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Regularity Properties of H-Convex Sets

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Abstract We study the first- and second-order regularity properties of the boundary of H-convex sets in the setting of a real vector space endowed with a suitable group structure: our starting point is indeed a step two Carnot group. We prove that, locally, the noncharacteristic part of the boundary has the intrinsic cone property and that it is foliated by intrinsic Lipschitz continuous curves that are twice differentiable almost everywhere.

Keywords Horizontal convexity · Boundary regularity · Vector spaces · Step two Carnot groups

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1 Introduction

We study the first- and second-order regularity properties of the boundary of H-convex sets in the setting of a real vector space endowed with a non-commutative group law. Our interest is motivated by the recent theory of H-convex functions in Carnot groups. We prove that, locally, the noncharacteristic part of the boundary of H-convex sets has the intrinsic cone property and that it is "foliated" by intrinsic Lipschitz continuous curves that are twice differentiable almost everywhere.

We fix our geometric framework. Let Z and T be two real vector spaces and let $G = Z \times T$ be the product space. We let $p = (z, t)$ denote a generic point of G with $z \in Z$ and $t \in T$. Let $\langle \cdot, \cdot \rangle$ be an inner product on G that makes Z and T orthogonal, and let $|\cdot|$ denote the corresponding norm. Let $Q : Z \times Z \to T$ be a mapping satisfying the following axioms:

(Q1) Q is bilinear and continuous;
(Q2) Q is skew-symmetric, i.e., $Q(z, \zeta) = -Q(\zeta, z)$ for all $z, \zeta \in Z$;
(Q3) for all $t \in T$ and $z \in Z$, $z \neq 0$, there exists a $\zeta \in Z$ such that $Q(z, \zeta) = t$.

Up to a normalization of the quadratic form Q, we can assume that we have

$$|Q(z, \zeta)| \leq |z|\,|\zeta|, \quad \text{for all } z, \zeta \in Z. \tag{1.1}$$

We introduce the binary operation $\cdot : G \times G \to G$

$$(z, t) \cdot (\zeta, \tau) = \big(z + \zeta, t + \tau + Q(z, \zeta)\big). \tag{1.2}$$

By the axioms (Q1) and (Q2), the operation $\cdot$ is a group law. The associativity property is a consequence of (Q1). By (Q2), the identity element is $0 \in G$ and the inverse of $p = (z, t)$ is $p^{-1} = (-z, -t)$. In general, the group is non-commutative. An example where all the axioms (Q1)–(Q3) are satisfied is the n-th Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \in \mathbb{N}$, with

$$Q(z, \zeta) = \mathrm{Im}(z\bar{\zeta}), \quad z, \zeta \in \mathbb{C}^n,$$

where $z\bar{\zeta} = z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n$. In the setting of Lie algebras, the axioms (Q1)–(Q3) identify the Métivier's algebras.

The left translation by the element $p \in G$ is the map $\tau_p : G \to G$, $\tau_p(q) = p \cdot q$. It is an affine map of G onto itself as a vector space, i.e., it is the composition of a translation w.r.t. the sum operation and a linear mapping.

We define the *horizontal plane at* $p = 0$ as the linear subspace $Z_0 := Z \times \{0\} \subseteq G$, and the *horizontal plane at* $p \in G$ as the affine subspace $Z_p := \tau_p(Z_0) = p \cdot Z_0$. A *horizontal line through* $0 \in G$ is a 1-dimensional linear subspace of Z_0. A *horizontal line through* $p \in G$ is an affine line through p contained in Z_p, i.e., a line of the form $r := \tau_p(s) = p \cdot s$ for some horizontal line s through 0. We denote by $\mathcal{R}_p$ the set of all horizontal lines through p and by $\mathcal{R} = \bigcup_{p \in G} \mathcal{R}_p$ the set of all horizontal lines in G. Finally, we say that two points $p, q \in G$ are *horizontally aligned* if there exists an $r \in \mathcal{R}$ such that $p, q \in r$.

Definition 1.1 (H-convex set) A set $C \subseteq G$ is H-convex if $(1-\lambda)p + \lambda q \in C$ for any pair of horizontally aligned points $p, q \in C$ and for any $0 \le \lambda \le 1$.

Equivalently, a set $C \subseteq G$ is H-convex if and only if the set $C \cap Z_p$ is star-shaped with respect to p, for any $p \in C$.

The family of mappings $\{\delta_\lambda\}_{\lambda>0}$, $\delta_\lambda(z,t) = (\lambda z, \lambda^2 t)$, is a one-parameter group of automorphisms of G. We call these automorphisms of G *dilations*. For $\lambda = 0$ we let $\delta_\lambda(z,t) = 0$. The class of H-convex sets is stable under dilations.

H-convex sets were introduced by Danielli, Garofalo, and Nhieu as *weakly H-convex sets* (see Sect. 7 of [10]). The notion of *strongly H-convex set* was also introduced in the same paper. A set $C \subseteq G$ is strongly H-convex if $q \cdot \delta_\lambda(q^{-1} \cdot p) \in C$ for all (not necessarily aligned) $p, q \in C$ and all $0 \le \lambda \le 1$. In [10] it was observed that this stronger notion of convexity is quite restrictive. This was confirmed by the description of strongly H-convex sets in $\mathbb{H}^1$ given in [6]. In the same setting, a different notion of convexity, the geodesic convexity, was studied in [22]: the class of geodetically convex sets is also quite poor.

On the other hand, any convex set of G in the standard sense is H-convex, and there are H-convex sets which are not convex (see Sect. 4). In fact, there are H-convex sets which are not even Lebesgue measurable, as shown by Rickly in his Ph.D. thesis [24]. H-convex sets also arise as sub-level sets of H-convex real-valued functions defined on Carnot groups. Such functions are of interest because of their connection with nonlinear partial differential equations of the sub-elliptic type (see [3–5, 7, 8, 11, 14–19, 25–27]).

To state our results, we need a few more definitions. For any horizontal line $r \in \mathcal{R}_0$, let $\pi_r : G \to r$ be the *orthogonal projection* of G onto r with respect to the fixed inner product and denote by $r^\perp$ the orthogonal complement of r in G. Define the *(left) group projection* $\pi_r^\perp : G \to r^\perp$ via the identity

$$p = \pi_r^\perp(p) \cdot \pi_r(p), \quad \text{for any } p \in G. \tag{1.3}$$

The group projection $\pi_r^\perp$ is not the orthogonal projection onto $r^\perp$ (see Sect. 2).

Next, for $p = (z,t) \in G$ define the *homogeneous (quasi-)norm*

$$\|p\| = \max\{|z|, |t|^{1/2}\}. \tag{1.4}$$

The norm is homogeneous in the sense that $\|\delta_\lambda(p)\| = \lambda \|p\|$ for all $\lambda \ge 0$. Using this norm we define, for any $p \in G$ and $\rho > 0$, the balls

$$B_\rho(p) = \left\{q \in G : \|p^{-1} \cdot q\| < \rho\right\}. \tag{1.5}$$

The topology on G induced by these (or equivalent) balls is the standard topology of G, i.e., the one induced by the inner product. We shall denote by $\mathrm{int}(C)$, $\bar{C}$, $\mathrm{ext}(C)$ and ∂C the topological interior, closure, exterior, and boundary of a given set $C \subseteq G$, respectively.

Definition 1.2 (Intrinsic cone)

(i) The *(left) open cone* with vertex $0 \in G$, axis $r \in \mathcal{R}_0$, aperture $\alpha > 0$, and height $h > 0$ is the set

$$C_L(0, r, \alpha, h) = \big\{p \in G : \|\pi_r^{\perp}(p)\| < \alpha\|\pi_r(p)\| < \alpha h\big\}. \tag{1.6}$$

Fix one of the two total orderings of r such that $0 \in r$ is the zero. Define the *positive and negative cones*

$$\begin{aligned} C_L^+(0, r, \alpha, h) &= C_L(0, r, \alpha, h) \cap \{p \in G : \pi_r(p) > 0\}, \\ C_L^-(0, r, \alpha, h) &= C_L(0, r, \alpha, h) \cap \{p \in G : \pi_r(p) < 0\}. \end{aligned} \tag{1.7}$$

(ii) The *(left) open cone* with vertex $p \in G$, axis $r \in \mathcal{R}_p$, aperture $\alpha > 0$, and height $h > 0$ is the set

$$C_L(p, r, \alpha, h) = \tau_p\big(C_L(0, \tau_{p^{-1}}(r), \alpha, h)\big).$$

The one-sided cones $C_L^+(p, r, \alpha, h)$ and $C_L^-(p, r, \alpha, h)$ are defined analogously.

Definition 1.3 (Non-characteristic point) Let $C \subseteq G$ be a set. A point $p \in \partial C$ is *non-characteristic* if there exists an $r \in \mathcal{R}_p$ such that $r \cap \mathrm{int}(C) \cap B_\rho(p) \neq \emptyset$ for all $\rho > 0$. In this case, we say that *r enters C at p*.

We let $\Sigma(C) \subseteq \partial C$ denote the set of *characteristic* points of C. Thus $p \in \partial C$ is non-characteristic if and only if $p \in \partial C \setminus \Sigma(C)$.

Theorem 1.4 *Assume* (Q1)–(Q3). *Let $C \subseteq G$ be an H-convex set and let $p \in \partial C \setminus \Sigma(C)$ with $\tau_p(r) \in \mathcal{R}_p$ entering C at p, for some $r \in \mathcal{R}_0$. Then there exist $\rho > 0$, $\alpha > 0$, and $h > 0$ such that we have for all $q \in \partial C \cap B_\rho(p)$*

$$C_L^+(q, \tau_q(r), \alpha, h) \subseteq \mathrm{int}(C), \tag{1.8}$$

$$C_L^-(q, \tau_q(r), \alpha, h) \subseteq \mathrm{ext}(C), \tag{1.9}$$

where the total ordering of r is possibly changed.

Axiom (Q3) is used in the construction of the interior cone: see *Step 3* in the proof of Theorem 1.4. It is not clear whether this axiom can be dropped or weakened. The interior cone property (1.8) is sharp, because the cone $C_L^+(q, \tau_q(r), \alpha, h)$ is itself H-convex. We prove this in the Heisenberg group (Example 4.2).

No regularity for the boundary can be expected at characteristic points. In fact, for any $\alpha > 0$ there is a $\beta > 0$ such that the set

$$\big\{(z, t) \in \mathbb{C} \times \mathbb{R} = \mathbb{H}^1 : |z| < t^\alpha < \beta\big\} \tag{1.10}$$

is H-convex in $\mathbb{H}^1$ (see Example 4.3).

Conditions (1.8) and (1.9) express the intrinsic cone property for H-convex sets. This property was introduced in [12] and [2] in order to define intrinsic Lipschitz continuous graphs inside Carnot groups (see also [21] for a different construction of metric cones). The same property appears in the theory of sets with finite horizontal perimeter and controlled normal (see [23]).

In the case of the Heisenberg group, Franchi, Serapioni, and Serra Cassano have recently proved in [13] that the cone property implies an intrinsic version of Rademacher's theorem. Theorem 1.4 is thus the counterpart of the first-order regularity of H-convex functions established under various a priori assumptions in [10, 17–19, 26], and then in [3] and [25] in full generality. So far, the best known regularity of H-convex sets is that they have locally finite horizontal perimeter (see [20]).

Our results on the second-order regularity of H-convex sets have a partial character. Roughly speaking, the non-characteristic boundary $\partial C \setminus \Sigma(C)$ is "foliated" by Lipschitz continuous curves in G which are convex in a suitable sense and which are twice differentiable almost everywhere in the standard sense. We say that a curve $\gamma : I \to G$, where $I \subseteq \mathbb{R}$ is an interval, is *Lipschitz continuous in* G if there is a constant $L > 0$ such that for all $s, \sigma \in I$

$$\|\gamma(\sigma)^{-1} \cdot \gamma(s)\| \le L|s - \sigma|. \tag{1.11}$$

The "*convex horizontal sections*" of $\partial C \setminus \Sigma(C)$ are found inside (cosets of) Heisenberg subgroups of G which are horizontally transversal to the given H-convex set.

We say that a subgroup H of G is a *Heisenberg subgroup* if it is of the form

$$H = \operatorname{span}\big\{(z, 0), (\zeta, 0), (0, t)\big\}, \tag{1.12}$$

for some $z, \zeta \in Z$ and $t \in T$ such that $Q(z, \zeta) = t \neq 0$. Finally, a Heisenberg subgroup H of G is *horizontally transversal* to a set $C \subseteq G$ at the point $0 \in \partial C$ if there exists an $r \in \mathcal{R}_0$ such that $r \subset H$ and r enters C at 0.

Theorem 1.5 *Assume* (Q1)–(Q3). *Let $C \subseteq G$ be an H-convex set, and let $H \subseteq G$ be a Heisenberg subgroup of G that is horizontally transversal to C at* 0, *with $r \in \mathcal{R}_0$ entering C at* 0 *and $r \subset H$. Then there exists a nonconstant curve $\gamma : I \to \partial C \cap H$, for some interval $I = [0, \delta]$, $\delta > 0$, with the following properties*:

(1) *γ is Lipschitz continuous in G and $\gamma(0) = 0$*;
(2) *$\gamma = (\zeta, \tau)$ for curves $\zeta : I \to Z$ and $\tau : I \to T$ such that ζ is twice differentiable a.e. on I and τ is three times differentiable a.e. on I*;
(3) *the function $s \mapsto \pi_r(\gamma(s)) \in r$, $s \in [0, \delta]$, is either convex or concave on I.*

In general, the curve γ provided by Theorem 1.5 is not unique: there can exist a curve $\bar{\gamma} : [0, \bar{\delta}] \to \partial C \cap H$ satisfying (1), (2), and (3) but such that $\gamma([0, s]) \neq \bar{\gamma}([0, \sigma])$ for all $s > 0$ and $\sigma > 0$ (see Example 4.1).

Theorem 1.5 is the counterpart of the second-order regularity of H-convex functions. In fact, H-convex functions in $\mathbb{H}^n$, $n = 1, 2$, have second-order horizontal derivatives almost everywhere (see [1, 11, 16, 19], and [14] for related results). Our result should be compared with Theorem 1.1 in [9]. In their article, Capogna, Pauls, and Tyson prove, under a C^2-type regularity assumption, that the epigraph of real-valued functions in Carnot groups is H-convex if and only if the symmetrized horizontal second fundamental form of the boundary is nonnegative.

An overview of the paper is now in order. Section 2 is devoted to the cone property. In Sect. 3, we construct the convex and Lipschitz sections. The examples are discussed in Sect. 4.

2 Intrinsic Cone Property

Let $r \in \mathcal{R}_0$ be a horizontal line through 0. For some $z_0 \in Z$ with $|z_0| = 1$ we then have $r = \{(\lambda z_0, 0) : \lambda \in \mathbb{R}\}$. We can identify r with $\mathbb{R}$ and give r a natural total ordering. Here and in the following, we restrict $\langle \cdot, \cdot \rangle$ to Z and T. Let $r^\perp$ denote the orthogonal complement of r in G. Then we have

$$G = r^\perp \cdot r, \quad r^\perp \cap r = \{0\}, \ r^\perp \text{ is normal in } G,$$

i.e., G is the semi-direct product of the subgroups $r^\perp$ and r.

The *orthogonal projection* $\pi_r : G \to r$ of G onto r is $\pi_r(p) = (\langle z, z_0\rangle z_0, 0)$, for any $p = (z,t) \in G$. The *left group projection* $\pi_r^\perp : G \to r^\perp$ defined via (1.3) is given by the formula

$$\pi_r^\perp(z,t) = \big(z - \langle z, z_0\rangle z_0, t - \langle z, z_0\rangle Q(z, z_0)\big).$$

This is not the orthogonal projection onto $r^\perp$.

Analogously, we can define the *right group projection* $\widehat{\pi}_r^\perp : G \to r^\perp$ via the identity $p = \pi_r(p) \cdot \widehat{\pi}_r^\perp(p)$. In this case we have the formula

$$\widehat{\pi}_r^\perp(z,t) = \big(z - \langle z, z_0\rangle z_0, t + \langle z, z_0\rangle Q(z, z_0)\big).$$

We introduced the left cones in Definition 1.2. Now let the *right open cone* with vertex $0 \in G$, axis $r \in \mathcal{R}_0$, aperture $\alpha > 0$, and height $h > 0$ be the set

$$C_R(0, r, \alpha, h) = \big\{p \in G : \|\widehat{\pi}_r^\perp(p)\| < \alpha \|\pi_r(p)\| < \alpha h\big\}. \tag{2.1}$$

Finally, let the positive and negative right cones be the sets

$$\begin{aligned} C_R^+(0, r, \alpha, h) &= C_R(0, r, \alpha, h) \cap \{p \in G : \pi_r(p) > 0\}, \\ C_R^-(0, r, \alpha, h) &= C_R(0, r, \alpha, h) \cap \{p \in G : \pi_r(p) < 0\}. \end{aligned} \tag{2.2}$$

The right cones with vertex $p \in G$ are defined by left translation.

Lemma 2.1 *The left and right cones are comparable in the following quantitative sense*:

$$\begin{aligned} &C_L(0, r, \alpha, h) \subseteq C_R(0, r, \alpha + \sqrt{2\alpha}, h) \quad \text{and} \\ &C_R(0, r, \alpha, h) \subseteq C_L(0, r, \alpha + \sqrt{2\alpha}, h). \end{aligned} \tag{2.3}$$

Proof Indeed, for $(z,t) \in C_L(0, r, \alpha, h)$ and with $b = \langle z, z_0\rangle$, we have

$$|z - bz_0| < \alpha|b| \quad \text{and} \quad |t - bQ(z, z_0)|^{1/2} < \alpha|b|. \tag{2.4}$$

From (2.4), we get

$$\begin{aligned}|t+bQ(z,z_0)|^{1/2} &\le |t-bQ(z,z_0)|^{1/2}+|2bQ(z,z_0)|^{1/2}\\ &< \alpha|b|+\sqrt{2|b|}|Q(z,z_0)|^{1/2},\end{aligned}$$

where, by (Q1) along with (1.1), and (Q2), $|Q(z,z_0)| = |Q(z-bz_0,z_0)| \le |z-bz_0| < \alpha|b|$, and thus

$$|t+bQ(z,z_0)|^{1/2} < \big(\alpha+\sqrt{2\alpha}\big)|b|.$$

This shows that $(z,t)\in C_R(0,r,\alpha+\sqrt{2\alpha},h)$.

The proof of the second inclusion in (2.3) is analogous. □

We now compare left cones having the same vertex and different axes. Let $r,s\in\mathcal{R}_0$ be horizontal lines associated with the points $z_0,\zeta_0\in Z$, with $|z_0|=|\zeta_0|=1$ and $\langle z_0,\zeta_0\rangle\ge 0$. Namely, let $r=\{(\lambda z_0,0)\in G:\lambda\in\mathbb{R}\}$ and $s=\{(\lambda\zeta_0,0)\in G:\lambda\in\mathbb{R}\}$. Let the distance between r and s be

$$\operatorname{dist}(r,s)=|z_0-\zeta_0|. \tag{2.5}$$

Lemma 2.2 *For any $k>1$ and $\varepsilon>0$ there exists an $\alpha_0>0$ such that for all $0<\alpha<\alpha_0$ and $r,s\in\mathcal{R}_0$ with $\operatorname{dist}(r,s)\le\alpha^{2+\varepsilon}$ we have $C_L(0,r,\alpha,1)\subseteq C_L(0,s,k\alpha,2)$.*

Proof With the notation introduced above, we have $(z,t)\in C_L(0,r,\alpha,1)$ if and only if

$$|z|<\sqrt{1+\alpha^2}|b|<\sqrt{1+\alpha^2}\quad\text{and}\quad |t-bQ(z,z_0)|^{1/2}<\alpha|b|, \tag{2.6}$$

where we let $b=\langle z,z_0\rangle$. Let $\delta>0$ be a real number to be fixed later and such that

$$\delta\sqrt{1+\alpha^2}<1. \tag{2.7}$$

Assume that $|z_0-\zeta_0|\le\delta$. Then from the first inequality in (2.6) and from $|b|\le|\langle z,\zeta_0\rangle|+\delta|z|$ it follows that

$$|z|<\beta|\langle z,\zeta_0\rangle|\quad\text{with } \beta=\frac{\sqrt{1+\alpha^2}}{1-\delta\sqrt{1+\alpha^2}}. \tag{2.8}$$

By the triangle inequality, (Q1), and the right-hand side of (2.6), we obtain

$$\begin{aligned}\big|t-\langle z,\zeta_0\rangle Q(z,\zeta_0)\big|^{1/2} &\le \big|t-bQ(z,z_0)\big|^{1/2}+\big|\langle z,z_0-\zeta_0\rangle Q(z,z_0)\big|^{1/2}\\ &\quad+\big|\langle z,\zeta_0\rangle Q(z,z_0-\zeta_0)\big|^{1/2}\\ &< \alpha|b|+2\sqrt{\delta}|z|\\ &\le \alpha|\langle z,\zeta_0\rangle|+\big(\alpha\delta+2\sqrt{\delta}\big)|z|.\end{aligned}$$

By (2.8), we finally get

$$\left|t - \langle z, \zeta_0\rangle Q(z, \zeta_0)\right|^{1/2} < \gamma|\langle z, \zeta_0\rangle|$$
$$\text{with } \gamma = \alpha + \frac{\sqrt{1+\alpha^2}(\alpha\delta + 2\sqrt{\delta})}{1 - \delta\sqrt{1+\alpha^2}}, \tag{2.9}$$

and the claim follows from (2.8) and (2.9). In fact, with the choice $\delta = \alpha^{2+\varepsilon}$, for any $k > 1$ there is an $\alpha_0 > 0$ such that $\beta \le \sqrt{1 + k^2\alpha^2}$ and $\gamma \le k\alpha$ for all $0 < \alpha < \alpha_0$. □

In the proof of Theorem 1.4 and in the following sections, we shall make free use of the following observation. For any pair of points $p = (z, t)$ and $q = (\zeta, \tau)$ in G, the following statements are equivalent: (1) p and q are horizontally aligned; (2) $p \in Z_q$; (3) $q \in Z_p$; (4) $q^{-1} \cdot p \in Z_0$; (5) $\tau - t = Q(z, \zeta)$.

Proof of Theorem 1.4 The proof is divided into a number of steps. The central step is *Step 4*, where we study the 3-dimensional case (i.e., the Heisenberg group). We first prove (1.8) and we show in the last step that (1.9) follows from (1.8).

Step 1. Ler $r \in \mathcal{R}_0$ be a horizontal line such that $\tau_p(r)$ enters C at p. We claim without proof that the set $\mathrm{int}(C)$ is H-convex. The proof is a routine argument based on axioms (Q1) and (Q2). The set $\tau_p(r) \cap \mathrm{int}(C)$ is therefore a nonempty open interval. One endpoint of this interval is $p \in \partial C$. An easy continuity argument shows that there is a $\rho > 0$ such that $\tau_q(r) \cap \mathrm{int}(C)$ is a nonempty open interval for all $q \in \partial C \cap B_\rho(p)$.

We claim that, possibly changing the orientation of r, there exist an $\alpha > 0$ and an $h > 0$ such that $C_L^+(q, \tau_q(r), \alpha, h) \subset \mathrm{int}(C)$ for all $q \in \mathrm{int}(C) \cap B_\rho(p)$. Let $q \in \partial C \cap B_\rho(p)$ and assume without loss of generality that $q = 0$. This can be achieved by a left translation. The horizontal line $r \in \mathcal{R}_0$ is of the form $r = \{(\lambda z_0, 0) \in G : \lambda \in \mathbb{R}\}$ for some $z_0 \in Z$ with $|z_0| = 1$. We assume without loss of generality that $(z_0, 0) \in \mathrm{int}(C)$. This can be achieved by a dilatation. We also agree that r is oriented in such a way that $(z_0, 0) \in r$ is positive.

There exists $0 < \sigma < 1$ such that $B_\sigma(z_0, 0) \subset \mathrm{int}(C)$. The number σ does not depend on our initial choice of $\rho > 0$ (possibly take a smaller ρ).

Step 2. Consider the sets $E, F \subset G$

$$E = \left\{(z, t) \in G : \max\{|z - z_0|, |t + Q(z, z_0)|^{1/2}\} \le \sigma,\ \langle z, z_0\rangle = 1\right\}, \tag{2.10}$$
$$F = \left\{(z, 0) \in G : |z - \langle z, z_0\rangle z_0| < \sigma^2\langle z, z_0\rangle \le \sigma^2\right\}. \tag{2.11}$$

The set E is a certain "vertical" section of $B_\sigma(z_0, 0)$ and thus it is contained in $\mathrm{int}(C)$. The set E is H-convex. The set F is a truncated, positive cone in $Z \times \{0\}$. By (Q1)–(Q3), we have $|Q(z, z_0)| = |Q(z - z_0, z_0)| \le |z - z_0|$ and then $F \cap \{\langle z, z_0\rangle = 1\} \subset E \subset \mathrm{int}(C)$. The set F is also H-convex. Then, from the H-convexity of $\mathrm{int}(C)$ it follows that $F \subset \mathrm{int}(C)$.

Step 3. Let A be the first H-convex envelope of $E \cup F$. Namely,

$$A = \left\{(1-\lambda)p_1 + \lambda p_2 : p_1 \in E,\ p_2 \in F,\ p_1 \text{ and } p_2 \text{ horizontally aligned},\ 0 \le \lambda \le 1\right\}.$$

Because $E \cup F \subset \mathrm{int}(C)$, from the H-convexity of $\mathrm{int}(C)$ it follows that $A \subseteq \mathrm{int}(C)$. We claim there exists an $\alpha > 0$ depending only on ρ such that $C_L^+(0, r, \alpha, 1) \subset A$. The claim will be proved, as soon as we show that given $(\zeta, \tau) \in C_L^+(0, r, \alpha, 1)$ there are points $p_1 = (z_1, t_1) \in E$, $p_2 = (z_2, 0) \in F$, and $\lambda \in (0, 1)$ such that:

(a) the points p_1 and p_2 are horizontally aligned;
(b) $\lambda p_2 + (1 - \lambda) p_1 = (\zeta, \tau)$.

Observe that p_1 and p_2 are horizontally aligned if and only if $t_1 = Q(z_2, z_1)$. Statements (a) and (b) are thus equivalent to the system of equations $\lambda z_2 + (1 - \lambda) z_1 = \zeta$ and $(1 - \lambda) Q(z_2, z_1) = \tau$. Inserting the first equation into the second one, we find the equation $Q(z_2, \zeta) = \tau$. By (Q3), this equation has a solution $z_2 \in Z$ for any given $\zeta \in Z$, $\zeta \neq 0$. Conditions (a) and (b) are thus equivalent to the system of equations

$$\begin{cases} \lambda z_2 + (1 - \lambda) z_1 = \zeta, \\ Q(z_2, \zeta) = \tau, \\ Q(z_2, z_1) = t_1. \end{cases} \tag{2.12}$$

We solve system (2.12) with the restrictions $p_1 \in E$ and $p_2 \in F$ in the 3-dimensional case first.

Step 4. Let $Z = \mathbb{C}$, $T = \mathbb{R}$, and $Q : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$, $Q(z, z') = \mathrm{Im}(z\bar{z}')$. Let $r \in \mathcal{R}_0$ be the horizontal line identified by $z_0 \in \mathbb{C}$. We can assume that $z_0 = 1 \in \mathbb{C}$. This can be achieved by a rotation in $\mathbb{C}$. Let $\alpha > 0$ be a real number such that

$$8(\alpha^2 + \alpha) < \sigma^2. \tag{2.13}$$

Finally, let $(\zeta, \tau) \in C_L^+(0, r, \alpha, 1)$, i.e., with $\zeta = \xi + i\eta$,

$$|\eta| < \alpha\xi < \alpha \quad \text{and} \quad |\tau - \xi\eta|^{1/2} < \alpha\xi. \tag{2.14}$$

We claim that for all $\zeta = \xi + i\eta, \tau$ satisfying (2.14) we can find $\lambda \in (0, 1)$, $p_1 = (z_1, t_1) \in E$, and $p_2 = (z_2, 0) \in F$ solutions to the system (2.12). The point $p_1 = (z_1, t_1)$, $z_1 = x_1 + iy_1$, belongs to the set E in (2.10) if and only if

$$x_1 = 1, \qquad |y_1| \le \sigma \quad \text{and} \quad |t_1 + y_1| \le \sigma^2. \tag{2.15}$$

The point $p_2 = (z_2, 0)$, $z_2 = x_2 + iy_2$, belongs to the "triangle" F in (2.11) if and only if

$$|y_2| < \sigma^2 x_2 \le \sigma^2. \tag{2.16}$$

The parameter λ can be determined through the first equation in (2.12), i.e.,

$$\lambda z_2 + (1 - \lambda) z_1 = \zeta. \tag{2.17}$$

In particular, using the first scalar equation in the vector equation (2.17) we get

$$\lambda = \frac{1 - \xi}{1 - x_2}. \tag{2.18}$$

With the restriction $0 < x_2 < \xi \leq 1$, we have $\lambda \in (0,1)$. In particular, we fix x_2 in the following way

$$x_2 = \frac{\xi}{2} < 1. \tag{2.19}$$

We solve the second equation in (2.17) and the second equation in (2.12), $Q(z_2, \zeta) = \tau$, in y_1 and y_2. Using (2.18), we find:

$$\begin{cases} y_1 = \frac{1}{\xi}(\eta - \frac{1-\xi}{\xi - x_2}\tau), \\ y_2 = \frac{1}{\xi}(\tau + x_2\eta). \end{cases} \tag{2.20}$$

We check condition (2.16). By the triangle inequality, (2.14), (2.19), and (2.13)

$$|y_2| = \frac{|\tau + x_2\eta|}{\xi} \leq \frac{|\tau - \xi\eta| + |\eta|(\xi + x_2)}{\xi} \leq 3(\alpha^2 + \alpha)x_2 < \sigma^2 x_2. \tag{2.21}$$

We check the second condition in (2.15). By the triangle inequality, (2.14), (2.19), (2.13), and $0 < \sigma \leq 1$ we find

$$\begin{aligned} |y_1| &= \frac{1}{\xi}\left|\eta - \tau\frac{1-\xi}{\xi - x_2}\right| \leq \frac{1}{\xi(\xi - x_2)}((1-\xi)|\tau - \eta\xi| + (\xi^2 + x_2)|\eta|) \\ &\leq 4(\alpha^2 + \alpha) \leq \sigma^2 \leq \sigma. \end{aligned} \tag{2.22}$$

Finally, we check the condition on the right-hand side of (2.15). Notice that t_1 is determined by the third equation in (2.12) along with (2.20). We find:

$$\xi|y_1 + t_1| = \xi|y_2 + (1 - x_2)y_1| = \frac{1}{\xi - x_2}|\eta(\xi - x_2) + \tau(2\xi - x_2\xi - 1)|,$$

and then by the triangle inequality, (2.14), (2.19), and (2.13)

$$\begin{aligned} |y_1 + t_1| &\leq \frac{1}{\xi(\xi - x_2)}(|2\xi - x_2\xi - 1||\tau - \eta\xi| + |\eta||2\xi^2 - x_2\xi^2 - x_2|) \\ &\leq 8(\alpha^2 + \alpha) \leq \sigma^2 \leq \sigma. \end{aligned}$$

This estimates finishes the proof of *Step 4*.

Step 5. In this step, we reduce the general case to the case discussed in *Step 4*. With the reduction made in *Step 1*, we show that there exists an $\alpha > 0$ such that

$$C_L^+(0, r, \alpha^3, 1/2) \subset \mathrm{int}(C), \tag{2.23}$$

where α does not depend on the vertex 0, but it is uniform in a neighborhood of $0 \in \partial C$. Given $(\zeta, \tau) \in G$ with $\zeta \neq 0$, by (Q3) there exists a $\zeta' \in Z$ such that $Q(\zeta, \zeta') = \tau$. We can assume that $\tau \neq 0$, otherwise our claim is clear. Fix such a ζ' depending on ζ (it need not be unique). The linear span

$$G_{\zeta,\tau} = \mathrm{span}\{(\zeta, 0), (\zeta', 0), (0, \tau)\}$$

is a 3-dimensional linear subspace of G that is also a subgroup of G isomorphic to the Heisenberg group $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$. We denote by $s_\zeta \in \mathcal{R}_0$ the horizontal line identified by $\zeta \in Z$, $\zeta \neq 0$. By Lemma 2.2, there exists an $0 < \alpha_0 \leq 1/4$ such that for all $0 < \alpha < \alpha_0$ we have

$$\begin{aligned} &\operatorname{dist}(r, s_\zeta) \leq \alpha^{5/2} \\ &\quad \Longrightarrow \quad C_L^+(0, r, 2\alpha^3, 1/2) \subseteq C_L^+(0, r, \alpha, 1/2) \subseteq C_L^+(0, s_\zeta, 2\alpha, 1). \end{aligned} \tag{2.24}$$

If $(\zeta, \tau) \in C_L^+(0, r, \alpha^3, 1/2)$ then we have $\operatorname{dist}(r, s_\zeta) \leq 2\alpha^3 \leq \alpha^{5/2}$. Moreover, by *Step 4*, we have

$$G_{\zeta,\tau} \cap C_L^+(0, s_\zeta, 2\alpha, 1) \subset G_{\zeta,\tau} \cap \operatorname{int}(C), \tag{2.25}$$

as soon as α is small enough, independently from $(\zeta, \tau) \in C_L^+(0, r, \alpha^3, 1/2)$. From (2.24) and (2.25), we deduce that

$$\begin{aligned} C_L^+(0, r, \alpha^3, 1/2) &= \bigcup_{(\zeta,\tau) \in C_L^+(0,r,\alpha^3,1/2)} G_{\zeta,\tau} \cap C_L^+(0, r, \alpha^3, 1/2) \\ &\subset \bigcup_{(\zeta,\tau) \in C_L(0,r,\alpha^3,1/2)} G_{\zeta,\tau} \cap C_L^+(0, s_\zeta, 2\alpha, 1) \\ &\subset \operatorname{int}(C). \end{aligned}$$

Step 6. We prove that (1.9) follows from (1.8). We assume that we have $p = 0$ in the statement of Theorem 1.4. Let $\iota : G \to G$ be the mapping $\iota(p) := p^{-1} = -p$. Then we have the relations

$$\begin{aligned} \iota(C_R^+(0, r, \alpha, h)) &= C_L^-(0, r, \alpha, h), \\ \iota(C_R^-(0, r, \alpha, h)) &= C_L^+(0, r, \alpha, h). \end{aligned} \tag{2.26}$$

Let $r \in \mathcal{R}_0$, $\alpha > 0$ and assume that $C_L^+(p, \tau_p(r), \alpha, 1) \subseteq \operatorname{int}(C)$ for all $p \in \partial C \cap B_\rho(0)$ for some $\rho > 0$. Let $\beta > 0$ be such that $\beta + \sqrt{2\beta} = \alpha$. For any $p, q \in \partial C \cap B_\rho(0)$, by (2.26) it follows that

$$\begin{aligned} q \notin C_L^+(p, \tau_p(r), \alpha, 1) \quad &\Longleftrightarrow \quad p^{-1} \cdot q \notin C_L^+(0, r, \alpha, 1) \\ &\Longleftrightarrow \quad q^{-1} \cdot p \notin C_R^-(0, r, \alpha, 1). \end{aligned}$$

Now, from (2.3) we deduce that $q^{-1} \cdot p \notin C_L^-(0, r, \beta, 1)$. This is equivalent to $p \notin C_L^-(q, \tau_q(r), \beta, 1)$ and the claim follows. $\square$

3 Horizontal Second-Order Regularity

In this section, we fix the factorization $G = r^\perp \cdot r$ for some $r \in \mathcal{R}_0$. On identifying r and $\mathbb{R}$, we have a natural total ordering on r, which is inherited by any $\tau_p(r) \in \mathcal{R}_p$, with $p \in G$.

Let $\phi : W \to r$ be a continuous function on some open set $W \subseteq r^\perp$. The *intrinsic graph* of ϕ is the subset of G

$$\mathrm{gr}(\phi) = \{q \cdot \phi(q) \in G : q \in W\}. \tag{3.1}$$

The function ϕ is *intrinsic Lipschitz continuous* if there exists a constant $L > 0$ such that for all $p \in \mathrm{gr}(\phi)$

$$C_L(p, \tau_p(r), 1/L, +\infty) \cap \mathrm{gr}(\phi) = \emptyset. \tag{3.2}$$

Analogously, the *intrinsic epigraph* of ϕ is the subset of G

$$\mathrm{epi}(\phi) = \{p \cdot q \in G : p \in W,\ q \in \tau_p(r),\ q > \tau_p(\phi(p))\}. \tag{3.3}$$

Any curve $\gamma : I \to \mathrm{gr}(\phi)$, where $I \subseteq \mathbb{R}$ is an interval, has the factorization

$$\gamma = \kappa \cdot \phi(\kappa), \tag{3.4}$$

where $\kappa : I \to W$ and $\kappa(s) = (\zeta(s), \tau(s))$ with $\zeta(s) \in Z$ and $\tau(s) \in T$ for all $s \in I$.

Definition 3.1 We say that a function $\phi : W \to r$ is convex along the curve $\kappa : I \to W$, $I \subseteq \mathbb{R}$ an interval, if the function $\phi \circ \kappa : I \to r = \mathbb{R}$ is convex.

Theorem 3.2 *Let $C \subseteq G$ be a closed H-convex set, $W \subseteq r^\perp$ be an open set with $0 \in W$, and $\phi : W \to r$ be such that* $\mathrm{epi}(\phi) \cap U = \mathrm{int}(C) \cap U$ *for some open set U. For any $\zeta \in Z$ with $(\zeta, 0) \in W$, there exists a curve $\gamma : I \to$* $\mathrm{gr}(\phi)$, *for some interval $I = [0, \delta]$ with $\delta > 0$, such that*:

(i) *γ is Lipschitz continuous in G*;
(ii) *$\gamma = \kappa \cdot \phi(\kappa)$ for a curve $\kappa : I \to W$ such that ϕ is convex along κ*;
(iii) *$\kappa(s) = (s\zeta, \tau(s))$, $s \in I$, for some curve $\tau : I \to T$ such that $\tau(0) = 0$.*

Proof Without loss of generality, we assume that $W = \{p = (z, t) \in r^\perp : |z| < 4, |t| < 4\}$, and $\phi(0) = 0$. This is possible by a dilatation and a left translation. By Theorem 1.4, there exists a constant $\alpha > 0$ such that for all $p \in \mathrm{gr}(\phi) \cap U$

$$C_L(p, \tau_p(r), \alpha, +\infty) \cap \mathrm{gr}(\phi) = \emptyset. \tag{3.5}$$

In particular, ϕ is continuous and bounded. We can assume that $\mathrm{gr}(\phi) \cap U = \mathrm{gr}(\phi)$ and $|\phi| \le 1$ on W.

Fix $\zeta \in Z$ such that $|\zeta| = 1$ and $(\zeta, 0) \in W$. We set $\delta = 1$ and we construct a curve $\gamma : [0, 1] \to \mathrm{gr}(\phi)$ such that $\gamma(0) = 0$ and (i), (ii), and (iii) hold. The curve γ is obtained as the limit of a sequence of polygonal curves. Each polygonal curve is made up by horizontal segments and is contained in the set C. By the H-convexity of C, these curves enjoy a suitable convexity property which will be explained along in the proof. This property passes to the limit, yielding the convexity of ϕ along the component κ of γ.

We need a preliminary remark. Let $h \in \mathbb{N}$. For some $q = (z_0, t_0) \in W$, with $|z_0| < 2$ and $|t_0| < 2$, we look for a point $p = (z, t) \in W$ such that

$$(q \cdot \phi(q))^{-1} \cdot (p \cdot \phi(p)) = (\zeta/h + \phi(p) - \phi(q), 0). \tag{3.6}$$

By (3.6), the points $q \cdot \phi(q)$ and $p \cdot \phi(p)$ are horizontally aligned. This is equivalent to solving the system

$$\begin{cases} z - z_0 = \zeta/h, \\ t - t_0 = Q(\phi(z,t) + \phi(z_0, t_0), z - z_0). \end{cases} \tag{3.7}$$

Plugging the first equation into the second one, we get the equation for t

$$t = t_0 + \frac{1}{h} Q(\phi(z,t) + \phi(z_0,t_0), \zeta) =: \Phi(t), \tag{3.8}$$

where $\Phi : \{t \in T : |t| < 4\} \to T$ is the mapping defined in the right-hand side of (3.8). By (1.1), $|\zeta| = 1$, and $|\phi| \le 1$, we have $|\Phi(t) - t_0| \le 2/h$. Then Φ is continuous from the closed ball $B \subset T$ centered at t_0 with radius $2/h$ into itself, and therefore it has at least one fixed point $t \in B$, i.e., there is a solution $t \in B$ to (3.8). Notice that (3.8) is essentially one-dimensional. Then $p = (z, t)$ is a solution to the system (3.7).

For any $h \in \mathbb{N}$, we define by induction points $p_0, p_1, \dots, p_h \in W$. Each of these points depends on h. We let $p_0 = 0$ and assume that $p_0, p_1, \dots, p_{j-1} \in W$ satisfy $p_i = (z_i, t_i)$ with $|z_i| \le i/h$ and $|t_i| \le 2i/h$ for $i = 0, 1, \dots, j-1$. Denote by $P_j^h \subset W$ the set of the points $p = (z,t) \in W$ that are solutions to the system (3.7) with data $(z_0, t_0) = p_{j-1}$ and such that $|z - z_{j-1}| \le 1/h$ and $|t - t_{j-1}| \le 2/h$. The previous argument proves that $P_j^h \neq \emptyset$. Choose one $p_j \in P_j^h$. This choice is not unique, in general.

Let us define the curve $\gamma^h : I \to G$, $I = [0,1]$, in the following way. Let $I_j^h = [(j-1)/h, j/h]$, $j = 1, \dots, h$. Then we have $I = I_1^h \cup \dots \cup I_h^h$. For $s \in I_j^h$ we let

$$\gamma^h(s) = h\left\{\left(\frac{j}{h} - s\right)\big(p_{j-1} \cdot \phi(p_{j-1})\big) + \left(s - \frac{j-1}{h}\right)\big(p_j \cdot \phi(p_j)\big)\right\}. \tag{3.9}$$

The sequence of curves $(\gamma^h)_{h \in \mathbb{N}}$ has the following properties:

(a) $\gamma^h = (\zeta^h, \tau^h)$ for curves $\zeta^h : I \to Z$ and $\tau^h : I \to T$ such that

$$\zeta^h(s) = s\zeta + h\left\{\left(\frac{j}{h} - s\right)\phi(p_{j-1}) + \left(s - \frac{j-1}{h}\right)\phi(p_j)\right\}, \quad s \in I_j^h; \tag{3.10}$$

(b) $\gamma^h = \pi_r^\perp(\gamma^h) \cdot \pi_r(\gamma^h)$ for curves $\pi_r^\perp(\gamma^h) : I \to W$ and $\pi_r(\gamma^h) : I \to r$ such that $s \mapsto \pi_r(\gamma^h(s)) \in r$ is convex;

(c) the sequence $(\gamma^h)_{h \in \mathbb{N}}$ is equi-Lipschitz continuous and equi-bounded in G endowed with the quasi-distance induced by the norm (1.4).

Formula (3.10) follows from (3.9) and from the recursive definition of p_j. In particular, we have

$$\pi_r(\gamma^h(s)) = h\left\{\left(\frac{j}{h} - s\right)\phi(p_{j-1}) + \left(s - \frac{j-1}{h}\right)\phi(p_j)\right\}, \quad s \in I_j^h. \tag{3.11}$$

Let $g_j = p_j \cdot \phi(p_j)$. As the points $g_{j-1}, g_j \in \partial C$ are horizontally aligned, the segment joining them is contained in some horizontal line and thus it is contained in C, by the H-convexity of C. From the H-convexity of C, it also follows that

$$\langle g_j^{-1} \cdot g_{j-1}, g_j^{-1} \cdot g_{j+1} \rangle \geq 0, \quad j = 1, \ldots, h-1. \tag{3.12}$$

The curve ζ^h in (3.10) is thus a convex polygonal contained in a 2-dimensional plane (the plane spanned by ζ and r). This proves that $\pi_r(\gamma^h)$ is convex.

In order to prove (c), we show that there exists a constant $L > 0$ independent of $h \in \mathbb{N}$ such that for all $s, \sigma \in I$

$$\|\gamma^h(\sigma)^{-1} \cdot \gamma^h(s)\| \leq L|s - \sigma|. \tag{3.13}$$

It then follows that the sequence is also equi-bounded, because $\gamma^h(0) = 0$ for all $h \in \mathbb{N}$. In the case $s = j/h$ and $\sigma = (j-1)/h$, we have by (3.6)

$$\begin{aligned}
\gamma^h(\sigma)^{-1} \cdot \gamma^h(s) &= (p_{j-1} \cdot \phi(p_{j-1}))^{-1} \cdot (p_j \cdot \phi(p_j)) \\
&= (\zeta/h + \phi(p_j) - \phi(p_{j-1}), 0) \\
&= \big(\zeta/h, Q(\phi(p_j) - \phi(p_{j-1}), \zeta/h)\big) \cdot \big(\phi(p_j) - \phi(p_{j-1}), 0\big).
\end{aligned} \tag{3.14}$$

From the second line of (3.14), it follows that

$$\|\gamma^h(\sigma)^{-1} \cdot \gamma^h(s)\| \leq 1/h + |\phi(p_j) - \phi(p_{j-1})|. \tag{3.15}$$

On the other hand, by (3.5) we have

$$\gamma^h(\sigma)^{-1} \cdot \gamma^h(s) \notin C_L(0, r, \alpha, +\infty). \tag{3.16}$$

By the third line of (3.14), (3.16) is equivalent to

$$\alpha|\phi(p_j) - \phi(p_{j-1})| \leq \max\big\{|\zeta/h|, |Q(\phi(p_j) - \phi(p_{j-1}), \zeta/h)|^{1/2}\big\}. \tag{3.17}$$

If the maximum on the right-hand side of (3.17) is $|\zeta/h|$, we get (3.13) with $L = 1 + 1/\alpha$ and $|s - \sigma| = 1/h$. If this is not the case, then by (1.1) we have

$$|\phi(p_j) - \phi(p_{j-1})|^2 \leq \frac{1}{\alpha^2}|Q(\phi(p_j) - \phi(p_{j-1}), \zeta/h)| \leq \frac{1}{h\alpha^2}|\phi(p_j) - \phi(p_{j-1})|,$$

and we get (3.13) with $L = 1 + 1/\alpha^2$. If $s \in I_j^h$ then $\gamma^h(s)$ is a linear convex combination of the points $p_{j-1} \cdot \phi(p_{j-1})$ and $p_j \cdot \phi(p_j)$. Thus, the same argument as above proves that (3.13) holds whenever $s, \sigma \in I_j^h$. Finally, by the triangle inequality, (3.13) holds for $s, \sigma \in [0, 1]$. Concerning the triangle inequality, we need the following observation. If G is a finite-dimensional vector space, then there is a constant $c > 1$ such that for all $p, q \in G$ we have $c^{-1}\|q^{-1} \cdot p\| \leq d(p, q) \leq c\|q^{-1} \cdot p\|$, where d stands for the Carnot–Carathéodory distance of G, which satisfies the triangle inequality. The curves γ^h lie in a finite-dimensional subgroup of G and thus the quasi-metric induced by the norm $\|\cdot\|$ is equivalent to a metric, in this subgroup.

By the Ascoli–Arzelà theorem, the sequence $(\gamma^h)_{h\in\mathbb{N}}$ has a subsequence, which is still denoted by $(\gamma^h)_{h\in\mathbb{N}}$, which converges uniformly to a curve $\gamma : I \to C$. The curve γ is Lipschitz continuous in G and, in fact, we have $\gamma(I) \subset \mathrm{gr}(\phi)$, because $\gamma^h(j/h) \in \mathrm{gr}(\phi)$ for all $0 \le j \le h$. Then we have $\gamma = \kappa \cdot \phi(\kappa)$ for some curve $\kappa : I \to W$. From the (pointwise and in fact uniform) convergence $\pi_r(\gamma^h) \to \pi_r(\gamma) = \phi(\kappa)$ and from (b), we deduce that $s \mapsto \phi(\kappa(s))$ is convex. □

The proof of Theorem 1.5 follows from Theorem 3.2 and from the following proposition.

Proposition 3.3 *Let $\phi : W \to r$ be continuous and let $\gamma : I \to \mathrm{gr}(\phi)$ be a Lipschitz continuous curve in G with the factorization $\gamma = \kappa \cdot \phi(\kappa)$. Then $\kappa = (\zeta, \tau) : I \to W$ is a Lipschitz continuous curve w.r.t. $|\cdot|$ that solves the differential equation*

$$\dot{\tau}(s) + Q\big(\dot{\zeta}(s), \zeta(s) + 2\phi(\kappa(s))\big) = 0 \tag{3.18}$$

for a.e. $s \in I$. Here, $\phi(\kappa)$ is thought of as an element of Z.

Proof By (1.4), the Lipschitz condition (1.11) is equivalent to the inequalities

$$|\zeta(s) - \zeta(\sigma) + \phi(\kappa(s)) - \phi(\kappa(\sigma))| \le L|s - \sigma|, \tag{3.19}$$

$$|\tau(s) - \tau(\sigma) + Q(\zeta(s) - \zeta(\sigma), \zeta(\sigma) + \phi(\kappa(s)) + \phi(\kappa(\sigma)))|^{1/2} \le L|s - \sigma|. \tag{3.20}$$

In order to get the expressions on the left-hand side of (3.19) and (3.20), we used the group law (1.2) three times and the properties (Q1) and (Q2) several times, along with (3.4). The inequality (3.19) implies that the curves ζ and $\phi \circ \kappa$ are both Lipschitz continuous in the standard sense, because the vectors $\zeta(s) - \zeta(\sigma)$ and $\phi(\kappa(s)) - \phi(\kappa(\sigma))$ are orthogonal. The line (3.20) implies that τ is Lipschitz continuous as well. Moreover, on dividing by $|s - \sigma|^{1/2}$ and letting $\sigma \to s$ for $s \in I$ a differentiability point of ζ, we get (3.18). □

Proof of Theorem 1.5 Let $r \in \mathcal{R}_0$ be of the form $r = \{(\lambda z_0, 0) \in G : \lambda \in \mathbb{R}\}$ for some $z_0 \in Z$ with $|z_0| = 1$. The Heisenberg subgroup H is then of the form

$$H = \mathrm{span}\{(z_0, 0), (\zeta_0, 0), (0, t)\}$$

for some $\zeta_0 \in Z$ and $t \in T$ such that $Q(z_0, \zeta_0) = t \neq 0$. Moreover, we can assume that $\langle z_0, \zeta_0 \rangle = 0$. By Theorem 1.4 there are $W \subseteq r^\perp$, $U \subset G$ a neighborhood of $0 \in G$, and $\phi : W \to r$ intrinsic Lipschitz continuous such that $\partial C \cap U = \mathrm{gr}(\phi) \cap U$. Without loss of generality, we can also assume that $\mathrm{int}(C) \cap U = \mathrm{epi}(\phi) \cap U$ and that C is relatively closed in U.

Let $\gamma : I \to \mathrm{gr}(\phi)$, $I = [0, \delta]$, be the curve provided by Theorem 3.2. This curve is Lipschitz continuous in G and $\gamma = \kappa \cdot \phi(\kappa)$ with $\kappa(s) = (s\zeta_0, \tau(s))$ for some $\tau : I \to T$. By Proposition 3.3, the curve τ is Lipschitz continuous and

$$\dot{\tau}(s) = 2Q(\phi(\kappa(s)), \zeta_0), \tag{3.21}$$

where $\phi(\kappa(s))$ is thought of as an element of Z. As $\tau(0)=0$, it follows that $\tau(s)$ is a multiple of $Q(z_0,\zeta_0)$ and thus $\gamma(s)\in H$ for all $s\in I$. By Theorem 3.2, $s\mapsto\phi(\kappa(s))$ is convex and thus twice differentiable a.e. on I. Equation (3.21) implies then that τ is of class $C^{1,1}$ with second derivative differentiable almost everywhere. □

4 Examples

In this section we discuss various examples of H-convex sets in the Heisenberg group $\mathbb{H}^1=\mathbb{C}\times\mathbb{R}=\mathbb{R}^3$. We use the coordinates $z=x+iy\in\mathbb{C}$ and $t\in\mathbb{R}$. The bilinear form $Q:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$ is $Q(z,\zeta)=\mathrm{Im}(z\bar{\zeta})$.

4.1 Convex Cone

Let $r\in\mathcal{R}_0$ be the x-axis, which is identified with $\mathbb{R}$ via $x\equiv(x,0)\in\mathbb{C}\times\mathbb{R}$. We use the standard inner product of $\mathbb{R}^3$. Then $r^\perp$ is the yt-plane. Consider the function $\phi:r^\perp\to r$ defined by $\phi(y,t)=\sqrt{|t|}$. The intrinsic graph of ϕ is the subset of $\mathbb{C}\times\mathbb{R}$

$$\mathrm{gr}(\phi)=\left\{\left(\sqrt{|t|}+iy,t+y\sqrt{|t|}\right)\in\mathbb{C}\times\mathbb{R}:y,t\in\mathbb{R}\right\},$$

and the closed intrinsic epigraph of ϕ is

$$C=\left\{(x+iy,t+xy)\in\mathbb{C}\times\mathbb{R}:x,y,t\in\mathbb{R},\ x\geq\sqrt{|t|}\right\}. \tag{4.1}$$

The set C is a cone in the sense that $\delta_\lambda(C)=C$ for all $\lambda>0$.

Proposition 4.1 *The set C in (4.1) is H-convex.*

Before proving this proposition, let us observe that $0\in\partial C$ is non-characteristic. Then, by Theorem 1.5, there is a nonconstant Lipschitz curve passing through 0 and contained in ∂C. This curve is of the form $\gamma(s)=\kappa(s)\cdot\phi(\kappa(s))$, $s\in I\subset\mathbb{R}$, where $\kappa:I\to r^\perp=\mathbb{R}^2$ is $\kappa(s)=(s,\tau(s))$ for some function $\tau:I\to\mathbb{R}$ such that $\tau(0)=0$. The function τ can be determined by the necessary condition (3.21) (with $\zeta=i$ and $\phi(\kappa(s))=|\tau(s)|^{1/2}$). In particular, we get the Cauchy problem

$$\dot{\tau}(s)=-2|\tau(s)|^{1/2}\quad\text{with }\tau(0)=0, \tag{4.2}$$

for which the solution is not unique. The function ϕ is convex along any curve $s\mapsto\kappa(s)=(s,\tau(s))$, $s\in\mathbb{R}$, with τ solving (4.2).

Proof of Proposition 4.1 It is sufficient to prove that if $p_0,p_1\in\partial C=\mathrm{gr}(\phi)$ are horizontally aligned then $(1-\lambda)p_0+\lambda p_1\in C$ for all $0\leq\lambda\leq1$. The points $p_0=(x_0+iy_0,t_0)$ and $p_1=(x_1+iy_1,t_1)$ belong to $\mathrm{gr}(\phi)$ if and only if

$$x_0^2=|t_0-x_0y_0|\quad\text{and}\quad x_1^2=|t_1-x_1y_1|,\quad\text{with }x_0,x_1\geq0. \tag{4.3}$$

The points are horizontally aligned if and only if

$$t_1-t_0=y_0x_1-x_0y_1. \tag{4.4}$$

With the notation $x_\lambda = (1-\lambda)x_0 + \lambda x_1$, $y_\lambda = (1-\lambda)y_0 + \lambda y_1$, and $t_\lambda = (1-\lambda)t_0 + \lambda t_1$, we have to show that for all $0 \le \lambda \le 1$

$$x_\lambda^2 \ge |t_\lambda - x_\lambda y_\lambda|. \tag{4.5}$$

By a short computation, we preliminarily notice that

$$\begin{aligned} t_\lambda - x_\lambda y_\lambda = (1-\lambda)^2(t_0 - x_0 y_0) + \lambda^2(t_1 - x_1 y_1) \\ + \lambda(1-\lambda)(t_0 + t_1 - x_0 y_1 - x_1 y_0). \end{aligned} \tag{4.6}$$

We distinguish two cases.

Case 1: $(t_0 - x_0 y_0)(t_1 - x_1 y_1) \ge 0$. In particular, we can assume that $t_0 - x_0 y_0 \ge 0$ and $t_1 - x_1 y_1 \ge 0$. This is without loss of generality, because the map $(z, t) \mapsto (\bar{z}, -t)$ preserves H-convexity and maps both C and ∂C onto itself. Then we have

$$x_0^2 = t_0 - x_0 y_0 \quad \text{and} \quad x_1^2 = t_1 - x_1 y_1. \tag{4.7}$$

From (4.7) and (4.4), we deduce that $x_0 + y_0 = x_1 + y_1$. Using this piece of information along with (4.7), we finally get $t_0 + t_1 - x_0 y_1 - x_1 y_0 = 2x_0 x_1$. The right-hand side of (4.6) is then a square and we have equality in (4.5) for all $0 \le \lambda \le 1$. In other words, $(1-\lambda)p_0 + \lambda p_1 \in \partial C$ for all $0 \le \lambda \le 1$.

Case 2: $(t_0 - x_0 y_0)(t_1 - x_1 y_1) < 0$. As above, we can without loss of generality assume that

$$x_0^2 = t_0 - x_0 y_0 \quad \text{and} \quad x_1^2 = x_1 y_1 - t_1. \tag{4.8}$$

From (4.8) and (4.4), we get $x_0^2 + x_1^2 = (x_0 + x_1)(y_1 - y_0)$, and then using this information we find

$$t_0 + t_1 - x_0 y_1 - x_1 y_0 = 2x_0(x_0 + y_0 - y_1). \tag{4.9}$$

Moreover, the identity after (4.8) and $x_0 x_1 > 0$ imply

$$|x_0 + y_0 - y_1| < x_1. \tag{4.10}$$

By the triangle inequality, (4.6), (4.8), (4.9), and (4.10)

$$\begin{aligned} |t_\lambda - x_\lambda y_\lambda| &\le (1-\lambda)^2|t_0 - x_0 y_0| + \lambda^2|t_1 - x_1 y_1| \\ &\quad + 2\lambda(1-\lambda)x_0|x_0 + y_0 - y_1| \\ &\le (1-\lambda)^2 x_0^2 + \lambda^2 x_1^2 + 2\lambda(1-\lambda)x_0 x_1, \end{aligned}$$

where the last inequality is strict if $0 < \lambda < 1$. This proves (4.5), and we have $(1-\lambda)p_0 + \lambda p_1 \in \mathrm{int}(C)$ for $0 < \lambda < 1$. $\square$

4.2 Cone

Let $r \in \mathcal{R}_0$ be the x-axis in $\mathbb{H}^1 = \mathbb{C} \times \mathbb{R}$ and let $C \subset \mathbb{H}^1$ be the H-convex set defined in (4.1). Then the positive cone with vertex 0, axis r, aperture $\alpha = 1$, and height $h = +\infty$ is

$$C_L^+(0, r, 1, +\infty) = C \cap \big\{(x + iy, t) \in \mathbb{C} \times \mathbb{R} : |y| < x\big\}.$$

The set $C_L^+(0, r, 1, +\infty)$ is H-convex because it is the intersection of C with a convex set in the standard sense.

4.3 Thin Cusp

In this subsection, we study the thin cusp in (1.10). We preliminarily state without proof the following proposition. The proof is an easy computation.

Proposition 4.2 *Let $f : \mathbb{R} \to [0, +\infty)$ be a function such that*

$$(1-\lambda)^2 f(t)^2 + \lambda^2 f(\tau)^2 + 2\lambda(1-\lambda)\sqrt{f(t)^2 f(\tau)^2 - (\tau - t)^2} \le f((1-\lambda)t + \lambda\tau)^2 \tag{4.11}$$

for all $\lambda \in [0, 1]$ and for all $t, \tau \in \mathbb{R}$ such that

$$|\tau - t| \le f(t) f(\tau). \tag{4.12}$$

Then the set $\{(z, t) \in \mathbb{C} \times \mathbb{R} : |z| \le f(t)\}$ is H-convex.

Now we prove that for any $\alpha > 0$ there is a constant $\beta > 0$ such that the set

$$C = \left\{(z, t) \in \mathbb{C} \times \mathbb{R} : |z| < t^\alpha < \beta\right\}$$

is H-convex. In the case $\alpha \in (0, 1]$, the claim holds even with $\beta = +\infty$, because the resulting set is convex in the ordinary sense. For $\alpha > 1$, we can, for example, choose

$$\beta = \left(\frac{\sqrt{2}}{\alpha(2\alpha - 1)}\right)^{\frac{\alpha}{2\alpha-1}}. \tag{4.13}$$

This choice is not optimal.

Let $0 < t < \tau \le \beta$ be such that $(t\tau)^{2\alpha} \ge (\tau - t)^2$. By (4.11) with $f(t) = t^\alpha$ and $s = (1-\lambda)t + \lambda\tau$, we have to show that for all $s \in [t, \tau]$

$$\Phi(s) = s^{2\alpha} - \frac{1}{(\tau - t)^2}\Big\{(\tau - s)^2 t^{2\alpha} + (s - t)^2 \tau^{2\alpha} + 2(s - t)(\tau - s)\sqrt{(\tau t)^{2\alpha} - (\tau - t)^2}\Big\} \ge 0.$$

Because $\Phi(t) = \Phi(\tau) = 0$, it is sufficient to show that $\Phi''(s) \le 0$ for $s \in [t, \tau]$, where

$$\Phi''(s) = 2\alpha(2\alpha - 1)s^{2\alpha-2} - \frac{2}{(\tau - t)^2}\Big\{t^{2\alpha} + \tau^{2\alpha} - 2\sqrt{(\tau t)^{2\alpha} - (\tau - t)^2}\Big\}.$$

The inequality $\Phi''(s) \le 0$ for $s \in [t, \tau]$ is implied by $\Phi''(\tau) \le 0$, i.e.,

$$\alpha(2\alpha - 1)\tau^{2\alpha-2}(\tau - t)^2 + 2\sqrt{(\tau t)^{2\alpha} - (\tau - t)^2} \le t^{2\alpha} + \tau^{2\alpha}. \tag{4.14}$$

By the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$, (4.14) is implied by the stronger inequality

$$2\alpha^2(2\alpha-1)^2\tau^{4\alpha-4}(\tau-t)^4 + 4\big((\tau t)^{2\alpha} - (\tau-t)^2\big) \le (t^{2\alpha}+\tau^{2\alpha})^2,$$

that is, $2\alpha^2(2\alpha-1)^2\tau^{4\alpha-4}(\tau-t)^4 \le (t^{2\alpha}-\tau^{2\alpha})^2 + 4(\tau-t)^2$. This inequality is satisfied for $0 < t < \tau \le \beta$, with β as in (4.13).

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