by the Mean Value Theorem there exists \(\tau > 0\) such that
\[
\frac{1}{\varepsilon} \int_{\{x \in \Omega : \varepsilon(x) < 2_t\varepsilon\}} |W(u_\varepsilon) - W(v_\varepsilon)| dx \leq \frac{|\eta_\varepsilon|}{\varepsilon} |\{x \in \Omega : |\varepsilon(x)| < 2t_\varepsilon\}| \sup_{s \in [0, 1 + \tau]} |W'(s)|,
\]
and the last quantity approaches to zero as \(\varepsilon \downarrow 0\).

**Step 2.** We prove the thesis under the only assumption \(X_1, \ldots, X_m \in \text{Lip}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)\). Thanks to Proposition 2.5 \(X = (X_1, \ldots, X_m)\) may be assumed to satisfy (35) and (36). For \(\sigma > 0\) let \(X_\sigma\) be the family of vector fields defined in (38), i.e.
\[
X_\sigma = (X_1^{\sigma_1}, \ldots, X_m^{\sigma_m}, \sigma \partial_1, \ldots, \sigma \partial_n) \equiv (X_1^{\sigma}, \ldots, X_m^{\sigma})
\]
Now, \(X_j^{\sigma} \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)\) for all \(j = 1, \ldots, m + n\), these vector fields are bounded on \(\mathbb{R}^n\) and by (37)
\[
\sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \leq \sum_{j=1}^{m+n} \langle X_j^{\sigma}(x), \xi \rangle^2 \quad \text{for all } x, \xi \in \mathbb{R}^n.
\]
The cc distance $d_\sigma$ induced on $\mathbb{R}^n$ by $X_\sigma$ is a Riemannian metric and since the vector fields are bounded ($\mathbb{R}^n, d_\sigma$) is a complete metric space. Hypothesis $(X_\sigma c)$ holds, and by [43, Theorem 3.1] the family $X_\sigma$ satisfies the Eikonal hypothesis $(Ek)$.

Therefore the first step of the proof does apply to the functionals $G_\sigma^\varepsilon : L^1(\Omega) \rightarrow [0, +\infty]$

$$G_\sigma^\varepsilon(u) = \begin{cases} \varepsilon \int_\Omega |X_\sigma u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx & \text{if } u \in H^1_{X_\sigma}(\Omega) \cap A_V \\ +\infty & \text{otherwise.} \end{cases}$$

(55)

Precisely, for all $\sigma > 0$

$$\Gamma(L^1(\Omega))^{-} \lim_{\varepsilon \downarrow 0} G_\sigma^\varepsilon = G_\sigma^\varepsilon,$$

(56)

where $G_\sigma : L^1(\Omega) \rightarrow [0, +\infty]$ is the functional

$$G_\sigma(u) = \begin{cases} 2\alpha|\partial E|_{X_\sigma}(\Omega) & \text{if } u = \chi_E \in BV_{X_\sigma}(\Omega) \cap A_V \\ +\infty & \text{otherwise.} \end{cases}$$

(57)

By the vector fields’ form

$$H^1_{X_\sigma}(\Omega) = H^1(\Omega) \subset H^1_{X}(\Omega), \quad \text{for all } \sigma > 0,$$

and then by (54)

$$G_\varepsilon(u) \leq G_\varepsilon^\sigma(x), \quad \text{for all } u \in L^1(\Omega) \text{ and for all } \varepsilon, \sigma > 0.$$

(58)

Let $G', G'' : L^1(\Omega) \rightarrow [0, +\infty]$ be respectively the lower and upper $\Gamma-$limits of $(G_\varepsilon)_{\varepsilon > 0}$ (see [19, Chapter 4]), i.e. if $u \in L^1(\Omega)$

$$G'(u) = \Gamma(L^1(\Omega))^{-} \liminf_{\varepsilon \downarrow 0} G_\varepsilon(u),$$

$$G''(u) = \Gamma(L^1(\Omega))^{-} \limsup_{\varepsilon \downarrow 0} G_\varepsilon(u).$$

Then, from [19, Proposition 6.7], (58) and (56)

$$G'(u) \leq G''(u) \leq G_\sigma(u) \quad \text{for all } u \in L^1(\Omega) \text{ and for all } \sigma > 0.$$

(59)

We claim that

$$G(u) \leq G'(u) \quad \text{for all } u \in L^1(\Omega).$$

(60)

Indeed, by [19, Proposition 8.1] we have to prove that for every $u \in L^1(\Omega)$, for every sequence $(u_h)_{h \in \mathbb{N}} \subset L^1(\Omega)$ strongly converging to $u$ in $L^1(\Omega)$ and for every sequence $(\varepsilon_h)_{h \in \mathbb{N}}$ of real numbers such that $\varepsilon_h \downarrow 0$

$$G(u) \leq \liminf_{h \rightarrow \infty} G_{\varepsilon_h}(u_h),$$

and this can be done exactly as in [43, Theorem 6.5, proof] where only the coarea formula (32) is involved.

Define

$$\mathcal{D} = \{ \chi_E : E \subset \mathbb{R}^n \text{ bounded open set, } \partial E \in C^\infty, |E \cap \Omega| = V, \mathcal{H}^{n-1}(\partial E \cap \partial \Omega) = 0 \}.$$
and notice that $D \subset BV_{X_\sigma}(\Omega)$ for all $\sigma > 0$. If $u = \chi_E \in D$ then from (25)

$$G^\sigma(u) = 2\alpha |\partial E|_{X_\sigma}(\Omega) = 2\alpha \int_{\partial E \cap \Omega} |C^\sigma n| d\mathcal{H}^{n-1},$$

(61)

where $C^\sigma(x)$ is the $(m+n) \times n$ matrix of the coefficients of the vector fields $X_j^\sigma$’s as in (18), and $n$ is the Euclidean normal to $\partial E$.

In particular, from (61) we get for all $u = \chi_E \in D$

$$\lim_{\sigma \to 0} G^\sigma(u) = 2\alpha \int_{\partial E \cap \Omega} |C n| d\mathcal{H}^{n-1} = G(u),$$

(62)

being $C(x)$ the matrix of the coefficients of the vector fields $X_j$’s. On the other hand, from (60), (59) and (62)

$$G(u) \leq G'(u) \leq G''(u) \leq G(u) \text{ for all } u \in D,$$

whence

$$G(u) = \Gamma(L^1(\Omega))-\lim_{\varepsilon \to 0} G_{\varepsilon}(u) \text{ for all } u \in D.$$

(63)

Applying (60), (63), Proposition 3.3 and the Reduction Lemma [42, Lemma IV] we finally find

$$G = \Gamma(L^1(\Omega))-\lim_{\varepsilon \to 0} G_{\varepsilon}.$$

The last result in this section deals with the $\Gamma$–convergence of functionals defined with degenerate quadratic forms. Let $A(x)$ be a symmetric, non negative matrix and consider the functionals $Q, Q_{\varepsilon} : L^1(\Omega) \to [0, +\infty]$ defined as

$$Q_{\varepsilon}(u) = \left\{ \begin{array}{ll}
\varepsilon \int_{\Omega} \langle ADu, Du \rangle \, dx + \frac{1}{\varepsilon} \int_{\Omega} W(u) \, dx & \text{if } u \in C^1(\Omega) \cap A_V \\
+\infty & \text{otherwise},
\end{array} \right.$$  

(64)

and

$$Q(u) = \left\{ \begin{array}{ll}
2\alpha |\partial E|_{A}(\Omega) & \text{if } u = \chi_E \in BV_{A}(\Omega) \cap A_V \\
+\infty & \text{otherwise},
\end{array} \right.$$  

(65)

where $V, A_V, W$ and $\alpha$ are as in Theorem 3.1.

**Theorem 3.5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^2$ boundary and let $A(x)$ be a symmetric, non negative $n \times n$–matrix, i.e. $\langle A(x)\xi, \xi \rangle \geq 0$ for all $x, \xi \in \mathbb{R}^n$. Suppose that $A$ has $C^2$ entries and satisfies (30). Moreover, assume that there exist $C \geq 1$, $u_0 > 0$ and $p \geq 1$ such that

$$C^{-1}|u|^p \leq W(u) \leq C|u|^p \text{ for all } |u| \geq u_0.$$  

(66)

Then

$$Q = \Gamma(L^1(\Omega))-\lim_{\varepsilon \to 0} Q_{\varepsilon}.$$  

(67)
Remark 3.6. When the matrix $A$ is positive definite on $\Omega$, i.e. there exists $\lambda_0 > 0$ such that $\langle A(x)\xi, \xi \rangle \geq \lambda_0|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$ Theorem 3.5 is well known under the only hypothesis of continuity of the matrix entries (see [14] and [12]).

In the degenerate setting we are dealing with, requiring the matrix $A$ to be of class $C^2$ is necessary in order to assure the factorization $A = C^TC$ as in Lemma 2.2. Actually, the assumptions on $A$ in Theorem 3.5 can be weakened requiring only $A(x) = C(x)^TC(x)$ for all $x \in \Omega$ and for some $m \times n$ matrix $C(x)$ with Lipschitz continuous entries. Without such a factorization we do not know if Theorem 3.5 still holds.

Proof of Theorem 3.5. By Lemma 2.2 there exists a $n \times n$ matrix $C(x)$ with Lipschitz continuous entries such that $A(x) = C(x)^TC(x)$ for all $x \in \mathbb{R}^n$. Let $X_1,...,X_n$ be the family of vector fields whose coefficients are the rows of the matrix $C(x)$ (see (18)). By Proposition 2.1 we can write the functionals $Q_\varepsilon$ and $Q$ as follows

$$Q_\varepsilon(u) = \begin{cases} \varepsilon \int_{\Omega} |Xu|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} W(u) \, dx & \text{if } u \in C^1(\Omega) \cap A_V \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$Q(u) = \begin{cases} 2\alpha|\partial E|_X(\Omega) & \text{if } u = \chi_E \in BV_X(\Omega) \cap A_V \\ +\infty & \text{otherwise.} \end{cases}$$

By a general $\Gamma-$convergence result (see [19, Proposition 6.11]) (67) holds if and only if

$$Q = \Gamma(L^1(\Omega)) \lim_{\varepsilon \to 0} \text{sc}^-(L^1(\Omega))Q_\varepsilon, \tag{68}$$

where $\text{sc}^-(L^1(\Omega))Q_\varepsilon : L^1(\Omega) \to [0, +\infty]$ is the relaxed functional of $Q_\varepsilon$ with respect to the topology of $L^1(\Omega)$.

Recalling Theorem 3.1 we only have to prove that for every $\varepsilon > 0$

$$\text{sc}^-(L^1(\Omega))Q_\varepsilon(u) = G_\varepsilon(u) = \begin{cases} \varepsilon \int_{\Omega} |Xu|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} W(u) \, dx & \text{if } u \in H^1_X(\Omega) \cap A_V \\ +\infty & \text{otherwise.} \end{cases} \tag{69}$$

The inequality $\text{sc}^-(L^1(\Omega))Q_\varepsilon(u) \geq G_\varepsilon(u)$ follows at once by a well known characterization of the relaxed functional (see, for instance, [19, Proposition 3.6]) and by the lower semicontinuity of $G_\varepsilon$ with respect to the topology of $L^1(\Omega)$. We claim that

$$\text{sc}^-(L^1(\Omega))Q_\varepsilon(u) \leq G_\varepsilon(u) \quad \text{for all } u \in L^1(\Omega). \tag{70}$$

If $G_\varepsilon(u) = +\infty$ there is nothing to prove. Let $u \in H^1_X(\Omega) \cap A_V$ be such that $G_\varepsilon(u) < +\infty$. The growth condition (66) implies $u \in L^p(\Omega)$. Since $u \in H^1_X(\Omega)$ by [28, Theorem 1.2.3] there exists a sequence $(v_h)_{h \in \mathbb{N}} \subset C^1(\Omega) \cap H^1_X(\Omega)$ such that $v_h \to u$ in $H^1_X(\Omega)$. Moreover, as $u \in L^p(\Omega)$ and the technique of approximation by convolution is involved, it is not restrictive to assume that $v_h \to u$ in $L^p(\Omega)$. Let $c_h = \int_{\Omega} u \, dx / \int_{\Omega} v_h \, dx$ and define $u_h = c_h v_h$. Then $u_h \in H^1_X(\Omega) \cap A_V$, $u_h \to u$ in $H^1_X(\Omega)$ and

$$u_h \to u \quad \text{in } L^p(\Omega). \tag{71}$$
By (66), (71) and Carathéodory continuity Theorem (see [19, Example 1.22])
\[
\lim_{h \to \infty} \int_\Omega W(u_h) \, dx = \int_\Omega W(u) \, dx.
\]

Eventually
\[
\text{sc}^{-}(L^1(\Omega))Q_\varepsilon(u) \leq \liminf_{h \to \infty} \left( \varepsilon \int_\Omega |Xu_h|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u_h) \, dx \right)
\leq \varepsilon \int_\Omega |Xu|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx = G_\varepsilon(u).
\]
This proves (70). As a consequence, (69) and (68) do hold.

4. Convergence of minima and minimizers

In this section we study existence and asymptotic behavior of minima and minimizers of the functionals \(G_\varepsilon\) and \(Q_\varepsilon\) defined in (41) and (64). To this purpose we recall the following fundamental variational property of \(\Gamma\)–convergence (see [19, Corollary 7.20]).

**Theorem 4.1.** Let \((M, \varrho)\) be a metric space and let \(F, F_h : M \to [0, +\infty]\) be such that \(F = \Gamma(M)\text{-lim}_{h \to \infty} F_h\). Let \((\varepsilon_h)_{h \in \mathbb{N}}\) be a sequence of real numbers such that \(\varepsilon_h \downarrow 0\), and let \((u_h)_{h \in \mathbb{N}} \subset M\) be a relatively compact sequence of \(\varepsilon_h\)–minimizers, i.e. \(F_h(u_h) \leq \inf_M F_h + \varepsilon_h\) for all \(h \in \mathbb{N}\). Then
\begin{enumerate}[(i)]
    
(i) \(\min_{u \in M} F(u) = \lim_{h \to \infty} \inf_{u \in M} F_h(u)\);

(ii) every cluster point \(u \in M\) of \((u_h)_{h \in \mathbb{N}}\) is a minimum of \(F\), i.e. \(F(u) = \min_{v \in M} F(v)\).
\end{enumerate}

In order to apply Theorem 4.1 a fundamental tool will be the compact embedding of \(H^1_p(\Omega)\) in \(L^p(\Omega)\) which will be discussed more in detail in Section 5. An open set \(\Omega \subset \mathbb{R}^n\) will be said to support the \(H^1_p(\Omega)\)–compact embedding, \(1 \leq p \leq +\infty\), if
\[(C)_p\ \text{the embedding} \ H^1_p(\Omega) \hookrightarrow L^p(\Omega) \text{ is compact.}\]
In the Euclidean case the compact embedding is known to imply a Poincaré inequality (see, for instance, [32]). Following the same proof an analogous result for vector fields can be obtained.

**Proposition 4.2.** Let \(X = (X_1, \ldots, X_m)\) be a family of Lipschitz vector fields on \(\mathbb{R}^n\) satisfying \((Xc)\). Let \(\Omega \subset \mathbb{R}^n\) be a connected bounded open set. If \((C)_p\) holds for \(1 \leq p < +\infty\) then there exists \(C > 0\) such that
\[
\int_\Omega |u - u_\Omega|^p \, dx \leq C \int_\Omega |Xu|^p \, dx
\]
for all \(u \in H^1_p(\Omega)\), where \(u_\Omega := \int_\Omega u \, dx\).

Let \(G_\varepsilon\) be as in (41). The first result of this section is the existence of minima for the functionals \(G_\varepsilon\) and the compactness of the family of such minima.
Theorem 4.3. Let $X = (X_1, ..., X_m)$ be a family of Lipschitz vector fields on $\mathbb{R}^n$ satisfying (Xc), let $\Omega \subset \mathbb{R}^n$ be a connected, bounded open set such that the compact embedding (C)$_2$ holds, and finally let $W : \mathbb{R} \to \mathbb{R}$ be a function satisfying (66) for some $p > 2$. Then for all $\varepsilon > 0$ there exists $u_\varepsilon \in A_V$ such that

$$G_\varepsilon(u_\varepsilon) = \min_{u \in L^1(\Omega)} G_\varepsilon(u). \quad (73)$$

If, in addition, $\Omega$ supports the compact embedding (C)$_1$, then the family $\{u_\varepsilon : \varepsilon > 0\}$ is relatively compact in $L^1(\Omega)$.

Let $G$ be the functional defined in (41). Choosing $M = L^1(\Omega)$, $F_h = G_{\varepsilon_h}$ and $F = G$ in Theorem 4.1 and taking into account Theorem 3.1 and Theorem 4.3 we get the following Corollary.

Corollary 4.4. Let $X$, $\Omega$ and $W$ be as in Theorem 4.3. Moreover, assume that $\Omega$ is of class $C^2$ and $W$ satisfies (39). Let $(\varepsilon_h)_{h \in \mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$. Then:

(i) there exists $\min_{u \in L^1(\Omega)} G(u) = \lim_{h \to \infty} \min_{u \in L^1(\Omega)} G_{\varepsilon_h}(u);

(ii) if $(u_h)_{h \in \mathbb{N}}$ is a sequence of minimizers of $(G_{\varepsilon_h})_{h \in \mathbb{N}} (G_{\varepsilon_h}(u_h) = \min_{u \in L^1(\Omega)} G_{\varepsilon_h}(u))$ then there exist a subsequence $(u_{h_j})_{j \in \mathbb{N}}$ and a function $u_0 = \chi_E \in BV_X(\Omega)$ such that $u_{h_j} \to u_0$ in $L^1(\Omega)$ and $G(u_0) = \min_{u \in L^1(\Omega)} G(u)$.

Proof of Theorem 4.1. The proof can be essentially carried out as in [41] and we shall only sketch the main steps.

The existence of $u_\varepsilon \in A_V$ such that (73) holds can be proved by the direct method of Calculus of Variations. To this aim we have to check that $G_\varepsilon : L^1(\Omega) \to [0, +\infty]$ is lower semicontinuous and coercive (see, for instance, [19, Theorem 1.15]). The lower semicontinuity and the coerciveness follow as in the classic case by the compact embedding (C)$_2$, by the Poincaré inequality (72) and by Fatou Lemma.

Let us prove that the family of minima $\{u_\varepsilon : \varepsilon > 0\}$ is relatively compact in $L^1(\Omega)$. Define $\varphi \in C^1(\mathbb{R})$ by $\varphi(t) = \int_0^t \sqrt{W(s)} \, ds$, and let $v_\varepsilon(x) := \varphi(u_\varepsilon(x)) \in H^1_X(\Omega)$. By (66) and arguing as in [41, Proposition 3, proof] we get the existence of two positive constants $c_3, c_4$ such that

$$\int_\Omega v_\varepsilon \, dx \leq c_3 |\Omega| + c_4 G_\varepsilon(u_\varepsilon) \quad \text{for all } \varepsilon \in (0, 1),$$

and moreover

$$\int_\Omega |Xv_\varepsilon| \, dx = \int_\Omega \varphi'(u_\varepsilon)|Xu_\varepsilon| \, dx \leq \frac{1}{2} \int_\Omega \left( \varepsilon |Xu_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) \, dx = \frac{1}{2} G_\varepsilon(u_\varepsilon).$$

If we show that $G_\varepsilon(u_\varepsilon) \leq C < +\infty$ for all $\varepsilon > 0$ and for some $C > 0$, then the set $\{v_\varepsilon : \varepsilon > 0\}$ is bounded in $H^1_X(\Omega)$ and hence relatively compact in $L^1(\Omega)$ by the compact embedding (C)$_1$. The function

$$w_\varepsilon(x) = \begin{cases} 
\frac{1}{\varepsilon} + \frac{1}{2\varepsilon^2}(x_1 - \delta_\varepsilon) & \text{if } x_1 \leq \delta_\varepsilon - \varepsilon \\
\frac{1}{2} + \frac{1}{\varepsilon}(x_1 - \delta_\varepsilon) & \text{if } \delta_\varepsilon - \varepsilon < x_1 < \delta_\varepsilon + \varepsilon \\
0 & \text{if } x_1 \geq \delta_\varepsilon + \varepsilon
\end{cases}$$

satisfies

$$\frac{1}{\varepsilon} |Xv_\varepsilon| \leq \varphi'(u_\varepsilon) |Xu_\varepsilon| \quad \text{in } \Omega \setminus \{x_1 = \delta_\varepsilon\}.$$
belongs to $H^1_X(\Omega)$ for all $\varepsilon > 0$ and for all $\delta_\varepsilon \in \mathbb{R}$. Since $0 < V < |\Omega|$, $\delta_\varepsilon \in \mathbb{R}$ can be chosen in such a way that $w_\varepsilon \in \mathcal{A}_V$. If $x \in (\delta_\varepsilon - \varepsilon, \delta_\varepsilon + \varepsilon) \times \mathbb{R}^{n-1} \cap \Omega$ then
\[
|Xw_\varepsilon(x)|^2 = \sum_{j=1}^{m} (X_jw_\varepsilon(x))^2 = \frac{1}{4\varepsilon^2} \sum_{j=1}^{m} (c_{j1}(x))^2 \leq C/\varepsilon^2.
\]
Moreover $W(w_\varepsilon) \leq \sup_{t \in [0,1]} W(t)$ and thus
\[
G_\varepsilon(w_\varepsilon) = \int_{\Omega \cap \{\delta_\varepsilon - \varepsilon < x_1 < \delta_\varepsilon + \varepsilon\}} \left(\varepsilon |Xw_\varepsilon|^2 + \frac{1}{\varepsilon} W(w_\varepsilon)\right) \, dx
\]
\[
\leq C \varepsilon |\Omega \cap \{\delta_\varepsilon - \varepsilon < x_1 < \delta_\varepsilon + \varepsilon\}| \leq C < +\infty.
\]
This proves that $G_\varepsilon(u_\varepsilon) \leq C < +\infty$ for all $\varepsilon > 0$.

Since the set $\{v_\varepsilon \in L^1(\Omega) : \varepsilon > 0\}$ is relatively compact there exist $v \in L^1(\Omega)$ and $\varepsilon_\varepsilon \downarrow 0$ such that $v_{\varepsilon_\varepsilon} \to v$ in $L^1(\Omega)$. The function $\varphi$ is strictly increasing and thus there exists $\psi = \varphi^{-1} \in C^1(\mathbb{R})$. Define $u(x) := \psi(v(x))$ and notice that $u_{\varepsilon_\varepsilon} = \psi(v_{\varepsilon_\varepsilon})$. Arguing as in [41] we finally get $u_{\varepsilon_\varepsilon} \to u$ in $L^1(\Omega)$. □

Let $V$ and $\mathcal{A}_V$ be as in (40) and let $Q_\varepsilon$ be the functionals defined in (64). The second result of this section deals with the compactness of $Q_\varepsilon$'s minimizers.

**Theorem 4.5.** Let $\Omega$ be a connected, bounded open set, let $A(x)$ be a symmetric matrix of functions on $\mathbb{R}^n$ and let $Y = (Y_1, ..., Y_r)$ be a family of Lipschitz continuous vector fields on $\mathbb{R}^n$ satisfying the connectivity hypothesis (Yc). Assume that:

(i) $A(x)$ has entries of class $C^2(\mathbb{R}^n)$ and satisfies (30);

(ii) $\langle A(x)\xi, \xi \rangle \geq \sum_{j=1}^{r} (Y_j(x), \xi)^2$ for all $x, \xi \in \mathbb{R}^n$;

(iii) the compact embeddings (C)_1 and (C)_2 hold with $X \equiv Y$ relatively to $\Omega$;

(iv) the function $W$ in the functional $Q_\varepsilon$ satisfies (39) and (66).

Let $(\varepsilon_\varepsilon)_{\varepsilon \in \mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_\varepsilon \downarrow 0$. Then every sequence $(u_{\varepsilon_\varepsilon})_{\varepsilon \in \mathbb{N}}$ of $\varepsilon_\varepsilon$–minimizers of $Q_\varepsilon$ (i.e. $Q_{\varepsilon_\varepsilon}(u_{\varepsilon_\varepsilon}) \leq \inf_{u \in \mathcal{A}_V} Q_{\varepsilon_\varepsilon}(u) + \varepsilon_\varepsilon$) is relatively compact in $L^1(\Omega)$.

**Remark 4.6.** The requirement on $A$ to be of class $C^2$ is necessary in order to assure the factorization $A = C^T C$ as in Lemma 2.2. Actually, assumption (i) in Theorem 4.5 can be weakened requiring only $A(x) = C(x)^T C(x)$ for all $x \in \Omega$ for some $m \times n$ matrix $C(x)$ with Lipschitz continuous entries (see also Remark 3.6).

Let $Q$ be the functional defined in (65). Choosing $M = L^1(\Omega)$, $F_\varepsilon = Q_{\varepsilon_\varepsilon}$ and $F = Q$ from Theorem 4.1 and Theorem 4.5 we get the following Corollary.

**Corollary 4.7.** Let $\Omega$, $A$ and $Y$ be as in Theorem 4.5. Assume that $\Omega$ has $C^2$ boundary and that $W$ satisfies (39) and (66). Let $(\varepsilon_\varepsilon)_{\varepsilon \in \mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_\varepsilon \downarrow 0$. Then:

(i) there exists $\min_{u \in L^1(\Omega)} Q(u) = \lim_{\varepsilon \to 0} \inf_{u \in L^1(\Omega)} Q_{\varepsilon_\varepsilon}(u)$;

(ii) if $(u_{\varepsilon_\varepsilon})_{\varepsilon \in \mathbb{N}}$ is a sequence of $\varepsilon_\varepsilon$–minimizers of $(Q_{\varepsilon_\varepsilon})_{\varepsilon \in \mathbb{N}}$ then there exist a subsequence $(u_{\varepsilon_\varepsilon})_{\varepsilon \in \mathbb{N}}$ and a function $u_0 = \chi_E \in BV(\Omega)$ such that $u_{\varepsilon_\varepsilon} \to u_0$ in $L^1(\Omega)$ and $Q(u_0) = \min_{u \in L^1(\Omega)} Q(u)$.
Proof of Theorem 4.5. By assumption (i) Lemma 2.2 can be applied and arguing as in the proof of Theorem 3.5 we conclude that

$$Q_\varepsilon(u) = \begin{cases} \int_\Omega \left( \varepsilon |Xu|^2 + \frac{1}{\varepsilon} W(u) \right) dx & \text{if } u \in C^1(\Omega) \cap A_V \\ +\infty & \text{otherwise,} \end{cases}$$

for a suitable family $X = (X_1, \ldots, X_n)$ of Lipschitz continuous vector fields. Moreover, for every $\varepsilon > 0$ and for all $u \in L^1(\Omega)$

$$\text{sc}^{-}(L^1(\Omega))Q_\varepsilon(u) = G_\varepsilon(u),$$

being $\text{sc}^{-}(L^1(\Omega))Q_\varepsilon$ the relaxed functional of $Q_\varepsilon$ with respect to the $L^1(\Omega)$ topology and $G_\varepsilon$ the functional defined in (41).

On the other hand, by assumptions (ii) $X$ can be assumed to satisfy $(Xc)$, and by (iii) $(C_1)$ and $(C_2)$ can be assumed to hold relatively to $X$ and $\Omega$. Theorem 4.3 can be applied. As pointed out in the first part of the proof of Theorem 4.3 $G_\varepsilon$ is coercive with respect to the $L^1(\Omega)$ topology and from a well-known result of relaxation theory (see, for instance, [19, Theorem 3.8]) there exists

$$\min_{u \in L^1(\Omega)} G_{\varepsilon_h}(u) = \inf_{u \in L^1(\Omega)} Q_{\varepsilon_h}(u).$$

The thesis follows.

5. Examples and applications

In this section we give some important examples of families of vector fields to which our results of Sections 3 and 4 apply. Moreover, we study in detail a couple of examples that often play a paradigmatic role in the theory of cc spaces.

Example 5.1 (Hörmander vector fields). Let $X = (X_1, \ldots, X_m)$ with $X_j \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and denote by $\mathcal{L}(X_1, \ldots, X_m)$ the Lie algebra generated by such vector fields by iterated commutators. If the Chow-Hörmander condition

$$\text{rank} \mathcal{L}(X_1, \ldots, X_m)(x) = n \quad \text{for every } x \in \mathbb{R}^n,$$

(74)

holds, then $X$ satisfies $(Xc)$. Vector fields of this type were introduced in [36] and a deep study of the induced cc metric can be found in [46].

Example 5.2 (Grushin’s type vector fields). Let $X = (X_1, \ldots, X_n)$ and $X_j = \lambda_j(x) \partial_j$, $j = 1, \ldots, n$ with $\lambda_j \in \text{Lip}_\text{loc}(\mathbb{R}^n)$, $\lambda_j \geq 0$. Assume that:

(i) $\lambda_1 \equiv 1$ and $\lambda_j(x) = \lambda_j(x_1, \ldots, x_{j-1})$, $j = 2, \ldots, n$;

(ii) $\lambda_j \in C^1(\mathbb{R}^n \setminus \Pi_j)$ where $\Pi_j = \{ x \in \mathbb{R}^n : x_1 \cdots x_{j-1} = 0 \}$;

(iii) $\lambda_j(x) = \lambda_j(x_1, \ldots, |x_k|, \ldots, x_{j-1})$ for all $k = 1, \ldots, j-1$ and $j = 2, \ldots, n$;

(iv) there exist positive constants $\alpha_{ijk}$ such that $0 \leq x_k \partial_k \lambda_j(x) \leq \alpha_{ijk} \lambda_j(x)$ for all $x \in \mathbb{R}^n \setminus \Pi_j$.

The vector fields $X$ satisfy $(Xc)$. Vector fields of this type were introduced in [25], [26], [23]. In the special case $\lambda_1 = \ldots = \lambda_r = 1$, $\lambda_{r+1} = \ldots = \lambda_n \equiv \lambda$ they were studied in [24] even under less restrictive assumptions on the regularity of $\lambda$. This class can be considered as a “weak-extension” to the non smooth case of Hörmander vector fields given in Example 5.1.
We introduce some basic notions on regular domains in metric spaces (see [18] for an account of recent results in this argument and see also [51]). The following definition gives a generalization of the well known “interior cone property” of domains in Euclidean spaces to domains in a general metric space (see, for instance, [37, §9] and [17]).

**Definition 5.3.** Let \((M, d)\) be a metric space. A bounded open set \(\Omega \subset M\) is a John domain if there exist \(x_0 \in \Omega\) and \(C > 0\) such that for every \(x \in \Omega\) there exists a continuous rectifiable curve parameterized by arclength \(\gamma: [0, T] \to \Omega, T \geq 0\), such that \(\gamma(0) = x, \gamma(T) = x_0\) and \(\text{dist}(\gamma(t), \partial \Omega) \geq Ct\).

If \(B\) is a ball in the metric space \((M, d)\) and \(\lambda \geq 0\) with \(\lambda B\) we denote the ball with same center as \(B\) and radius \(\lambda\) times that of \(B\). The following definition extends the “Boman chain condition” to metric spaces (see [30, Definition 1.4] and see also [37, §9]).

**Definition 5.4.** Let \((M, d)\) be a metric space and \(\mu\) a positive Borel measure on \(M\). A bounded open set \(\Omega \subset M\) is a Boman domain if there exists a covering \(\{B: B \in \mathcal{F}\}\) of \(\Omega\) with balls, and there exist \(N \geq 1, \lambda > 1\) and \(\nu \geq 1\) such that:

1. \(\lambda B \subset \Omega\) for all \(B \in \mathcal{F}\);
2. every point of \(\Omega\) belongs to at most \(N\) balls \(\lambda B\) with \(B \in \mathcal{F}\);
3. there exists a central ball \(B_0 \in \mathcal{F}\) such that for any \(B \in \mathcal{F}\) there exists a chain of balls \(B_1, \ldots, B_k\) such that \(B_k = B, B_i \cap B_{i+1} \neq \emptyset, \mu(B_i \cap B_{i+1}) \geq 1/N \max\{\mu(B_i), \mu(B_{i+1})\}\) and \(B \subset \nu B_i\) for all \(i = 0, 1, \ldots, k\).

Definitions 5.3 and 5.4 turn out to identify the same class of domains in homogeneous metric spaces with geodesics (see [15], [30, Theorem 1.30] and [37, Proposition 9.6]).

**Theorem 5.5.** Let \((M, d)\) be a metric space endowed with a positive Borel measure \(\mu\). Assume that:

1. every couple of points can be connected by a geodesic;
2. there exists a constant \(\delta > 0\) such that \(0 < \mu(B(x, 2r)) \leq \delta \mu(B(x, r)) < +\infty\) for all \(x \in M\) and \(r \geq 0\).

Then, the class of John domains equals that of Boman domains.

In the examples we shall consider condition (i) is true, and condition (ii) is also true choosing \(\mu\) to be the Lebesgue measure. Boman domains are of special interest because of the following Compactness Theorem which is proved in [30]. The metric space is a cc space \((\mathbb{R}^n, d)\) endowed with Lebesgue measure.

**Theorem 5.6.** Let \((\mathbb{R}^n, d)\) be the cc space induced by a family \(X = (X_1, \ldots, X_m)\) of Hörmander or Grushin’s type vector fields (see Examples 5.1 and 5.2). If \(\Omega \subset \mathbb{R}^n\) is a Boman domain then for all \(1 \leq p < +\infty\) the embedding \(H^{1,p}_X(\Omega) \hookrightarrow L^p(\Omega)\) is compact.

**Remark 5.7.** Theorem 5.6 may fail if \(\Omega\) is not a Boman domain in \((\mathbb{R}^n, d)\). Indeed, it is known that even for open sets \(\Omega \subset \mathbb{R}^n\) with boundary of class \(C^{\infty}\) the Poincaré inequality (72) is no longer true (see, for instance, [35] and Remark 5.18).

From Theorems 5.6 and 4.5, Remark 4.6 and Corollary 4.7 we get the following result. Let \(Q_\varepsilon, Q\) be as in (64) and (65) and let \(W\) be a function which satisfies (39) and (66).
Theorem 5.8. Let $\Omega$ be a connected, bounded open set of class $C^2$, let $A(x)$ be a matrix of functions on $\mathbb{R}^n$ and let $Y = (Y_1, ..., Y_r)$ be a family of Hörmander or Grushin’s type vector fields on $\mathbb{R}^n$. Assume that:

(i) $A(x) = C^T(x)C(x)$ for all $x \in \Omega$ where $C(x)$ is a $m \times n$ matrix with Lipschitz continuous entries on $\mathbb{R}^n$;

(ii) $\langle A(x)\xi, \xi \rangle \geq \sum_{j=1}^r \langle Y_j(x), \xi \rangle^2$ for all $x, \xi \in \mathbb{R}^n$;

(iii) $\Omega$ is a Boman domain in $(\mathbb{R}^n, d)$, where $d$ is the cc metric induced by the family of vector fields $Y$.

If $(u_h)_{h \in \mathbb{N}}$ is a sequence of $\varepsilon_h$-minimizers of $Q_{\varepsilon_h}$ ($Q_{\varepsilon_h}(u_h) \leq \inf_u \in A \ V Q_{\varepsilon_h}(u) + \varepsilon_h$ with $\varepsilon_h \downarrow 0$) then there exists a subsequence $(u_{h_j})_{j \in \mathbb{N}}$ and a function $u_0 = \chi_E \in BV_A(\Omega)$ such that $u_{h_j} \to u_0$ in $L^1(\Omega)$ and $Q(u_0) = \min_{u \in L^1(\Omega)} Q(u)$.

We shall now study in detail two examples of vector fields which are respectively of Hörmander and Grushin’s type. In particular, we shall see that in these cases a suitable Euclidean regularity of the domain $\Omega$ also provides its intrinsic regularity with respect to the induced cc metric (see Theorems 5.10 and 5.17).

The first example is the Heisenberg group, a Lie group whose origins can be found in quantum mechanics (see [50, §11]). The quadratic form associated with the Heisenberg vector fields is degenerate at every point of the manifold.

The second example is the so called Grushin’s space where the degeneration of the quadratic form is concentrated on a small set but the coefficients of the vector fields may not be regular of class $C^k$ for $k \geq 1$. Moreover, there is no Lie structure compatible with the cc metric of the Grushin space (see also Remark 5.19).

Example 5.9 (Heisenberg group). In $\mathbb{R}^{2n+1}$ we shall write the coordinates $(x, y, t) \in \mathbb{R}^{2n+1}$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The Heisenberg vector fields are

$$X_j = \partial_{x_j} + 2y_j \partial_t, \quad Y_j = \partial_{y_j} - 2x_j \partial_t \quad j = 1, ..., n,$$

which satisfy the commutation relations

$$[X_i, X_j] = 0 \quad \text{and} \quad [Y_i, Y_j] = 0, \quad \text{for all} \ i, j = 1, ..., n,$$

$$[X_i, Y_j] = 0, \quad \text{for all} \ i, j = 1, ..., n, \ i \neq j \quad \text{(76)}$$

$$[X_i, Y_i] = -4\partial_t \quad \text{for all} \ i = 1, ..., n.$$

The vector fields $X_1, ..., X_n, Y_1, ..., Y_n$ are a system of generators of the left invariant Lie algebra of $\mathbb{R}^{2n+1}$ when endowed with the Lie group product

$$(x, y, t) \cdot (\xi, \eta, \tau) = (x + \xi, y + \eta, t + \tau + 2(\langle y, \xi \rangle - \langle x, \eta \rangle)). \quad \text{(77)}$$

The group $(\mathbb{R}^{2n+1}, \cdot)$ is usually called the Heisenberg group and denoted by $\mathbb{H}^n$. This group is homogeneous in the sense that it admits a one parameter family of automorphisms $\delta_\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}, \lambda > 0$, given by $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. Lebesgue measure is the Haar measure of the Heisenberg group. Moreover, for any measurable set $E \subset \mathbb{R}^{2n+1}$ and $\lambda > 0$ we have $|\delta_\lambda(E)| = \lambda^N |E|$ with $N := 2n + 2$. Here $|.|$ stands for the $2n+1$-dimensional Lebesgue measure on $\mathbb{R}^{2n+1}$. The integer $N$ is called the homogeneous dimension of $\mathbb{H}^n$.

The vector fields (75) satisfy Hörmander condition (74) and therefore induce on $\mathbb{R}^{2n+1}$ a
The Carnot-Carathéodory metric \( d \) verifying (Xc). \( (\mathbb{R}^{2n+1}, d) \) is a metric space with Hausdorff dimension equal to \( N \) (see [40]).

The Heisenberg gradient is \( \nabla_H = (X_1, ..., X_n, Y_1, ..., Y_n) \), and if \( f \in C^1(\mathbb{R}^{2n+1}) \) we can write

\[
|\nabla_H f|^2 = \sum_{j=1}^n (X_j f)^2 + (Y_j f)^2 = \langle B \nabla f, \nabla f \rangle,
\]

where \( B \) is the \((2n+1)\times(2n+1)\) square matrix with variable entries

\[
B(x, y, t) = \begin{pmatrix}
I_n & 0 & 2y^T \\
0 & I_n & -2x^T \\
2y & -2x & 4(|x|^2 + |y|^2)
\end{pmatrix},
\] (78)

and \( I_n \) is the identity \((n \times n)\)-matrix. Notice that \( \det(B(x, y, t)) = 0 \) for all \((x, y, t) \in \mathbb{R}^{2n+1}: \) the degeneration of the quadratic form \( B \) is distributed at every point of the space.

Let now \( \Omega \subset \mathbb{R}^{2n+1} \) be an open set. According to definition (19) denote by \( H^1_\mathbb{H}(\Omega) = H^1_X(\Omega) \) the Sobolev space associated with the family of vector fields \( X = \nabla_H \).

Examples of Boman domains in the Heisenberg group are provided in [30] and [17]. The following Theorem proved in [44] and Theorem 5.5 give a general sufficient condition for a domain to be Boman.

**Theorem 5.10.** Let \( \Omega \subset \mathbb{R}^{2n+1} \) be a bounded open set with boundary of class \( C^2 \). Then \( \Omega \) is a John domain.

**Remark 5.11.** There are open sets which are John domains without being of class \( C^2 \). Examples are Carnot-Carathéodory balls which are always John domains in general cc spaces (see [30]).

Let \( E \subset \mathbb{H}^n \) be a measurable set. According to definition (23) denote by \( |\partial E|_\mathbb{H}(\Omega) := |\partial E|_X(\Omega) \) and by \( BV_\mathbb{H}(\Omega) = BV_X(\Omega) \) respectively the Heisenberg perimeter of \( E \) and the space of the functions with bounded variation associated with the family of vector fields \( X = \nabla_H \).

**Remark 5.12.** The space \( BV(\Omega) \), i.e. the space of the functions with bounded variation in \( \Omega \), is strictly contained in \( BV_\mathbb{H}(\Omega) \) (see [29, Example 1]).

The measure of a surface in the Heisenberg group can also be computed by means of suitable Hausdorff measures. Define the \((N-1)\)-dimensional spherical measure of a set \( A \subset \mathbb{R}^{2n+1} \) as

\[
S_d^{N-1}(A) = \lim_{\delta \to 0} \inf \left\{ \gamma(N-1) \sum_{i=1}^{+\infty} (\text{diam}(B_i))^{N-1} : A \subset \bigcup_{i=1}^{+\infty} B_i, \text{diam}(B_i) \leq \delta, \right. \\
\left. B_i \subset \mathbb{R}^{2n+1} \right\},
\]

where \( \gamma(N-1) \) is a geometric constant, \( B_i \) are closed balls in \( (\mathbb{R}^{2n+1}, d) \) and \( \text{diam}(B_i) \) is the diameter of \( B_i \) with respect to \( d \). We already noticed that the metric space \( (\mathbb{R}^{2n+1}, d) \) has Hausdorff dimension \( N = 2n + 2 \) and thus \( N-1 \) is the correct “surface dimension”. The
link between perimeter and spherical Hausdorff measure is given in the following theorem first proved in [29] when the Heisenberg group is equipped with a metric equivalent to the cc metric \( d \).

**Theorem 5.13.** Let \( E \subset \mathbb{R}^{2n+1} \) be a measurable set and let \( \Omega \subset \mathbb{R}^{2n+1} \) be an open set. Assume that \( |\partial E|_\mathbb{H}(\omega) < +\infty \) for every open set \( \omega \subset \Omega \). Then there exists a Borel set \( \partial^* E \subset \partial E \) (called the \( \mathbb{H} \)-reduced boundary of \( E \) in \( \Omega \)) such that \( |\partial E|_\mathbb{H}(A) = S_{d-1}^\mathbb{H}(\partial^* E \cap A) \) for every Borel set \( A \subset \Omega \). Moreover, if \( E \) is an open set with boundary of class \( C^1 \) then \( |\partial E|_\mathbb{H}(\Omega) = S_{d-1}^\mathbb{H}(\partial E \cap \Omega) \).

We finally come to the applications of the results obtained in Section 3 and Section 4. Let \( W \in C^2(\mathbb{R}) \) be a function which satisfies (39) and (66) and let \( V \) and \( A_V \) be as in (40). For \( \varepsilon > 0 \) consider the functionals \( G_\varepsilon, Q_\varepsilon : L^1(\Omega) \to [0, +\infty) \)

\[
G_\varepsilon(u) = \begin{cases} 
\int_{\Omega} \left( \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dxdydt & \text{if } u \in H^1_\mathbb{H}(\Omega) \cap A_V \\
+\infty & \text{otherwise},
\end{cases}
\]

and

\[
Q_\varepsilon(u) = \begin{cases} 
\int_{\Omega} \left( \varepsilon ADu, Du \right) dxdydt + \frac{1}{\varepsilon} W(u) dxdydt & \text{if } u \in C^1(\Omega) \cap A_V \\
+\infty & \text{otherwise},
\end{cases}
\]

where \( A = A(x,y,t) \) is a symmetric, non negative definite \((n \times n)\)-matrix of \( C^2 \) functions on \( \mathbb{R}^n \) verifying (30) and for some positive constant \( C > 0 \)

\[
\langle A(x,y,t)\xi,\xi \rangle \geq C\langle B(x,y,t)\xi,\xi \rangle \quad \text{for all } (x,y,t), \xi \in \mathbb{R}^{2n+1},
\]

and \( B \) is the matrix (78).

Analogously, \( G, Q : L^1(\Omega) \to [0, +\infty] \) are the functionals defined by

\[
G(u) = \begin{cases} 
2\alpha |\partial E|_\mathbb{H}(\Omega) & \text{if } u = \chi_E \in BV_\mathbb{H}(\Omega) \cap A_V \\
+\infty & \text{otherwise},
\end{cases}
\]

and

\[
Q(u) = \begin{cases} 
2\alpha |\partial E|_A(\Omega) & \text{if } u = \chi_E \in BV_A(\Omega) \cap A_V \\
+\infty & \text{otherwise},
\end{cases}
\]

where \( |\partial E|_A \) is the perimeter measure defined in (11) and \( \alpha = \int_0^1 \sqrt{W(s)} ds \).

Then from Theorems 3.1, 3.5, 4.3, 4.5, 5.6, 5.10 and from Corollaries 4.4 and 4.7 we get at once the following result.

**Theorem 5.14.** Let \( \Omega \subset \mathbb{R}^{2n+1} \) be a connected, bounded open set of class \( C^2 \). Then:

(i) \( \Gamma(L^1(\Omega)) - \lim_{\varepsilon \downarrow 0} G_\varepsilon = G \);
(ii) \( \Gamma(L^1(\Omega)) - \lim_{\varepsilon \downarrow 0} Q_\varepsilon = Q \);
(iii) for all \( \varepsilon > 0 \) there exists \( u_\varepsilon \in H^1_\mathbb{H}(\Omega) \cap A_V \) such that \( G_\varepsilon(u_\varepsilon) = \min_{u \in L^1(\Omega)} G_\varepsilon(u) \).

Moreover, let \( (\varepsilon_h)_{h \in \mathbb{N}} \) be a sequence of real numbers such that \( \varepsilon_h \downarrow 0 \). Then:

(iv) the sequence \( (u_{\varepsilon_h})_{h \in \mathbb{N}} \) is relatively compact in \( L^1(\Omega) \);
Let $\Omega$ be a connected open set with Lipschitz boundary such that $\partial \Omega$ is of class $C^1$ in a neighborhood of every point $(0, y) \in \partial \Omega$. A point $(0, y) \in \partial \Omega$ will be said flat if there exist a neighborhood $\mathcal{V}$ of $(0, y)$ and a neighborhood $\mathcal{U}$ of the origin in $\mathbb{R}^{n-1}$ such that $\partial \Omega \cap \mathcal{V} = \{(x, \varphi(x)) : x \in \mathcal{U}\}$ for some $\varphi \in C^1(\mathcal{U}; \mathbb{R})$ with $\nabla \varphi(0) = 0$. A flat point $(0, y) \in \partial \Omega$ will be said $\beta$-admissible if there exists a constant $C > 0$ such that

$$|\nabla \varphi(x)| \leq C|x|^\beta$$

for all $x \in \mathcal{U}$. (81)

Finally, $\Omega$ will be said $\beta$-admissible if flat points in $\partial \Omega$ are $\beta$-admissible or if $\Omega$ has no flat points.
For example, the cube \( I = \{ (x, y) \in \mathbb{R}^n : |y|, |x| < 1, \text{ for } i = 1, ..., n-1 \} \) is \( \beta \)-admissible for all \( \beta > 0 \). Condition (81) states that in a neighborhood of the singular line \( \{ (x, y) \in \mathbb{R}^n : x = 0 \} \) the boundary \( \partial \Omega \) is suitably flat in connection with the power of degeneration of the quadratic form \( B(x) \).

The following theorem, which is a special case of the results proved in [45], and Theorem 5.5 show that \( \beta \)-admissible domains support the compact embedding \( H^{1,p}_X(\Omega) \hookrightarrow L^p(\Omega) \).

**Theorem 5.17.** If \( \Omega \subset \mathbb{R}^n \) is a \( \beta \)-admissible domain then it is a John domain.

**Remark 5.18.** If \( \Omega \) is a \( \beta \)-admissible domain then by Theorems 5.17, 5.5, 5.6 and by Proposition 4.2 it supports the Poincaré inequality (72) for all \( 1 < p < +\infty \). Fix \( n = 2, \beta = 3 \) and \( \Omega = \{ (x, y) \in \mathbb{R}^2 : x^2 < y < 1 \} \). Then \( \Omega \) is of class \( C^\infty \) in a neighborhood of the origin but it is not \( \beta \)-admissible. Taking \( u(x, y) = y^{-3/4} \) it can be easily checked that

\[
\int_{\Omega} |Xu|^2 \, dx \, dy = \int_{\Omega} |x|^{2\beta} |\partial_y u|^2 \, dx \, dy < +\infty \quad \text{but} \quad \int_{\Omega} |u|^2 \, dx \, dy = +\infty,
\]

and the Poincaré inequality (72) with \( p = 2 \) does not hold (see [35]).

The space \( \text{BV}_X(\Omega) \) of the function with bounded \( X \)-variation is defined as in (21) and (22). As usual, \( |\partial E|_X(\Omega) \) denotes the \( X \)-perimeter of a measurable set \( E \). If \( E \subset \mathbb{R}^n \) has Lipschitz boundary then by (25)

\[
|\partial E|_X(\Omega) = \int_{\partial E \cap \Omega} |Xn| \, d\mathcal{H}^{n-1},
\]

where

\[
|Xn(x, y)| := \left( |n_x(x, y)|^2 + |x|^{2\beta} |n_y(x, y)|^2 \right)^{1/2}
\]

and \( n = (n_x, n_y) \in \mathbb{R}^{n-1} \times \mathbb{R} \) is the unit normal to \( \partial E \) which is defined \( \mathcal{H}^{n-1} \)-almost everywhere on \( \partial E \).

**Remark 5.19.** Differently from the Heisenberg group (see Theorem 5.13) the metric space \( (\mathbb{R}^n, d) \) has not a metric dimension constant at every point and a representation of the perimeter in terms of a unique intrinsic Hausdorff measures is not available. Indeed, let \( n = 2 \) and \( X = (X_1, X_2) \) with \( X_1 = \partial_x \) and \( X_2 = x \partial_y \). Then it is easy to see that the Hausdorff dimension of \( (\mathbb{R}^2, d) \) is \( N = 2 \). The set \( E = \{ (x, y) \in \mathbb{R}^2 : 0 < x, y < 1 \} \) has Lipschitz boundary and \( |\partial E|_X(\mathbb{R}^2) < +\infty \). On the other hand, it is easy to see that the Hausdorff dimension of \( \partial E \cap \{ (x, y) \in \mathbb{R}^n : x = 0 \} \) is 2 whereas the one of \( \partial E \cap \{ (x, y) \in \mathbb{R}^n : x > 0 \} \) is 1 (see also [43]).

We finally come to the applications of the main results of Section 3 and Section 4 to this example. Let \( W, V, A_\alpha \) and \( \alpha \) be as in Example 5.9. The functionals \( G, G_\varepsilon : L^1(\Omega) \to [0, +\infty] \) are as in (41) but considering the Grushin vector fields \( X = (\partial_{x_1}, ..., \partial_{x_{n-1}}, |x|^{\beta} \partial_y) \). Let \( A(x, y) \) be a matrix as in Example 5.9 such that

\[
\langle A(x, y)\xi, \xi \rangle \geq C \langle B(x)\xi, \xi \rangle \quad \text{for some } C > 0 \text{ and for all } (x, y), \xi \in \mathbb{R}^n,
\]

where \( B(x) \) is the matrix (80). Let \( Q, Q_\varepsilon : L^1(\Omega) \to [0, +\infty] \) be defined as above, as well.
Theorem 5.20. Let \( \Omega \subset \mathbb{R}^n \) be a connected bounded open set of class \( C^2 \) and assume that it is \( \beta^- \)-admissible. Then all statements (i) – (vii) of Theorem 5.14 hold replacing \( H^1_\Omega \) with \( H^1_X(\Omega) \) and \( BV^1_\Omega \) with \( BV^1_X(\Omega) \).

References


