Non-tangentially Accessible Domains for Vector Fields

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ABSTRACT. We study non-tangentially accessible (NTA) domains for diagonal vector fields. We introduce a geometric notion of "admissible boundary" ensuring the NTA property. For general Hörmander vector fields, we prove that a domain with non characteristic boundary is NTA.

1. INTRODUCTION

Analysis of second order elliptic degenerate PDEs has been characterized in recent years by the prominence of metric aspects. Distances associated with second order operators appeared in the late 70s in the work of Nagel and Stein [50], and then in the work of Fefferman and Phong [19] on subellipticity of operators of the form $\sum_{j,k=1}^{n} \partial/\partial x_j(a_{jk}\partial/\partial x_k)$, and in the work of Franchi and Lanconelli [25] on Hölder regularity for weak solutions of equations of the form $\sum_{j=1}^{n} \lambda_j^2(\partial^2/\partial x_j^2)u = 0$ in $\mathbb{R}^n$. The *control distance* associated with a system of vector fields $X_1, \dots, X_m$ also played a central role in the work by Jerison [38] on the Poincaré inequality and in the estimates of Sánchez Calle [53] for the fundamental solution of Hörmander operators. Deep structure theorems for such metrics were proved by Nagel, Stein and Wainger [51]. Finally, integral curves of the vector fields X_j also played a role in Bony's paper [8].

After these seminal papers, the local and global theory of second order PDEs has been intensively studied from a metric point of view. The boundary behavior in Hölder spaces for the Dirichlet problem in the Heisenberg group has been studied in [37]. Wiener criterion for Hörmander sum of squares has been studied in [52], [15], and [35]. After the paper [38], Poincaré inequalities for vector

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fields and functional analysis for Sobolev and BV functions have been studied in [42], [43], [23], [29], [26], and [34]. Jones' theorems on extension of functions have been generalized to the Carnot-Carathéodory setting in [55], [30] and [31]. Properties of the trace at the boundary have been studied in [16], [7], [17], [47], [1]. Properties of subelliptic harmonic measures have been studied in [9], [11], [12], [20], and [21]. Finally, Fatou type theorems for positive subelliptic harmonic functions have been proved in [9]. Several more references could be enumerated concerning non linear PDEs.

All the difficulties in the analysis at the boundary of a set Ω stem from characteristic points, i.e., points $x \in \partial\Omega$ where all the given vector fields $X_1, \dots, X_m$ are tangent to the boundary. Note that in the Euclidean case ($X_j = \partial_j$, $j = 1, \dots, n$) there are no such points. If $x \in \partial\Omega$ is characteristic, then any integral curve of the vector fields starting from x is tangent to the boundary at x. On the other hand, if x is noncharacteristic, then there exist integral curves transversal to the surface at x. This difference has a great influence on the size of control balls and in their interplay with $\partial\Omega$. The quantitative understanding of this phenomenon is the key point in problems at the boundary for degenerate PDEs. In general, nontrivial assumptions are expected to be added to the Euclidean regularity. This is suggested by the work [38] (see also [48]), where examples of smooth sets whose boundary has a "cuspidal behavior" in the control metric are exhibited.

There are several definitions of regular domain which can be formulated in metric spaces, for instance (somehow from the weakest to the strongest) domain with the interior corkscrew property, domain with the twisted cone property (or John domain), (ε, δ)-domain (also called uniform domain), and non-tangentially accessible (briefly NTA) domain. Properties of PDE's which can be established starting from these notions have been studied in many papers (see the list below). In the framework of vector fields the problem is that only few examples of such regular domains are known, and most of them are in the setting of homogeneous groups. In groups of step 2, bounded open sets with boundary of class C^2 are known to be NTA (see [48], [9], [13]). In groups of step 3, the cone property has been studied in [48]. In the specific case of the Heisenberg group, a C^1 condition does not even guarantee the boundary accessibility through rectifiable curves (see [3]); Carnot-Carathéodory balls are uniform (see [55]) but not NTA (see [9]), cubes centered at the origin are uniform (see [32]); finally, the uniformity is preserved under quasi-conformal mappings (see [14]). When no group structure is available no general result is known, except the easy fact that Carnot-Carathéodory balls are John domains. In the case of Grushin vector fields, a class of regular domains (called φ-Harnack domains) has been recently studied in [20]. A partial survey on such results can be found in [10].

In Section 3 we begin our investigation by considering a general system of Hörmander vector fields. We prove the following result.

Theorem 1.1. *A smooth, bounded open domain which is noncharacteristic for a system of Hörmander vector fields is NTA for the control distance.*

This result, which answers a question raised in [17], is natural but it was known only for step 2 homogeneous groups (see [9] and see also [17] for examples of noncharacteristic sets in groups of Heisenberg type). The class of noncharacteristic open sets is believed not to be very rich. On the other hand, it is known that the characteristic set has vanishing surface measure (see [18], [28], and see also the recent references [2] and [44]), and regularity properties related to the noncharacteristic part of the boundary have been widely studied by several authors, see [41], [18], [6], and [27]. In Example 3.4, we give examples of noncharacteristic sets for vector fields of step greater than 2, which naturally arise in the study of solutions of sublaplacians at the boundary of complex domains of the form $\{(z_1, z_2) \in \mathbb{C}^2 \mid \Im(z_2) > |f(z_1)|^2\}$.

In Section 4, which is the central part of the paper, we tackle the problem of characteristic points. We study NTA domains in a class of metric spaces generated by vector fields with no underlying group structure, and with an arbitrarily high order of degeneration. We consider a system of diagonal vector fields in $\mathbb{R}^n$ of the form

$$X_1 = \lambda_1(x)\partial_1,\ X_2 = \lambda_2(x)\partial_2,\ \ldots,\ X_n = \lambda_n(x)\partial_n,$$

whose control metric, under suitable assumptions on the functions λ_j, is known in detail (see [24]). The basic model case we shall study can be exemplified in $\mathbb{R}^3$ by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = x_1^{\alpha_1}\frac{\partial}{\partial x_2}, \quad X_3 = x_1^{\alpha_1}x_2^{\alpha_2}\frac{\partial}{\partial x_3}, \quad \alpha_1, \alpha_2 \in \mathbb{N}. \tag{1.1}$$

Consider an open set in $\mathbb{R}^3$ of the form $\Omega = \{x_3 > \varphi(x')\}$, $x' = (x_1, x_2) \in \mathbb{R}^2$, where $\varphi \in C^1(\mathbb{R}^2)$. By the results of [24], control balls can be written as $Q(x', r) \times]x_3 - F_3(x, r), x_3 + F_3(x, r)[$, where $Q(x', r)$ are suitable rectangles in the plane and $F_3(x, r) > 0$. We say that the boundary $\partial\Omega$ is *admissible* if for all $x' \in \mathbb{R}^2$ and $r > 0$

$$\sum_{i=1,2} \operatorname{osc}(X_i\varphi, Q(x', r)) \le C\Big(r \sum_{i=1,2} |X_i\varphi(x')|^m + \operatorname{osc}(\lambda_3; Q(x', r))\Big), \tag{1.2}$$

where m is a power suitably dependent on the numbers α_1 and α_2 in (1.1). This inequality is a requirement on the oscillation of the derivatives of the function φ along the vector fields X_1 and X_2. The first term in the right hand side vanishes exactly in the characteristic set, while the second one gives an amount of oscillation admitted also at characteristic points. This latter is determined by the oscillation of the function $\lambda_3(x) = x_1^{\alpha_1}x_2^{\alpha_2}$, which is strictly related to the size of control balls in the vertical direction. The balance between the two terms is a very delicate point and it turns out that the correct choice of the power is $m = (\alpha_1 + \alpha_2 + \alpha_1\alpha_2 - 1)/(\alpha_1 + \alpha_2 + \alpha_1\alpha_2)$. In Definition 4.9, generalizing (1.2), we introduce a class of *domains with admissible boundary* in the n-dimensional setting. The main result of the paper is the following theorem.

Theorem 1.2. *Domains with admissible boundary are NTA.*

The proof partially relies on some results in [49], where we prove that admissible domains are John domains. In the paper [49], we also show that the "homogeneous ball" $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (|x_1|^{2(\alpha_1+1)} + x_2^2)^{1+\alpha_2} + x_3^2 < 1\}$ has admissible boundary for the vector fields in (1.1), and we give a criterion for checking the admissibility of surfaces of the form $x_3 = g(|x_1|^{2(\alpha_1+1)} + x_2^2)$. Our examples of NTA domains are the first ones in a setting different from homogeneous groups of step 2 and from diagonal vector fields in the plane. In Example 4.15, we also show that Carnot-Carathéodory balls not necessarily are uniform domains.

Notation. If u, $v \geq 0$, we write $u \lesssim v$ for $u \leq Cv$, where $C \geq 1$ is an absolute constant. Analogously, $u \simeq v$ stands for $u \lesssim v$ and $v \lesssim u$. By d we denote the control metric induced on $\mathbb{R}^n$ by a system of vector fields. For $K \subset \mathbb{R}^n$ we write $\operatorname{diam}(K) = \sup_{x,y\in K} d(x,y)$ and $\operatorname{dist}(x,K) = \inf_{y\in K} d(x,y)$. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ will be denoted by $|E|$. If $\gamma : [0,1] \to \mathbb{R}^n$ is a curve and $0 \leq a \leq b \leq 1$, we denote by $\gamma|_{[a,b]}$ the restriction of γ to the interval $[a,b]$.

2. BASIC DEFINITIONS

In this section we recall all basic definitions and we prove some preliminary propositions that will be used later. We begin with the definition of the control metric associated with a family of vector fields.

Let $X = (X_1, \dots, X_m)$ be a system of vector fields $X_j = \sum_{i=1}^n a_{ij}\partial/\partial x_i$, $j = 1, \dots, m$, where the functions a_{ij} are locally Lipschitz continuous in $\mathbb{R}^n$. A Lipschitz curve $\gamma : [0,T] \to \mathbb{R}^n$, $T \geq 0$, is *X-subunit* if there exists a measurable function $h = (h_1, \dots, h_m) : [0,T] \to \mathbb{R}^m$ such that $\dot\gamma(t) = \sum_{j=1}^m h_j(t)X_j(\gamma(t))$ for a.e. $t \in [0,T]$ with $|h(t)| \leq 1$ a.e. Define $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty]$ by setting

$$d(x,y) = \inf\{T \geq 0 \mid \text{there exists a subunit curve } \gamma : [0,T] \to \mathbb{R}^n \text{ such that } \gamma(0) = x \text{ and } \gamma(T) = y\}. \tag{2.1}$$

If $d(x,y) < +\infty$ for all x, $y \in \mathbb{R}^n$, then d is a metric on $\mathbb{R}^n$, sometimes called *control distance* (or Carnot-Carathéodory, or sub-Riemannian metric). By Chow theorem, the function d is finite if the vector fields $X_j \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfy the Hörmander condition, see e.g. [51]. We denote by $B(x,r)$ the balls in $\mathbb{R}^n$ defined by the metric d.

Now we introduce domains with the corkscrew property, John domains, (ε,δ)-domains, and non-tangentially accessible domains.

Definition 2.1. An open set $\Omega \subset (\mathbb{R}^n, d)$ satisfies the *interior* (*exterior*) *corkscrew condition* if there exist $r_0 > 0$ and $\varepsilon > 0$ such that for all $r \in (0, r_0)$ and $x \in \partial\Omega$ the set $B(x,r) \cap \Omega$ (the set $B(x,r) \cap (\mathbb{R}^n \setminus \overline{\Omega})$) contains a ball of radius εr. An open set Ω satisfies the *corkscrew condition* if it satisfies both the interior and the exterior corkscrew condition.

Definition 2.2. An open set $\Omega \subset (\mathbb{R}^n, d)$ is a *John domain* (or a *domain with the interior cone property*) if there exist $x_0 \in \Omega$ and $\sigma > 0$ such that for all $x \in \Omega$ there exists a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$B(\gamma(t), \sigma \operatorname{diam}(\gamma|_{[0,t]})) \subset \Omega. \tag{2.2}$$

A curve satisfying (2.2) will be called a *John curve of parameter* σ.

Remark 2.3. If both Ω and $\mathbb{R}^n \setminus \bar{\Omega}$ are John domains, then Ω satisfies the corkscrew condition.

Definition 2.4. An open set $\Omega \subset (\mathbb{R}^n, d)$ is a *uniform domain* if there exists $\varepsilon > 0$ such that for every $x, y \in \Omega$ there exists a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$,

$$\operatorname{diam}(\gamma) \leq \frac{1}{\varepsilon} d(x, y), \tag{2.3}$$

and for all $t \in [0,1]$

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq \varepsilon \min\{\operatorname{diam}(\gamma|_{[0,t]}), \operatorname{diam}(\gamma|_{[t,1]})\}. \tag{2.4}$$

It is known that for bounded domains the uniform property is equivalent to the (ε, δ)-property. Recall that the (ε, δ)-property has been introduced in [40] in the Euclidean case, and in [30] for vector fields. This property requires that (2.3) and (2.4) hold only for all pairs of points x, y such that $d(x, y) \leq \delta$, where δ is a suitable positive number.

Remark 2.5. In the definition of John and uniform domain the curves γ are usually required to be rectifiable, and the diameter is replaced by their length (see, for instance, [54]). Anyway, in metric spaces of homogeneous (doubling) type and with geodesics, as the metric spaces we are working with, these stronger definitions are equivalent to the weaker ones we are giving here (this is proved in [45, Theorem 2.7]).

The notion of non-tangentially accessible domain was introduced in the Euclidean case by Jerison and Kenig in [39], and then generalized to the setting of vector fields in [9]. Let $\Omega \subset (\mathbb{R}^n, d)$ be an open set and $\alpha \geq 1$. A sequence of balls $B_0, B_1, \ldots, B_k \subset \Omega$ is an *α-Harnack chain* in Ω if $B_i \cap B_{i-1} \neq \emptyset$ for all $i = 1, \ldots, k$, and $\alpha^{-1} \operatorname{dist}(B_i, \partial\Omega) \leq r(B_i) \leq \alpha \operatorname{dist}(B_i, \partial\Omega)$, where $\operatorname{dist}(B_i, \partial\Omega) = \inf_{x \in B_i, y \in \partial\Omega} d(x, y)$ and $r(B_i)$ is the radius of B_i.

Definition 2.6. A bounded open set Ω is a *NTA domain* in the metric space $(\mathbb{R}^n, d)$ if the following conditions hold:

(i) there exists $\alpha \geq 1$ such that for all $\eta > 0$ and for all $x, y \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) \geq \eta$, $\mathrm{dist}(y, \partial\Omega) \geq \eta$ and $d(x, y) \leq C\eta$ for some $C > 0$, there exists an α-Harnack chain $B_0, B_1, \ldots, B_k \subset \Omega$ such that $x \in B_0$, $y \in B_k$ and k depends on C but not on η;

(ii) Ω satisfies the corkscrew condition.

Remark 2.7. If Ω is a uniform domain according to Definition 2.4, then condition (i) in Definition 2.6 is fulfilled (see [14, Proposition 4.2]).

The following lemma gives a useful sufficient condition for an open set to be uniform. Roughly speaking, we prove that a domain is uniform if, for any pair of points x and y, there exist curves γ_x and γ_y moving far away from the boundary but not from each other.

Lemma 2.8. *Let $\Omega \subset (\mathbb{R}^n, d)$ be an open set. Assume that there exist constants σ, C_3, $C_2 > 0$ such that for all $x, y \in \Omega$ there are John curves $\gamma_x : [0, t_x] \to \Omega$ and $\gamma_y : [0, t_y] \to \Omega$ of parameter σ, with $\gamma_x(0) = x$ and $\gamma_y(0) = y$, and such that*

$$\mathrm{diam}(\gamma_x) \geq C_3 d(x, y), \tag{2.5}$$

$$d(\gamma_x(t_x), \gamma_y(t_y)) \leq \frac{\sigma}{2} C_3 d(x, y), \tag{2.6}$$

and

$$\max\{\mathrm{diam}(\gamma_x), \mathrm{diam}(\gamma_y)\} \leq C_2 d(x, y). \tag{2.7}$$

Then Ω is a uniform domain.

Proof. There exists a continuous curve $\tilde{\gamma}$ joining the point $\gamma_x(t_x)$ to the point $\gamma_y(t_y)$ and satisfying the condition $\mathrm{diam}(\tilde{\gamma}) \leq d(\gamma_x(t_x), \gamma_y(t_y))$. Consider the sum path $\gamma = -\gamma_y + \tilde{\gamma} + \gamma_x$, where $-\gamma_y$ stands for a reverse parameterization. We first show condition (2.3):

$$\begin{aligned}\mathrm{diam}(\gamma) &\leq \mathrm{diam}(\gamma_x) + \mathrm{diam}(\tilde{\gamma}) + \mathrm{diam}(\gamma_y)\\ &\leq C_2 d(x, y) + \frac{\sigma}{2} C_3 d(x, y) + C_2 d(x, y)\\ &\leq \left(\frac{\sigma}{2} C_3 + 2C_2\right) d(x, y).\end{aligned}$$

Now we check (2.4). The proof also shows that Ω is arcwise connected. Take a point $\gamma_x(t)$ with $t \leq t_x$. Since γ_x is a John curve of parameter σ we have

$$\begin{aligned}\mathrm{dist}(\gamma_x(t), \partial\Omega) &\geq \sigma\, \mathrm{diam}\,(\gamma_x|_{[0,t]})\\ &\geq \sigma \min\{\mathrm{diam}\,(\gamma_x|_{[0,t]}), \mathrm{diam}\,(-\gamma_y + \tilde{\gamma} + \gamma_x|_{[t,t_x]})\}.\end{aligned}$$

The same argument works for a point $\gamma_y(t)$, $t \le t_y$. Finally, given a point $w \in \tilde{\gamma}$, by the triangle inequality, (2.5) and (2.6) we get

$$\begin{aligned}\operatorname{dist}(w,\partial\Omega) &\ge \operatorname{dist}(\gamma_x(t_x),\partial\Omega) - d(w,\gamma_x(t_x))\\ &\ge \sigma \operatorname{diam}(\gamma_x) - \frac{\sigma}{2} C_3 d(x,y)\\ &\ge \sigma \operatorname{diam}(\gamma_x) - \frac{\sigma}{2}\operatorname{diam}(\gamma_x) = \frac{\sigma}{2}\operatorname{diam}(\gamma_x).\end{aligned}$$

In order to provide a lower bound for the last term it is enough to note that the hypotheses of the lemma ensure that $\operatorname{diam}(\gamma_x) \simeq \operatorname{diam}(\gamma)$ through constants depending on σ, C_3 and C_2. □

3. Non Characteristic Boundary for Hörmander Vector Fields

In this section we show that a bounded smooth domain without characteristic points is NTA with respect to the control metric induced by a system of Hörmander vector fields $X = (X_1, \dots, X_m)$. Recall that the system X is of Hörmander type in $\mathbb{R}^n$ if the vector fields are smooth and for some $p \in \mathbb{N}$

$$\operatorname{span}\{X_{j_1}(x), [X_{j_1}, X_{j_2}](x), \dots [X_{j_1}, [X_{j_2}, \dots, [X_{j_{p-1}}, X_{j_p}]] \cdots](x) \mid j_k = 1, \dots, m\}$$

has dimension n for any $x \in \mathbb{R}^n$. Here, $[X_j, X_k]$ denotes the commutator of X_j and X_k.

A point $x \in \partial\Omega$ is *characteristic* if all the vector fields $X_1, \dots, X_m$ are tangent to $\partial\Omega$ at x. We say that Ω is *non characteristic* if all its boundary points are non characteristic.

If $\bar{x} \in \partial\Omega$ is non characteristic and ν is a normal vector to $\partial\Omega$ at x, then we can find a vector field, say X_m, such that $\langle X_m(x), \nu\rangle \ne 0$. By a standard argument, it can be shown that for a suitable neighborhood U of $\bar{x}$, there exists a diffeomorphism $\Phi : U \to \Phi(U)$ such that $d\Phi(x) X_m(x) = \partial_n$ for all $x \in U$, $\Phi(\bar{x}) = 0$, and $\Phi(\partial\Omega \cap U) \subset \{y_n = 0\}$. Therefore, possibly performing such a change of variable, the vector fields can be assumed to be of the form

$$Y_j = b_j(y)\partial_{y_n} + \sum_{i=1}^{n-1} a_{ij}(y)\partial_{y_i}, \quad j = 1, \dots, m-1, \; Y_m = \partial_{y_n},$$

and we can consequently assume that $\Omega = \{y_n > 0\}$ in a neighborhood of the origin. The vector fields $Y_1, \dots, Y_m$ still satisfy Hörmander condition and induce the control metric d_Y. It is now easy to check that the new family of vector fields

$$X_j = \sum_{i=1}^{n-1} a_{ij}(y)\partial_i, \quad j = 1, \dots, m-1, \; X_m = \partial_n, \tag{3.1}$$

still satisfies Hörmander condition. Moreover, if d_X is the corresponding control metric, it is not difficult to show (see [28] for a proof) that there exist two constants c_1 and c_2 such that in a neighborhood of the origin we have $c_1 d_Y \le d_X \le c_2 d_Y$.

We give now an easy lemma.

Lemma 3.1. *Let $X_1, \dots, X_m$ be Hörmander vector fields of the form (3.1) with $a_{ij} \in C^\infty(\mathbb{R}^n)$. Then, for all $(x', x_n), (y', y_n) \in \mathbb{R}^n$,*

$$d((x',x_n),(y',y_n)) \ge |y_n - x_n| = d((x',x_n),(x',y_n)).$$

Proof. Let $\gamma : [0,T] \to \mathbb{R}^n$ be a subunit curve connecting $(x', x_n) = \gamma(0)$ and $(y', y_n) = \gamma(T)$, for some $T > 0$. Then $\dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t))$ with $|h| \le 1$ almost everywhere. Since all the vector fields $X_1, \dots, X_{m-1}$ lie in $\mathbb{R}^{n-1}$, we have

$$|y_n - x_n| = \left| \int_0^T h_m(t)\,dt \right| \le T.$$

Taking the infimum over all possible such curves we get $d((x',x_n),(y',y_n)) \ge |y_n - x_n|$.

Moreover, the subunit curve $\gamma(t) = (x', x_n + t\,\mathrm{sgn}(y_n - x_n))$, with $0 \le t \le |y_n - x_n|$, is a geodesic between (x', x_n) and (x', y_n). Thus $|y_n - x_n| = d((x',x_n),(x',y_n))$. ◻

Now we recall a deep result due to Nagel, Stein and Wainger. Given a system of Hörmander vector fields $X_1, \dots, X_m$ and a compact set $K \subset \mathbb{R}^n$, denote by $Y_1, \dots, Y_q$ a family of commutators which are of maximal rank at every point $x \in K$. Assign to any commutator Y in this family a degree equal to its length, that is, write $d(Y) = k$ if Y has length $k \ge 1$. Given a n-tuple $I = (i_1, \dots, i_n) \in \{1, \dots, q\}^n$, write

$$B_2(x,r) = \{\exp(u_1 Y_1 + \cdots + u_q Y_q)(x) : |u_j| < r^{d(Y_j)}\}.$$

Theorem 3.2 ([51])**.** *Let $d_2(x,y) = \inf\{r > 0 \mid y \in B_2(x,r)\}$. Then d_2 is locally equivalent to the control distance d.*

Now we are ready to prove the main theorem of this section.

Theorem 3.3. *A smooth non characteristic bounded domain with respect to a family of Hörmander vector fields is NTA for the control distance.*

Proof. We prove that Ω is a uniform domain in the sense of Definition 2.4, and by Remark 2.7 condition (i) in Definition 2.6 will be satisfied. Moreover, the proof will show that both Ω and $\mathbb{R}^n \setminus \bar\Omega$ are John domains and condition (ii) will be satisfied, as well.

By a general result of Väisälä [54, Theorem 4.1], it is sufficient to prove the uniform condition in a neighborhood of a fixed point $x \in \partial\Omega$ (see also [48, Proposition 2.5], where the same localization argument is described). Then, without loss of generality we assume that the vector fields are of the form (3.1) and $\Omega = \{x \in \mathbb{R}^n \mid x_n > 0\}$. Consider two points $x = (x', x_n)$ and $y = (y', y_n)$ with $x_n, y_n > 0$ and assume for instance $y_n \geq x_n$. First of all, we define two John curves starting from x and y, in the following way

$$\begin{aligned} \gamma_x(t) &= (x', x_n + t), \quad 0 < t \leq y_n - x_n + d(x,y) := t_x, \\ \gamma_y(t) &= (y', y_n + t), \quad 0 < t \leq d(x,y) := t_y. \end{aligned}$$

It is easy to check, by Lemma 3.1, that γ_x and γ_y are John curves of parameter $\sigma = 1$.

Denote by $\tilde{x} = (x', y_n + d(x,y))$ and $\tilde{y} = (y', y_n + d(x,y))$ the endpoints of γ_x and γ_y. Let $W_1, \ldots, W_{q-1}, W_q = X_n$ be the family of all the commutators of sufficiently high length in order to apply Theorem 3.2. Note that each commutator W_j, $j = 1, 2, \ldots, q-1$, has n-th component equal to zero. Then, by Theorem 3.2, we can write

$$\tilde{y} = \exp(u_1 W_1 + \cdots + u_{q-1} W_{q-1} + u_n X_n)(\tilde{x})$$

for some $u \in \mathbb{R}^q$ with $|u_j| \leq C d(\tilde{x}, \tilde{y})^{\deg(W_j)}$, $|u_q| \leq C d(\tilde{x}, \tilde{y})$. Since $\tilde{x}$ and $\tilde{y}$ have the same n-th coordinate, it must be $u_q = 0$. Define

$$\tilde{\gamma}(t) = \exp(t(u_1 W_1 + \cdots + u_{q-1} W_{q-1}))(\tilde{x}), \quad 0 \leq t \leq 1.$$

By Theorem 3.2, we have $\operatorname{diam}(\gamma) \leq C d(\tilde{x}, \tilde{y})$. Moreover, by the triangle inequality and Lemma 3.1, $d(\tilde{x}, \tilde{y}) \leq d(\tilde{x}, x) + d(x,y) + d(y, \tilde{y}) \lesssim d(x,y)$. Thus, condition (2.3) is satisfied. Finally, again by Lemma 3.1, we have $\operatorname{dist}(\tilde{\gamma}(t), \partial\Omega) \geq y_n + d(x,y)$. The path $\gamma = -\gamma_y + \tilde{\gamma} + \gamma_x$ satisfies all requirements of Definition 2.4. $\square$

Example 3.4. Consider in $\mathbb{R}^3$ the vector fields

$$X = \frac{\partial}{\partial x} + 2k|z|^{2k-2} y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2k|z|^{2k-2} x \frac{\partial}{\partial t}, \tag{3.2}$$

where $(z,t) = (x,y,t) \in \mathbb{R}^3$. The vector fields X and Y naturally arise in the analysis of the sublaplacian of the boundary of a domain in $\mathbb{C}^2$. Moreover, X and Y satisfy, for any $k \in \mathbb{N}$, the Hörmander condition. When $k = 1$ we have the Heisenberg vector fields.

The open set $\Omega = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid (|z|^k - 2)^2 + t^2 < 1\}$ is bounded and has boundary of class C^∞. We show that Ω is non characteristic for X and Y (see [17] for the same example in the setting of the Heisenberg group $k = 1$). Thus, by Theorem 3.3, Ω is a NTA domain in the associated metric space. A defining function $\Phi = 0$ for the boundary of Ω is $\Phi(z,t) = (|z|^k - 2)^2 + t^2 - 1$. Since

$$\begin{aligned} X\Phi(z,t) &= 2kx(|z|^k - 2)|z|^{k-2} + 2ky|z|^{2k-2}t, \\ Y\Phi(z,t) &= 2ky(|z|^k - 2)|z|^{k-2} - 2kx|z|^{2k-2}t, \end{aligned}$$

we find

$$|X\Phi(z,t)|^2 + |Y\Phi(z,t)|^2 = 4k^2|z|^{2k-2}\{(|z|^k - 2)^2 + |z|^{2k}t^2\}.$$

The last expression never vanishes when $(z,t) \in \partial\Omega$.

The vector fields X and Y appear in subelliptic analysis as follows. Let $f(z) = z^k$, where $k \in \mathbb{N}$ is a fixed integer and $z \in \mathbb{C}$. We write $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2$. Consider the domain $D = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Im(z_2) > |f(z_1)|^2\} \subset \mathbb{C}^2$. The holomorphic tangent vector field to the boundary of D is

$$Z = \frac{\partial}{\partial z_1} + 2if'(z_1)\overline{f(z_1)}\frac{\partial}{\partial z_2}, \quad \text{where } \frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right).$$

In the tangential coordinates $z = z_1$ and $t = \Re(z_2)$ we have

$$Z = \frac{\partial}{\partial z} + if'(z)\overline{f(z)}\frac{\partial}{\partial t}.$$

Writing $Z = \frac{1}{2}(X - iY)$ we get the vector fields in (3.2). The subelliptic Laplacian arising from this situation is studied in [33] and [4].

4. Non-tangentially Accessible Domains for Diagonal Vector Fields

In this section we describe the geometry of diagonal vector fields, we introduce a class of admissible domains, and we show that they are NTA for the related control metric. Consider

$$X_j = \lambda_j(x)\frac{\partial}{\partial x_j}, \quad j = 1, \dots, n, \tag{4.1}$$

where

$$\lambda_1(x) = 1 \text{ and } \lambda_j(x) = \prod_{i=1}^{j-1} |x_i|^{\alpha_i}, \quad j = 2, \dots, n. \tag{4.2}$$

We assume that

$$\alpha_i = 0 \text{ or } \alpha_i \in [1, \infty[\,, \quad i = 1, \dots, n. \tag{4.3}$$

This condition ensures that the functions λ_j, and thus the vector fields X_j, are locally Lipschitz continuous.

Remark 4.1. If the numbers α_i are integers, then the functions λ_j in (4.2) can be defined writing $x_i^{\alpha_i}$ in place of $|x_i|^{\alpha_i}$. In this case the vector fields $X_1, \dots, X_n$ are smooth and satisfy the Hörmander condition. In this smooth case, all the definitions that follow remain unchanged and all the results still hold.

According to the definition in (2.1), the vector fields $X_1, \dots, X_n$ define a control metric d in $\mathbb{R}^n$. Thanks to the special form (4.2) of the functions λ_j, the metric balls $B(x, r)$ can be described rather explicitly. Following [24], for all $j = 1, \dots, n$ define inductively the functions $F_j : \mathbb{R}^n \times [0, +\infty) \to [0, +\infty)$ by

$$\begin{aligned} F_1(x, r) = r, \quad F_2(x, r) &= r\lambda_2(|x_1| + F_1(x, r)), \quad \dots, \\ F_j(x, r) &= r\lambda_j(|x_1| + r, |x_2| + F_2(x, r), \dots, |x_{j-1}| + F_{j-1}(x, r)). \end{aligned} \tag{4.4}$$

Equivalently, the definition can be also written in the following recursive way

$$F_{j+1}(x, r) = F_j(x, r)(|x_j| + F_j(x, r))^{\alpha_j}. \tag{4.5}$$

Note that $F_j(x, r)$ actually depends only on $x_1, \dots, x_{j-1}$. It is easy to check that $r \mapsto F_j(x, r)$ satisfies the following doubling property

$$F_j(x, 2r) \le CF_j(x, r), \quad x \in \mathbb{R}^n, \ 0 < r < +\infty, \tag{4.6}$$

for all $j = 1, \dots, n$. Here and in the sequel $C > 0$ is an absolute constant. Moreover, an inspection of the form (4.4) of the functions F_j shows that

$$F_j(x, \varrho r) \le \varrho F_j(x, r), \qquad \varrho \le 1, \ r > 0 \tag{4.7a}$$

$$(1 + \eta)F_j(x, r) \le F_j(x, (1 + \eta)r), \quad \eta \ge 0. \tag{4.7b}$$

Finally, since for any fixed $x \in \mathbb{R}^n$ the function $F_j(x, \cdot)$ is strictly increasing from $[0, \infty[$ onto itself, we denote its inverse by $G_j(x, \cdot) = F_j(x, \cdot)^{-1}$.

The following theorem proved in [24] shows that the structure of the control balls $B(x, r)$ can be described by means of the following boxes

$$\operatorname{Box}(x, r) = \{x + h : |h_j| < F_j(x, r), \ j = 1, \dots, n\}. \tag{4.8}$$

Theorem 4.2 ([24]). *There exists a constant $C > 0$ such that:*

$$\operatorname{Box}(x, C^{-1}r) \subset B(x, r) \subset \operatorname{Box}(x, Cr), \quad x \in \mathbb{R}^n, \ r \in \,]0, +\infty[\,, \tag{4.9a}$$

$$C^{-1}d(x, y) \le \sum_{j=1}^{n} G_j(x, |y_j - x_j|) \le Cd(x, y), \quad x, y \in \mathbb{R}^n. \tag{4.9b}$$

Remark 4.3. Looking at the form of the vector fields, it is easy to check that, for all $x \in \mathbb{R}^n$, $t \geq 0$, $k = 1, \dots, n$, $\operatorname{diam}\{x + se_k \mid 0 \leq s \leq t\} = d(x, x + te_k) = d((x_1, \dots, x_{k-1}, 0, \dots, 0), (x_1, \dots, x_{k-1}, t, 0 \dots, 0))$.

Before proceeding we introduce the following convention. If $j = 1, \dots, n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we write $\hat{x}_j = (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n)$ and we identify it with $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \in \mathbb{R}^{n-1}$.

Define

$$\operatorname{Box}_n(\hat{x}_n, r) = \{\hat{x}_n + \hat{h}_n : |h_i| < F_i(\hat{x}_n, r),\ i = 1, \dots, n-1\}, \tag{4.10}$$

and let

$$\Lambda_n(\hat{x}_n, r) = \sup_{\hat{y}_n \in \operatorname{Box}_n(\hat{x}_n, r)} |\lambda_n(\hat{y}_n) - \lambda_n(\hat{x}_n)|.$$

For each $j = 1, \dots, n$ define inductively the real number d_j by

$$d_1 = 1,\ d_2 = 1 + \alpha_1,\ \dots,\ d_j = 1 + \sum_{i=1}^{j-1} d_i \alpha_i = (1 + \alpha_1) \cdots (1 + \alpha_{j-1}). \tag{4.11}$$

We say that d_j is the degree of the variable x_j. Note that $F_j(0, r) = r^{d_j}$.

The following proposition is proved in [49].

Proposition 4.4. *There is $\eta > 0$ such that for all $r > 0$ and $\varrho \in]0, 1]$*

$$\Lambda_n(\hat{x}_n, \varrho r) \leq h(\varrho) \Lambda_n(\hat{x}_n, r), \quad \text{where } h(\varrho) = \frac{\varrho}{\varrho + \eta(1 - \varrho)}. \tag{4.12}$$

Moreover, there exists a constant $C > 0$ such that $\Lambda_n(\hat{x}_n, r) \leq (C/r) F_n(x, r)$ and $\Lambda_n(\hat{x}_n, r) \geq r^{d_n - 1}$.

Denote in the following by c_ϱ any positive constant depending on $\varrho > 0$ such that $c_\varrho \to 0$, as $\varrho \downarrow 0$. The following lemma holds.

Lemma 4.5. $\operatorname{Box}(y, r) \subset \operatorname{Box}(x, (1 + c_\varrho) r)$ *for all x, y, r satisfying $d(x, y) \leq \varrho r$.*

Proof. By definition, $z \in \operatorname{Box}(y, r)$ if and only if $|z_j - y_j| \leq F_j(y, r)$ for all $j = 1, \dots, n$. We need to prove

$$|z_j - x_j| \leq F_j(x, (1 + c_\varrho) r), \quad j = 1, \dots, n. \tag{4.13}$$

The assumptions of the lemma, Theorem 4.2, and the first inequality in (4.7) give

$$\begin{aligned} |z_j - x_j| &\leq |z_j - y_j| + |y_j - x_j| \\ &\leq F_j(y, r) + F_j(x, C d(x, y)) \leq F_j(y, r) + c_\varrho F_j(x, r). \end{aligned} \tag{4.14}$$

We claim that

$$F_k(y,r) \le F_k(x,(1+c_\varrho)r) \quad \text{for all } k = 1,\dots,n. \tag{4.15}$$

If the claim is proved, then inserting (4.15) in (4.14) we conclude

$$\begin{aligned} |z_j - x_j| &\le F_j(x,(1+c_\varrho)r) + c_\varrho F_j(x,r) \\ &\le (1+c_\varrho)F_j(x,(1+c_\varrho)r) \le F_j(x,(1+c_\varrho)^2 r), \end{aligned}$$

by (4.7) (in our notations $(1+c_\varrho)^2 = 1+c_\varrho$). Then the lemma is proved.

In order to show (4.15) we use induction on k. The statement is trivial for $k = 1$. If (4.15) holds for some k, then by (4.5)

$$\begin{aligned} F_{k+1}(y,r) &= F_k(y,r)(|y_k| + F_k(y,r))^{\alpha_k} \\ &\le F_k(x,(1+c_\varrho)r) \\ &\qquad \times (|x_k| + |y_k - x_k| + F_k(x,(1+c_\varrho)r))^{\alpha_k}. \end{aligned} \tag{4.16}$$

Recall that, by Theorem 4.2, $|y_k - x_k| \le F_k(x, Cd(x,y)) \le c_\varrho F_k(x,r)$, and

$$c_\varrho F_k(x,r) + F_k(x,(1+c_\varrho)r) \le (1+c_\varrho)F_k(x,(1+c_\varrho)r) \le F_k(x,(1+c_\varrho)^2 r),$$

by (4.7). Inserting the last inequality into the second line of (4.16) we immediately conclude the proof of (4.15). ◻

Now we introduce our definition of admissible surface with respect to the vector fields $X_1, \dots, X_n$ in (4.1), for surfaces of the type $\{x_n = \varphi(\hat{x}_n)\}$. We proceed as follows. First of all we give the definition of "admissible surface" for a graph of the form $x_n = \varphi(\hat{x}_n)$. This is the most degenerate case and contains all the difficulties of the problem. Then, we will show that a graph of the form $x_j = \varphi(\hat{x}_j)$, with $j \ne n$, can be studied reducing to the previous case. Finally, in Definition 4.9 we introduce the notion of open set with admissible boundary.

Definition 4.6. Let $\varphi \in C^1(\mathbb{R}^{n-1})$. The surface $\{x_n = \varphi(\hat{x}_n)\}$ is said to be *admissible* if there exist $C > 0$ and $r_0 > 0$ such that, for all $\hat{x}_n \in \mathbb{R}^{n-1}$ and $r \in]0, r_0]$,

$$\begin{aligned} &\sum_{i \ne n} \operatorname{osc}(X_i\varphi, \mathrm{Box}_n(\hat{x}_n, r)) \\ &\qquad \le C\Big(r \sum_{i\ne n} |X_i\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n, r)\Big). \end{aligned} \tag{4.17}$$

Note that the exponent $(d_n - 2)/(d_n - 1)$ is nonnegative as soon as at least one of the numbers α_i is non zero (otherwise we are in the Euclidean case).

In order to define admissible surfaces of the type $\{x_j = \varphi(\hat{x}_j)\}$ when $j \neq n$, we start with the following euristic remark. The variables $x_{j+1}, \dots, x_n$ are "more degenerate" than x_j: the size of the balls in their direction is larger than the size in the j-th direction. This suggests that the behavior of the function φ with respect to the mentioned variables does not need to be controlled in a careful way.

To implement this idea, consider the new functions and vector fields

$$\tilde{\lambda}_i(x) = \begin{cases} \lambda_i(x) & \text{if } i \leq j, \\ \lambda_j(x) & \text{if } i \geq j, \end{cases} \quad \text{and} \quad \tilde{X}_i = \tilde{\lambda}_i \partial_i, \quad i = 1, \dots, n. \tag{4.18}$$

The functions $\tilde{F}_j$ and $\tilde{\Lambda}_j$ are defined exactly as above, using $\tilde{X}_1, \dots, \tilde{X}_n$. Define the boxes $\widetilde{\mathrm{Box}}_j(\hat{x}_j, r) = \{\hat{x}_j + \hat{h}_j : |h_i| < \tilde{F}_i(\hat{x}_j, r),\ i \neq j\}$ and denote by $\tilde{d}$ the metric constructed as in (2.1) using subunit curves with respect to the vector fields $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_n)$, and let $\tilde{B}(x, r)$ be the corresponding balls. The new vector fields have the advantage that the variable x_j can be thought of as the n-th variable. In [49] we prove the following proposition describing some relations between d and $\tilde{d}$.

Proposition 4.7. *The following properties hold:*

(i) *for any $C_1 > 0$ there is $C_2 > 0$ such that if $|x_j|$, $|y_j|$, $r < C_1$, then $B(x, r) \subset \tilde{B}(x, C_2 r)$ and $\tilde{d}(x, y) \leq C_2 d(x, y)$;*

(ii) *using the notation $x' = (x_1, \dots, x_j)$ and $x'' = (x_{j+1}, \dots, x_n)$, we have $d((x', x''), (y', x'')) \simeq \tilde{d}((x', x''), (y', x''))$.*

Definition 4.8. Let $\varphi \in C^1(\mathbb{R}^{n-1})$. The surface $\{x_j = \varphi(\hat{x}_j)\}$ is said to be *admissible* if there exist $C > 0$ and $r_0 > 0$ such that for all $\hat{x}_j \in \mathbb{R}^{n-1}$ and $r \in\ ]0, r_0]$

$$\begin{aligned} &\sum_{i \neq j} \operatorname{osc}(\tilde{X}_i \varphi, \widetilde{\mathrm{Box}}_j(\hat{x}_j, r)) \\ &\qquad \leq C\Big(r \sum_{i \neq j} |\tilde{X}_i \varphi(\hat{x}_j)|^{(d_j - 2)/(d_j - 1)} + \tilde{\Lambda}_j(\hat{x}_j, r)\Big). \end{aligned} \tag{4.19}$$

Definitions 4.6 and 4.8 can be stated also for a bounded graph $x_j = \varphi(\hat{x}_j)$, where φ is defined on a bounded open set of $\mathbb{R}^{n-1}$.

Definition 4.9. A bounded open set $\Omega \subset \mathbb{R}^n$ is said to be *with admissible boundary* with respect to X if it is of class C^1, and for all $x \in \partial\Omega$ there exists a neighborhood U of x such that $\partial\Omega \cap U$ is an admissible surface according to Definitions 4.6 or 4.8.

Example 4.10. Consider in $\mathbb{R}^3$ the vector fields

$$X_1 = \partial_1, \quad X_2 = |x_1|^{\alpha_1} \partial_2, \quad X_3 = |x_1|^{\alpha_1} |x_2|^{\alpha_2} \partial_3, \tag{4.20}$$

with $\alpha_1, \alpha_2 \geq 1$. In [49] the open set

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (|x_1|^{2(\alpha_1+1)} + x_2^2)^{1+\alpha_2} + x_3^2 < 1\}$$

is proved to have admissible boundary with respect to $X = (X_1, X_2, X_3)$ according to Definition 4.9. A key tool in the proof is the following proposition.

Proposition 4.11 ([49]). *Let $N(x_1, x_2) = |x_1|^{2(\alpha_1+1)} + x_2^2$ and consider the function $\varphi(x_1, x_2) = g(N(x_1, x_2))$, where $g \in C^2(0, +\infty)$ is such that for some $C > 0$*

$$0 \leq g'(t) \leq Ct^{(\alpha_2-1)/2}, \; |g''(t)| \leq C\frac{g'(t)}{t}, \; g'(2t) \leq Cg'(t), \quad t > 0. \tag{4.21}$$

Then the surface $\{x_3 = \varphi(x_1, x_2)\}$ is admissible according to Definition 4.6.

Now we recall how to construct a John curve starting from a point in an open set with admissible boundary. The construction is taken from [49] and it relies on (4.17). Here, we study the uniform property, which is stronger than the cone condition. We need to deduce from (4.17) some deeper information describing how the John curve starting from a point x changes when the point x moves. This is done in Lemma 4.13.

Consider again an open set of the form $\Omega = \{x_n > \varphi(\hat{x}_n)\}$, take a point $x = \hat{x}_n + x_n e_n \in \Omega$, and introduce the following notation

$$\nu_i = \nu_i(\hat{x}_n) = -\partial_i \varphi(\hat{x}_n) \quad \text{and} \quad N_i = \frac{\nu_i}{|\nu_i|}, \quad \text{if } \nu_i \neq 0, \; i \neq n. \tag{4.22}$$

In order to construct a John curve $\gamma_x : [0, 1] \to \Omega$ starting from x, two different situations need to be distinguished:

$$\max_{i<n} |X_i \varphi(\hat{x}_n)| \leq \lambda_n(\hat{x}_n) \quad \text{(Case 1)}, \tag{4.23a}$$

$$\max_{i<n} |X_i \varphi(\hat{x}_n)| > \lambda_n(\hat{x}_n) \quad \text{(Case 2)}. \tag{4.23b}$$

In Case 1, the characteristic case, define the curve

$$\gamma_x(t) = x + te_n = \hat{x}_n + (x_n + t)e_n, \quad t \geq 0. \tag{4.24}$$

In Case 2, the curve γ_x is defined in two steps. First of all, take any $k = 1, \ldots, n-1$ such that $|X_k \varphi(\hat{x}_n)|$ is "maximal" in the following sense (this choice is not unique)

$$|X_k \varphi(\hat{x}_n)| \geq \frac{1}{2} \max_{i<n} |X_i \varphi(\hat{x}_n)| > \frac{1}{4} \lambda_n(\hat{x}_n), \tag{4.25}$$

and let $\delta_k(x)$ be the solution of the following equation in the variable δ

$$\Lambda_n(\hat{x}_n, \delta) = \varepsilon_0 |X_k \varphi(\hat{x}_n)|, \tag{4.26}$$

(the solution is unique because $\Lambda_n(\hat{x}_n, \cdot)$ is strictly increasing; here, $\varepsilon_0 > 0$ is a suitable constant which depends on the surface and whose choice is discussed in [49]). Finally, define the positive time $t(x) = t_k(x)$ by

$$t_k(x) = F_k(x, \delta_k(x)). \tag{4.27}$$

The first piece of γ_x is defined for $t \in [0, t_k(x)]$ by letting

$$\gamma_x(t) = x + tN_k e_k. \tag{4.28}$$

Here, $N_k = N_k(x)$ depends on x. The number $\delta_k(x)$ essentially represents the diameter of the first piece of the path. The second piece is

$$\gamma(t) = x + t_k(x)N_k e_k + (t - t_k(x))e_n, \quad t \ge t_k(x). \tag{4.29}$$

The following theorem is proved in [49].

Theorem 4.12. *Assume that $\varphi \in C^1(\mathbb{R}^{n-1})$ satisfies (4.17). Let $\Omega = \{x_n > \varphi(\hat{x}_n)\}$. Then there exists a constant $\sigma > 0$ such that: if $x \in \Omega$ and Case 1 holds, then the curve γ_x defined as in (4.24) is a John curve of parameter σ; if $x \in \Omega$ and Case 2 holds, then for any k such that (4.25) holds, the curve γ_x defined in (4.28)–(4.29) is a John curve of parameter σ.*

Now we start the core of our discussion. For any $x \in \Omega$ for which Case 2 in (4.23) holds, fix a $k = k(x) \in \{1, \dots, n-1\}$ such that $|X_k \varphi(\hat{x}_n)| = \max_{i<n} |X_i \varphi(\hat{x}_n)|$. Introduce now the parameter $\Delta(x)$ (equivalent to the diameter of the first piece of the path γ_x starting from x) as follows:

$$\Delta(x) = \begin{cases} 0 & \text{if } x \text{ satisfies (4.23), Case 1,} \\ \delta_k(x) & \text{if } x \text{ satisfies (4.23), Case 2,} \end{cases}$$

where, if x satisfies Case 2, $\delta_k(x)$ is given by (4.26).

Let $\varrho > 0$ be a constant that will be fixed later. Given a pair of points x and $y \in \Omega$, we distinguish two cases. The first case is

$$d(x, y) > \varrho \max\{\Delta(x), \Delta(y)\}. \tag{Case A}$$

If Case A does not hold, assuming for instance $\Delta(x) \ge \Delta(y)$, it should be $d(x, y) \le \varrho\Delta(x)$. Moreover, if $k = k(x)$ is the number selected above, we can write $\Delta(x) = \delta_k(x)$. Then the second case is

$$\begin{cases} |X_k \varphi(\hat{x}_n)| = \max_{i \ne n} |X_i \varphi(\hat{x}_n)| > \lambda_n(\hat{x}_n), \\ d(x, y) \le \varrho \delta_k(x). \end{cases} \tag{Case B}$$

Case B is the more delicate one. The problem here is that if the points x and y are very near and we want to connect them by a curve with total diameter comparable with $d(x, y)$, we have to use only the first piece of the paths γ_x and γ_y starting from x and y. The following lemma provides the suitable tools to prove that if y is near x (in other words, if we are in Case B and ϱ is small), then we can choose a John curve γ_y from y which starts in the *same direction* of the curve γ_x starting from x. This lemma gives the correct bound on the oscillation of the horizontal derivatives $X_i\varphi$ near characteristic points. The properties established in this lemma are crucial.

Lemma 4.13. *Let $\varphi \in C^1(\mathbb{R}^{n-1})$ satisfy (4.17). There are a constant $\varrho_0 > 0$ and a function $\varrho \mapsto c_\varrho$ from $(0, \varrho_0)$ to $\mathbb{R}^+$, with $\lim_{\varrho\downarrow 0} c_\varrho = 0$ and such that, if Case B holds for a pair of points $x, y \in \{x_n > \varphi(\hat{x}_n)\}$ and for a number $k = 1, \ldots, n-1$, then we have*

$$|X_i\varphi(\hat{x}_n) - X_i\varphi(\hat{y}_n)| \leq c_\varrho |X_k\varphi(\hat{x}_n)| \qquad \forall\, i = 1, \ldots, n-1, \tag{4.30}$$

$$|X_k\varphi(\hat{y}_n)| \geq (1 - c_\varrho)\lambda_n(\hat{y}_n), \tag{4.31}$$

and, denoting by $\delta_k(y)$ the solution of (4.26) with $\hat{y}_n$ instead of $\hat{x}_n$,

$$\delta_k(y) \geq \frac{1}{2}\delta_k(x). \tag{4.32}$$

Using Lemma 4.13, whose proof will be given later, we can prove the main theorem of this section.

Theorem 4.14. *If $\Omega \subset \mathbb{R}^n$ is an admissible domain for $X_1, \ldots, X_n$, then it is a NTA domain in the metric space $(\mathbb{R}^n, d)$.*

Proof. We show that Ω is a uniform domain in the sense of Definition 2.4, and this will prove condition (i) in Definition 2.6. Condition (ii) is a direct consequence of Theorem 4.12.

It will be enough to consider the case $\Omega = \{x_n > \varphi(\hat{x}_n)\}$, where $\varphi \in C^1(\mathbb{R}^{n-1})$ is a function satisfying (4.17). We start the discussion with Case B. Let $x, y \in \Omega$ and $k \in \{1, \ldots, n-1\}$ be as in Case B for some $\varrho > 0$. The estimates provided by Lemma 4.13 and a choice of ϱ small enough easily imply

$$|X_k\varphi(\hat{y}_n)| \geq \frac{1}{2}|X_i\varphi(\hat{y}_n)|, \quad \text{for all } i \neq n, \tag{4.33}$$

$$|X_k\varphi(\hat{y}_n)| > \frac{1}{2}\lambda_n(\hat{y}_n). \tag{4.34}$$

By Theorem 4.12 and (4.25) there are two John curves γ_x and γ_y of parameter $\sigma > 0$, starting respectively from x and y, which are of the form (compare (4.28))

$$\gamma_x(t) = x + tN_k e_k,\ t \leq t_k(x), \text{ and } \gamma_y(t) = y + tN_k e_k, \quad t \leq t_k(y). \tag{4.35}$$

The numbers $t_k(x)$ and $t_k(y)$ are respectively defined by $t_k(x) = F_k(x, \delta_k(x))$ and $t_k(y) = F_k(y, \delta_k(y))$, where $\delta_k(x)$ and $\delta_k(y)$ are solutions of equation (4.26) written in x and y, respectively. Moreover, note that γ_x and γ_y are parallel. This is a consequence of the fact that (4.33) and (4.34) give (4.25) with y instead of x. In addition, $X_k\varphi(\hat{x}_n)$ and $X_k\varphi(\hat{y}_n)$ must have the same sign by (4.30) and thus $N_k(x) = N_k(y)$ (recall (4.22)). We denoted both by N_k.

We claim that if $\varrho > 0$ is small enough, there exist constants C_2, $C_3 > 0$ (independent of x and y) and times $t_x \le t_k(x)$ and $t_y \le t_k(y)$ such that the curves γ_x and γ_y satisfy assumptions (2.5)–(2.7) of Lemma 2.8. This will show that Ω is a uniform domain.

Define the numbers

$$\delta^* = \frac{1}{2\varrho} d(x, y) \quad \text{and} \quad t^* = F_k(x, \delta^*). \tag{4.36}$$

Since we are in Case B, we trivially have $\delta^* \le \delta_k(x)/2$, and by (4.32), $\delta^* \le \delta_k(y)$. It follows that $t^* \le t_k(x)$, $t_k(y)$. We would like to apply Lemma 2.8 for the times $t_x = t_y = t^*$. This would require the estimate (2.6), i.e., $d(\gamma_x(t^*), \gamma_y(t^*)) \le \sigma C_3/2 d(x, y)$. Unfortunately, it may happen that $\gamma_x(t^*)$ belongs (or is very near) to the plane $\{x_k = 0\}$. In this case the size of the boxes may become too small (this can be seen letting $x_k = 0$ in (4.4)), and the estimate (2.6) does not seem to hold. To overcome this problem we operate as follows.

Consider the projection of x onto the k'th coordinate plane $x_k = 0$ and denote it by $\pi(x) = \sum_{i \ne k} x_i e_i$. We distinguish the following two cases:

$$d(x + t^* N_k e_k, \pi(x)) \ge \frac{1}{4} d(x, \pi(x)), \tag{4.37}$$

$$d(x + t^* N_k e_k, \pi(x)) < \frac{1}{4} d(x, \pi(x)). \tag{4.38}$$

We first study case (4.37). Case (4.38) can be reduced to the first one (this is discussed after equation (4.44)). Choose $t_x = t_y = t^*$, and let $\gamma_x : [0, t^*] \to \Omega$ and $\gamma_y : [0, t^*] \to \Omega$ be as in (4.35). We first check (2.5), which is easier. By Theorem 4.2

$$\operatorname{diam}(\gamma_x) \ge C_0 \delta^* = C_0 \frac{d(x, y)}{2\varrho}, \tag{4.39}$$

where $C_0 < 1$ is an absolute constant. Then (2.5) holds with

$$C_3 = \frac{C_0}{2\varrho}. \tag{4.40}$$

Now we have to check (2.6), which is

$$d(\gamma_x(t^*), \gamma_y(t^*)) = d(x + t^* N_k e_k, y + t^* N_k e_k) \le \frac{\sigma C_0}{4\varrho} d(x, y). \tag{4.41}$$

We claim that there exists a constant $C_4 > 0$, independent of ϱ, x, y, such that

$$d(\gamma_x(t^*), \gamma_y(t^*)) \le C_4 d(x, y), \tag{4.42}$$

whenever x satisfies (4.37). Then (4.41) follows choosing ϱ small enough to ensure $C_4 \le \sigma C_0/(4\varrho)$.

To prove (4.42), first of all notice that, by Theorem 4.2, condition (4.37) implies $G_k(\pi(x), |x_k + t^* N_k e_k|) \ge C G_k(\pi(x), |x_k|)$ and thus

$$\begin{aligned} |x_k + t^* N_k e_k| &\ge F_k(\pi(x), C G_k(\pi(x), |x_k|)) \\ &\ge C F_k(\pi(x), G_k(\pi(x), |x_k|)) = C|x_k|, \end{aligned}$$

for some absolute (small) constant C. This estimate together with the explicit form (4.2) and (4.4) of the vector fields also implies

$$F_i(x + t^* N_k e_k, s) \ge \varepsilon_1 F_i(x, s), \quad \forall s > 0,\ i = 1, \dots, n, \tag{4.43}$$

where $\varepsilon_1 > 0$ is a new absolute small constant. Then

$$\begin{aligned} |y_i - x_i| &= F_i(x, G_i(x, |y_i - x_i|)) \le F_i(x, C d(x, y)) \\ &\le \varepsilon_1^{-1} F_i(x + t^* N_k e_k, C d(x, y)). \end{aligned}$$

This is equivalent to saying that $y + t^* N_k e_k \in \mathrm{Box}(x + t^* N_k e_k, C d(x, y))$, which gives (4.42) (by Theorem 4.2) provided C_4 is large enough. Note that all such estimates do not depend on ϱ. This proves the claim (4.41).

We have proved hypotheses (2.5) and (2.6) of Lemma 2.8 under condition (4.37). We discuss later the turning condition (2.7).

Now we study case (4.38). We shall show that it can be essentially reduced to case (4.37). By continuity, there is $t^{**} < t^*$ such that

$$d(x + t^{**} N_k e_k, \pi(x)) = \frac{1}{4} d(x, \pi(x)). \tag{4.44}$$

In this case we choose $t_x = t_y = t^{**}$, and we define δ^{**} by $t^{**} = F_k(x, \delta^{**})$.

Now we are using shorter paths. We have to make sure that their diameter is large enough to ensure that (2.5) continues to hold. In order to check (2.5), notice that the triangle inequality and (4.38) give

$$d(x, \pi(x)) \ge d(x, \gamma_x(t^*)) - d(\pi(x), \gamma_x(t^*)) > d(x, \gamma_x(t^*)) - \frac{1}{4} d(x, \pi(x)),$$

which yields $d(x, \pi(x)) \ge \frac{4}{5} d(x, \gamma_x(t^*))$. Thus, by (4.44)

$$\begin{aligned} \mathrm{diam}\,(\gamma_x|_{[0,t^{**}]}) &\ge d(x, \pi(x)) - d(\gamma_x(t^{**}), \pi(x)) \\ &= \frac{3}{4} d(x, \pi(x)) \ge \frac{3}{5} d(x, \gamma_x(t^*)) \ge \frac{3}{5} C_3 d(x, y), \end{aligned}$$

where C_3 is given by (4.40). In other words, changing δ^* with δ^{**} does not give any problem in checking (2.5). We just have to modify slightly the constant C_3 in (4.40).

Moreover, since (4.44) holds, we can prove (4.42) and ultimately (4.41) with t^{**} instead of t^*. This shows that (2.6) holds in case (4.38), as well.

In order to finish the proof of the theorem in Case B, we have to check condition (2.7). We check the upper bound for t^*, which is greater than t^{**}. The estimate $\operatorname{diam}(\gamma_x|_{[0,t^*]}) \lesssim Cd(x,y)/\varrho$ follows from the definition of δ^*. It remains to estimate the diameter of γ_y. Since by Theorem 4.2 $\operatorname{diam}(\gamma_y|_{[0,t^*]}) \simeq G_k(y,t^*)$, the proof is concluded as soon as we show that $G_k(y,t^*) \le 2G_k(x,t^*)$. Since $t^* = F_k(x,\delta^*)$, the claim is equivalent to

$$G_k(y, F_k(x,\delta^*)) \le 2\delta^* \iff F_k(x,\delta^*) \le F_k(y, 2\delta^*),$$

which holds (also with $1 + c_\varrho$ instead of 2) in force of (4.15) (in the statement of Lemma 4.5 x and y can be interchanged). The proof of Case B is concluded.

Case A is the easy part. Denote by $\bar{x}$ and $\bar{y}$ the endpoints of the paths γ_x and γ_y at the end of their first piece, i.e.,

$$\begin{aligned}\bar{x} &= x + t_{k(x)}(x) N_{k(x)} e_{k(x)}\,,\\ \bar{y} &= y + t_{k(y)}(y) N_{k(y)} e_{k(y)}\,.\end{aligned}$$

Here $k(x)$ may be different from $k(y)$. This does not matter because the points are not too near. It could also be $\bar{x} = x$ or $\bar{y} = y$ if one or both of the points belong to Case 1 in (4.23). At any rate, we have

$$d(x,\bar{x}) \le \Delta(x) \le \frac{1}{\varrho} d(x,y).$$

The same estimate holds for $d(y,\bar{y})$ (we are assuming $\Delta(x) \ge \Delta(y)$). Here ϱ is small but has been fixed in the proof of Case B. We have the paths

$$\gamma_x(s) = \bar{x} + s e_n \quad \text{and} \quad \gamma_y(s) = \bar{y} + s e_n,$$

with $s \ge 0$. The proof of Case A can be concluded noting that by invariance with respect to translations along the n-th direction we have, independently of s,

$$\begin{aligned} d(\bar{x} + s e_n, \bar{y} + s e_n) &= d(\bar{x},\bar{y}) \le d(\bar{x},x) + d(x,y) + d(\bar{y},y)\\ &\le \left(\frac{1}{\varrho} + 1 + \frac{1}{\varrho}\right) d(x,y). \end{aligned}$$ □

Proof of Lemma 4.13. Fix $k \in \{1,\dots,n-1\}$ such that

$$|X_k\varphi(\hat{x}_n)| = \max_{i=1,\dots,n-1} |X_i\varphi(\hat{x}_n)|.$$

Then (4.17) gives

$$
\begin{aligned}
&|X_i\varphi(\hat{x}_n) - X_i\varphi(\hat{y}_n)| \\
&\qquad \le \operatorname{osc}(X_i\varphi, \mathrm{Box}_n(\hat{x}_n, d(x,y))) \\
&\qquad \le C(d(x,y)|X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n, d(x,y))) \\
&\qquad \le C(\varrho\delta_k(x)|X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} + C\varrho\Lambda_n(\hat{x}_n, \delta_k(x))),
\end{aligned}
$$

by Case B and Proposition 4.4. Now, in order to estimate the right hand side, note that, by (4.26), $\Lambda_n(\hat{x}_n, \delta_k(x)) = \varepsilon_0|X_k\varphi(\hat{x}_n)|$. Moreover, by Proposition 4.4

$$
\delta_k(x) \le \Lambda_n(\hat{x}_n, \delta_k(x))^{1/(d_n-1)} = (\varepsilon_0|X_k\varphi(\hat{x}_n)|)^{1/(d_n-1)}.
$$

Then (4.30) is proved. Letting $i = k$ in (4.30) we get

$$
|X_k\varphi(\hat{y}_n)| \ge (1 - c_\varrho)|X_k\varphi(\hat{x}_n)|. \tag{4.45}
$$

We are now ready to prove (4.31). By the definition of Λ_n we have

$$
\begin{aligned}
\lambda_n(\hat{y}_n) &\le \lambda_n(\hat{x}_n) + \Lambda_n(\hat{x}_n, d(x,y)) \le \lambda_n(\hat{x}_n) + \Lambda_n(\hat{x}_n, \varrho\delta_k(x)) \\
&\le \lambda_n(\hat{x}_n) + c_\varrho\varepsilon_0|X_k\varphi(\hat{x}_n)| \le (1 + c_\varrho)|X_k\varphi(\hat{y}_n)|,
\end{aligned}
$$

where we used Case B to estimate $\lambda_n(\hat{x}_n)$ and (4.45). Then (4.31) is proved.

We prove (4.32). By (4.45) and by the definition of δ_k, we have

$$
\begin{aligned}
\Lambda_n(\hat{y}_n, \delta_k(y)) &= \varepsilon_0|X_k\varphi(\hat{y}_n)| \ge \varepsilon_0(1 - c_\varrho)|X_k\varphi(\hat{x}_n)| \\
&\ge (1 - c_\varrho)\Lambda_n(\hat{x}_n, \delta_k(x)).
\end{aligned} \tag{4.46}
$$

Assume by contradiction that $\delta_k(y) < \frac{1}{2}\delta_k(x)$. Then, we have

$$
\mathrm{Box}_n(\hat{y}_n, \delta_k(y)) \subset \mathrm{Box}_n\left(\hat{y}_n, \frac{1}{2}\delta_k(x)\right) \subset \mathrm{Box}_n\left(\hat{x}_n, \frac{1}{2}(1 + c_\varrho)\delta_k(x)\right),
$$

by Lemma 4.5 (recall that $d(x,y) \le \varrho\delta_k(x)$, by Case B). Then

$$
\begin{aligned}
\Lambda_n(\hat{y}_n, \delta_k(y)) &= \sup_{\hat{z}_n \in \mathrm{Box}_n(\hat{y}_n, \delta_k(y))} |\lambda_n(\hat{z}_n) - \lambda_n(\hat{y}_n)| \\
&\le \Lambda_n\left(\hat{x}_n, \frac{1}{2}(1 + c_\varrho)\delta_k(x)\right) + |\lambda_n(\hat{x}_n) - \lambda_n(\hat{y}_n)| \\
&\le \Lambda_n\left(\hat{x}_n, \frac{1}{2}(1 + c_\varrho)\delta_k(x)\right) + \Lambda_n(\hat{x}_n, \varrho\delta_k(x)) \\
&\le \left(h\left(\frac{1}{2}(1 + c_\varrho)\right) + h(\varrho)\right)\Lambda_n(\hat{x}_n, \delta_k(x)),
\end{aligned}
$$

where h is the function introduced in Proposition 4.4. By the properties of h, we immediately see that the last chain of inequalities is in contradiction with (4.46), if ϱ is small enough. This finishes the proof of Lemma 4.13. □

Example 4.15. It is known that control balls defined by vector fields are always John domains (see [23] and [29]). We show that they not necessarily are uniform domains. Consider in $\mathbb{R}^2$ the vector fields $X_1 = \partial_1$ and $X_2 = x_1\partial_2$ and let $(\mathbb{R}^2, d)$ be the metric space with metric defined as in (2.1). Applying Theorem 4.2 to this special case it is not difficult to see that

$$d((x_1, x_2), (0, y_2)) \simeq |x_1| + |x_2 - y_2|^{1/2}. \tag{4.47}$$

The ball $B = B(0, 1)$ is a symmetric set with respect to x_1 and x_2, and can be computed explicitly (see for instance [22] and [5]). Precisely,

$$\begin{aligned}&\partial B \cap \{(x_1, x_2) \mid x_1, x_2 \geq 0\}\\ &\quad = \Big\{(x_1(\vartheta), x_2(\vartheta)) = \Big(\frac{\sin\vartheta}{\vartheta}, \frac{2\vartheta - \sin 2\vartheta}{4\vartheta^2}\Big) \mid 0 \leq \vartheta \leq \pi\Big\}.\end{aligned}$$

Notice that

$$(x_1(\pi), x_2(\pi)) = \Big(0, \frac{1}{2\pi}\Big), \quad (x_1'(\pi), x_2'(\pi)) = \Big(-\frac{1}{\pi}, -\frac{1}{\pi^2}\Big).$$

Then, all the points of the set $\{x \mid x_2 = (1/(2\pi))(1 + |x_1|)\}$ belong to B, if $|x_1|$ is small enough.

Take the points

$$x^+ = \Big(x_1, \frac{1}{2\pi}(1 + |x_1|)\Big) \quad \text{and} \quad x^- = \Big(-x_1, \frac{1}{2\pi}(1 + |x_1|)\Big),$$

where $x_1 > 0$ is small. If $\gamma : [0, 1] \to B$ is a continuous curve joining the point x^+ to the point x^-, then it must intersect the x_2 axis. Call $(0, y_2)$ this intersection point. It must be the case that $|y_2| < 1/(2\pi)$. Then by (4.47)

$$\begin{aligned}\operatorname{diam}(\gamma) &\geq d\Big(\Big(x_1, \frac{1}{2\pi}(1 + |x_1|)\Big), (0, y_2)\Big)\\ &\simeq |x_1| + \Big(\frac{1}{2\pi}(1 + |x_1|) - y_2\Big)^{1/2} \geq \Big(\frac{1}{2\pi}|x_1|\Big)^{1/2}.\end{aligned}$$

On the other hand, $d(x^+, x^-) = 2|x_1|$, and we find

$$\operatorname{diam}(\gamma) \geq \frac{C}{|x_1|^{1/2}} d\Big(\Big(x_1, \frac{1}{2\pi}(1 + |x_1|)\Big), \Big(-x_1, \frac{1}{2\pi}(1 + |x_1|)\Big)\Big),$$

for some absolute constant $C > 0$. Letting $x_1 \to 0$ we see that condition (2.3) can not hold uniformly.

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