JOHN DOMAINS FOR THE CONTROL DISTANCE OF DIAGONAL VECTOR FIELDS

By

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Abstract. We study John domains in the metric space associated with a system of diagonal vector fields.

1 Introduction

We study domains with the twisted cone property in $\mathbb{R}^n$ equipped with the control distance generated by a family of diagonal vector fields $X_1, ..., X_n$, where $X_j = \lambda_j(x)\partial_{x_j}$, $j = 1, \dots, n$. Bounded domains satisfying similar geometric conditions are relevant, for instance, in the global theory of second order PDEs related to subelliptic operators of the form $\mathcal{L} = \sum_{j=1}^n X_j^2$.

The twisted cone property was introduced in the Euclidean setting by John in his seminal paper [Joh] on the rigidity of quasiisometric maps in $\mathbb{R}^n$. Besides its importance in geometric function theory, this property plays a central role in the theory of first order Sobolev spaces; see, e.g., the papers by Reshetnyak [R], Besov [Be], Martio [M], Bojarski [Bo] and the monograph by Maz'ya and Poborchi [MP]. More recent references are [SS], [BK], [HK1], [KOT]. Other classes of domains appearing in more refined questions in harmonic analysis and PDEs can be defined through cone conditions: uniform and NTA domains are the most important examples (see [MS], [Jon] and [JK]).

Domains with the cone property can be defined in any metric space and, in particular, in Carnot–Carathéodory spaces. Given a metric space (M, d), a rectifiable path $\gamma : [0, 1] \to M$ and a positive number $\varepsilon > 0$, the twisted cone with core at γ and aperture ε is the set $\mathcal{C}_{\gamma,\varepsilon} = \bigcup_{0<t<1} B\big(\gamma(t), \varepsilon \operatorname{length}(\gamma|_{(0,t)})\big)$. A bounded domain $\Omega \subset M$ is a *John domain* with John constant ε and center $x_0 \in \Omega$ if, for any $x \in \Omega$,

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there is a cone $C_{\gamma,\epsilon} \subset \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = x_0$. John domains are also known as *domains with the twisted cone property*. In the Euclidean setting, the "rigid" cone property is more general than the "twisted" cone property, as the von Koch snowflake in the plane shows.

Several theorems involving the cone property have been generalized from the Euclidean to the Carnot–Carathéodory setting. If a bounded open set Ω satisfies an exterior cone property with respect to the distance generated by a family of Hörmander vector fields, then the Wiener criterion is satisfied at any boundary point and Ω is regular in the sense of Perron, Wiener, Brelot and Bony for the Dirichlet problem for the subelliptic Laplacian $\mathcal{L}$; see the papers by Negrini and Scornazzani [NS], Hueber [Hu] and Danielli [D].

Moreover, the interior cone property is related to chaining conditions that are useful in the proof of subelliptic Sobolev–Poincaré inequalities. This fact was recognized by Jerison in [Je], and later used by several authors; see the contributions by Lu [L], Franchi, Gutierrez and Wheeden [FGW], Garofalo and Nhieu [GN1], Franchi, Lu and Wheeden [FLW], Buckley, Koskela and Lu [BKL], Hajłasz and Koskela [HK2].

Despite all these results, no easy condition is known which ensures the John property in metric spaces associated with vector fields. This problem, which has somehow eluded study in the literature, requires a precise knowledge of the structure of control balls. To emphasize the nontrivial nature of the situation, we mention an example in [Je]: in the apparently simple situation of the vector fields $X_1 = \partial_1$, $X_2 = x_1^3\partial_2$ in the plane, the Poincaré inequality may fail for smooth domains, which, therefore, need not be John domains for the related control metric. Thus, in Carnot–Carathéodory spaces, Euclidean regularity is not sufficient to ensure the cone property. An even more striking example is that "gauge balls" in homogeneous groups may fail to be John domains if the group has step greater than 2 (see [MM2]). See also [HH], where examples of regular and irregular sets in Potential Theory are discussed.

The John property for a set Ω with respect to a system of vector fields may fail owing to the presence of *characteristic points*. A point $x \in \partial\Omega$ is characteristic if all the vector fields X_j are tangent to the boundary $\partial\Omega$ at x. Note that in the Euclidean case $\mathcal{L} = \Delta$ (equivalently $X_j = \partial_j$, $j = 1, \dots, n$), the characteristic set is empty. A relevant problem in the study of the regularity up to the boundary for elliptic degenerate operators is to understand how a boundary having characteristic points interacts with the vector fields. The purpose of this paper is to analyze the structure of John domains in Carnot–Carathéodory spaces for diagonal families of vector fields.

We briefly recall some known results. Capogna and Garofalo proved in [CG] that $C^{1,1}$-domains in step 2 homogeneous groups are John domains (see also [MM2], where the NTA property is proved under the same assumptions). In the same setting, examples of John and NTA domains are provided by Capogna and Tang [CT], Greshnov [G2] and Capogna, Garofalo and Pauls [CGP]. For the vector fields in the plane $X_1 = \partial_1$, $X_2 = |x_1|^\alpha \partial_2$, $\alpha > 0$, it is not difficult to check that α-admissible domains in the sense of [MM1] are John domains (compare Jerison's example mentioned above and see also [FF1] and the discussion in [DGN]). Finally, a sufficient condition ensuring the John property for open sets in step 3 homogeneous groups is given in [MM2]. The analysis becomes considerably harder in groups of higher step.

We study domains with the cone property in a class of Carnot–Carathéodory spaces with no underlying group structure. We consider a system of diagonal vector fields in $\mathbb{R}^n$ of the form $X_1 = \lambda_1(x)\partial_1, ..., X_n = \lambda_n(x)\partial_n$, whose control metric, under suitable assumptions on the functions λ_j, has been studied in detail by Franchi and Lanconelli in [FL]. Our basic model is given by the following vector fields on $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = x_1^{\alpha_1} \frac{\partial}{\partial x_2}, \quad X_3 = x_1^{\alpha_1} x_2^{\alpha_2} \frac{\partial}{\partial x_3}, \quad \alpha_1, \alpha_2 \in \mathbb{N}. \tag{1.1}$$

Consider an open set in $\mathbb{R}^3$ of the form $\Omega = \{x_3 > \varphi(x)\}$, $x = (x_1, x_2) \in \mathbb{R}^2$, where $\varphi \in C^1(\mathbb{R}^2)$. By the results of [FL], control balls are comparable with boxes of the form $Q(x,r) \times]x_3 - F_3(x,r), x_3 + F_3(x,r)[$, where $Q(x,r)$ are suitable rectangles in the plane and $F_3(x,r) > 0$. We say that the boundary $\partial\Omega$ is *admissible* if for all $x \in \mathbb{R}^2$, $r > 0$,

$$\sum_{i=1,2} \operatorname{osc}(X_i\varphi, Q(x,r)) \le C\Big(r \sum_{i=1,2} |X_i\varphi(x)|^m + \operatorname{osc}(\lambda_3; Q(x,r))\Big). \tag{1.2}$$

The oscillation of the derivatives of the function φ along the vector fields X_1 and X_2 should be bounded by a sum of two terms. The first term in the right hand side vanishes on the characteristic set, while the second one gives an amount of oscillation admitted also at characteristic points. The latter is determined by the oscillation on $Q(x,r)$ of the function $\lambda_3(y) = y_1^{\alpha_1} y_2^{\alpha_2}$. This oscillation is strictly related to the size of control balls in the vertical direction. The appropriate balance between the two terms is given by the power

$$m = \frac{\alpha_1 + \alpha_2 + \alpha_1\alpha_2 - 1}{\alpha_1 + \alpha_2 + \alpha_1\alpha_2}.$$

This choice is a key point. In Definition 2.8, generalizing (1.2), we introduce a class of *domains with admissible boundary* in the n-dimensional situation. Our main result is the following

Theorem 1.1. *Domains with admissible boundary are John domains.*

The class of domains with admissible boundary is nonempty. In Theorem 2.10, we show that the open set $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \left(|x_1|^{2(\alpha_1+1)} + x_2^2\right)^{1+\alpha_2} + x_3^2 < 1\}$ has admissible boundary for the vector fields in (1.1). The basic tool in the proof of this result is the following criterion for checking the admissibility condition (1.2).

Theorem 1.2. *Let $N(x) = |x_1|^{2(\alpha_1+1)} + x_2^2$ and assume that $\varphi(x) = g(N(x))$, where $g \in C^2(0, +\infty)$ is a function such that for some constant $C > 0$,*

$$0 \le g'(t) \le Ct^{(\alpha_2-1)/2}, \quad |g''(t)| \le C\frac{g'(t)}{t}, \quad g'(2t) \le Cg'(t), \quad t > 0.$$

Then the surface $\{x_3 = \varphi(x_1, x_2)\}$ is admissible.

Together with the "two dimensional" examples discussed in [FF1], our results are the only examples of regular domains for diagonal vector fields.

Finally, we mention some more papers dealing with regularity of domains. The notion of uniform domain (or (ε, δ)-domain) for a family of vector fields has been introduced and studied by Garofalo and Nhieu in [GN2], and by Vodop'yanov and Greshnov in [VG] and [G1]. They generalized the Jones extension theorem to the setting of subelliptic Sobolev spaces. Capogna and Garofalo [CG] introduced the notion of NTA domain in Carnot–Carathéodory spaces and studied the boundary behavior of positive $\mathcal{L}$-harmonic functions. The papers [CGN] and [G2] deal with examples of uniform and NTA domains in homogeneous groups of step 2. In [DGN] the problem of the trace of Sobolev functions in Carnot–Carathéodory spaces is studied (see also [MM1]). In [FF1] and [FF2], a new class of domains, called φ-Harnack domains, is introduced and studied. In the Heisenberg group, the problem of boundary accessibility through rectifiable curves is examined in [BM]. Finally, we mention [MM3], which is the continuation of the research initiated in the present work. In this paper we prove that "admissible domains" in the sense of Definition 2.8 are NTA domains.

The plan of the paper is the following. In Section 2, we introduce "admissible domains" and show that the class is nonempty. In Section 3, we prove that these domains are John domains.

Notation. If $u, v \ge 0$, we write $u \lesssim v$ for $u \le Cv$, where $C \ge 1$ is an absolute constant. Analogously, $u \simeq v$ stands for $u \lesssim v$ and $v \lesssim u$. By d we denote the control metric induced on $\mathbb{R}^n$ by a system of vector fields. For $K \subset \mathbb{R}^n$, we write $\operatorname{diam}(K) = \sup_{x,y \in K} d(x,y)$ and $\operatorname{dist}(x, K) = \inf_{y \in K} d(x,y)$. If $\gamma : [0,1] \to \mathbb{R}^n$ is a curve and $0 \le a \le b \le 1$, we denote by $\gamma|_{[a,b]}$ the restriction of γ to the interval $[a,b]$. The coordinate versors of $\mathbb{R}^n$ are denoted by $e_1, \dots, e_n$. Finally, the symbol

σ always denotes the "aperture" of a cone. In Section 3, we sometimes adopt a slight abuse of notation by writing σ instead of $f(\sigma)$, where f is a positive function whose relevant property is $\lim_{\sigma \longrightarrow 0_+} f(\sigma) = 0$.

2 Geometry of diagonal vector fields and flat surfaces

In this section, we describe the geometry of diagonal vector fields and introduce the basic definitions of regularity for surfaces in the metric space associated with them. We also provide examples of regular surfaces.

2.1 Preliminaries and definition of admissible surface. Consider the vector fields

$$X_j = \lambda_j(x)\frac{\partial}{\partial x_j}, \qquad j = 1, \dots, n, \tag{2.1}$$

where

$$\lambda_1(x) = 1 \quad \text{and} \quad \lambda_j(x) = \prod_{i=1}^{j-1} |x_i|^{\alpha_i}, \quad j = 2, ..., n. \tag{2.2}$$

Assume that the real numbers α_i satisfy

$$\alpha_i = 0 \quad \text{or} \quad \alpha_i \in [1, \infty[. \tag{2.3}$$

This condition ensures that the functions λ_j, and thus the vector fields X_j, are locally Lipschitz continuous. If the numbers α_i are integers, then we can change the functions λ_j in (2.2), writing $x_i^{\alpha_i}$ instead of $|x_i|^{\alpha_i}$. The vector fields then become a Hörmander system.

We introduce some notation, following [FL]. For all $j = 1, ..., n$, define inductively the functions $F_j : \mathbb{R}^n \times [0, +\infty) \to [0, +\infty)$ by

$$\begin{aligned} &F_1(x,r) = r, \qquad F_2(x,r) = r\lambda_2(|x_1| + F_1(x,r)), \quad \dots \\ &F_j(x,r) = r\lambda_j\big(|x_1| + r, |x_2| + F_2(x,r), \dots, |x_{j-1}| + F_{j-1}(x,r)\big). \end{aligned} \tag{2.4}$$

An inspection of the explicit form (2.2) of the functions λ_j shows that

$$F_{j+1}(x,r) = F_j(x,r)\big(|x_j| + F_j(x,r)\big)^{\alpha_j}, \quad j = 1, \dots, n-1. \tag{2.5}$$

Note that $F_j(x,r)$ actually depends only on $x_1, \dots, x_{j-1}$. Moreover, $r \mapsto F_j(x,r)$ is increasing and satisfies the doubling property

$$F_j(x, 2r) \le CF_j(x,r), \quad x \in \mathbb{R}^n, 0 < r < \infty \tag{2.6}$$

for all $j = 1, ..., n$. This also implies

(2.7) $$F_j(x, r+s) \le C(F_j(x,r) + F_j(x,s)), \quad 0 < r, s < \infty.$$

Finally, for any fixed $x \in \mathbb{R}^n$, the function $F_j(x,\cdot)$ is strictly increasing and maps $]0,+\infty[$ onto itself. We denote its inverse by $G_j(x,\cdot) = F_j(x,\cdot)^{-1}$.

Define inductively the real numbers d_j by

(2.8)
$$d_1 = 1, \quad d_2 = 1+\alpha_1, \quad \dots \quad , d_j = 1 + \sum_{i=1}^{j-1} d_i\alpha_i = (1+\alpha_1)\cdot \ldots \cdot (1+\alpha_{j-1}).$$

We say that d_j is the *degree of the variable* x_j. Note that $F_j(0,r) = r^{d_j}$.

It is well-known that the vector fields (2.1) induce on $\mathbb{R}^n$ a metric d in the following way (see [FL], [FP] and [NSW]). A Lipschitz continuous curve $\gamma : [0,T] \to \mathbb{R}^n$, $T \ge 0$, is *subunit* if there exists a vector of measurable functions $h = (h_1, ..., h_n) : [0,T] \to \mathbb{R}^n$ such that $\dot\gamma(t) = \sum_{j=1}^n h_j(t) X_j(\gamma(t))$, $|h(t)| \le 1$ for a.e. $t \in [0,T]$. Define $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ by setting

(2.9) $$\begin{aligned} d(x,y) = \inf\{T \ge 0 : &\text{ there exists a subunit curve } \gamma : [0,T] \to \mathbb{R}^n \\ &\text{ such that } \gamma(0) = x \text{ and } \gamma(T) = y\}. \end{aligned}$$

The definition of the metric d still makes sense for any system of smooth vector fields $X_1, ..., X_m$, even with $m < n$, provided $d(x,y)$ is finite for all x, y (this happens, for instance, if the vector fields satisfy the Hörmander condition; see [NSW]). We denote by $B(x,r)$ the balls in $\mathbb{R}^n$ defined by the metric d.

The structure of the control balls can be described by means of the boxes

(2.10) $$\mathrm{Box}(x,r) := \{x + h : |h_j| < F_j(x,r),\ j = 1, ..., n\}.$$

The following theorem is proved in [FL].

Theorem 2.1. *There exists a constant $C > 0$ such that*

(2.11) $$\begin{aligned} &\mathrm{Box}(x, C^{-1}r) \subset B(x,r) \subset \mathrm{Box}(x, Cr), \qquad x \in \mathbb{R}^n, r \in]0,+\infty[, \\ &C^{-1}d(x,y) \le \sum_{j=1}^n G_j(x, |y_j - x_j|) \le Cd(x,y), \qquad x, y \in \mathbb{R}^n. \end{aligned}$$

Theorem 2.1, the triangle inequality and the doubling property (2.6) (or a direct computation) give the estimate

(2.12) $$F_j(x + F_k(x,r)e_k, s) \le CF_j(x,s) \quad x \in \mathbb{R}^n, 0 < r \le s < +\infty.$$

Introduce the following convention. If $j = 1, ..., n$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we write $\hat{x}_j = (x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) \equiv (x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n)$.

In Section 3, we study regularity properties connected with the boundary of an open set $\Omega \subset \mathbb{R}^n$ of class C^1. Here, we introduce the basic assumptions on $\partial\Omega$. Given a point $x \in \partial\Omega$, we write $\partial\Omega$ locally as a graph of the form $x_j = \varphi(\hat{x}_j)$ for some $j = 1, \dots, n$. We first discuss the case $j = n$, which is the most significant.

Introduce the $(n-1)$-dimensional box

$$\text{(2.13)} \qquad \mathrm{Box}_n(\hat{x}_n, r) = \{\hat{x}_n + \hat{h}_n : |h_i| < F_i(\hat{x}_n, r),\ i = 1, \dots, n-1\},$$

and let $\Lambda_n(\hat{x}_n, r) = \sup_{\hat{y}_n \in \mathrm{Box}_n(\hat{x}_n, r)} |\lambda_n(\hat{y}_n) - \lambda_n(\hat{x}_n)|$.

The following proposition records some properties of Λ_n needed in the sequel. The proof is postponed to the Appendix. It relies on the simple fact that if $\alpha \geq 1$, then

$$\text{(2.14)} \qquad (t+r)^\alpha - t^\alpha \simeq \alpha r (t+r)^{\alpha-1}, \quad t \geq 0, \quad r \geq 0.$$

Proposition 2.2. *Assume that at least one of the numbers α_j, $j = 1, \dots, n$, is strictly positive. Then there exists a constant $\eta > 0$ such that for all $\hat{x}_n \in \mathbb{R}^{n-1}$, $R > 0$ and $a \in]0, 1]$,*

$$\text{(2.15)} \qquad \Lambda_n(\hat{x}_n, aR) \leq h(a)\Lambda_n(\hat{x}_n, R), \quad \textit{where } h(a) = \frac{a}{a + \eta(1-a)}.$$

Moreover, $\Lambda_n(\hat{x}_n, r)) \geq r^{d_n - 1}$, $\Lambda_n(\hat{x}_n, r) \leq (C/r)F_n(x, r)$ and $\Lambda_n(x, 2r) \leq C\Lambda_n(x, r)$ for some constant $C > 0$, and for all $r > 0$ and $\hat{x}_n \in \mathbb{R}^{n-1}$.

In order to introduce our notion of "admissible surface," we first give the definition for a graph of the form $x_n = \varphi(\hat{x}_n)$. This is the most degenerate case and contains all the difficulties of the problem. Then we show that a graph of the form $x_j = \varphi(\hat{x}_j)$ with $j \neq n$ can be studied by reducing to the previous case. Finally, in Definition 2.8, we introduce the notion of open set with admissible boundary. Recall the standard notation $\mathrm{osc}(f, A) := \sup_{x,y \in A} |f(x) - f(y)|$.

Definition 2.3. Let $\varphi \in C^1(\mathbb{R}^{n-1})$. The surface $\{x_n = \varphi(\hat{x}_n)\}$ is said to be *admissible* if there exist $C > 0$ and $r_0 > 0$ such that for all $\hat{x}_n \in \mathbb{R}^{n-1}$, $r \in]0, r_0]$,

$$\text{(2.16)} \qquad \sum_{i \neq n} \mathrm{osc}(X_i\varphi, \mathrm{Box}_n(\hat{x}_n, r)) \leq C\Big(r \sum_{i \neq n} |X_i\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n, r)\Big).$$

Note that if we are not in the Euclidean case, i.e., at least one of the numbers α_i is strictly positive, then $d_n \geq 2$ (compare (2.3)) and the exponent $(d_n - 2)/(d_n - 1)$ is nonnegative.

We are actually interested in surfaces which are the boundaries of bounded sets. Definition 2.3 can be stated also for a bounded graph $x_n = \varphi(\hat{x}_n)$, letting $\hat{x}_n$ belong to a bounded open set of $\mathbb{R}^{n-1}$.

Proposition 2.4. *Let $\varphi \in C^1(\mathbb{R}^{n-1})$ satisfy (2.16). Then there exists $C > 0$ such that for all $\hat{x}_n \in \mathbb{R}^{n-1}$ and $r \in]0, r_0]$,*

$$\operatorname{osc}(\varphi, \operatorname{Box}_n(\hat{x}_n, r)) \leq C\Big(r \sum_{i \neq n} |X_i\varphi(\hat{x}_n)| + r\Lambda_n(\hat{x}_n, r)\Big). \tag{2.17}$$

Proof. Fix $\hat{x}_n, \hat{y}_n \in \mathbb{R}^{n-1}$, and let $\delta = d(\hat{x}_n, \hat{y}_n)$. Then there is a subunit curve $\gamma : [0, \delta] \to \mathbb{R}^{n-1} \cong \mathbb{R}^{n-1} \times \{0\}$ such that $\gamma(0) = \hat{x}_n$ and $\gamma(\delta) = \hat{y}_n$. Then we have

$$|\varphi(\hat{x}_n) - \varphi(\hat{y}_n)| \leq \int_0^\delta \sum_{i \neq n} |X_i\varphi(\gamma(t))| dt \leq \delta \sup_{\operatorname{Box}_n(\hat{x}_n, \delta)} \sum_{i \neq n} |X_i\varphi|. \tag{2.18}$$

By (2.16),

$$\begin{aligned}
\sup_{\operatorname{Box}_n(\hat{x}_n, \delta)} \sum_{i \neq n} |X_i\varphi| &\leq \sum_{i \neq n} |X_i\varphi(\hat{x}_n)| + \sum_{i \neq n} \operatorname{osc}(X_i\varphi, \operatorname{Box}_n(\hat{x}_n, \delta)) \\
&\lesssim \sum_{i \neq n} |X_i\varphi(\hat{x}_n)| + \delta \sum_{i \neq n} |X_i\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n, \delta) \\
&\lesssim \sum_{i \neq n} |X_i\varphi(\hat{x}_n)| + \Lambda_n(\hat{x}_n, \delta).
\end{aligned}$$

Here we have used Hölder's inequality $\delta |X_i\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} \lesssim \delta^{d_n-1} + |X_i\varphi(\hat{x}_n)|$ and the inequality $\delta^{d_n-1} \leq \Lambda_n(\hat{x}_n, \delta)$ proved in Proposition 2.2. Now, (2.17) follows from Proposition 2.1 and from the doubling property $\Lambda_n(\hat{x}_n, 2r) \leq C\Lambda_n(\hat{x}_n, r)$, proved in Proposition 2.2. □

Remark 2.5. Let $\varphi \in C^2(\mathbb{R}^{n-1})$. Assume that

$$\sum_{i,j=1}^{n-1} |X_iX_j\varphi(\hat{x}_n)| \leq C \sum_{i=1}^{n-1} |X_i\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} \tag{2.19}$$

for all $\hat{x}_n \in \mathbb{R}^{n-1}$ with $x_1 x_2 \cdot \ldots \cdot x_{n-1} \neq 0$. Then the surface $\{x_n = \varphi(\hat{x}_n)\}$ is admissible. It is not difficult to see that condition (2.19) implies (2.16). This can be checked by a suitable adaptation of the Gronwall inequality. Condition (2.19) is also easier to check than (2.16). Surfaces of the form $x_n = |x_1|^m$, where m is large enough, satisfy (2.19). The drawback is that this condition is not refined enough to give examples of admissible surfaces which are boundaries of bounded sets (see Theorems 2.9 and 2.10).

Next we introduce admissible surfaces of the form $\{x_j = \varphi(\hat{x}_j)\}$, $j \neq n$. We would like to give a definition similar to Definition 2.3. The set $\mathrm{Box}_n(\hat{x}_n, r)$ is the intersection of $\mathrm{Box}(x,r)$ with the plane $\{y \in \mathbb{R}^n : y_n = x_n\}$. When $j \neq n$, the intersection of $\mathrm{Box}(x,r)$ with the plane $\{y \in \mathbb{R}^n : y_j = x_j\}$ depends on x_j. Thus (2.13) cannot be trivially generalized. But, roughly speaking, the vector fields $X_{j+1}, \dots, X_n$ are "more degenerate" than X_j; and this suggests that the dependence of the function $\varphi(\hat{x}_j)$ on $x_{j+1}, \dots, x_n$ needs a less careful control than the dependence on $x_1, \dots, x_{j-1}$. In order to make this remark rigorous, define new functions and vector fields

$$\widetilde{\lambda}_i(x) = \begin{cases} \lambda_i(x) & \text{if } i \le j, \\ \lambda_j(x) & \text{if } i \ge j, \end{cases} \qquad \text{and } \widetilde{X}_i = \widetilde{\lambda}_i \partial_i, \quad i = 1, \dots, n. \tag{2.20}$$

In this situation, we can view the variable x_j as the n-th variable with respect to the new vector fields. All previous results hold for these vector fields. The functions $\widetilde{F}_i(x,r)$ are defined exactly as in (2.4). Set

$$\widetilde{\mathrm{Box}}(x,r) = \{x + h : |h_i| < \widetilde{F}_i(x,r),\ i = 1, \dots, n\}$$

and denote by $\widetilde{d}$ the metric constructed as in (2.9) using subunit curves with respect to the vector fields $\widetilde{X}_j$. Let $\widetilde{B}(x,r)$ be the corresponding balls. In the following proposition, we list some easy relations between the distances d and $\widetilde{d}$.

Proposition 2.6. *For any $C_1 > 0$, there is $C_2 > 0$ such that*

(i) *if $|x|, |y|, r < C_1$ then $B(x,r) \subset \widetilde{B}(x, C_2 r)$ and $\widetilde{d}(x,y) \le C_2 d(x,y)$;*

(ii) *writing $x' = (x_1, \dots, x_j)$ and $x'' = (x_{j+1}, \dots, x_n)$, we have $d((x', x''), (y', x'')) \simeq \widetilde{d}((x', x''), (y', x'')$.*

Proof. We have $\widetilde{F}_i(\hat{x}_j, r) = \widetilde{F}_i(\hat{x}_j, r)$ if $i \le j$; while for $i > j$, $\widetilde{F}_i(x,r) = F_j(x,r)$. Then, if $i > j$,

$$\begin{aligned} F_i(x,r) &= F_j(x,r)\big(|x_j| + F_j(x,r)\big)^{\alpha_j} \cdots \big(|x_{i-1}| + F_{i-1}(x,r)\big)^{\alpha_{i-1}} \\ &\le C F_j(x,r) \le F_j(x, Cr) = \widetilde{F}_i(x, Cr), \end{aligned}$$

as soon as $|x|, r \le C$. Then $\mathrm{Box}(x,r) \subset \widetilde{\mathrm{Box}}(x, Cr)$. Thus (i) follows by Theorem 2.1.

In order to see (ii), recall that the function $\widetilde{G}_i(x, \cdot)$ is the inverse of $\widetilde{F}_i(x, \cdot)$. Moreover, if $i \le j$, then $\widetilde{F}_i(x,r) = F_i(x,r)$. Thus Theorem 2.1 gives

$$d\big((x', x''), (y', x'')\big) \simeq \sum_{i=1}^{j} G_i(x', |x_i - y_i|) = \sum_{i=1}^{j} \widetilde{G}_i(x', |x_i - y_i|) \simeq d\big((x', x''), (y', x'')\big).$$

This concludes the proof of (ii). □

The sections of the boxes $\widetilde{\mathrm{Box}}(x,r)$ with the planes $\{y \in \mathbb{R}^n : y_j = x_j\}$ do not depend on x_j. Thus we can set with unambiguous meaning

$$\widetilde{\mathrm{Box}}_j(\hat{x}_j, r) = \{\hat{x}_j + \hat{h}_j : |h_i| < \widetilde{F}_i(\hat{x}_j, r),\ i \neq j\} \tag{2.21}$$

and

$$\widetilde{\Lambda}_j(\hat{x}_j) = \sup_{\hat{y}_j \in \widetilde{\mathrm{Box}}_j(\hat{x}_j, r)} |\widetilde{\lambda}_j(\hat{y}_j) - \widetilde{\lambda}_j(\hat{x}_j)|. \tag{2.22}$$

The function $\widetilde{\Lambda}_j$ enjoys the properties of Proposition 2.2 (with the subscript n replaced by j).

We are now ready to give the general definition of admissible surface and of a set with admissible boundary.

Definition 2.7. Let $\varphi \in C^1(\mathbb{R}^{n-1})$. The surface $\{x_j = \varphi(\hat{x}_j)\}$ is said to be *admissible* if there exist $C > 0$ and $r_0 > 0$ such that for all $\hat{x}_j \in \mathbb{R}^{n-1}$ and $r \in]0, r_0]$,

$$\sum_{i \neq j} \mathrm{osc}(\widetilde{X}_i\varphi, \widetilde{\mathrm{Box}}_j(\hat{x}_j, r)) \leq C\Big(r \sum_{i \neq j} |\widetilde{X}_i\varphi(\hat{x}_j)|^{(d_j-2)/(d_j-1)} + \widetilde{\Lambda}_j(\hat{x}_j, r)\Big). \tag{2.23}$$

Definition 2.8 (Domain with admissible boundary). A connected bounded open set $\Omega \subset \mathbb{R}^n$ is said to have *admissible boundary* if it is of class C^1 and for all $x \in \partial\Omega$, there exists a neighborhood $\mathcal{U}$ of x such that $\partial\Omega \cap \mathcal{U}$ is an admissible surface according to Definition 2.3 or 2.7.

2.2 Examples of admissible domains in $\mathbb{R}^3$.

We study some examples of admissible surfaces and of sets with admissible boundary in $\mathbb{R}^3$. Consider the functions $\lambda_1 \equiv 1$, $\lambda_2 = |x_1|^{\alpha_1}$, $\lambda_3 = |x_1|^{\alpha_1}|x_2|^{\alpha_2}$ and the corresponding diagonal vector fields

$$X_1 = \partial_1, \quad X_2 = |x_1|^{\alpha_1}\partial_2, \quad X_3 = |x_1|^{\alpha_1}|x_2|^{\alpha_2}\partial_3. \tag{2.24}$$

We consider the case $\alpha_i \geq 1$, $i = 1, 2$. The degrees of the variables x_1, x_2 and x_3 are $d_1 = 1$, $d_2 = 1 + \alpha_1$, $d_3 = (1 + \alpha_1)(1 + \alpha_2)$, respectively.

We begin with the study of admissible surfaces of the form $\{x_3 = \varphi(x_1, x_2)\}$. We write $x = (x_1, x_2)$ and $|X\varphi| = |X_1\varphi| + |X_2\varphi|$. If $\varphi \in C^1(\mathbb{R}^2)$, condition (2.16) reads

$$\sum_{i=1,2} \mathrm{osc}(X_i\varphi, \mathrm{Box}_3(x,r)) \lesssim r|X\varphi(x)|^{(d_3-2)/(d_3-1)} + \Lambda_3(x,r), \tag{2.25}$$

where $\mathrm{Box}_3(x,r) = \{(x_1 + u_1F_1(x,r), x_2 + u_2F_2(x_1,r)) : |u_1|, |u_2| \leq 1\}$ and $\Lambda_3(x,r) = \sup_{\mathrm{Box}_3(x,r)} |\lambda_3 - \lambda_3(x)|$. Here $F_1(x,r) = r$ and $F_2(x,r) = r(|x_1| + r)^{\alpha_1}$.

We can write explicitly, by (2.14) (see also (A.2) in the Appendix),

(2.26)
$$\begin{aligned}\Lambda_3(x) &\gtrsim r\big(|x_1|+r\big)^{\alpha_1-1}\big(|x_2|+F_2(x_1,r)\big)^{\alpha_2}+|x_1|^{\alpha_1}F_2(x_1,r)\big(|x_2|+F_2(x_1,r)\big)^{\alpha_2-1}\\ &\gtrsim r\big(|x_1|+r\big)^{\alpha_1-1}\big(|x_2|+F_2(x_1,r)\big)^{\alpha_2}.\end{aligned}$$

Theorem 2.9. *Let $N(x)=|x_1|^{2d_2}+x_2^2$ and assume that $\varphi(x)=g(N(x))$, where $g\in C^2(0,+\infty)$ is a function such that for some constant $C>0$,*

(2.27)
$$0\le g'(t)\le Ct^{\frac{d_3}{2d_2}-1}\equiv Ct^{\frac{\alpha_2-1}{2}},\quad |g''(t)|\le C\frac{g'(t)}{t},\quad g'(2t)\le Cg'(t),\quad t>0.$$

Then the surface $\{x_3=\varphi(x_1,x_2)\}$ is admissible.

Proof. We check (2.25). Note that in this example, inequality (2.19) fails. Without loss of generality, we assume $x_1,x_2>0$. A short computation gives

$$\begin{aligned}|X_1\varphi(x)|&\simeq x_1^{2\alpha_1+1}g'(N(x))=x_1^{\alpha_1}\{x_1^{d_2}g'(N(x))\}:=x_1^{\alpha_1}h_1(x),\\ |X_2\varphi(x)|&\simeq x_1^{\alpha_1}\{x_2g'(N(x))\}:=x_1^{\alpha_1}h_2(x).\end{aligned}\tag{2.28}$$

Note that $|h(x)|=|(h_1(x),h_2(x))|=N(x)^{1/2}g'(x)$. Then

$$|X\varphi(x)|\simeq|x_1|^{\alpha_1}N(x)^{1/2}g'(N(x)).$$

Moreover,

$$\begin{aligned}&\operatorname{osc}(X_i\varphi,\mathrm{Box}_3(x,r))\\ &\quad\lesssim|(x_1+r)^{\alpha_1}h_i(x+F(x,r))-x_1^{\alpha_1}h_i(x)|\\ &\quad\le((x_1+r)^{\alpha_1}-x_1^{\alpha_1})h_i(x)+(x_1+r)^{\alpha_1}(h_i(x+F(x,r))-h_i(x))\\ &\quad\lesssim\alpha_1r(x_1+r)^{\alpha_1-1}h_i(x)+(x_1+r)^{\alpha_1}(h_i(x+F(x,r))-h_i(x)),\end{aligned}$$

where we have used (2.14). Writing $h=(h_1,h_2)$, we estimate the oscillation from above by

$$\begin{aligned}\sum_{i=1,2}\operatorname{osc}(X_i\varphi,\mathrm{Box}_3(x,r))&\lesssim r(x_1+r)^{\alpha_1-1}|h(x)|\\ &\quad+(x_1+r)^{\alpha_1}|h(x+F(x,r))-h(x)|.\end{aligned}\tag{2.29}$$

We already know that $|h(x)|\simeq N(x)^{1/2}g'(N(x))$. In order to estimate the last term on the right hand side, we use (as in (2.18)) the inequality

$$|h_i(x+F(x,r))-h_i(x)|\lesssim r\sup_{y\in\mathrm{Box}_3(x,r),\,k=1,2}|X_kh_i(y)|.$$

The computation of the second derivatives and condition $g''(t) \le Cg'(t)/t$ give

$$\begin{aligned}
X_1 h_1(x) &\simeq x_1^{\alpha_1}\{g'(N(x)) + x_1^{2(\alpha_1+1)} g''(N(x))\} \simeq x_1^{\alpha_1} g'(N(x)),\\
X_2 h_1(x) &\simeq x_2 x_1^{2\alpha_1+1} g''(N(x)) \lesssim \frac{x_1^{\alpha_1+1} x_2}{N(x)} x_1^{\alpha_1} g'(N(x)) \lesssim x_1^{\alpha_1} g'(N(x)),\\
X_1 h_2(x) &= x_2 x_1^{2\alpha_1+1} g''(N(x)) \lesssim x_1^{\alpha_1} g'(N(x)),\\
X_2 h_2(x) &= x_1^{\alpha_1}\{g'(N(x)) + x_2^2 g''(N(x))\} \simeq x_1^{\alpha_1} g'(N(x)).
\end{aligned}$$

Hence we find

$$|h(x+F(x,r)) - h(x)| \lesssim r(x_1+r)^{\alpha_1} g'\big(N(x+F(x,r))\big).$$

Coming back to (2.29), we see that condition (2.25) is guaranteed by

(2.30)
$$\begin{aligned}
&r(x_1+r)^{\alpha_1-1} N(x)^{1/2} g'(N(x)) + r(x_1+r)^{2\alpha_1} g'(N(x+F(x,r)))\\
&\quad \lesssim r\{x_1^{\alpha_1} N(x)^{1/2} g'(N(x))\}^{(d_3-2)/(d_3-1)} + r\big(x_1+r\big)^{\alpha_1-1}\big(x_2+F_2(x_1,r)\big)^{\alpha_2},
\end{aligned}$$

where the first term in the right hand side is provided by (2.28) and the second one comes from (2.26).

Now two cases need to be distinguished: (A) $x_2 \ge x_1^{\alpha_1+1}$; (B) $x_2 < x_1^{\alpha_1+1}$.

Study of Case (A). We ignore the contribution of the first term on the right hand side of (2.30) and we consider the second one only. Thus (2.30) is implied by

(2.31) $$N(x)^{1/2} g'(N(x)) + (x_1+r)^{\alpha_1+1} g'(N(x+F(x,r))) \lesssim (x_2+F_2(x_1,r))^{\alpha_2}.$$

Notice that in Case (A) $N(x) \simeq x_2^2$.

We distinguish the following two subcases: (A1) $x_2 \le r^{\alpha_1+1}$; (A2) $x_2 > r^{\alpha_1+1}$.

Case (A1). We majorize the left hand side of (2.31) using $x_1 \le r$ and $x_2 \le r^{\alpha_1+1}$ and set $x = 0$ in the right hand side, obtaining the stronger condition $r^{\alpha_1+1} g'(r^{2(\alpha_1+1)}) \lesssim r^{\alpha_2(\alpha_1+1)}$, which can be rewritten as $g'(r^{2d_2}) \lesssim r^{d_3-2d_2}$. This last inequality is satisfied by assumption (2.27).

Case (A2). Condition (2.31) is implied by

$$N(x)^{1/2} g'(N(x)) + (x_1+r)^{\alpha_1+1} g'(N(x+F(x,r))) \lesssim x_2^{\alpha_2}.$$

We can use $x_1^{\alpha_1+1}, r^{\alpha_1+1} \le x_2$ and $N(x) \simeq N(x+F(x,r)) \simeq x_2^2$. This gives $x_2 g'(x_2^2) \lesssim x_2^{\alpha_2}$, i.e., $g'(x_2^2) \lesssim x_2^{d_3/d_2-2}$. The latter inequality holds by assumption.

Study of Case (B). Here we have $N(x) \simeq x_1^{2(\alpha_1+1)}$. Two subcases must be distinguished: (B1) $x_1 \le r$; (B2) $x_1 > r$.

Case (B1). In this case we ignore the contribution of the first term on the right hand side of (2.30) and consider the second term only. Condition (2.30) is guaranteed by

$$(2.32)\quad N(x)^{1/2}g'(N(x)) + (x_1+r)^{\alpha_1+1}g'(N(x+F(x,r))) \lesssim (x_2+F_2(x_1,r))^{\alpha_2}.$$

Set $x=0$ in the right hand side of (2.32) and use $x_1 \le r$ and $x_2 < r^{\alpha_1+1}$. We find the stronger inequality $r^{\alpha_1+1}g'(r^{2(\alpha_1+1)}) \le r^{\alpha_2(\alpha_1+1)}$, i.e., $g'(r^{2d_2}) \le r^{d_3-2d_2}$.

Case (B2). We use here the contribution of the first term on the right hand side of (2.30). Then we get the stronger inequality

$$(x_1+r)^{\alpha_1-1}N(x)^{1/2}g'(N(x)) + (x_1+r)^{2\alpha_1}g'(N(x+F(x,r)))$$
$$\lesssim \left\{x_1^{\alpha_1}N(x)^{1/2}g'(N(x))\right\}^{(d_3-2)/(d_3-1)}.$$

Since $r \le x_1$ and $x_2 \le x_1^{\alpha_1+1}$, we finally find the stronger condition $x_1^{2\alpha_1}g'(x_1^{2d_2}) \lesssim \{x_1^{\alpha_1+d_2}g'(x_1^{2d_2})\}^{(d_3-2)/(d_3-1)}$, i.e. $g'(x_1^{2d_2}) \lesssim x_1^{d_3-2d_2}$. The proof is complete. □

Finally, we give an example in $\mathbb{R}^3$ of a bounded open set with admissible boundary.

Theorem 2.10. *The open set*

$$\Omega = \left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : \left(|x_1|^{2(\alpha_1+1)} + x_2^2\right)^{1+\alpha_2} + x_3^2 < 1\right\}$$

has admissible boundary.

Proof. Let $\varepsilon \in (0,1)$ be fixed. The surface $\partial\Omega \cap \{|x_3| > \varepsilon\}$ can be studied by means of Theorem 2.9. Indeed, the lower cap can be written in the form

$$x_3 = -\left(1 - N(x_1,x_2)^{1+\alpha_2}\right)^{1/2} = g(N(x_1,x_2)),$$

where $N(x_1,x_2) = |x_1|^{2(\alpha_1+1)} + x_2^2$; and, for any fixed $t_0 < 1$, it is easy to see that the function $g(t) = -\left(1-t^{1+\alpha_2}\right)^{1/2}$ satisfies conditions (2.27) for $t\in(0,t_0)$.

The surface $\partial\Omega \cap \{|x_3| < \varepsilon\}$ is noncharacteristic, and hence admissible, away from a neighborhood of its intersection with the plane $x_1 = 0$. To complete the proof of the theorem it is enough to show that $\partial\Omega$ is admissible in a neighborhood of $(0,1,0)$. Here, $\partial\Omega$ can be parameterized as follows:

$$x_2 = \left(\left(1-x_3^2\right)^{1/(1+\alpha_2)} - x_1^{2(\alpha_1+1)}\right)^{1/2} := \varphi(x_1,x_3).$$

We check that the function φ satisfies condition (2.23). To this end, as suggested by (2.20), we consider the vector fields $\widetilde{X}_1 = \partial_1$, $\widetilde{X}_2 = |x_1|^{\alpha_1}\partial_2$, $\widetilde{X}_3 = |x_1|^{\alpha_1}\partial_3$. We have to check

$$(2.33)\quad \sum_{i=1,3}\operatorname{osc}(\widetilde{X}_i\varphi, \widetilde{\mathrm{Box}}_2(\hat{x}_2,r)) \lesssim r(|\widetilde{X}_1\varphi(\hat{x}_2)| + |\widetilde{X}_3\varphi(\hat{x}_2)|)^{(d_2-2)/(d_2-1)} + \widetilde{\Lambda}_2(\hat{x}_2,r),$$

where $d_2 = 1 + \alpha_1$ and

$$\widetilde{\mathrm{Box}}_2(\hat{x}_2, r) = \{\hat{x}_2 + \hat{h}_2 : |h_1| < \widetilde{F}_1(\hat{x}_2, r) = r,\ |h_3| < \widetilde{F}_3(\hat{x}_2, r) = r(|x_1| + r)^{\alpha_1}\}.$$

Write $x = (x_1, x_3)$, $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_3)$. An easy computation yields

$$|\widetilde{X}_1\varphi(x)| = h_1(x)|x_1|^{2\alpha_1+1} \quad \text{and} \quad |\widetilde{X}_2\varphi(x)| = h_2(x)|x_3|\,|x_1|^{\alpha_1},$$

where h_1 and h_2 are positive Lipschitz continuous functions in a neighborhood of the origin (we do not need their explicit form here). Assume without loss of generality $x_1, x_3 \geq 0$.

We estimate the left hand side of (2.33):

$$\begin{aligned}
\mathrm{osc}(\widetilde{X}_1\varphi, \widetilde{\mathrm{Box}}_2(\hat{x}_2, r)) &\lesssim \left|h_1(x + \widetilde{F}(x,r))(x_1 + r)^{2\alpha_1+1} - h_1(x)x_1^{2\alpha_1+1}\right| \\
&\lesssim \left|h_1(x + \widetilde{F}(x,r)) - h_1(x)\right| x_1^{2\alpha_1+1} \\
&\qquad + h_1(x + \widetilde{F}(x,r))\left|(x_1 + r)^{2\alpha_1+1} - x_1^{2\alpha_1+1}\right| \\
&\lesssim r x_1^{2\alpha_1+1} + r(x_1 + r)^{2\alpha_1} \lesssim r(x_1 + r)^{2\alpha_1}.
\end{aligned} \tag{2.34}$$

Here, we have used the Lipschitz continuity of h_1 and the estimate $|\widetilde{F}(x,r)| \lesssim r$. Moreover,

$$\begin{aligned}
\mathrm{osc}(\widetilde{X}_3\varphi, \widetilde{\mathrm{Box}}(\hat{x}_2, r)) &\lesssim \left|h_2(x + \widetilde{F}(x,r))(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - h_2(x)x_1^{\alpha_1}x_3\right| \\
&\lesssim \left|h_2(x + \widetilde{F}(x,r)) - h_2(x)\right| x_1^{\alpha_1}x_3 \\
&\qquad + h_2(x + \widetilde{F}(x,r))\left|(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - x_1^{\alpha_1}x_3\right| \\
&\lesssim r x_1^{\alpha_1}x_3 + \left|(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - x_1^{\alpha_1}x_3\right|.
\end{aligned} \tag{2.35}$$

The last term can be evaluated as follows:

$$\begin{aligned}
&(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - x_1^{\alpha_1}x_3 \\
&\qquad \lesssim (x_1 + r)^{\alpha_1} - x_1^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) + x_1^{\alpha_1}\left((x_3 + \widetilde{F}_3(x_1, r)) - x_3\right) \\
&\qquad \lesssim r(x_1 + r)^{\alpha_1 - 1} + \widetilde{F}_3(x_1, r)x_1^{\alpha_1} \lesssim r(x_1 + r)^{\alpha_1 - 1}.
\end{aligned} \tag{2.36}$$

Taking into account (2.34), (2.35), (2.36) and the equivalence $\widetilde{\Lambda}_2(\hat{x}_2, r) \simeq r(x_1 + r)^{\alpha_1 - 1}$, we conclude that condition (2.33) is implied by

$$(x_1 + r)^{2\alpha_1} + x_1^{\alpha_1}x_3 + (x_1 + r)^{\alpha_1 - 1} \lesssim (x_1 + r)^{\alpha_1 - 1},$$

which is trivially satisfied. □

3 Domains with admissible boundary are John domains

In this section, we prove that admissible domains are John domains.

Definition 3.1. A bounded open set $\Omega \subset (\mathbb{R}^n, d)$ is a *John domain* if there exist $x_0 \in \Omega$ and $\sigma > 0$ such that for all $x \in \Omega$, there exists a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$\mathrm{dist}\big(\gamma(t), \partial\Omega\big) \geq \sigma \mathrm{diam}\big(\gamma|_{[0,t]}\big). \tag{3.1}$$

A curve satisfying (3.1) is a *John curve*, x_0 is the *center* and σ the *John constant* of Ω.

In order to avoid possible confusion concerning definitions, we stress that, in general metric spaces, the definition of John domain is given with length$(\gamma|_{[0,t]})$ replacing diam$(\gamma|_{[0,t]})$ in (3.1). By a general result due to Martio and Sarvas [MS, Theorem 2.7], such definitions are in fact equivalent in doubling metric spaces with geodesics. Moreover, in our proofs we shall always work with John curves γ satisfying diam$(\gamma|_{[0,t]}) \simeq d(\gamma(t), \gamma(0))$.

For the proof of the following easy proposition, we refer to [MM2].

Proposition 3.2. *Let $\Omega \subset (\mathbb{R}^n, d)$ be a bounded open set, and for any $r > 0$ define $\Omega_r = \{y \in \Omega : \mathrm{dist}(y, \partial\Omega) > r\}$. Assume that there exist $r > 0$ and $\sigma > 0$ such that Ω_r is arcwise connected and such that for any $x \in \Omega$, there is a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) \in \Omega_r$ and*

$$\mathrm{dist}(\gamma(t), \partial\Omega) \geq \sigma \mathrm{diam}(\gamma|_{[0,t]}) \tag{3.2}$$

for all $t \in [0,1]$. Then Ω is a John domain.

Now we are able to state our main theorem.

Theorem 3.3. *If Ω is an open set with admissible boundary according to Definition 2.8, then it is a John domain in the metric space $(\mathbb{R}^n, d)$.*

Proof. We use Proposition 3.2. Given $\bar{x} \in \partial\Omega$, we show that there exists a neighborhood $\mathcal{U}$ of $\bar{x}$ and $\sigma > 0$ such that, for all $x \in \Omega \cap \mathcal{U}$, there exists a curve γ starting from x and satisfying (3.2). The claim follows on choosing by compactness a finite covering of $\partial\Omega$.

Fix $\bar{x} \in \Omega$ and write $\partial\Omega$ locally as a graph of the form $x_j = \varphi(\hat{x}_j)$ for some $j = 1, \dots, n$, where φ is a C^1 function. We begin with the basic case $j = n$. Let $\varphi \in C^1(\mathbb{R}^{n-1})$ be a function satisfying the admissibility condition (2.16) and assume, for the sake of simplicity, that $\Omega = \{x_n > \varphi(\hat{x}_n)\}$.

We have to construct a John curve starting from a point $x = \hat{x}_n + x_n e_n \in \Omega$. To this end, two different situations need to be distinguished:

$$
\begin{aligned}
&\max_{i<n} |X_i\varphi(\hat{x}_n)| \le \lambda_n(\hat{x}_n) \qquad \text{(Case 1)},\\
&\max_{i<n} |X_i\varphi(\hat{x}_n)| > \lambda_n(\hat{x}_n) \qquad \text{(Case 2)}.
\end{aligned}
\tag{3.3}
$$

In Case 1, the characteristic case, we construct a John curve starting from x of the form $x + te_n$, $t \ge 0$. In Case 2, the path must be split into two pieces. The first one starts from x in the coordinate direction e_k, where $k < n$ is such that the derivative $|X_k\varphi(\hat{x}_n)|$ is "maximal" among all the $|X_i\varphi(\hat{x}_n)|$, $i = 1, \dots, n-1$, and moves in this direction for a time $\bar{t} = \bar{t}(x)$ which must be established in a careful way (compare (3.7)). The second part of the path will be of the form $\gamma(\bar{t}) + (t - \bar{t})e_n$.

First of all, we introduce the following notation:

$$
\nu_i = \nu_i(\hat{x}_n) = -\partial_i\varphi(\hat{x}_n), \quad N_i = \frac{\nu_i}{|\nu_i|}, \quad \text{if } \nu_i \ne 0, \quad i \ne n,
$$

$$
\text{and} \quad w(\hat{x}_n) = \sum_{i\ne n} |X_i\varphi(\hat{x}_n)|.
$$

Case 1. Define

$$
\gamma(t) = x + te_n = \hat{x}_n + (x_n + t)e_n \quad \text{and} \quad \delta \equiv \delta(t) = G_n(\hat{x}_n, t) \simeq d(\gamma(t), \gamma(0)).
\tag{3.4}
$$

Consider for small $\sigma > 0$

$$
\mathrm{Box}(\gamma(t), \sigma\delta) = \Big\{ \big(x_n + t + u_n F_n(x, \sigma\delta)\big)e_n + \hat{x}_n + \hat{u}_n \hat{F}_n(x, \sigma\delta) : |u_i| \le 1,\ i = 1, \dots, n \Big\}.
$$

Here, $\hat{u}_n = (u_1, u_2, \dots, u_{n-1}, 0)$. We used $F_i(x + te_n, \delta) = F_i(x, \delta)$.

We claim that there exists $\sigma > 0$ independent of x such that $\mathrm{Box}(\gamma(t), \sigma\delta) \subset \Omega$, i.e., such that

$$
x_n + t + u_n F_n(x, \sigma\delta) > \varphi\big(\hat{x}_n + \hat{u}_n \hat{F}_n(x, \sigma\delta)\big), \quad \delta > 0,\ |u_i| \le 1.
\tag{3.5}
$$

This is the John condition (3.2). Since $x \in \Omega$, $x_n - \varphi(\hat{x}_n) \ge 0$. Take the worst case $u_n = -1$ in (3.5). Moreover, $F_n(x, \sigma\delta) \le F_n(x, \delta) = F_n(x, G_n(x, t)) = t$. Thus, condition (3.5) is easily seen to be implied by

$$
\big|\varphi\big(\hat{x}_n + \hat{u}_n \hat{F}_n(x, \sigma\delta)\big) - \varphi(\hat{x}_n)\big| \le (1 - \sigma)t, \quad \delta > 0,\ |\hat{u}_n| \le 1.
$$

Using the control (2.17) for the oscillation of φ, Case 1 and Proposition 2.2, we may estimate the left hand side by

$$
\begin{aligned}
\big|\varphi\big(\hat{x}_n + \hat{u}_n \hat{F}_n(x, \sigma\delta)\big) - \varphi(\hat{x}_n)\big| &\le \mathrm{osc}(\varphi, \mathrm{Box}_n(\hat{x}_n, \sigma\delta))\\
&\lesssim \sigma\delta w(x) + \sigma\delta\Lambda_n(\hat{x}_n, \sigma\delta))\\
&\lesssim \sigma\delta\lambda_n(\hat{x}_n) + \sigma F_n(\hat{x}_n, \delta) \simeq \sigma F_n(\hat{x}_n, \delta).
\end{aligned}
$$

In the last equivalence, we have used the trivial estimate $\delta\lambda_n(\hat{x}_n) \le F_n(x,\delta)$. Thus, (3.5) is implied by $\sigma F_n(x,\delta) \le \sigma_0 t$, where σ_0 is a small but absolute constant. Since $t = F_n(x,\delta)$, this inequality holds as soon as $\sigma \le \sigma_0$.

Case 2. Assume that x satisfies Case 2 in (3.3). Take any $k = 1, \dots, n-1$ such that

$$|X_k\varphi(\hat{x}_n)| \ge \frac{1}{2}\max_{i<n}|X_i\varphi(\hat{x}_n)| > \frac{1}{4}\lambda_n(\hat{x}_n). \tag{3.6}$$

(The factors $\frac{1}{2}$ and $\frac{1}{4}$ will become relevant in [MM3], where we prove that admissible domains are non-tangentially accessible.)

Take $\varepsilon_0 > 0$ and let $\bar{\delta} = \bar{\delta}(x)$ be the solution of the equation

$$\Lambda_n(\hat{x}_n, \bar{\delta}) = \varepsilon_0 |X_k\varphi(\hat{x}_n)|. \tag{3.7}$$

(The function $\Lambda_n(x,\cdot)$ is a homeomorphism of $[0,\infty[$ onto itself.) The number ε_0 will be fixed and becomes an absolute constant in (3.16). Finally, set $\bar{t} = F_k(x,\bar{\delta})$.

Define the first piece of the John curve, letting for $t \in [0,\bar{t}]$

$$\gamma(t) = x + tN_k e_k, \qquad \text{and} \quad \delta = \delta(t) = G_k(x,t) \simeq d(\gamma(0),\gamma(t)).$$

For $\sigma > 0$, consider the box

$$\mathrm{Box}(\gamma(t),\sigma\delta) = \Big\{\big(tN_k e_k + \sum_{i=1}^{n}\big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big)e_i : |u_i| \le 1,\ i = 1,\dots,n\Big\}.$$

We claim that there exist $\varepsilon_0, \sigma > 0$ independent of x such that the John condition

$$\mathrm{Box}(\gamma(t),\sigma\delta(t)) \subset \Omega, \quad t \in [0,\bar{t}] \tag{3.8}$$

holds. Points of the box belong to Ω as soon as for all u, $|u_i| \le 1$, $i = 1, \dots, n$, we have

$$x_n + u_n F_n(x + tN_k e_k, \sigma\delta) > \varphi\Big(tN_k e_k + \sum_{i\ne n}\big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big)e_i\Big).$$

Take the worst case $u_n = -1$ and use $x_n > \varphi(\hat{x}_n)$. The inequality above is implied by

$$\varphi\Big(tN_k e_k + \sum_{i\ne n}\big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big)e_i\Big) - \varphi(\hat{x}_n) + F_n(x + tN_k e_k, \sigma\delta) \le 0,$$

which can be rewritten as

$$\mathrm{I} + \mathrm{II} + F_n(x + tN_k e_k, \sigma\delta) \le 0, \tag{3.9}$$

where we set

$$\mathrm{I} = \varphi\Big(tN_k e_k + \sum_{i\neq n}\big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big)e_i\Big) - \varphi(tN_k e_k + \hat{x}_n),$$

$$\mathrm{II} = \varphi(tN_k e_k + \hat{x}_n) - \varphi(\hat{x}_n).$$

We claim that ε_0 in (3.7) can be fixed independently from x in such a way that

$$\mathrm{II} \le -\frac{1}{2}t|\nu_k| \quad \text{for all } t \in [0, \bar{t}]. \tag{3.10}$$

Indeed, by the mean value theorem, there exists $\vartheta \in [0,1]$ such that

$$\begin{aligned}\varphi(\hat{x}_n + tN_k e_k) - \varphi(\hat{x}_n) &= \partial_k\varphi(\hat{x}_n + \vartheta tN_k e_k)tN_k \\ &= \partial_k\varphi(\hat{x}_n)tN_k + \{\partial_k\varphi(\hat{x}_n + \vartheta tN_k e_k) - \partial_k\varphi(\hat{x}_n)\}tN_k \\ &= -|\nu_k|t + \{\partial_k\varphi(\hat{x}_n + \vartheta tN_k e_k) - \partial_k\varphi(\hat{x}_n)\}tN_k.\end{aligned}$$

Notice that Case 2 in (3.3) ensures $\nu_k \neq 0$ and $\lambda_k(\hat{x}_n) \neq 0$. The curly brackets can be estimated by (2.16) as follows (note that λ_k does not depend on x_k and $t = F_k(x,\delta)$):

$$\begin{aligned}\big|\{\partial_k\varphi(\hat{x}_n + \vartheta tN_k e_k) - \partial_k\varphi(\hat{x}_n)\}\big| &= \frac{1}{\lambda_k(\hat{x}_n)}\big|X_k\varphi(\hat{x}_n + \vartheta tN_k e_k) - X_k\varphi(\hat{x}_n)\big| \\ &\le \frac{1}{\lambda_k(\hat{x}_n)}\operatorname{osc}(X_k\varphi, \mathrm{Box}_n(\hat{x}_n,\delta)) \\ &\lesssim \frac{1}{\lambda_k(\hat{x}_n)}\big(\delta w(\hat{x}_n)^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n,\delta)\big) \\ &\lesssim \frac{1}{\lambda_k(\hat{x}_n)}\big(\delta|X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n,\delta)\big).\end{aligned}$$

In the last inequality, we have used (3.6). Now, (3.10) is guaranteed by

$$C_0\big(\delta|X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n,\delta)\big) \le \frac{1}{2}|\nu_k|\lambda_k(\hat{x}_n) = \frac{1}{2}|X_k\varphi(\hat{x}_n)|, \tag{3.11}$$

where C_0 is a large but absolute constant. By Hölder's inequality and Proposition 2.2,

$$C_0\delta|X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} \le \frac{1}{4}|X_k\varphi(\hat{x}_n)| + C_1\delta^{d_n-1} \le \frac{1}{4}|X_k\varphi(\hat{x}_n)| + C_2\Lambda_n(\hat{x}_n,\delta),$$

where C_2 is a new large absolute constant. Using $\Lambda_n(\hat{x}_n,\delta) \le \Lambda_n(\hat{x}_n,\bar{\delta}) = \varepsilon_0|X_k\varphi(\hat{x}_n)|$ (this is (3.7)), we see that (3.11) is guaranteed by a choice of $\varepsilon_0 > 0$ such that $4(C_0 + C_2)\varepsilon_0 \le 1$.

Now, by the estimate on II, inequality (3.9) is implied by

$$\mathrm{I} + F_n(x + tN_k e_k, \sigma\delta) \le \frac{1}{2}t|\nu_k|, \quad t \in [0,\bar{t}]. \tag{3.12}$$

We claim that this inequality holds as soon as $\sigma > 0$ is small enough, independent of x.

First of all, by (2.17) we find

$$\begin{aligned}
\mathrm{I} &\leq \Big|\varphi\Big(tN_k e_k + \sum_{i\neq n}\big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big)e_i\Big) - \varphi(tN_k e_k + \hat{x}_n)\Big| \\
&\leq \operatorname{osc}\big(\varphi, \operatorname{Box}_n(\hat{x}_n + tN_k e_k, \sigma\delta)\big) \\
&\lesssim \sigma\delta w(\hat{x}_n + tN_k e_k) + \sigma\delta\Lambda_n(\hat{x}_n + tN_k e_k, \sigma\delta),
\end{aligned}$$

and by (2.16),

$$\begin{aligned}
w(\hat{x}_n + tN_k e_k) &= \sum_{i\neq n}|X_i\varphi(\hat{x}_n + tN_k e_k)| \\
&\leq \sum_{i\neq n}|X_i\varphi(\hat{x}_n)| + \sum_{i\neq n}|X_i\varphi(\hat{x}_n + tN_k e_k) - X_i\varphi(\hat{x}_n)| \\
&\leq w(\hat{x}_n) + \sum_{i\neq n}\operatorname{osc}(X_i\varphi, \operatorname{Box}_n(\hat{x}_n, \delta)) \\
&\lesssim w(\hat{x}_n) + \delta w(\hat{x}_n)^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n, \delta) \\
&\lesssim w(\hat{x}_n) + \Lambda_n(\hat{x}_n, \delta).
\end{aligned} \tag{3.13}$$

We have once again used Hölder's inequality and Proposition 2.2. Since $\Lambda_n(\hat{x}_n + tN_k e_k, \sigma\delta) \lesssim \Lambda_n(\hat{x}_n, \delta)$, using $w(\hat{x}_n) \leq |X_k\varphi(\hat{x}_n)|$, we get finally

$$\mathrm{I} \lesssim \sigma\delta(|X_k\varphi(\hat{x}_n)| + \Lambda_n(\hat{x}_n, \delta)). \tag{3.14}$$

Now we show that the second term in the left hand side of (3.12) satisfies the same estimate (we will need (2.12)):

$$\begin{aligned}
F_n(x + tN_k e_k, \sigma\delta) &\leq \sigma F_n(x + tN_k e_k, \delta) \lesssim \sigma F_n(x, \delta) \\
&= \sigma\delta\lambda_n(\hat{x}_n + \hat{F}_n(x, \delta)) \leq \sigma\delta(\Lambda_n(\hat{x}_n, \delta) + \lambda_n(\hat{x}_n)) \\
&\lesssim \sigma\delta(\Lambda_n(\hat{x}_n, \delta) + |X_k\varphi(\hat{x}_n)|).
\end{aligned} \tag{3.15}$$

Here we have used (3.6).

Taking into account (3.14) and (3.15), and recalling that $\Lambda_n(\hat{x}_n, \delta) \leq \Lambda_n(\hat{x}_n, \bar{\delta}) = \varepsilon_0|X_k\varphi(\hat{x}_n)|$, we see that condition (3.12) holds as soon as $\sigma\delta|X_k\varphi(\hat{x}_n)| \lesssim t|\nu_k|$, i.e., $C_0\sigma\delta|X_k\varphi(\hat{x}_n)| \leq t|\nu_k|$, i.e., $C_0\sigma\delta\lambda_k(\hat{x}_n) \leq t$ for all $t > 0$. Here $C_0 > 0$ is a large but absolute constant. This inequality holds if $\sigma > 0$ is chosen in such a way that $\sigma C_0 \leq 1$, because $\delta\lambda_k(\hat{x}_n) \leq F_k(\hat{x}_n, \delta) = t$. This proves (3.12) and hence claim (3.8), as well.

Thus far, we have defined a John curve γ starting from a point $x \in \Omega$ satisfying Case 2 in (3.3) for a time $t \in [0, \bar{t}]$, where

$$\bar{t} = F_k(x, \bar{\delta}) \quad \text{and } \bar{\delta} \text{ solves} \quad \Lambda_n(\hat{x}_n, \bar{\delta}) = \varepsilon_0|X\varphi(\hat{x}_n)|. \tag{3.16}$$

The constant $\varepsilon_0 > 0$ is from now on fixed. Now we define γ for times $t \geq \bar{t}$. Let

$$\gamma(t) = x + \bar{t}N_k e_k + (t - \bar{t})e_n, \qquad t \geq \bar{t}. \tag{3.17}$$

We write $s = t - \bar{t}$. Set

$$\delta = \delta(t) = \bar{\delta} + G_n(x, t - \bar{t}) = \bar{\delta} + G_n(x, s) \simeq d(\gamma(0), \gamma(t)). \tag{3.18}$$

For $\sigma > 0$, consider the box

$$\mathrm{Box}(\gamma(t), \sigma\delta) = \Big\{ se_n + \bar{t}N_k e_k + \sum_{i=1}^{n} (x_i + u_i F_i(x + \bar{t}N_k e_k, \sigma\delta))e_i : |u_i| \leq 1 \Big\}.$$

Since $\varphi(\hat{x}_n) \leq x_n$, taking the worst case $u_n = -1$, the John condition $\mathrm{Box}(\gamma(t), \sigma\delta) \subset \Omega$ is implied by

$$J + JJ + F_n(x + \bar{t}N_k e_k, \sigma\delta) \leq s, \tag{3.19}$$

where

$$J = \varphi\Big(\bar{t}N_k e_k + \sum_{i \neq n} (x_i + u_i F_i(x + \bar{t}N_k e_k, \sigma\delta))e_i\Big) - \varphi(\bar{t}N_k e_k + \hat{x}_n),$$

$$JJ = \varphi(\bar{t}N_k e_k + \hat{x}_n) - \varphi(\hat{x}_n).$$

By (3.10) with $t = \bar{t}$, we have $JJ \leq -\frac{1}{2}|\nu_k|\bar{t}$. Hence, (3.19) is guaranteed by

$$J + F_n(x + \bar{t}N_k e_k, \sigma\delta) \leq s + \frac{1}{2}|\nu_k|\bar{t}. \tag{3.20}$$

We begin with the estimate of J. By (2.17),

$$\begin{aligned} J &\leq \mathrm{osc}\left(\varphi, \mathrm{Box}_n(\hat{x}_n + \bar{t}N_k e_k, \sigma\delta)\right) \\ &\lesssim \sigma\delta\big(w(\hat{x}_n + \bar{t}e_k N_k) + \Lambda_n(\hat{x}_n + \bar{t}e_k N_k, \sigma\delta)\big); \end{aligned}$$

and by (3.13), (3.6) and Proposition 2.2 (use $\bar{\delta} \leq \delta$),

$$\delta w(\hat{x}_n + \bar{t}e_k N_k) \lesssim \delta w(\hat{x}_n) + \delta\Lambda_n(\hat{x}_n, \bar{\delta}) \lesssim \delta|\nu_k|\lambda_k(\hat{x}_n) + F_n(\hat{x}_n, \delta).$$

On the other hand, by Proposition 2.2, (2.12), (3.18) and (2.7),

$$\begin{aligned} \delta\Lambda_n(\hat{x}_n + \bar{t}e_k N_k, \delta) &\lesssim F_n(\hat{x}_n + \bar{t}e_k N_k, \delta) \\ &\lesssim F_n(\hat{x}_n, \delta) = F_n(\hat{x}_n, \bar{\delta} + G_n(\hat{x}_n, s)) \\ &\lesssim F_n(\hat{x}_n, \bar{\delta}) + F_n(\hat{x}_n, G_n(\hat{x}_n, s)) = F_n(\hat{x}_n, \bar{\delta}) + s. \end{aligned}$$

Thus (3.20) is ensured by the inequality

$$\sigma(\delta|\nu_k|\lambda_k(\hat{x}_n) + F_n(\hat{x}_n, \bar{\delta}) + s) \lesssim s + \frac{1}{2}|\nu_k|\bar{t},$$

which reduces (recall that $\delta - \bar{\delta} = G_n(x, \delta)$, by (3.18)) to

$$\sigma C_0\big(\bar{\delta}|\nu_k|\lambda_k(\hat{x}_n) + G_n(\hat{x}_n, s)\lambda_k(\hat{x}_n) + F_n(\hat{x}_n, \bar{\delta})\big) \leq s + |\nu_k|\bar{t} \tag{3.21}$$

for some large absolute constant $C_0 > 0$. If we put $s = 0$ in (3.21), we get

$$\sigma C_0(\bar{\delta}|\nu_k|\lambda_k(\hat{x}_n) + F_n(\hat{x}_n, \bar{\delta}))) \leq |\nu_k|\bar{t},$$

which holds for σ small enough (we have already proved this when we proved (3.12); see also (3.14), (3.15) and (3.7)).

To complete the estimate (3.21), it is enough to show that

$$\sigma G_n(\hat{x}_n, s)\lambda_k(\hat{x}_n)|\nu_k| \leq s + |\nu_k|\bar{t}, \quad \text{for all } s > 0, \tag{3.22}$$

as soon as $\sigma > 0$ is small enough, independently of x. Now, (3.22) is equivalent to

$$G_n(\hat{x}_n, s) \leq \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k|\nu_k|} \quad \Leftrightarrow \quad s \leq F_n\Big(\hat{x}_n, \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k|\nu_k|}\Big).$$

Notice that the function $f_n(\hat{x}_n, r) = \frac{1}{r}F_n(\hat{x}_n, r)$ is increasing in the variable r. From

$$s + |\nu_k|\bar{t} \geq |\nu_k|\bar{t} = |\nu_k|F_k(\hat{x}_n, \bar{\delta}) \geq |\nu_k|\bar{\delta}\lambda_k(\hat{x}_n) \geq \sigma|\nu_k|\bar{\delta}\lambda_k(\hat{x}_n)$$

it follows that

$$f_n\Big(\hat{x}_n, \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k|\nu_k|}\Big) \geq f_n(x, \bar{\delta}) \gtrsim \Lambda_n(\hat{x}_n, \bar{\delta}),$$

by Proposition 2.2. Finally, recalling (3.7), we find that (3.22) is implied by

$$s \leq \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k(\hat{x}_n)|\nu_k|}\Lambda_n(\hat{x}_n, \bar{\delta}) = \frac{\varepsilon_0}{\sigma}(s + |\nu_k|\bar{t}),$$

which holds for all $s > 0$ as soon as $\sigma \leq \varepsilon_0$. This proves (3.19) and completes the discussion of Case 2 and of the parameterization $x_n = \varphi(\hat{x}_n)$.

Now assume that $\bar{x} \in \partial\Omega$ is a point such that for a neighborhood $\mathcal{U}$ of $\bar{x}$ the piece of boundary $\partial\Omega \cap \mathcal{U}$ is a surface of type $\{x_j = \varphi(\hat{x}_j)\}$ for some $j \neq n$ and for some function φ of class C^1 which satisfies the admissibility condition (2.23). We explain how to construct a John curve starting from points $x \in \mathcal{U} \cap \{x_j > \varphi(\hat{x}_j)\}$.

The functions $\widetilde{\lambda}_i$ and the vector fields $\widetilde{X}_i$ are defined in (2.20). By $\widetilde{d}$ we denote the metric induced on $\mathbb{R}^n$ by the vector fields $\widetilde{X}_i$. The boxes $\widetilde{\mathrm{Box}}_j(\hat{x}_j, r)$ and the function $\widetilde{\Lambda}_j$ have been defined in (2.21) and (2.22), respectively. Without loss of generality, we can assume $\mathcal{U} \subset \{|x_i| < 1 : i = 1, ..., n\}$ and $|\partial_i\varphi(\hat{x}_j)| \leq 1$, $i > j$. Then $|\widetilde{X}_i\varphi(\hat{x}_j)| \leq \widetilde{\lambda}_j(\hat{x}_j)$, for all $i > j$. Thus the distinction of cases (3.3) simply is

$$\max_{i<j}|\widetilde{X}_i\varphi(\hat{x}_j)| \leq \widetilde{\lambda}_j(\hat{x}_j) \qquad \text{(Case 1)},$$

$$\max_{i<j}|\widetilde{X}_i\varphi(\hat{x}_j)| > \widetilde{\lambda}_j(\hat{x}_j) \qquad \text{(Case 2)}.$$

In Case 1, we define a curve γ moving directly in the direction e_j, analogously to (3.4). In Case 2, we first define $\gamma(t) = x + tN_k$, where $k = 1, ..., j-1$ is any index such that

$$|\widetilde{X}_k \varphi(\hat{x}_j)| \geq \frac{1}{2} \max_{i<j} |\widetilde{X}_i \varphi(\hat{x}_j)| > \frac{1}{4} \widetilde{\lambda}_j(\hat{x}_j),$$

and $t \in [0, \bar{t}]$, where now $\bar{t} = F_k(\hat{x}_j, \bar{\delta})$ and $\bar{\delta}$ solves $\widetilde{\Lambda}_j(\hat{x}_j, \bar{\delta}) = \varepsilon_0 |\widetilde{X}_k \varphi(\hat{x}_j)|$ instead of (3.7). Then we let γ move in the direction e_j, analogously to (3.17).

The curve γ so defined satisfies, for some $\sigma > 0$ independent of x, the John condition with respect to the metric $\widetilde{d}$, i.e., $\widetilde{B}(\gamma(t), \sigma \widetilde{\mathrm{diam}}(\gamma|_{[0,t]})) \subset \Omega, t \in [0,1]$, where $\widetilde{B}$ denote balls in the metric $\widetilde{d}$. The proof of this is exactly the same as for the case $j = n$. By Proposition 2.6 (ii) it follows that $\widetilde{\mathrm{diam}}(\gamma|_{[0,t]}) \simeq \mathrm{diam}(\gamma|_{[0,t]})$ and by (i) it also follows that $B(\gamma(t), \sigma \mathrm{diam}(\gamma|_{[0,t]})) \subset \Omega$. This remark ends the proof of the theorem. □

Appendix

Proof of Proposition 2.2. Let $\hat{x}_n \in \mathbb{R}^{n-1}$ and assume without loss of generality $x_i \geq 0$, $i = 1, ..., n-1$. Take $a < 1$. The "reverse doubling" property (2.15) is equivalent to

$$\frac{\Lambda_n(x, R) - \Lambda_n(x, aR)}{\Lambda_n(x, aR)} \geq \frac{\eta(1-a)}{a}.$$

It is easy to see that $\Lambda_n(x,t) = \prod_{j=1}^{n-1} \left(x_j + F_j(x,t)\right)^{\alpha_j} - \prod_{j=1}^{n-1} x_j^{\alpha_j}$. Then

$$\begin{aligned} \frac{\Lambda_n(x, R) - \Lambda_n(x, aR)}{\Lambda_n(x, aR)} &= \frac{\prod_{j=1}^{n-1} \left(x_j + F_j(x,R)\right)^{\alpha_j} - \prod_{j=1}^{n-1} \left(x_j + F_j(x,aR)\right)^{\alpha_j}}{\prod_{j=1}^{n-1} \left(x_j + F_j(x,aR)\right)^{\alpha_j} - \prod_{j=1}^{n-1} \left(x_j\right)^{\alpha_j}} \\ &=: \frac{N}{D}. \end{aligned}$$

To write N, recall that given nonnegative numbers $m_j \leq M_j$, $j = 1, \ldots, p$, the difference of their products can be written as

$$M_1 M_2 \cdots M_p - m_1 m_2 \cdots m_p = \sum_{k=1}^{p} (M_k - m_k) \prod_{i=1}^{k-1} M_i \prod_{i=k+1}^{p} m_i. \tag{A.1}$$

Then

$$N = \sum_{k=1}^{n-1} \left\{ \left(x_k + F_k(x,R)\right)^{\alpha_k} - \left(x_k + F_k(x,aR)\right)^{\alpha_k} \right\}$$
$$\cdot \prod_{i=1}^{k-1} \left(x_i + F_i(x,R)\right)^{\alpha_i} \prod_{i=k+1}^{n-1} \left(x_i + F_i(x,aR)\right)^{\alpha_i}$$
$$\gtrsim \sum_{k=1}^{n-1} \alpha_k \left(F_k(x,R) - F_k(x,aR)\right)\left(x_k + F_k(x,R)\right)^{\alpha_k - 1}$$
$$\cdot \prod_{i=1}^{k-1} \left(x_i + F_i(x,aR)\right)^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i}.$$

Now note that, letting $F_k(x,t) = t f_k(x,t)$ (see (2.4)), we get

$$F_k(x,R) - F_k(x,aR) = R f_k(x,R) - aR f_k(x,aR)$$
$$\geq R(1-a) f_k(x,aR) = \frac{1-a}{a} F_k(x,aR).$$

Then

$$N \gtrsim \frac{1-a}{a} \sum_{k=1}^{n-1} \alpha_k F_k(x,aR)\left(x_k + F_k(x,aR)\right)^{\alpha_k - 1} \prod_{i=1}^{k-1} \left(x_i + F_i(x,aR)\right)^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i}$$
$$\simeq \frac{1-a}{a} D,$$

again by (A.1) and (2.14). Thus $N/D \geq \sigma \frac{1-a}{a}$. Therefore the proof of (2.15) is concluded.

To prove the remaining statements, use again (A.1) and (2.14) to write

(A.2)
$$\Lambda_n(x,R) \simeq \sum_{k=1}^{n-1} \alpha_k F_k(x,R)\left(x_k + F_k(x,R)\right)^{\alpha_k - 1} \prod_{i=1}^{k-1} \left(x_i + F_i(x,R)\right)^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i}.$$

Now, for any $k = 1, ..., n-1$,

$$\alpha_k F_k(x,R)(x_k + F_k(x,R))^{\alpha_k - 1} \prod_{i=1}^{k-1} (x_i + F_i(x,R))^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i}$$
$$\leq \alpha_k \prod_{i=1}^{n-1} (x_i + F_i(x,R))^{\alpha_i} = \frac{1}{R} F_n(x,R).$$

Then the second statement follows. Incidentally, note that the explicit estimate of $\Lambda_n(x,r)$ given in (A.2), together with (2.6), shows the doubling property $\Lambda_n(x,2r) \leq C\Lambda_n(x,r)$.

Finally, in order to prove that $\Lambda_n(\hat{x}_n, R)) \geq R^{d_n - 1}$ it is enough to estimate from below the right hand side of (A.2) using $x_j + F_j(x, R) \geq F_j(x, R) \geq F_j(0, R)$, $j = 1, \ldots, n$:

$$\Lambda_n(x, R) \gtrsim \sum_{k=1}^{n-1} (F_k(0, R))^{\alpha_k} \prod_{i=1}^{k-1} (F_i(0, R))^{\alpha_i} \prod_{i=k+1}^{n-1} (F_i(0, R))^{\alpha_i} = CR^{d_n - 1}.$$

This ends the proof. □

REFERENCES

[BM] Z. Balogh and R. Monti, *Accessible domains in the Heisenberg group*, Proc. Amer. Math. Soc. **132** (2004), 37–106.

[Be] O. V. Besov, *Integral representations of functions in a domain with the flexible horn condition, and imbedding theorems* (Russian), Dokl. Akad. Nauk SSSR **273** (1983), 1294–1297.

[Bo] B. Bojarski, *Remarks on Sobolev imbedding inequalities*, in *Proc. of the Conference on Complex Analysis (Joensu 1987)*, Lecture Notes in Math. **1351**, Springer, Berlin, 1988, pp. 52–68.

[BK] S. Buckley and P. Koskela, *Sobolev–Poincaré implies John*, Math. Res. Lett. **2** (1995), 577–593.

[BKL] S. Buckley, P. Koskela and G. Lu, *Boman equals John*, in *XVIth Rolf Nevanlinna Colloquium (Joensuu, 1995)*, de Gruyter, Berlin, 1996, pp. 91–99.

[CG] L. Capogna and N. Garofalo, *Boundary behavior of nonnegative solutions of subelliptic equations in NTA domains for Carnot–Carathéodory metrics*, J. Fourier Anal. Appl. **4** (1998), 403–432.

[CGN] L. Capogna, N. Garofalo and D. M. Nhieu, *Examples of uniform and NTA domains in Carnot groups*, in *Proceedings on Analysis and Geometry* (Russian) (Novosibirsk Akademgorodok, 1999), Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000, pp. 103–121.

[CGP] L. Capogna, N. Garofalo and S. W. Pauls, *On a class of NTA domains in the Heisenberg groups*, preprint.

[CT] L. Capogna and P. Tang, *Uniform domains and quasiconformal mappings on the Heisenberg group*, Manuscripta Math. **86** (1995), 267–281.

[D] D. Danielli, *Regularity at the boundary for solutions of nonlinear subelliptic equations*, Indiana Univ. Math. J. **44** (1995), 269–285.

[DGN] D. Danielli, N. Garofalo and D. M. Nhieu, *Non-doubling Ahlfors measures, perimeter measures, and the characterization of the trace spaces of Sobolev functions in Carnot–Carathéodory spaces*, preprint.

[FP] C. Fefferman and D. H. Phong, *Subelliptic eigenvalue problems*, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, Vol. I, II (Chicago, Ill., 1981), Wadsworth, Belmont, CA, 1983, pp. 590–606.

[FF1] F. Ferrari and B. Franchi, *Geometry of the boundary and doubling property of the harmonic measure for Grushin type operators*, Rend. Sem. Mat. Univ. Politec. Torino **58** (2000), 281–299.

[FF2] F. Ferrari and B. Franchi, *A local doubling formula for harmonic measure associated with subelliptic operators*, Comm. Partial Differential Equations **28** (2003), 1–60.

[FGW] B. Franchi, C. E. Gutiérrez and R. L. Wheeden, *Weighted Sobolev–Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations **19** (1994), 523–604.

[FL] B. Franchi and E. Lanconelli, *Une métrique associée à une classe d'opérateurs elliptiques dégénérés*, in *Conference on Linear Partial and Pseudodifferential Operators (Torino, 1982)*, Rend. Sem. Mat. Univ. Politec. Torino, Special Issue (1983), 105–114.

[FL1] B. Franchi and E. Lanconelli, *Hölder regularity theorem for a class of non-uniformly elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa **10** (1983), 523–541.

[FLW] B. Franchi, G. Lu and R. L. Wheeden, *A relationship between Poincaré-type inequalities and representation formulas in spaces of homogeneous type*, Internat. Math. Res. Notices (1996), no. 1, 1–14.

[FSSC] B. Franchi, R. Serapioni and F. Serra Cassano, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. **321** (2001), 479–531.

[GN1] N. Garofalo and D. M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), 1081–1144.

[GN2] N. Garofalo and D. M. Nhieu, *Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot–Carathéodory spaces*, J. Analyse Math. **74** (1998), 67–97.

[G1] A. V. Greshnov, *Extension of differentiable functions beyond the boundary of the domain on Carnot Groups*, Siberian Adv. Math. **7** (1997), 20–62.

[G2] A. V. Greshnov, *Uniform and NTA-domains on Carnot groups* (Russian), Sibirsk. Mat. Zh. **42** (2001), 1018–1035.

[HK1] P. Hajłasz and P. Koskela, *Isoperimetric inequalities and embedding theorems in irregular domains*, J. London Math. Soc. **58** (1998), 425–450.

[HK2] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **688** (2000).

[HH] W. Hansen and H. Hueber, *The Dirichlet problem for sublaplacians on nilpotent groups — Geometric criteria for regularity*, Math. Ann. **276** (1987), 537–547.

[Ho] L. Hörmander, *Hypoelliptic second-order differential equations*, Acta Math. **121** (1968), 147–171.

[Hu] H. Hueber, *Wiener's criterion in potential theory with applications to nilpotent Lie groups*, Math. Z. **190** (1985), 527–542.

[Je] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), 503–523.

[JK] D. Jerison and C. E. Kenig, *Boundary behavior of harmonic functions in non-tangentially accessible domains*, Adv. Math. **46** (1982), 80–147.

[Joh] F. John, *Rotation and strain*, Comm. Pure Appl. Math. **14** (1961), 391–413.

[Jon] P. W. Jones, *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math. **147** (1981), 71–88.

[KOT] P. Koskela, J. Onninen and J. T. Tyson, *Quasihyperbolic boundary conditions and capacity: Poincaré domains*, Math. Ann. **323** (2002), 811–830.

[L] G. Lu, *The sharp Poincaré inequality for free vector fields: an endpoint result*, Rev. Mat. Iberoamericana **10** (1994), 453–466.

[M] O. Martio, *John domains, bilipschitz balls and Poincaré inequality*, Rev. Roumaine Math. Pures Appl. **33** (1988), 107–112.

[MS] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1978/1979), 383–401.

[MP] V. G. Maz'ya and S. V. Poborchi, *Differentiable Functions on Bad Domains*, World Scientific Publishing Co., River Edge, NJ, 1997.

[MM1] R. Monti and D. Morbidelli, *Trace theorems for vector fields*, Math. Z. **239** (2002), 747–776.

[MM2] R. Monti and D. Morbidelli, *Regular domains in homogeneous groups*, preprint.

[MM3] R. Monti and D. Morbidelli, *Non-tangentially accessible domains for vector fields*, Indiana Univ. Math. J., to appear.

[MSC] R. Monti and F. Serra Cassano, *Surface measures in Carnot–Carathéodory spaces*, Calc. Var. Partial Differential Equations **13** (2001), 339–376.

[NSW] A. Nagel, E. M. Stein and S. Wainger, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math. **155** (1985), 103–147.

[NS] P. Negrini and V. Scornazzani, *Wiener criterion for a class of degenerate elliptic operators*, J. Differential Equations **66** (1987), 151–164.

[R] Yu. G. Reshetnyak, *Integral representations of differentiable functions in domains with nonsmooth boundary* (Russian), Sibirsk. Mat. Sb. **21** (1980), 108–116.

[SS] W. Smith and D. Stegenga, *Hölder domains and Poincaré domains*, Trans Amer. Math. Soc. **319** (1990), 67–100.

[V] J. Väisälä, *Uniform domains*, Tohoku Math. J. **40** (1988), 101–118.

[VG] S. K. Vodop'yanov and A. V. Greshnov, *On extension of functions of bounded mean oscillation from domains in a space of homogeneous type with intrinsic metric*, Siberian Math. J. **36** (1995), 873–901.

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