KELVIN TRANSFORM FOR GRUSHIN OPERATORS AND CRITICAL SEMILINEAR EQUATIONS

ROBERTO MONTI and DANIELE MORBIDELLI

Abstract
We study positive entire solutions $u = u(x, y)$ of the critical equation

$$\Delta_x u + (\alpha + 1)^2 |x|^{2\alpha} \Delta_y u = -u^{(Q+2)/(Q-2)} \quad \text{in } \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k, \tag{1}$$

where $(x, y) \in \mathbb{R}^m \times \mathbb{R}^k$, $\alpha > 0$, and $Q = m + k(\alpha + 1)$. In the first part of the article, exploiting the invariance of the equation with respect to a suitable conformal inversion, we prove a "spherical symmetry" result for solutions. In the second part, we show how to reduce the dimension of the problem using a hyperbolic symmetry argument. Given any *positive solution u of (1), after a suitable scaling and a translation in the variable y, the function $v(x) = u(x, 0)$ satisfies the equation*

$$\operatorname{div}_x(p \nabla_x v) - qv = -pv^{(Q+2)/(Q-2)}, \quad |x| < 1, \tag{2}$$

with a mixed boundary condition. Here, p and q are appropriate radial functions. In the last part, we prove that if $m = k = 1$, the solution of (2) is unique and that for $m \geq 3$ and $k = 1$, problem (2) has a unique solution in the class of x-radial functions.

1. Introduction and results

In this article, we study entire positive solutions of the semilinear equation

$$\Delta_x u + (\alpha + 1)^2 |x|^{2\alpha} \Delta_y u = -u^{(Q+2)/(Q-2)} \quad \text{in } \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k, \tag{1.1}$$

where Δ_x and Δ_y are Laplace operators in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, respectively. Here, $\alpha > 0$ is a positive real number and $Q = m + k(\alpha + 1)$ is the appropriate homogeneous dimension. The partial differential operator $\mathscr{L} := \Delta_x + (\alpha + 1)^2 |x|^{2\alpha} \Delta_y$ is known as the Grushin operator. The power $(Q + 2)/(Q - 2)$ in the nonlinear term is the corresponding critical exponent. We prove symmetry and uniqueness results for entire positive solutions to equation (1.1).

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If $\alpha = 0$, then $Q = n$ and (1.1) is the Yamabe equation in $\mathbb{R}^n$. A positive solution u yields a Riemannian metric $ds^2 = u^{4/(n-2)}|dz|^2$ conformal to the standard metric in $\mathbb{R}^n$ and with constant scalar curvature equal to $4(n-1)/(n-2)$. Positive solutions are radial functions about some point in $\mathbb{R}^n$. This was proved by Gidas, Ni, and Nirenberg in [GNN] under some assumptions on the behavior at infinity of solutions and by Caffarelli, Gidas, and Spruck [CGS] in the general case. (See also [CL] and [LZ] for simpler proofs.)

In the case where $\alpha = 1$, the nonlinear equation (1.1) already appeared in connection with the Cauchy-Riemann Yamabe problem solved by Jerison and Lee in [JL1] and [JL2]. The model space for this problem is the Heisenberg group $\mathbb{C}^n \times \mathbb{R}$, and in this setting, the Yamabe equation for Webster scalar curvature is

$$\Delta_b u = -u^{(Q+2)/(Q-2)}, \quad (z,t) \in \mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n} \times \mathbb{R},\ Q = 2n+2, \tag{1.2}$$

where Δ_b is the Heisenberg sub-Laplacian. If the function u is radial in the variable z, $u = u(|z|, t)$, then $\Delta_b u = \Delta_z u + 4|z|^2 \partial_{tt} u$ is a Grushin operator with $\alpha = 1$. In this radial case, Jerison and Lee found in [JL1] a method for solving equation (1.1) with $\alpha = 1$, m even integer, and $k = 1$. This method has been also generalized to more general choices of m and k by Garofalo and Vassilev in [GV]. These techniques, however, seem very much to depend on the choice $\alpha = 1$, and the problem of finding explicit solutions of (1.1) for any $\alpha > 0$ is, to the authors' knowledge, still open.

There are two main motivations for our interest in equation (1.1). First, a typical example of weakly pseudoconvex domain in the complex space is the generalized Siegel domain $\Omega_p = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im}(z_2) > |z_1|^{2p}\}$ with $p > 1$, the case where $p = 1$ being strictly pseudoconvex. Under a radiality assumption in the variable z_1, the natural boundary sub-Laplacian on $\partial\Omega_p$ takes the form of a Grushin operator with $\alpha > 1$. Thus, understanding equation (1.1) seems to be the first step in the study of semilinear equations with geometric relevance at the boundary of weakly pseudoconvex domains.

Second, equation (1.1) also results as the Euler equation for the Sobolev inequality

$$\Big(\int_{\mathbb{R}^n} |u|^{2Q/(Q-2)}\,dx\,dy\Big)^{(Q-2)/(2Q)} \le C\Big(\int_{\mathbb{R}^n} (|\nabla_x u|^2 + (\alpha+1)^2|x|^{2\alpha}|\nabla_y u|^2)\,dx\,dy\Big)^{1/2} \tag{1.3}$$

for functions in the Sobolev space $D^1(\mathbb{R}^n)$, the completion of $C_0^\infty(\mathbb{R}^n)$ in the seminorm appearing on the right-hand side. Inequality (1.3) follows from the Poincaré inequality in [FL] and from the representation formula in [FLW]. The search for extremal functions, our original motivation for this problem, naturally leads to the Grushin semilinear critical equation. The problem of extremal functions in the case $\alpha = 1$ and for suitable m, k has been discussed by Beckner in [Be].

Equation (1.1) with $\alpha > 0$ is not invariant under x-translations. This introduces a new difficulty that does not appear in the case where $\alpha = 0$. In particular, the classical moving-planes method no longer works if $\alpha > 0$. We can, however, develop a moving-spheres method, exploiting the conformal invariance of equation (1.1). We introduce a spherical inversion and a Kelvin transform preserving the equation, and we prove a "spherical symmetry" for solutions.

For $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$, let

$$\|z\| = (|x|^{2(\alpha+1)} + |y|^2)^{1/(2(\alpha+1))}.$$

The function $\Gamma(z) = \|z\|^{2-Q}$ solves the equation $\mathscr{L}\Gamma(z) = 0$, $z \neq 0$. Actually, Γ is a constant multiple of the fundamental solution for $\mathscr{L}$ with pole at the origin. Formulas representing the fundamental solution with pole at arbitrary points of $\mathbb{R}^n$ have been computed by Beals, Greiner, and Gaveau in [BGG].

The "norm" $\|z\|$ is 1-homogeneous for the group of anisotropic dilations $(x, y) \mapsto \delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y)$, $\lambda > 0$. Using such dilations, a spherical inversion $\mathscr{I}$ can be defined by

$$\mathscr{I}(z) = \delta_{\|z\|^{-2}}(z), \quad z \neq 0.$$

In the case where $\alpha = 0$, the map $\mathscr{I}$ becomes the Möbius inversion $z \mapsto z/|z|^2$.

The inversion $\mathscr{I}$ is a conformal map in the following sense.

THEOREM 1
Let $D_\alpha = (\nabla_x, (\alpha+1)|x|^\alpha \nabla_y)$. Then, for any $u \in C^1(\mathbb{R})$ and $z \neq 0$,

$$|D_\alpha(u \circ \mathscr{I})(z)|^2 = |J_{\mathscr{I}}(z)|^{2/Q} \big|(D_\alpha u)\big(\mathscr{I}(z)\big)\big|^2, \tag{1.4}$$

where $J_{\mathscr{I}} = \det \frac{\partial \mathscr{I}}{\partial z}$ is the Jacobian of $\mathscr{I}$.

The Jacobian of $\mathscr{I}$ satisfies $|J_{\mathscr{I}}(z)| = \|z\|^{-2Q}$, and therefore, $|J_{\mathscr{I}}(z)|^{(Q-2)/(2Q)} = \Gamma(z)$ is an $\mathscr{L}$-harmonic function (see Lem. 2.2). This generalizes the analogous Euclidean phenomenon with the appropriate "dimension" Q (see [IM, Sec. 2.7]). Equation (1.4) can be seen as a Cauchy-Riemann system, and it can also be described from a metric point of view. Consider the (singular) Riemannian metric $ds^2 = |dx|^2 + (\alpha+1)^{-2}|x|^{-2\alpha}|dy|^2$. (This actually generates the sub-Riemannian distance of the operator $\mathscr{L}$ studied by Franchi and Lanconelli [FL].) Then Theorem 1 essentially says that the map $\mathscr{I}$ is conformal in the metric ds^2 (see the precise statement in Th. 2.3).

The conformal inversion $\mathscr{I}$ induces the following Kelvin transform. Given a function $u : \mathbb{R}^n \to \mathbb{R}$, define $u^* : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$u^*(z) = \Gamma(z) u\big(\mathscr{I}(z)\big), \quad z \neq 0.$$

Using the conformal property (1.4), we prove that the Kelvin transform preserves equation (1.1).

THEOREM 2
If $u \in C^2(\mathbb{R}^n)$ is a positive entire solution to equation (1.1), then u^* is a solution of the same equation in $\mathbb{R}^n \setminus \{0\}$.

The function u^* has, a priori, a singularity at $z = 0$. This singularity, however, turns out to be removable, as a by-product of Theorem 3.

Concerning the notion of inversion, Korányi [K] seems to have been the first to introduce a Kelvin transform in the Heisenberg group. More recently, the existence of a "conformal inversion" has played a substantial role in the classification of the so-called Heisenberg-type groups (see the work of Cowling, Dooley, Korányi, and Ricci [CDKR]).

A function u can be scaled according to the rule $\delta_\lambda u(z) = \lambda^{Q/2-1} u(\delta_\lambda(z))$, $\lambda > 0$. Then u solves (1.1) if and only if the scaled function $\delta_\lambda u$ does. Adapting some ideas of Li and Zhang [LZ] and using a suitable Hopf lemma for $\mathscr{L}$, we prove the following.

THEOREM 3
Let $u \in C^2(\mathbb{R}^n)$ be a positive solution of equation (1.1). Then for a suitable $\lambda > 0$, the function $\delta_\lambda u$ satisfies $\delta_\lambda u = (\delta_\lambda u)^*$.

Any translated function $u^{(b)}(x, y) := u(x, y + b)$, $b \in \mathbb{R}^k$, solves (1.1), if u does. Then Theorem 3 applies to any solution $u^{(b)}$, and by means of the family of the spherical identities so obtained, the solution u can be determined, at least on the subspace $x = 0$, by an argument due, in the Euclidean case, to Li and Zhu [LZh]. Precisely, if u is a positive solution with $u = u^*$, then there exists $y_0 \in \mathbb{R}^k$ such that

$$u(0, y) = u(0, y_0)(1 + |y - y_0|^2)^{-(Q-2)/(2(\alpha+1))}, \quad y \in \mathbb{R}^k. \tag{1.5}$$

This identity and Theorem 3 yield a symmetry result for solutions which is best described in the hyperbolic geometry.

To any function $u = u(x, y)$, associate a function U of the variables $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^k$ by letting

$$U(\xi, \eta) = |\xi|^{(Q-2)/(2(\alpha+1))} u\Big(|\xi|^{1/(\alpha+1)} \frac{\xi}{|\xi|}, \eta\Big). \tag{1.6}$$

In order to explain this functional change, for a moment, fix $m = 1$. In this case, if u solves (1.1), then U solves the equation

$$\Delta_H U + \frac{Q(Q-2)}{4(\alpha+1)^2} U = -\frac{1}{(\alpha+1)^2} U^{(Q+2)/(Q-2)}, \quad \xi > 0, \tag{1.7}$$

where $\Delta_H = \xi^2 \Delta + (1-k)\xi \partial_\xi$ with $\Delta = \partial_\xi^2 + \sum_{i=1}^k \partial_{\eta_i}^2$ is the $(k+1)$-dimensional hyperbolic Laplacian (see Prop. 3.2). Incidentally, notice that the Kelvin transform for the Grushin operator corresponds, via (1.6), to the hyperbolic isometry $\zeta \mapsto \zeta/|\zeta|^2$ (see Rem. 3.3). Equation (1.7) is invariant under the group of hyperbolic isometries. This suggests that the symmetry of solutions to (1.7), if any, should be of hyperbolic type.

Now, let $m \geq 1$ again. For any $v \in \mathbb{R}^m$ with $|v| = 1$ and for any $r \in (0, 1)$, define the k-dimensional sphere

$$\Sigma(v, r) = \left\{(x, y) \in \mathbb{R}^n : \frac{x}{|x|} = v, \ \frac{(1-|x|)^2 + |y|^2}{4|x|} = \frac{r^2}{1-r^2}\right\}.$$

This is a hyperbolic sphere in the half-space $\{(tv, y) : t > 0, \ y \in \mathbb{R}^k\}$ centered at $t = 1$, $y = 0$.

THEOREM 4
Let $u \in C^2(\mathbb{R}^n)$ be a positive solution to equation (1.1) such that $u = u^*$ and $y_0 = 0$ in (1.5). Then the function U in (1.6) is constant on $\Sigma(v, r)$ for any v with $|v| = 1$ and $r \in (0, 1)$.

This essentially says that a solution u satisfying the assumptions in Theorem 4 is determined by its values on the m-dimensional ball $\{|x| < 1\} \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^k$. Then it is natural to look for an equation satisfied by the function $v(x) := u(x, 0)$. In fact, it turns out that the dimension of the original equation (1.1) can be reduced and that the function $v(x)$ must solve the problem

$$\begin{cases} \mathrm{div}_x(p\nabla_x v) - qv = -pv^{(Q+2)/(Q-2)}, & |x| < 1, \\ v > 0, & |x| \leq 1, \\ \frac{\partial v}{\partial \nu} + \left(\frac{Q}{2} - 1\right), v = 0, & |x| = 1, \end{cases} \tag{1.8}$$

where $p(x) = (1 - |x|^{2(\alpha+1)})^k$ and $q(x) = k(\alpha+1)(Q-2)(1-|x|^{2(\alpha+1)})^{k-1}|x|^{2\alpha}$. Here, ν denotes the exterior normal to the unit ball in $\mathbb{R}^m$. The boundary condition in (1.8) is produced by the reduction procedure. The partial differential equation (PDE) in (1.8) is of variational type, and it is clearly related to the Sobolev-Hardy inequality

$$\left(\int_B |v|^{2Q/(Q-2)}(1-|x|)^k \, dx\right)^{(Q-2)/Q} \leq C \int_B \left\{|\nabla_x v|^2 (1-|x|)^k + |v|^2 |x|^{2\alpha} (1-|x|)^{k-1}\right\} dx$$

for functions $v \in C^1(\bar{B})$, where $B = \{x \in \mathbb{R}^m : |x| < 1\}$. We plan to discuss this inequality in a future article.

The last part of the article is devoted to the study of uniqueness. The existence of solutions to (1.1) can be proved by Lions's concentration-compactness method, and we do not address the problem here. It is natural to conjecture that the solution of (1.1) is unique up to scaling and translations in y. This statement reduces to proving uniqueness for problem (1.8). In the case where $m > 1$, we expect the solution v to be x-radial. The proof of this property seems to require new ideas, and we have not yet been able to achieve them.

We can, however, prove the conjecture in the following case.

THEOREM 5
Let $m = k = 1$, and let $\alpha > 0$. Up to a scaling and a vertical translation, there exists a unique positive solution $u \in C^2(\mathbb{R}^2)$ of equation (1.1).

If $m = k = \alpha = 1$ (so that $Q = 3$), the function in $\mathbb{R}^2$,

$$u(x, y) = \frac{1}{((1 + x^2)^2 + y^2)^{1/4}},$$

solves equation (1.1). By Theorem 5, this is the unique solution up to a scaling and a vertical translation. Beckner proved in [Be] that this function is an extremal for the Sobolev inequality (1.3) in the plane.

In order to prove Theorem 5, we show that problem (1.8) with $m = k = 1$ and $Q = \alpha + 2$ has a unique solution. The problem is

$$\begin{cases} (pu')' - qu + pu^{(Q+2)/(Q-2)} = 0 & \text{in } (-1, 1), \\ u > 0 & \text{in } [-1, 1], \\ \alpha u(x) + 2xu'(x) = 0, & x = \pm 1, \end{cases} \tag{1.9}$$

where $p(x) = (1 - |x|^{2(\alpha+1)})$, $q(x) = \alpha(\alpha + 1)|x|^{2\alpha}$.

The main step in the proof is showing that solutions are even functions.

THEOREM 6
If $u \in C^1([-1, 1]) \cap C^2(-1, 1)$ is a solution to problem (1.9), then $u'(0) = 0$, and u is even.

The uniqueness then follows by estimating how many times two even solutions can intersect (see the proof of Th. 4.5).

If $m > 1$, assuming the function v to be x-radial transforms problem (1.8) into a problem in one variable. We prove that the solution is unique, and thus we have the following theorem.

THEOREM 7
Let $m \geq 3$, let $k = 1$, let $n = m + 1$, and let $\alpha > 0$. Up to a scaling and a vertical translation, there exists, in the class of x-radial functions, a unique positive solution $u \in C^2(\mathbb{R}^n)$ of problem (1.1).

The problem of uniqueness of (radial) solutions of nonlinear PDEs is in general quite delicate. In our study of problem (1.8), after several attempts with different techniques, we have found out how to adapt the energy method of Kwong and Li [KL]. Unfortunately, the arguments are not simple, and moreover, in the case where $m = 2$ and $k = 1$, there is a clearly technical problem, and the proof does not work (see Rem. 4.2). In the perspective of the present work, it remains an interesting question to find a more natural and less complicated proof of Theorems 6 and 7.

A short description of the article is now in order. In Section 2 we study the conformal inversion, prove Theorems 1 and 2, and finally establish Theorem 3 and its corollary, identity (1.5). In Section 3 we study the hyperbolic symmetry of solutions, prove Theorem 4, and show how the reduced problem (1.8) can be obtained. Section 4 is devoted to uniqueness results.

Notation. If a, b are vectors in $\mathbb{R}^d$ for some $d \in \mathbb{N}$, we denote by $\langle a, b\rangle = \sum_{j=1}^{d} a_j b_j$ the standard inner product and by $|a| = \langle a, a\rangle^{1/2}$ the Euclidean norm. We denote by ∇_x and ∇_y Euclidean gradients with respect to the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$. Moreover, $D_\alpha = (\nabla_x, (\alpha + 1)|x|^\alpha \nabla_y)$ denotes the Grushin gradient, and

$$\operatorname{div}_\alpha(f, g) = \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} + (\alpha + 1)|x|^\alpha \sum_{j=1}^{k} \frac{\partial g_j}{\partial y_j}, \quad (f, g) \in C^1(\mathbb{R}^n; \mathbb{R}^m \times \mathbb{R}^k),$$

is the Grushin divergence. With this notation, $\mathscr{L}u = \operatorname{div}_\alpha D_\alpha u$. Finally, we let $2^* = 2Q/(Q - 2)$, so that $(Q + 2)/(Q - 2) = 2^* - 1$.

2. Kelvin transform and spherical symmetry of solutions

In this section, we study the Kelvin transform for the operator $\mathscr{L}$, and we prove the main spherical symmetry result for solutions to problem (1.1). For $z \in \mathbb{R}^n$ and $\lambda > 0$, we let

$$\|z\| = (|x|^{2(\alpha+1)} + |y|^2)^{1/(2(\alpha+1))} \qquad \text{and} \qquad \delta_\lambda(z) = (\lambda x, \lambda^{\alpha+1} y).$$

PROPOSITION 2.1
The singular function

$$\Gamma(z) = \|z\|^{2-Q}, \quad Q = m + k(\alpha + 1), \tag{2.1}$$

satisfies $\mathscr{L}\Gamma = 0$ in $\mathbb{R}^n \setminus \{0\}$.

Proof
The proof is a short computation. Letting $N(z) = |x|^{2(\alpha+1)} + |y|^2$ and $\beta = (2 - Q)/(2(\alpha + 1))$, we have, for $z \neq 0$, $\nabla_x \Gamma = 2(\alpha + 1)\beta N^{\beta-1}|x|^{2\alpha}x$ and $\nabla_y \Gamma = 2\beta N^{\beta-1} y$. Then

$$\Delta_x \Gamma = \mathrm{div}_x \nabla_x \Gamma = 2(\alpha + 1)\beta N^{\beta-2}|x|^{2\alpha}\big\{2(\alpha + 1)(\beta - 1)|x|^{2(\alpha+1)} + (2\alpha + m)N\big\},$$
$$\Delta_y \Gamma = \mathrm{div}_y \nabla_y \Gamma = 2\beta N^{\beta-2}\big\{2(\beta - 1)|y|^2 + kN\big\},$$

and thus

$$\mathscr{L}\Gamma = 2(\alpha + 1)\beta N^{\beta-1}|x|^{2\alpha}\big\{Q + 2(\alpha + 1)(\beta - 1) + 2\alpha\big\}.$$

Since $Q + 2(\alpha + 1)(\beta - 1) + 2\alpha = 0$, this shows that $\mathscr{L}\Gamma(z) = 0$ for $z \neq 0$. □

Define the inversion $\mathscr{I} : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n\backslash\{0\}$ by

$$\mathscr{I}(z) = \delta_{\|z\|^{-2}}(z), \quad z \neq 0.$$

Clearly, $\mathscr{I}^2$ is the identity. In Lemma 2.2, Theorem 2.3, and Corollary 2.4, we prove some basic properties of $\mathscr{I}$. We denote by $J_{\mathscr{I}}(z) = \det \frac{\partial \mathscr{I}(z)}{\partial z}$ the Jacobian of $\mathscr{I}$ at the point $z \neq 0$.

LEMMA 2.2
For all $z \neq 0$, we have $|J_{\mathscr{I}}(z)| = \Gamma(z)^{2Q/(Q-2)} = \|z\|^{-2Q}$.

Proof
We give a sketch of the proof. Let $\Phi(z) = \|z\|$, and consider a relatively open set $A \subset \{\Phi = 1\}$. The coarea formula and a dilation argument yield, for $t > 0$,

$$\int_{\delta_t(A)} \frac{d\mathscr{H}^{n-1}(z)}{|\nabla\Phi(z)|} = t^{Q-1}\,\mu(A), \quad \text{where } \mu(A) = \int_A \frac{d\mathscr{H}^{n-1}(z)}{|\nabla\Phi(z)|}, \tag{2.2}$$

where $\mathscr{H}^{n-1}$ denotes the $(n - 1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

Now fix $r > 0$, and for $\delta > 0$, define $\Omega_\delta = \{\delta_t(z) : z \in A,\ r < t < r + \delta\}$. The inverted set is $\mathscr{I}(\Omega_\delta) = \{\delta_t(z) : z \in A, r < 1/t < r + \delta\}$. By the coarea formula and by (2.2),

$$|\Omega_\delta| = \int_r^{r+\delta}\int_{\delta_t(A)} \frac{1}{|\nabla\Phi|}\, d\mathscr{H}^{n-1}\, dt = \mu(A)\int_r^{r+\delta} t^{Q-1}\, dt,$$

and analogously, $|\mathscr{I}(\Omega_\delta)| = \mu(A)\int_{1/(r+\delta)}^{1/r} t^{Q-1}\, dt$.

Note that $|\mathscr{I}(\Omega_\delta)|/|\Omega_\delta|$ does not depend on A. If $z \in \mathbb{R}^n$ is a point such that $\|z\| = r > 0$, then by the area formula, $|J_{\mathscr{I}}(z)| = \lim_{\delta\to 0} |\mathscr{I}(\Omega_\delta)|/|\Omega_\delta| = r^{-2Q} = \|z\|^{-2Q}$. The proof of Lemma 2.2 is concluded. $\square$

The next theorem and the following corollary describe a remarkable conformality property of $\mathscr{I}$.

THEOREM 2.3
Let $z = (x, y) \in \mathbb{R}^n$ with $x \neq 0$, and let $|\zeta|_z = \sqrt{|\xi|^2 + (\alpha+1)^{-2}|x|^{-2\alpha}|\eta|^2}$, $\zeta = (\xi, \eta) \in \mathbb{R}^n$. Then

$$\lim_{\zeta\to z} \frac{|\mathscr{I}(\zeta) - \mathscr{I}(z)|_{\mathscr{I}(z)}}{|\zeta - z|_z} = |J_{\mathscr{I}}(z)|^{1/Q}. \tag{2.3}$$

Proof
Fix $z \in \mathbb{R}^n$ with $x \neq 0$. Define $\mathscr{I}_x(z) \in \mathbb{R}^m$ and $\mathscr{I}_y(z) \in \mathbb{R}^k$ by the relation $\mathscr{I}(z) = (\mathscr{I}_x(z), \mathscr{I}_y(z))$, and let $N(z) = |x|^{2(\alpha+1)} + |y|^2$. Then

$$\begin{aligned}
&|\mathscr{I}(\zeta) - \mathscr{I}(z)|^2_{\mathscr{I}(z)} \\
&\quad = |\mathscr{I}_x(\zeta) - \mathscr{I}_x(z)|^2 + (\alpha+1)^{-2}|\mathscr{I}_x(z)|^{-2\alpha}|\mathscr{I}_y(\zeta) - \mathscr{I}_y(z)|^2 \\
&\quad = \left|\frac{\xi}{N(\zeta)^{1/(\alpha+1)}} - \frac{x}{N(z)^{1/(\alpha+1)}}\right|^2 + \frac{N(z)^{2\alpha/(\alpha+1)}}{(\alpha+1)^2|x|^{2\alpha}}\left|\frac{\eta}{N(\zeta)} - \frac{y}{N(z)}\right|^2 \\
&\quad = N(z)^{-2/(\alpha+1)}\left\{\left|\xi\Big(\frac{N(\zeta)}{N(z)}\Big)^{-1/(\alpha+1)} - x\right|^2 + \frac{1}{(\alpha+1)^2|x|^{2\alpha}}\left|\eta\Big(\frac{N(\zeta)}{N(z)}\Big)^{-1} - y\right|^2\right\}.
\end{aligned}$$

By a Taylor expansion of the function $N(\zeta)$ at the point z,

$$\frac{N(\zeta)}{N(z)} = 1 + \frac{\Omega(z,\zeta)}{N(z)} + o(|z-\zeta|), \tag{2.4}$$

where we let $\Omega(z,\zeta) = \{2(\alpha+1)|x|^{2\alpha}\langle x, \xi - x\rangle + 2\langle y, \eta - y\rangle\}$. Notice that $\Omega(z,\zeta) = O(|z-\zeta|)$. Therefore

$$\begin{aligned}
\Big(\frac{N(\zeta)}{N(z)}\Big)^{-1} &= 1 - \frac{\Omega(z,\zeta)}{N(z)} + o(|z-\zeta|), \\
\Big(\frac{N(\zeta)}{N(z)}\Big)^{-1/(\alpha+1)} &= 1 - \frac{\Omega(z,\zeta)}{(\alpha+1)N(z)} + o(|z-\zeta|).
\end{aligned}$$

In the following, N replaces $N(z)$. By Lemma 2.2, $N^{-2/(\alpha+1)} = |J_{\mathscr{I}}(z)|^{2/Q}$, and thus

$$
\begin{aligned}
&|J_{\mathscr{I}}(z)|^{-2/Q}|\mathscr{I}(\zeta) - \mathscr{I}(z)|^2_{\mathscr{I}(z)} \\
&\quad = \left|\xi - x - \frac{\Omega(z,\zeta)}{(\alpha+1)N}\xi\right|^2 + \frac{1}{(\alpha+1)^2|x|^{2\alpha}}\left|\eta - y - \frac{\Omega(z,\zeta)}{N}\eta\right|^2 + o(|z-\zeta|^2) \\
&\quad = |\zeta - z|_z^2 - \frac{2\langle \xi - x, \xi\rangle}{(\alpha+1)N}\Omega(z,\zeta) + \frac{|\xi|^2}{(\alpha+1)^2N^2}\Omega(z,\zeta)^2 \\
&\qquad - \frac{2\langle \eta - y, \eta\rangle}{(\alpha+1)^2|x|^{2\alpha}N}\Omega(z,\zeta) + \frac{|\eta|^2}{(\alpha+1)^2|x|^{2\alpha}N^2}\Omega(z,\zeta)^2 + o(|z-\zeta|^2) \\
&\quad = |\zeta - z|_z^2 + \Omega(z,\zeta)R(z,\zeta),
\end{aligned}
$$

where R is defined by the last equality. The proof of Theorem 2.3 is completed as soon as we prove that $R(z,\zeta) = o(|\zeta - z|)$. Using the explicit form of Ω,

$$
\begin{aligned}
R(z,\zeta) = {} & \frac{2}{N^2(\alpha+1)^2}\Big[-(\alpha+1)N\langle \xi - x, \xi\rangle + |\xi|^2(\alpha+1)|x|^{2\alpha}\langle x, \xi - x\rangle + |\xi|^2\langle y, \eta - y\rangle \\
& \qquad - \frac{N}{|x|^{2\alpha}}\langle \eta - y, \eta\rangle + (\alpha+1)|\eta|^2\langle x, \xi - x\rangle + \frac{|\eta|^2}{|x|^{2\alpha}}\langle y, \eta - y\rangle\Big] \\
= {} & \frac{2}{N^2(\alpha+1)}\langle \xi - x, \xi\rangle\big(-(|x|^{2(\alpha+1)} + |y|^2) + |x|^{2\alpha}|\xi|^2 + |\eta|^2\big) \\
& + \frac{2}{N^2(\alpha+1)^2}\langle \eta - y, y\rangle\left(|\xi|^2 - \frac{|x|^{2(\alpha+1)} + |y|^2}{|x|^{2\alpha}} + \frac{|\eta|^2}{|x|^{2\alpha}}\right) + o(|z-\zeta|).
\end{aligned}
$$

In the last equality, we replaced $\langle \xi - x, x\rangle$ with $\langle \xi - x, \xi\rangle$ (we did the same for η), and we consequently added an $o(|z-\zeta|)$. Now the claim follows because both the round brackets in the last two lines tend to zero as $\zeta \to z$. $\square$

COROLLARY 2.4
Let $u, v \in C^1(\mathbb{R}^n)$. Then for $z \neq 0$,

$$
\langle D_\alpha(u\circ\mathscr{I})(z), D_\alpha(v\circ\mathscr{I})(z)\rangle = |J_{\mathscr{I}}(z)|^{2/Q}\big\langle D_\alpha u\big(\mathscr{I}(z)\big), D_\alpha v\big(\mathscr{I}(z)\big)\big\rangle. \tag{2.5}
$$

Proof
We first prove the corollary for $x \neq 0$. Then the proof for any $z \neq 0$ follows by continuity. Notice that for $x \neq 0$,

$$
|D_\alpha u(z)| = \limsup_{\zeta\to z}\frac{|u(\zeta) - u(z)|}{|\zeta - z|_z}. \tag{2.6}
$$

The inequality $\geq$ follows from $|u(\zeta) - u(z)| \leq |D_\alpha u(z)||\zeta - z|_z + o(|\zeta - z|)$, and equality is achieved by choosing $\zeta_\varepsilon = z + \varepsilon(\nabla_x u(z), (\alpha+1)^2|x|^{2\alpha}\nabla_y u(z))$ and letting $\varepsilon \to 0$.

By Theorem 2.3 and (2.6), we get

$$\begin{aligned} |D_\alpha(u \circ \mathscr{I})(z)| &= \lim_{\zeta \to z} \frac{|\mathscr{I}(\zeta) - \mathscr{I}(z)|_{\mathscr{I}(z)}}{|\zeta - z|_z} \limsup_{\zeta \to z} \frac{|u(\mathscr{I}(\zeta)) - u(\mathscr{I}(z))|}{|\mathscr{I}(\zeta) - \mathscr{I}(z)|_{\mathscr{I}(z)}} \\ &= |J_{\mathscr{I}}(z)|^{1/Q} \big| D_\alpha u\big(\mathscr{I}(z)\big)\big|. \end{aligned}$$

Developing this identity for the function $u \circ \mathscr{I} + v \circ \mathscr{I}$, we find (2.5). □

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a function. The Kelvin transform $u^* : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ of u is defined by

$$u^*(z) = \Gamma(z) u\big(\mathscr{I}(z)\big), \quad z \neq 0. \tag{2.7}$$

THEOREM 2.5
If $u \in C^2(\mathbb{R}^n)$ is a positive solution of (1.1), then u^* is a solution of the same equation in $\mathbb{R}^n \backslash \{0\}$.

Proof
We first prove that if $u \in C^1(\mathbb{R}^n)$ and $\varphi \in C^1_0(\mathbb{R}^n \backslash \{0\})$, then

$$\int_{\mathbb{R}^n} \langle D_\alpha u, D_\alpha \varphi \rangle \, dz = \int_{\mathbb{R}^n} \langle D_\alpha u^*, D_\alpha \varphi^* \rangle \, dz. \tag{2.8}$$

In order to prove (2.8), let $v = u \circ \mathscr{I}$, and let $\psi = \varphi \circ \mathscr{I}$. Then

$$\begin{aligned} \langle D_\alpha u^*, D_\alpha \varphi^* \rangle &= \langle D_\alpha(\Gamma v), D_\alpha(\Gamma \psi) \rangle \\ &= \Gamma^2 \langle D_\alpha v, D_\alpha \psi \rangle + v\psi |D_\alpha \Gamma|^2 + \Gamma \langle D_\alpha(\psi v), D_\alpha \Gamma \rangle \\ &= \Gamma^2 \langle D_\alpha v, D_\alpha \psi \rangle + \mathrm{div}_\alpha(v\psi \Gamma D_\alpha \Gamma) \end{aligned}$$

because, by Proposition 2.1, $\mathscr{L}\Gamma = 0$ in $\mathbb{R}^n \backslash \{0\}$. On the other hand, by (2.5),

$$\langle D_\alpha v(z), D_\alpha \psi(z) \rangle = |J_{\mathscr{I}}(z)|^{2/Q} \big\langle D_\alpha u\big(\mathscr{I}(z)\big), D_\alpha \varphi\big(\mathscr{I}(z)\big)\big\rangle.$$

By Lemma 2.2, $\Gamma(z)^2 = |J_{\mathscr{I}}(z)|^{(Q-2)/Q}$, and then

$$\begin{aligned} \int_{\mathbb{R}^n} \langle D_\alpha u^*, D_\alpha \varphi^* \rangle \, dz &= \int_{\mathbb{R}^n} \big(\Gamma^2 \langle D_\alpha v, D_\alpha \psi \rangle + \mathrm{div}_\alpha(v\psi \Gamma D_\alpha \Gamma) \big) \, dz \\ &= \int_{\mathbb{R}^n} \Gamma^2 \langle D_\alpha v, D_\alpha \psi \rangle \, dz \\ &= \int_{\mathbb{R}^n} \big\langle D_\alpha u\big(\mathscr{I}(z)\big), D_\alpha \varphi\big(\mathscr{I}(z)\big)\big\rangle |J_{\mathscr{I}}(z)| \, dz \\ &= \int_{\mathbb{R}^n} \langle D_\alpha u(z), D_\alpha \varphi(z) \rangle \, dz. \end{aligned}$$

The proof of (2.8) is concluded.

In order to conclude the proof of Theorem 2.5, it suffices to note that

$$\begin{aligned}\int_{\mathbb{R}^n} (u^*)^{(Q+2)/(Q-2)}\varphi^*\,dz &= \int_{\mathbb{R}^n} u\big(\mathscr{I}(z)\big)^{(Q+2)/(Q-2)}\varphi\big(\mathscr{I}(z)\big)^*|J_{\mathscr{I}}(z)|\,dz\\ &= \int_{\mathbb{R}^n} u^{(Q+2)/(Q-2)}\varphi\,dz\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$. □

Now we apply the moving-spheres method to equation (1.1). Given a function $u : \mathbb{R}^n \to \mathbb{R}$ and $\lambda > 0$, define the function $\delta_\lambda u(z) = \lambda^{(Q/2)-1}u(\delta_\lambda(z))$. Then let

$$u_\lambda(z) = (\delta_{\lambda^2}u)^*(z) = \Big(\frac{\|z\|}{\lambda}\Big)^{2-Q} u\Big(\frac{\lambda^2 x}{\|z\|^2}, \frac{\lambda^{2(\alpha+1)}y}{\|z\|^{2(\alpha+1)}}\Big), \quad z \neq 0, \lambda > 0.$$

If $\mathscr{L}u = -u^{2^*-1}$, $2^* = 2Q/(Q-2)$, then $\delta_\lambda u$ and u_λ solve the same equation. The statement concerning $\delta_\lambda u$ is a simple computation. The statement concerning u_λ is a consequence of Theorem 2.5.

The knowledge of the singular solution (2.1) easily provides the following characteristic Hopf lemma.

LEMMA 2.6 (Hopf-type lemma)
Let $v \in \mathbb{R}^k$ with $|v| = 1$, $t \in \mathbb{R}$, $\Omega = \{(x, y) \in \mathbb{R}^n : \langle y, v\rangle > t\}$, and $(0, y_0) \in \partial\Omega$. If a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfies $u > 0$ in Ω, $u(0, y_0) = 0$, and $\mathscr{L}u \leq 0$ in Ω, then $\langle \nabla_y u(0, y_0), v\rangle > 0$.

Proof
The proof is a short computation. Let $y_1 = y_0 + v$, $z_0 = (0, y_0)$, $z_1 = (0, y_1)$, and $z = (x, y)$. The point $z_1 = z_0 + (0, v)$ belongs to Ω. The function $\Lambda(z) = \Gamma(z - z_1) - \Gamma(z_0 - z_1)$ satisfies $\mathscr{L}\Lambda(z) = 0$ for $z \neq z_1$, $\Lambda(z) = 0$ for $\|z - z_1\| = \gamma_1 := \|z_0 - z_1\|$, and $\Lambda(z) = 1$ for $\|z - z_1\| = \gamma_0$, for a suitable $\gamma_0 \in (0, \gamma_1)$.

Let $R = \{\gamma_0 < \|z - z_1\| < \gamma_1\} \subset \Omega$. If $\varepsilon > 0$ is small enough, the function $u_\varepsilon(z) = u(z) - \varepsilon\Lambda(z)$ is strictly positive on $\|z - z_1\| = \gamma_1$. Moreover, $u_\varepsilon(z) = u(z) \geq 0$ on $\|z - z_1\| = \gamma_0$. Since $u_\varepsilon \geq 0$ on ∂R and $\mathscr{L}u_\varepsilon(z) = \mathscr{L}u(z) - \varepsilon\mathscr{L}\Lambda(z) = \mathscr{L}u(z) \leq 0$ on R, by the weak maximum principle, it follows that $u \geq \varepsilon\Lambda$ on R. Thus, using $u(0, y_0) = 0$, we find

$$\begin{aligned}\langle \nabla_y u(0, y_0), v\rangle &= \lim_{t\to 0}\frac{u(0, y_0 + tv)}{t} \geq \varepsilon \lim_{t\to 0}\frac{\Lambda(0, y_0 + tv)}{t}\\ &= \varepsilon \lim_{t\to 0}\frac{1}{t}\big\{|y_0 - y_1 + tv|^{(2-Q)/(\alpha+1)} - |y_0 - y_1|^{(2-Q)/(\alpha+1)}\big\}\\ &= \varepsilon \lim_{t\to 0}\frac{1}{t}\big\{(1 - t)^{(2-Q)/(\alpha+1)} - 1\big\} = \varepsilon\frac{Q - 2}{\alpha + 1} > 0.\end{aligned}$$

This ends the proof of Lemma 2.6. □

THEOREM 2.7
Let $u \in C^2(\mathbb{R}^n)$ be a positive solution of equation (1.1). Then there exists $\lambda > 0$ such that $u = u_\lambda$.

Proof
The proof is articulated in three steps. Here, the scheme of the elegant proof by Li and Zhang in [LZ, Sec. 2] can be adapted without significant changes.

Step 1. If $u \in C^2(\mathbb{R}^n)$ is a positive solution of (1.1), then there exists $\lambda_0 > 0$ such that

$$u_\lambda(z) \le u(z), \quad \lambda \in (0, \lambda_0),\ \|z\| \ge \lambda.$$

Step 2. Define

$$\bar{\lambda} = \sup\big\{\lambda_0 > 0 : u_\lambda(z) \le u(z),\ \|z\| \ge \lambda,\ 0 < \lambda \le \lambda_0\big\}. \tag{2.9}$$

If $\bar{\lambda} < +\infty$, then $u \equiv u_{\bar{\lambda}}$ on $\mathbb{R}^n \setminus \{0\}$.

Step 3. For $b \in \mathbb{R}^k$, let $u^{(b)}(z) = u(z + (0, b))$, $z \in \mathbb{R}^n$, and let $\bar{\lambda}_b$ be defined as in (2.9), relatively to $u^{(b)}$. If there exists $b \in \mathbb{R}^k$ such that $\bar{\lambda}_b = +\infty$, then $u \equiv 0$.

Proof of Step 1
Let $z \in \mathbb{R}^n$ with $\|z\| \le 1$, and for $\lambda > 0$, define the function $\varphi(\lambda) = \lambda^{(Q-2)/2} u(\delta_\lambda(z))$. It is easy to check that

$$\varphi'(\lambda) = \lambda^{(Q-4)/2}\Big[\frac{Q-2}{2} u\big(\delta_\lambda(z)\big) + \lambda\big\langle \nabla_x u\big(\delta_\lambda(z)\big), x\big\rangle + (\alpha+1)\lambda^{\alpha+1}\big\langle \nabla_y u\big(\delta_\lambda(z)\big), y\big\rangle\Big].$$

Since $\lim_{\lambda \to 0}[\cdots] = ((Q-2)/2)u(0) > 0$ uniformly in $\|z\| \le 1$, we have $\varphi'(\lambda) > 0$ for $\lambda \in (0, \lambda_1)$ and $\|z\| \le 1$, where λ_1 is small enough. Then for any z with $\|z\| \le 1$ and $0 < \lambda < \lambda_1$, we have $\varphi(\lambda_1) - \varphi(\lambda) > 0$, by the mean-value theorem. Letting $\mu = \lambda/\lambda_1$, this yields $u(z) \ge \mu^{(Q-2)/2} u(\delta_\mu(z))$ for $\|z\| \le \lambda_1$ and $0 < \mu < 1$. Thus, choosing $\mu = \lambda^2/\|z\|^2 < 1$,

$$\Big(\frac{\lambda}{\|z\|}\Big)^{Q-2} u\big(\delta_{\lambda^2\|z\|^{-2}}(z)\big) \le u(z), \quad 0 < \lambda \le \|z\| \le \lambda_1.$$

Now note that

$$u(z) - \Big(\frac{\lambda_1}{\|z\|}\Big)^{Q-2} \min_{\|\zeta\| = \lambda_1} u(\zeta) \ge 0, \quad \|z\| \ge \lambda_1. \tag{2.10}$$

This follows from Bony's maximum principle [Bo]. Indeed, the function on the left-hand side is superharmonic for the operator $\mathscr{L}$ and nonnegative on $\|z\| = \lambda_1$ and as $\|z\| \to \infty$.

Now fix $\lambda_0 \in (0, \lambda_1)$ such that

$$\lambda_0^{Q-2} \max_{\|z\| \le \lambda_0} u(z) \le \lambda_1^{Q-2} \min_{\|z\| = \lambda_1} u(z).$$

If $\|z\| \ge \lambda_1$ and $\lambda \in (0, \lambda_0)$, we have $\|\delta_{\lambda^2 \|z\|^{-2}}(z)\| = \lambda^2 / \|z\| \le \lambda_0$. Thus

$$\left(\frac{\lambda}{\|z\|}\right)^{Q-2} u\big(\delta_{\lambda^2\|z\|^{-2}}(z)\big) \le \left(\frac{\lambda_0}{\|z\|}\right)^{Q-2} \max_{\|\zeta\| \le \lambda_0} u(\zeta) \le \left(\frac{\lambda_1}{\|z\|}\right)^{Q-2} \min_{\|\zeta\| = \lambda_1} u(\zeta) \le u(z).$$

□

Proof of Step 2
Let $w_\lambda = u - u_\lambda$. Then $w_\lambda(z) = 0$ for $\|z\| = \lambda$. Moreover, if $0 < \lambda \le \bar\lambda$,

$$w_\lambda(z) \ge 0, \qquad \mathscr{L} w_\lambda(z) \le 0 \quad \text{for } \|z\| \ge \lambda.$$

If $w_{\bar\lambda} \equiv 0$ on $\{\|z\| \ge \bar\lambda\}$, the proof is finished. Assume that this does not hold. Then by the maximum principle, it must be $w_{\bar\lambda} > 0$ on $\|z\| > \bar\lambda$. By the elliptic Hopf lemma and by Lemma 2.6, for all z with $\|z\| = \bar\lambda$, we have $\langle \nu(z), \nabla w_{\bar\lambda}(z)\rangle > 0$, where $\nu(z)$ is the unit normal at the point z to the surface $\{\zeta \in \mathbb{R}^n : \|\zeta\| = \|z\|\}$.

By continuity, for some $r > \bar\lambda$ suitably close to $\bar\lambda$ and for all $\lambda \in [\bar\lambda, r]$, we have

$$w_\lambda(z) > 0, \quad \lambda < \|z\| \le r. \tag{2.11}$$

We are going to extend this inequality on $\|z\| \ge r$, at least for $\lambda \in [\bar\lambda, \bar\lambda + \varepsilon_0)$ for a small $\varepsilon_0 < r - \bar\lambda$.

Using the maximum principle as in the proof of (2.10), we get

$$u(z) - u_{\bar\lambda}(z) \ge \left(\frac{r}{\|z\|}\right)^{Q-2} \min_{\|\zeta\| = r} \big(u(\zeta) - u_{\bar\lambda}(\zeta)\big) > 0, \quad \|z\| \ge r.$$

Then

$$u(z) - u_\lambda(z) \ge u_{\bar\lambda}(z) - u_\lambda(z) + \left(\frac{r}{\|z\|}\right)^{Q-2} \min_{\|\zeta\| = r} \big(u(\zeta) - u_{\bar\lambda}(\zeta)\big), \quad \|z\| \ge r.$$

Now

$$|u_{\bar\lambda}(z) - u_\lambda(z)| = \frac{1}{\|z\|^{Q-2}} \left| \bar\lambda^{Q-2} u\big(\delta_{\bar\lambda^2\|z\|^{-2}}(z)\big) - \lambda^{Q-2} u\big(\delta_{\lambda^2\|z\|^{-2}}(z)\big)\right|.$$

If $\|z\| \geq r$, the points $\delta_{\bar{\lambda}^2\|z\|^{-2}}(z)$ and $\delta_{\lambda^2\|z\|^{-2}}(z)$ are in a compact set. Then, by uniform continuity, there exists $\varepsilon_0 > 0$ such that for $\|z\| \geq r$ and $\lambda \in (\bar{\lambda}, \bar{\lambda} + \varepsilon_0)$, we have

$$\left|\bar{\lambda}^{Q-2} u\big(\delta_{\bar{\lambda}^2\|z\|^{-2}}(z)\big) - \lambda^{Q-2} u\big(\delta_{\lambda^2\|z\|^{-2}}(z)\big)\right| \leq \frac{1}{2} r^{Q-2} \min_{\|\zeta\|=r} \big(u(\zeta) - u_{\bar{\lambda}}(\zeta)\big).$$

Therefore, for all z with $\|z\| \geq r$ and $\lambda \in (\bar{\lambda}, \bar{\lambda} + \varepsilon_0)$, it holds that

$$w_\lambda(z) = u(z) - u_\lambda(z) \geq \frac{1}{2}\Big(\frac{r}{\|z\|}\Big)^{Q-2} \min_{\|\zeta\|=r} \big(u(\zeta) - u_{\bar{\lambda}}(\zeta)\big) > 0,$$

which together with (2.11) contradicts the definition of $\bar{\lambda}$ and finishes the proof of Step 2. □

Proof of Step 3

If u is a solution, then $u^{(b)}(z) = u(z + (0, b))$ is still a solution, and there exists a maximal $\bar{\lambda}_b > 0$ such that

$$(u^{(b)})_\lambda(z) \leq u^{(b)}(z), \quad \lambda \in (0, \bar{\lambda}_b),\ \|z\| \geq \lambda.$$

We first show that if $\bar{\lambda}_b = +\infty$ for some b, then $\bar{\lambda}_b = +\infty$ for all $b \in \mathbb{R}^k$. Let b be such that $\bar{\lambda}_b = +\infty$. Letting $z_b = z - (0, b)$, we have

$$u(z) \geq \Big(\frac{\lambda}{\|z_b\|}\Big)^{Q-2} u\big(\delta_{\lambda^2\|z_b\|^{-2}}(z_b) + (0, b)\big)$$

for all $\lambda > 0$ and $\|z_b\| \geq \lambda$. For any fixed $\lambda > 0$, it follows that $\liminf\limits_{\|z\|\to\infty} \|z\|^{Q-2} u(z) \geq \lambda^{Q-2} u(0, b) > 0$. Letting $\lambda \to +\infty$, this implies

$$\lim_{\|z\|\to\infty} \|z\|^{Q-2} u(z) = +\infty. \tag{2.12}$$

Assume now that there is $a \neq b$ such that $\bar{\lambda}_a < +\infty$. Then by Step 2, it is $u^{(a)} \equiv (u^{(a)})_{\bar{\lambda}_a}$; that is,

$$u(z) \equiv \Big(\frac{\bar{\lambda}_a}{\|z_a\|}\Big)^{Q-2} u\big(\delta_{\bar{\lambda}_a^2\|z_a\|^{-2}}(z_a) + (0, a)\big), \quad z_a = z - (0, a) \neq 0.$$

This gives $\lim_{\|z\|\to\infty} \|z\|^{Q-2} u(z) = \bar{\lambda}_a^{Q-2} u(0, a) < +\infty$, which contradicts (2.12).

Now we know that $\bar{\lambda}_b = +\infty$ for any b. Let

$$g_{\lambda,b}(z) := u\big(z + (0, b)\big) - \Big(\frac{\lambda}{\|z\|}\Big)^{Q-2} u\big(\delta_{\lambda^2\|z\|^{-2}}(z) + (0, b)\big)$$

for $\lambda > 0$, $\|z\| \geq \lambda$, $b \in \mathbb{R}^k$. Since $g_{\|z\|,b}(z) = 0$ and $g_{\|z\|,b}(\delta_r(z)) \geq 0$, $r \geq 1$, it follows that

$$\frac{d}{dr} g_{\|z\|,b}\big(\delta_r(z)\big) \geq 0 \quad \text{for } r = 1.$$

After a short computation, this furnishes

$$2\big\langle \nabla u\big(z + (0, b)\big), \big(x, (\alpha + 1)y\big)\big\rangle + (Q - 2)u\big(z + (0, b)\big) \geq 0, \quad z \in \mathbb{R}^n, b \in \mathbb{R}^k,$$

or, equivalently,

$$2\big\langle \nabla u(z), \big(x, (\alpha + 1)(y - b)\big)\big\rangle + (Q - 2)u(z) \geq 0, \quad z \in \mathbb{R}^n, b \in \mathbb{R}^k.$$

Dividing by $|b|$ and letting $|b| \to \infty$ with $b/|b| \to b_0$, $|b_0| = 1$, we get

$$\langle \nabla_y u(z), b_0 \rangle \leq 0 \quad \text{for } |b_0| = 1, z \in \mathbb{R}^n.$$

This implies that $\nabla_y u \equiv 0$, and therefore u does not depend on y and ultimately solves

$$\Delta_x u(x) = -u(x)^{(Q+2)/(Q-2)}, \quad x \in \mathbb{R}^m.$$

This implies that $u \equiv 0$. If $m \geq 3$, this is a well-known result on entire subcritical semilinear equations (see, e.g., [CL]). Note that $Q > m + 1 > m$, and thus $(Q + 2)/(Q - 2) < (m + 3)/(m - 1) < (m + 2)/(m - 2)$. If $m = 2$, the previous argument still works; only add one mute variable to $\mathbb{R}^2$. If $m = 1$, the function $u = u(x)$ is positive and strictly concave on the real line, and this is not possible.

This ends the proof of Step 3. □

Thus the proof of Theorem 2.7 is completed. □

In the classical moving-spheres method, the solution of the Yamabe equation in $\mathbb{R}^n$ can be determined explicitly on the whole space (see [LZh]). Here, the argument provides the explicit form of the solution only on the set $\{x = 0\}$.

COROLLARY 2.8

Let $u \in C^2(\mathbb{R}^n)$ be a positive solution of equation (1.1) such that $u = u^*$. Then there exists $y_0 \in \mathbb{R}^k$ such that for all $y \in \mathbb{R}^k$,

$$u(0, y) = u(0, y_0)(1 + |y - y_0|^2)^{(2-Q)/(2(\alpha+1))}. \tag{2.13}$$

Proof

The assumption $u = u^*$ reads

$$u(z) = \|z\|^{2-Q} u\big(\delta_{1/\|z\|^2}(z)\big), \tag{2.14}$$

where $z = (x, y)$.

For a fixed $b \in \mathbb{R}^k$, define $u^{(b)}(z) = u(z + (0, b))$. Clearly, $\mathscr{L}u^{(b)} = -(u^{(b)})^{2^*-1}$, and hence, by Theorem 2.7, there exists $\lambda_b > 0$ such that $u^{(b)} = (u^{(b)})_{\lambda_b}$; that is,

$$u\big(z + (0, b)\big) = \Big(\frac{\|z\|}{\lambda_b}\Big)^{2-Q} u\big(\delta_{\lambda_b^2/\|z\|^2}(z) + (0, b)\big).$$

Letting $z_b = z - (0, b)$ for all z, this identity becomes

$$u(z) = \Big(\frac{\|z_b\|}{\lambda_b}\Big)^{2-Q} u\big(\delta_{\lambda_b^2/\|z_b\|^2}(z_b) + (0, b)\big) \tag{2.15}$$

for all $z \neq (0, b)$. Multiplying (2.15) by $\|z\|^{Q-2}$ and letting $\|z\| \to \infty$, we find

$$\begin{aligned} u^*(0) = \lim_{\|z\|\to\infty} \|z\|^{Q-2} u(z) &= \lambda_b^{Q-2} \lim_{\|z\|\to\infty} \Big(\frac{\|z_b\|}{\|z\|}\Big)^{2-Q} u\big(\delta_{\lambda_b^2/\|z_b\|^2}(z_b) + (0, b)\big) \\ &= \lambda_b^{Q-2} u(0, b), \end{aligned}$$

and using $u(0) = u^*(0)$, we get

$$\lambda_b^{Q-2} = \frac{u(0, 0)}{u(0, b)}. \tag{2.16}$$

From (2.14) and (2.15), we also have

$$\|z\|^{2-Q} u\big(\delta_{1/\|z\|^2}(z)\big) = \Big(\frac{\|z_b\|}{\lambda_b}\Big)^{2-Q} u\big(\delta_{\lambda_b^2/\|z_b\|^2}(z_b) + (0, b)\big).$$

Now, let $f(y) = u(0, y)$. Setting $x = 0$ in the last identity and using (2.16), we obtain

$$|y|^{(2-Q)/(\alpha+1)} f\Big(\frac{y}{|y|^2}\Big) = \frac{f(0)}{f(b)} |y - b|^{(2-Q)/(\alpha+1)} f\Big(\frac{\lambda_b^{2(\alpha+1)}(y - b)}{|y - b|^2} + b\Big),$$

and by a first-order Taylor approximation with $|y| \to \infty$,

$$\begin{aligned} &|y|^{(2-Q)/(\alpha+1)} \Big\{ f(0) + \Big\langle \nabla f(0), \frac{y}{|y|^2} \Big\rangle + o\Big(\frac{1}{|y|}\Big) \Big\} \\ &\quad = \frac{f(0)}{f(b)} |y - b|^{(2-Q)/(\alpha+1)} \Big\{ f(b) + \lambda_b^{2(\alpha+1)} \Big\langle \nabla f(b), \frac{y - b}{|y - b|^2} \Big\rangle + o\Big(\frac{1}{|y - b|}\Big) \Big\}. \end{aligned} \tag{2.17}$$

The function f has a maximum point $y_0 \in \mathbb{R}^k$ because u is infinitesimal at infinity. Without loss of generality, we can assume that $y_0 = 0$ and $\nabla f(0) = 0$. Again using

(2.16) and rearranging terms in (2.17), we get

$$f(0)^{-2(\alpha+1)/(Q-2)}\Big\{1-\Big(\frac{|y|}{|y-b|}\Big)^{(Q-2)/(\alpha+1)}\Big\}$$
$$=\Big(\frac{|y|}{|y-b|}\Big)^{(Q-2)/(\alpha+1)} f(b)^{-(Q+2\alpha)/(Q-2)}\Big\langle\nabla f(b),\,\frac{y-b}{|y-b|^2}\Big\rangle+o\Big(\frac{1}{|y|}\Big).$$

We multiply this identity by y_i, $i=1,\dots,k$, and let $y_i\to\infty$. Notice that

$$\lim_{y_i\to+\infty} y_i\Big\{1-\Big(\frac{|y|}{|y-b|}\Big)^{(Q-2)/(\alpha+1)}\Big\}=-\frac{Q-2}{\alpha+1}b_i$$

and

$$\lim_{y_i\to+\infty} y_i\Big\langle\nabla f(b),\,\frac{y-b}{|y-b|^2}\Big\rangle=\partial_i f(b),$$

whence

$$\begin{aligned}f(0)^{-2(\alpha+1)/(Q-2)}\nabla(1+|b|^2)&=-\frac{2(\alpha+1)}{Q-2}f(b)^{-(Q+2\alpha)/(Q-2)}\nabla f(b)\\&=\nabla\big(f(b)^{-2(\alpha+1)/(Q-2)}\big),\end{aligned}$$

where ∇ is the gradient with respect to b. This finally gives, for $b\in\mathbb{R}^k$,

$$f(b)=f(0)(1+|b|^2)^{-(Q-2)/(2(\alpha+1))}.$$

This is (2.13) with $y_0=0$. □

3. Hyperbolic symmetry

In this section, we prove radial symmetry properties of solutions to equation (1.1). After a suitable functional change, such solutions become radial functions in the hyperbolic space.

Definition 3.1

Given a function $u=u(x,y)$ with $x\in\mathbb{R}^m$ and $y\in\mathbb{R}^k$, define the function $U=U(\xi,\eta)$ by

$$U(\xi,\eta)=|\xi|^{\beta}u\Big(|\xi|^{1/(\alpha+1)}\frac{\xi}{|\xi|},\eta\Big),\quad \beta=\frac{Q-2}{2(\alpha+1)}. \tag{3.1}$$

We let $U=T(u)$ and $u=T^{-1}(U)$.

In the case where $m=k=\alpha=1$, this function change was introduced by Beckner [Be] in his study of extremal functions for the Sobolev inequality in the Grushin plane.

In order to explain the functional transformation T, we choose $m = 1$, and we introduce the hyperbolic space. Let $H = \{\zeta = (\xi, \eta) \in \mathbb{R} \times \mathbb{R}^k : \xi > 0\}$ be the $(n = k+1)$-dimensional hyperbolic spaces with the metric $ds_H^2 = \xi^{-2}(d\xi^2 + |d\eta|^2)$. Denote by d_H the hyperbolic distance on H. The hyperbolic gradient is $D_H U = \xi \nabla U$, and the natural measure is $d\mu = \xi^{-k-1}\, d\xi\, d\eta$. The Laplace-Beltrami operator Δ_H is defined by the integration-by-parts formula

$$\int_H \langle D_H U, D_H \Phi \rangle \, d\mu = -\int_H \Phi \Delta_H U \, d\mu$$

for test functions $\Phi \in C_0^\infty(H)$.

We briefly recall the unit-ball model for the hyperbolic space. Let $B = \{z = (x, y) \in \mathbb{R} \times \mathbb{R}^k : |z| < 1\}$ be the $(n = k+1)$-dimensional unit ball endowed with the metric $ds_B^2 = 4(dx^2 + |dy|^2)/(1 - |z|^2)^2$. It is known that the Möbius map $S : B \to H$ defined by $S(x, y) = (-1, 0) + 2((x+1, y)/|(x+1, y)|^2)$ is a hyperbolic isometry between B and H. Moreover, S takes the geodesic spheres $\{(x, y) \in B : x^2 + |y|^2 = r^2\}$, $r < 1$, onto the spheres

$$\left\{ (\xi, \eta) \in H : \frac{(1-\xi)^2 + |\eta|^2}{4\xi} = \frac{r^2}{1 - r^2} \right\}.$$

The critical semilinear equation (1.1) for the Grushin operator is related to a semilinear equation in the hyperbolic space.

PROPOSITION 3.2
Let $m = 1$. If u is a positive solution to equation (1.1) in $\mathbb{R}^+ \times \mathbb{R}^k$, then $U = T(u)$ is a solution to the equation in H,

$$\Delta_H U + \frac{Q(Q-2)}{4(\alpha+1)^2} U = -\frac{1}{(\alpha+1)^2} U^{(Q+2)/(Q-2)}, \tag{3.2}$$

where Δ_H is the hyperbolic Laplacian.

Proof
The proof is an easy computation. Integrating equation (1.1) against a test function $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^k)$, we obtain

$$\int_{x>0} \left\{ \langle D_\alpha u, D_\alpha \varphi \rangle - u^{(Q+2)/(Q-2)} \varphi \right\} dz = 0. \tag{3.3}$$

From (3.1), we find $\partial_x u = (\alpha+1)\xi^{\alpha/(\alpha+1)}[-\beta \xi^{-\beta-1} U + \xi^{-\beta} \partial_\xi U]$ and $\nabla_y u = \xi^{-\beta} \nabla_\eta U(\xi, \eta)$, where $\xi = x^{\alpha+1}$ and $\beta = (Q-2)/(2(\alpha+1))$. Analogous formulas hold for the test function $\Phi = T(\varphi)$.

Performing the change of variable $\xi = x^{\alpha+1}$, so that $dz = dx\,dy = (1/(\alpha+1))\xi^{-\alpha/(\alpha+1)}\,d\xi\,d\eta$, we obtain

$$\int_{x>0}\langle D_\alpha u, D_\alpha\varphi\rangle\,dz=(\alpha+1)\int_{\xi>0}\big\{\beta^2 U\Phi-\beta\xi\partial_\xi[U\Phi]+\xi^2\langle\nabla_{\xi,\eta}U,\nabla_{\xi,\eta}\Phi\rangle\big\}\frac{d\xi\,d\eta}{\xi^{k+1}}.$$

We used the relation $\alpha/(\alpha+1)-2\beta = 1-k$. After an integration by parts of the term $\xi^{-k}\partial_\xi[U\Phi]$ and the introduction of hyperbolic gradient and measure, we get

$$\begin{aligned}\int_{x>0}\langle D_\alpha u, D_\alpha\varphi\rangle\,dz &= (\alpha+1)\int_H\Big\{-\frac{Q(Q-2)}{4(\alpha+1)^2}U\Phi+\langle D_H U, D_H\Phi\rangle\Big\}\,d\mu\\ &= -(\alpha+1)\int_H\Big\{\frac{Q(Q-2)}{4(\alpha+1)^2}U+\Delta_H U\Big\}\Phi\,d\mu.\end{aligned}$$

An even simpler computation shows that

$$\int_{x>0}u^{(Q+2)/(Q-2)}\varphi\,dz=\frac{1}{\alpha+1}\int_H U^{(Q+2)/(Q-2)}\Phi\,d\mu.$$

Comparing the last two formulas with (3.3), we get the proof of Proposition 3.2. □

Remark 3.3
Equation (3.2) is invariant under hyperbolic isometries. Indeed, via the functional transformation T, translations $u(x,y)\mapsto u(x,y+b)$, $b\in\mathbb{R}^k$, correspond to translations (hyperbolic isometries) $U(\xi,\eta)\mapsto U(\xi,\eta+b)$, and dilations $u(x,y)\mapsto\lambda^{(Q/2)-1}u(\lambda x,\lambda^{\alpha+1}y)$, $\lambda>0$, correspond to the hyperbolic isometries $U(\xi,\eta)\mapsto U(\lambda^{\alpha+1}\xi,\lambda^{\alpha+1}\eta)$. This observation suggests how to construct the Kelvin transform u^* introduced in (2.7). Consider the hyperbolic isometry of (H,d_H), $(\xi,\eta)=\zeta\mapsto\zeta/|\zeta|^2$, and for $U: H\to\mathbb{R}$, let

$$U^\dagger(\xi,\eta)=U\Big(\frac{\xi}{\xi^2+|\eta|^2},\frac{\eta}{\xi^2+|\eta|^2}\Big).$$

Then it holds that $T(u^*)=(T(u))^\dagger$ for any $u:\mathbb{R}^+\times\mathbb{R}^k\to\mathbb{R}$. Indeed, since $T(u)(\xi,\eta)=U(\xi,\eta)=\xi^{(Q-2)/(2(\alpha+1))}u(\xi^{1/(\alpha+1)},\eta)$,

$$\begin{aligned}\big(T(u)\big)^\dagger(\xi,\eta)&=U^\dagger(\xi,\eta)\\ &=\Big(\frac{\xi}{\xi^2+|\eta|^2}\Big)^{(Q-2)/(2(\alpha+1))}u\bigg(\Big(\frac{\xi}{\xi^2+|\eta|^2}\Big)^{1/(\alpha+1)},\Big(\frac{\eta}{\xi^2+|\eta|^2}\Big)\bigg),\end{aligned}$$

and therefore

$$\begin{aligned}
T^{-1}(U^\dagger)(x, y) &= x^{-(Q-2)/2} U^\dagger(x^{\alpha+1}, y) \\
&= x^{-(Q-2)/2} \Big(\frac{x^{\alpha+1}}{x^{2(\alpha+1)} + |y|^2} \Big)^{(Q-2)/(2(\alpha+1))} \\
&\quad \times u\Big(\frac{x}{[x^{2(\alpha+1)} + |y|^2]^{1/(\alpha+1)}}, \frac{y}{x^{2(\alpha+1)} + |y|^2} \Big) \\
&= \|z\|^{2-Q} u\big(\delta_{\|z\|^{-2}}(z)\big).
\end{aligned}$$

Then the Kelvin transform in the Grushin space stems from a hyperbolic reflection. The construction not only produces the correct form for the inversion $z \mapsto \delta_{1/\|z\|^2}(z)$, but it also yields the form of the singular solution $\Gamma(z) = \|z\|^{2-Q}$ for $\mathscr{L}$ appearing in the definition of u^*.

Now we prove the main hyperbolic symmetry theorem. Let $m, k \geq 1$, and for $v \in \mathbb{R}^m$ with $|v| = 1$, consider the half-space

$$H_v = \big\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^k : x = tv,\ t > 0\big\}.$$

H_v carries a natural structure of $(k+1)$-dimensional hyperbolic space. We use the coordinates (t, y) on H_v; by abuse of notation, $(t, y) \in H_v$ stands for $(tv, y) \in H_v$. The metric here is $ds^2 = t^{-2}(dt^2 + |dy|^2)$.

For any $v \in \mathbb{R}^m$ with $|v| = 1$ and for any $r \in (0, 1)$, define the k-dimensional sphere

$$\Sigma(v, r) = \Big\{(t, y) \in H_v : \frac{(1-t)^2 + |y|^2}{4t} = \frac{r^2}{1-r^2}\Big\}. \tag{3.4}$$

Let $B = \{z = (x, y) \in \mathbb{R} \times \mathbb{R}^k : |z| < 1\}$ be the unit ball endowed with the hyperbolic metric. The map $S_v : B \to H_v$, defined by

$$S_v(z) = \frac{1 - x^2 - |y|^2}{(1+x)^2 + |y|^2}(v, 0) + \frac{2}{(1+x)^2 + |y|^2}(0, y), \tag{3.5}$$

is an isometry, and it transforms the spheres $\{z \in \mathbb{R}^{k+1} : |z| = r\}$, $r \in (0, 1)$, into the spheres (3.4). Thus, $\Sigma(v, r)$ is a hyperbolic sphere in H_v centered at $t = 1$ and $y = 0$.

Introduce the class of functions $u : \mathbb{R}^n \to \mathbb{R}$:

$$\mathscr{S} = \big\{u\ :\ U = T(u) \text{ is constant on each } \Sigma(v, r),\ |v| = 1,\ \text{and } r \in (0, 1)\big\}.$$

THEOREM 3.4
Let $m, k \geq 1$, and let $n = m + k$. If $u \in C^2(\mathbb{R}^n)$ is a positive solution to equation (1.1) with $u = u^*$ and $y_0 = 0$ in (2.13), then $u \in \mathscr{S}$.

Proof
Let $z = (x, y) \in \mathbb{R}^n$, $b \in \mathbb{R}^k$, and $z_b = z - (0, b)$. By (2.15) in the proof of Corollary 2.8, there exists $\lambda_b > 0$ such that

$$\begin{aligned} u(z) &= \Big(\frac{\|z_b\|}{\lambda_b}\Big)^{2-Q} u\big(\delta_{\lambda_b^2/\|z_b\|^2}(z_b) + (0, b)\big) \\ &= \Big(\frac{\|(x, y - b)\|}{\lambda_b}\Big)^{2-Q} u\Big(\frac{\lambda_b^2 x}{\|(x, y - b)\|^2}, \frac{\lambda_b^{2(\alpha+1)} y}{\|(x, y - b)\|^{2(\alpha+1)}} + b\Big). \end{aligned} \tag{3.6}$$

Moreover, by (2.16), λ_b is determined by $u(0)\lambda_b^{2-Q} = u(0, b)$, and this, by (2.13) with $y_0 = 0$, gives

$$\lambda_b = (1 + |b|^2)^{1/(2(\alpha+1))}.$$

Let $\zeta = (\xi, \eta)$, $\zeta_b = \zeta - (0, b)$, and $|\zeta_b| = (|\xi|^2 + |\eta - b|^2)^{1/2}$. By Definition 3.1 and (3.6), we have

$$U(\zeta) = |\xi|^{(Q-2)/(2(\alpha+1))}\Big(\frac{|\zeta_b|^{1/(\alpha+1)}}{\lambda_b}\Big)^{2-Q} u\Big(\frac{\lambda_b^2|\xi|^{1/(\alpha+1)}}{|\zeta_b|^{2/(\alpha+1)}}\frac{\xi}{|\xi|}, \frac{\lambda_b^{2(\alpha+1)}}{|\zeta_b|^2}(\eta - b) + b\Big),$$

and using $u(x, y) = |x|^{1-(Q/2)}U(|x|^\alpha x, y)$, we finally get

$$U(\zeta) = U\Big(\frac{(1 + |b|^2)\xi}{|\zeta_b|^2}, \frac{(1 + |b|^2)(\eta - b)}{|\zeta_b|^2} + b\Big), \quad b \in \mathbb{R}^k. \tag{3.7}$$

In order to prove the theorem, it suffices to choose $m = 1$ and to consider the case $\xi > 0$. Let H and B be the hyperbolic half-space and ball, respectively. The map $I_b : \mathbb{R}^n \setminus \{(0, b)\} \to \mathbb{R}^n \setminus \{(0, b)\}$, given by

$$I_b(\xi, \eta) = \Big(\frac{(1 + |b|^2)\xi}{|\zeta_b|^2}, \frac{(1 + |b|^2)(\eta - b)}{|\zeta_b|^2} + b\Big),$$

is a spherical inversion with respect to the sphere

$$\Sigma_b = \big\{(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^k : \xi^2 + |\eta - b|^2 = 1 + |b|^2\big\}.$$

Clearly, $(1, 0) \in \Sigma_b$ for any $b \in \mathbb{R}^k$. Let $\Sigma_b^+ = \Sigma_b \cap \{\xi > 0\}$. Since the choice of b is arbitrary, the function U is symmetric with respect to any reflection fixing the point $(1, 0)$. This clearly means that U is radial.

To realize more concretely this fact, consider the map $S : B \to H$ defined in (3.5) with $v \in \mathbb{R}$, $v = 1$. S takes the plane $\pi_b = \{(x, y) \in B : x + \langle b, y\rangle = 0\}$ onto the half-sphere Σ_b^+, and $S(0) = (1, 0)$. By [R, Th. 4.3.7], the points $S(\zeta)$ and $S(I_b(\zeta))$ in B are symmetric with respect to the plane π_b for any $\zeta \in H$. Therefore, by (3.7), the function $U_B : B \to \mathbb{R}$ defined by $U_B(x, y) = U(S(x, y))$ is symmetric with respect to the plane π_b. Since $b \in \mathbb{R}^k$ is arbitrary, the function U_B is radial about the origin. Now the claim follows from the fact that S transforms the spheres $\{(x, y) \in B : x^2 + |y|^2 = r^2\}$, $r \in (0, 1)$, into the spheres (3.4). $\square$

Again, let $m = 1$ and $n = k + 1$. Theorem 3.4 has the following corollary, which is actually a hyperbolic symmetry result. Consider the class of functions

$$\mathscr{A} = \big\{U \in C^2(H) : u(x, y) = |x|^{(2-Q)/2} U(|x|^{\alpha+1}, y) \in C^2(\mathbb{R}^n)\big\}.$$

COROLLARY 3.5
If $U \in \mathscr{A}$ is a positive solution to equation (3.2), then U is a radial function about some point in H for the hyperbolic metric.

The condition $u \in C^2(\mathbb{R}^n)$ prescribes a suitable vanishing behavior of U on ∂H (i.e., at infinity). It is not clear whether this condition is precise. However, any attempt to directly prove Corollary 3.5 without requiring any similar condition must face the difficult task of applying a Hopf lemma at boundary points that could be, in principle, at infinity. In our case, this tool is provided by Lemma 2.6.

The following theorem shows how to reduce equation (1.1) to a lower-dimensional equation.

THEOREM 3.6
Let $m, k \geq 1$, and let $n = m + k$. If $u \in C^2(\mathbb{R}^n)$ is a positive solution to equation (1.1) with $u = u^*$ and $y_0 = 0$ in (2.13), then the function $v(x) = u(x, 0)$, $x \in \mathbb{R}^m$, is a solution of the problem

$$\begin{cases} \operatorname{div}_x(p\nabla_x v) - qv = -pv^{2^*-1}, & |x| < 1, \\ v > 0, & |x| \leq 1, \\ \frac{\partial v}{\partial \nu} + \big(\frac{Q}{2} - 1\big)v = 0, & |x| = 1, \end{cases} \tag{3.8}$$

where $p(x) = (1 - |x|^{2(\alpha+1)})^k$ and $q(x) = k(\alpha + 1)(Q - 2)(1 - |x|^{2(\alpha+1)})^{k-1}|x|^{2\alpha}$.

Proof
Let $x \in \mathbb{R}^m$ be a point such that $0 < |x| < 1$. By Theorem 3.4, the function $y \mapsto u(x, y)$ is radial, and therefore $\nabla_y u(x, 0) = 0$. Then for any $i = 1, \ldots, k$,

$$\frac{\partial^2 u}{\partial y_i^2}(x, 0) = \lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2}\big(u(x, \varepsilon e_i) - u(x, 0)\big),$$

where $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^k$ with 1 in the ith coordinate.

Let $U = T(u)$, and let $\xi = |x|^\alpha x$. By Theorem 3.4, for any $\varepsilon > 0$, there is a unique point $\xi_\varepsilon \in \mathbb{R}^m$ of the form $\xi_\varepsilon = t\xi$ with $t \in (0, 1)$ and such that $U(\xi, \varepsilon e_i) = U(\xi_\varepsilon, 0)$.

By (3.4), ξ_ε is determined by the condition

$$\frac{(1-|\xi_\varepsilon|)^2}{|\xi_\varepsilon|} = \frac{(1-|\xi|)^2+\varepsilon^2}{|\xi|},$$

which gives

$$|\xi_\varepsilon| = \frac{1}{2|\xi|}\big(1+|\xi|^2+\varepsilon^2-\sqrt{(1+|\xi|^2+\varepsilon^2)^2-4|\xi|^2}\big).$$

Letting $\varphi(\varepsilon) = |\xi_{\sqrt{\varepsilon}}|$, we get $\varphi(0) = |\xi|$ and $\varphi'(0) = |\xi|/(|\xi|^2-1)$. From (3.1), we find

$$u(x,\varepsilon e_i) = \frac{1}{|\xi|^\beta}U(\xi,\varepsilon e_i) = \frac{1}{|\xi|^\beta}U(\xi_\varepsilon,0) = \Big(\frac{|\xi_\varepsilon|}{|\xi|}\Big)^\beta u\Big(|\xi_\varepsilon|^{1/(\alpha+1)}\frac{x}{|x|},0\Big).$$

Therefore

$$\begin{aligned}\frac{\partial^2 u}{\partial y_i^2}(x,0) &= \lim_{\varepsilon\to 0}\frac{2}{\varepsilon}\bigg(\frac{\varphi(\varepsilon)^\beta}{|\xi|^\beta}u\Big(\varphi(\varepsilon)^{1/(\alpha+1)}\frac{x}{|x|},0\Big)-u(x,0)\bigg)\\ &= \frac{2}{|\xi|^\beta}\frac{d}{d\varepsilon}\bigg(\varphi(\varepsilon)^\beta u\Big(\varphi(\varepsilon)^{1/(\alpha+1)}\frac{x}{|x|},0\Big)\bigg)\bigg|_{\varepsilon=0}\\ &= -\frac{1}{(\alpha+1)(1-|x|^{2(\alpha+1)})}\big((Q-2)u(x,0)+2\langle\nabla_x u(x,0),x\rangle\big).\end{aligned}$$

The left-hand side is a continuous function on $|x|\le 1$, and thus it must be

$$(Q-2)u(x,0)+2\langle\nabla_x u(x,0),x\rangle = 0 \quad \text{for } |x| = 1.$$

Moreover,

$$\mathscr{L}u(x,0) = \Delta_x u(x,0) - \frac{k(\alpha+1)|x|^{2\alpha}}{1-|x|^{2(\alpha+1)}}\big((Q-2)u(x,0)+2\langle\nabla_x u(x,0),x\rangle\big).$$

Multiplying the equation $\mathscr{L}u(x,0) = -u(x,0)^{2^*-1}$ by $p(x) = (1-|x|^{2(\alpha+1)})^k$ and letting $q(x) = k(\alpha+1)(Q-2)(1-|x|^{2(\alpha+1)})^{k-1}|x|^{2\alpha}$, we finally get

$$\operatorname{div}_x\big(p(x)\nabla_x u(x,0)\big) - q(x)u(x,0) = -p(x)u(x,0)^{2^*-1}.$$

The proof of Theorem 3.6 is concluded. □

4. Uniqueness of solutions

In this section we prove the uniqueness theorems (Ths. 5, 7). By Theorem 3.6, after a scaling and a vertical translation, the problem of uniqueness for (1.1) is reduced to the problem of uniqueness for solutions to (1.8). In the case where $m = k = 1$, the function $u = u(x)$ with $x \in (-1, 1)$ solves problem (1.9). In the case where $m > 1$, for a radial function $u = u(|x|) = u(r)$, $r \in (0, 1)$, the partial differential equation in (1.8) becomes an ordinary equation. Adopting the notation

$$\begin{gathered} \langle r \rangle = 1 - |r|^{2(\alpha+1)}, \qquad s(r) = r^{m-1}, \\ p(r) = \langle r \rangle^k, \qquad q(r) = k(\alpha+1)(Q-2)\langle r \rangle^{k-1} r^{2\alpha}, \end{gathered} \tag{4.1}$$

this ordinary equation takes the form

$$(psu')' - qsu + psu^{2^*-1} = 0, \quad r \in (0, 1). \tag{4.2}$$

In order to study our uniqueness problem, we discuss a functional change suitable for the energy method introduced by Kwong and Li in [KL]. The discussion of the functional change is presented for the radial case $m > 1$, but with straightforward adaptations, it also works for $m = 1$, where no radiality is assumed (but the relevant interval is $(-1, 1)$ instead of $(0, 1)$).

Consider the auxiliary functions

$$h(r) = \big(s(r)p(r)\big)^{\vartheta}, \quad \vartheta = \frac{1}{2}\frac{Q-2}{Q-1}, \tag{4.3}$$

and

$$G(r) = h(r)^{2^*-2}(c_1 r^{2\alpha}\langle r \rangle^{-2} + c_2 r^{-2}), \tag{4.4}$$

where c_1 and c_2 are constants depending on m, k, α, and precisely

$$\begin{aligned} c_1 &= 2\vartheta k(\alpha+1)\left\{\frac{Q(m-1)}{Q-1} - Q + 2(\alpha+1)\right\}, \\ c_2 &= \vartheta(m-1)\left\{1 - \frac{(m-1)Q}{2(Q-1)}\right\}. \end{aligned} \tag{4.5}$$

Associate to the function u the function z by letting

$$z(r) = h(r)u(r), \tag{4.6}$$

and introduce the energy

$$E\big(z(r)\big) = h(r)^{2^*-2}z'(r)^2 + \frac{2}{2^*}z(r)^{2^*} + G(r)z(r)^2. \tag{4.7}$$

The reason for introducing z, G, and E is described by the following.

PROPOSITION 4.1
If the function u solves equation (4.2), then the function $z = hu$ solves the equation

$$\frac{d}{dr} E\big(z(r)\big) = G'(r) z(r)^2, \quad r \in (0,1). \tag{4.8}$$

Proof
The function z satisfies the equation

$$h^{1/\vartheta-1} z'' + \Big(\frac{1}{\vartheta} - 2\Big) h^{1/\vartheta-2} h' z' + Fz + h^{1/\vartheta+1-2^*} z^{2^*-1} = 0,$$

where

$$F = \Big(2 - \frac{1}{\vartheta}\Big) h^{1/\vartheta-3} h'^2 - h^{1/\vartheta-2} h'' - qsh^{-1}.$$

Multiplying the equation by $2h^{2^*-1-1/\vartheta} z'$, we get

$$\big(h^{2^*-2}(z')^2\big)' + (z^2)' G + \frac{2}{2^*}(z^{2^*})' = 0,$$

where

$$G = h^{2^*-2}\Big[\Big(2 - \frac{1}{\vartheta}\Big)\Big(\frac{h'}{h}\Big)^2 - \frac{h''}{h} - \frac{qs}{h^{1/\vartheta}}\Big].$$

Using (4.3), it can be checked by a rather long computation (we omit the details here) that G is the function in (4.4) with c_1 and c_2, as in (4.5). □

Equation (4.8) proves to be useful in comparing solutions of equation (4.2). Now we are going to discuss the sign of the function G' in the case where $k = 1$.

If $m = k = 1$, we have $Q = \alpha + 2$ and $2^* = 2Q/(Q-2) = 2(\alpha+2)/\alpha$. Problem (1.8) becomes problem (1.9) with

$$p(x) = (1 - |x|^{2(\alpha+1)}) \qquad \text{and} \qquad q(x) = \alpha(\alpha+1)|x|^{2\alpha}, \quad |x| < 1. \tag{4.9}$$

Letting $m = 1$ and $k = 1$ in (4.4)–(4.6), we find

$$\begin{gathered} z(x) = p(x)^{\alpha/(2(\alpha+1))} u(x), \qquad G(x) = \frac{\alpha^2 |x|^{2\alpha}}{(1 - |x|^{2(\alpha+1)})^{2\alpha/(\alpha+1)}}, \\ E\big(z(x)\big) = p(x)^{2/(\alpha+1)} z'(x)^2 + \frac{2}{2^*} z(x)^{2^*} + G(x) z(x)^2, \quad |x| < 1. \end{gathered} \tag{4.10}$$

Clearly, $G' > 0$ on $(0,1)$ because G is increasing.

In the case where $m > 1$ and $k = 1$, (4.4) becomes

$$G(r) = c_1 r^{2\alpha+2(m-1)/(Q-1)} \langle r\rangle^{2/(Q-1)-2} + c_2 r^{2(m-1)/(Q-1)-2} \langle r\rangle^{2/(Q-1)} = G_1(r) + G_2(r),$$

where c_1 and c_2 are as in (4.5) with $k = 1$. If $m \geq 3$, then both G_1 and G_2 are increasing. Indeed, for all $m \geq 1$ and $k = 1$, the constant c_1 and $2 - 2/(Q-1)$ are both positive. Therefore the function G_1 is increasing with $G_1(0) = 0$, and $G_1(1) = +\infty$. The function G_2 is identically zero if $m = 1$ because in this case, $c_2 = 0$. The exponent $2(m-1)/(Q-1) - 2$ is negative for all $m \geq 1$. The constant c_2 is positive as soon as $m \geq 3$. Then if $m \geq 3$, the function G_2 is increasing on $(0, 1)$, and $G_2(0) = -\infty$, $G_2(1) = 0$.

Thus, for $m \geq 3$ and $k = 1$, G is increasing on $(0, 1)$, $G(0) = -\infty$, and $G(1) = +\infty$. The singularities at 0 and 1 are carefully examined in the following.

Remark 4.2
For $m = 2$ and $k = 1$, we have $c_2 < 0$. The function G fails to enjoy the properties needed in the uniqueness argument.

We need the following lemma.

LEMMA 4.3
Let $u, v \in C^2(0,1) \cap C^1([0,1])$ be positive solutions of

$$\begin{cases} (psu')' - qsu + psu^{2^*-1} = 0, \\ u'(1) + \frac{Q-2}{2} u(1) = 0, \end{cases} \tag{4.11}$$

where p, q, s are as in (4.1) and $k = 1$. Letting $\eta = u(1)/v(1)$, we have $u(x) - \eta v(x) = O(1-x)^2$ and $u'(x) - \eta v'(x) = O(1-x)$ as $x \to 1$.

Proof
The Wronskian $w = uv' - vu'$ satisfies $(psw)' = (vu^{2^*-1} - uv^{2^*-1})ps$. Since $(psw)(1) = 0$, we have

$$w(x) = \frac{1}{p(x)s(x)} \int_x^1 (v^{2^*-2} - u^{2^*-2}) psuv \, dt. \tag{4.12}$$

The functions u and v are in $C^1([0,1])$ and $v > 0$ on $[0, 1]$. Then $u/v \in C^1([0,1])$, and

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} = -\frac{w}{v^2}.$$

By the mean-value theorem, there exists $\xi \in (x, 1)$ such that

$$\eta - \frac{u(x)}{v(x)} = \frac{u(1)}{v(1)} - \frac{u(x)}{v(x)} = \Big(\frac{u}{v}\Big)'(\xi)(1-x) = -\frac{w(\xi)}{v(\xi)^2}(1-x).$$

Now since $p(x) = O(1-x)$, as $x \to 1$, (4.12) and l'Hôpital's rule give $w(\xi) = (1-\xi)(\gamma_1 + o(1))$ for $\xi \to 1$ and for some constant $\gamma_1 \in \mathbb{R}$. Then $u(x) - \eta v(x) = (1-x)^2(\gamma_2 + o(1))$ for $x \to 1$ and for some new constant $\gamma_2 \in \mathbb{R}$.

Integrating (4.11) for u and v on $(x, 1)$ yields

$$u'(x) - \eta v'(x) = \frac{1}{(ps)(x)} \int_x^1 [-(u-\eta v)qs + (u^{2^*-1} - \eta v^{2^*-1})ps]\,dt.$$

Using $s(x) \to 1$, $p(x)/(1-x) \to 2\alpha + 2$, and $u(x) - \eta v(x) = O(1-x)^2$ as $x \to 1$, l'Hôpital's rule gives

$$\lim_{x\to 1} \frac{u'(x) - \eta v'(x)}{1-x} = \frac{1}{2}\big(u(1)^{2^*-1} - \eta v(1)^{2^*-1}\big).$$

This finishes the proof of Lemma 4.3. □

Now we show that for $m = k = 1$, all solutions of (1.9) are even functions and satisfy $u'(0) = 0$. In the higher-dimensional case, this is a consequence of the radiality assumption.

THEOREM 4.4
If $u \in C^2(-1, 1) \cap C^1([-1, 1])$ *solves problem (1.9), then* $u'(0) = 0$.

Proof
Assume by contradiction that $u'(0) < 0$. The function $v(x) = u(-x)$ is a new solution to problem (1.9) because p and q in (4.9) are even functions. Clearly, $v(0) = u(0)$ and $v'(0) > 0$.

We claim that $u(x) < v(x)$ for all $x \in (0, 1]$. Indeed, letting

$$r(x) = \frac{u'(x)}{u(x)} \qquad \text{and} \qquad R(x) = \frac{v'(x)}{v(x)}, \tag{4.13}$$

we have $r(0) < R(0)$. Assume by contradiction that there exists a point $\xi \in (0, 1]$ such that $u(\xi) = v(\xi)$, and let ξ be the smallest point. It cannot be $\xi = 1$ because the Cauchy problem with data at 1 has a unique solution. (This fact, although not completely trivial because of the singular terms, can be proved by the contraction principle.) It must be $u'(\xi) > v'(\xi)$ because if $u'(\xi) = v'(\xi)$, then $u \equiv v$ by uniqueness for the Cauchy problem with data at ξ. Therefore $r(\xi) > R(\xi)$. Then, by continuity, there exists a point $b \in (0, \xi)$ such that $r(b) = R(b)$. Let b be the smallest point. Then

$r(x) < R(x)$ for $x \in (0, b)$; that is, $u'/u < v'/v$ on the same interval. This condition is equivalent to $(u/v)' < 0$ on $(0, b)$, and thus the function u/v is strictly decreasing on this interval. Let

$$\omega = \frac{u(b)}{v(b)}. \tag{4.14}$$

Since $u(0) = v(0)$, we have $\omega \in (0, 1)$.

According to (4.10), define the functions $z = p^{\alpha/(2(\alpha+1))}u$ and $\zeta = p^{\alpha/(2(\alpha+1))}v$. Then $\omega = z(b)/\zeta(b)$. Moreover, since $r(b) = R(b)$, we also have $u'(b)/v'(b) = \omega$. Thus

$$\frac{z'(b)}{\zeta'(b)} = \frac{(p(b)^{\alpha/(2(\alpha+1))})'\, u(b) + p(b)^{\alpha/(2(\alpha+1))}u'(b)}{(p(b)^{\alpha/(2(\alpha+1))})'\, v(b) + p(b)^{\alpha/(2(\alpha+1))}v'(b)} = \omega. \tag{4.15}$$

Let $E(z)$ be the energy associated with z, as in (4.10). Integrating (4.8) over $(0, b)$ and using $G(0) = 0$, $p(0) = 1$, and $p'(0) = 0$ (this ensures $\zeta(0) = u(0)$ and $z'(0) = u'(0)$), we get

$$p(b)^{2/(\alpha+1)}\big(z'(b)\big)^2 + \frac{2}{2^*}z(b)^{2^*} + G(b)z(b)^2 = u'(0)^2 + \frac{2}{2^*}u(0)^{2^*} + \int_0^b G'(x)z(x)^2\,dx. \tag{4.16}$$

Recall now that $u(0) = v(0)$ and that $u'(0)^2 = v'(0)^2$ by definition of v. Subtract from (4.16) the same identity for ζ multiplied by ω^2, obtaining

$$\frac{2}{2^*}(1-\omega^{2-2^*})z(b)^{2^*} = (1-\omega^2)\Big(u'(0)^2 + \frac{2}{2^*}u(0)^{2^*}\Big) + \int_0^b G'(x)\big(z(x)^2 - \omega^2\zeta(x)^2\big)\,dx.$$

This is a contradiction. Indeed, the right-hand side is strictly positive because $G' > 0$ on $(0, 1)$, $z^2 - \omega^2\zeta^2 > 0$ on $(0, b)$, and $\omega \in (0, 1)$. On the other hand, $2^* > 2$, and therefore the left-hand side is strictly negative. We have proved that $u < v$ on $(0, 1]$.

Next we show that $r < R$ on the entire interval $(0, 1)$. Let $w = uv' - vu'$, and note that $R - r = w/uv$. We have to show that $w > 0$ on $(0, 1)$. Since u and v satisfy (1.9), we have $(pw)' = (u^{2^*-2} - v^{2^*-2})puv$. Integrating over $(x, 1)$ and using $p(1) = 0$, for $x \in (0, 1)$, we get

$$w(x) = \frac{1}{p(x)}\int_x^1 (v^{2^*-2} - u^{2^*-2})puv\,dt > 0,$$

and hence $r < R$ on $(0, 1)$.

Then the function u/v is strictly decreasing on $(0, 1)$. Let

$$\eta = \frac{u(1)}{v(1)} = \frac{u'(1)}{v'(1)}.$$

The last equality follows from the boundary conditions. By (4.10) and (4.8), for any $x \in (0, 1)$, we have

$$E\big(z(x)\big)-\eta^2 E\big(\zeta(x)\big) = E\big(z(0)\big)-\eta^2 E\big(\zeta(0)\big)+\int_0^x G'(t)\big(z(t)^2-\eta^2\zeta(t)^2\big)\,dt. \quad (4.17)$$

We are going to let $x \to 1$ in this identity. There is some singular term to estimate.

First, we show that the left-hand side tends to zero. Recalling (4.10), compute

$$\lim_{x\to 1} G(x)\big(z(x)^2 - \eta^2\zeta(x)^2\big) = \alpha^2 \lim_{x\to 1}(1 - x^{2(\alpha+1)})^{-\alpha/(\alpha+1)}\big(u(x)^2 - \eta^2 v(x)^2\big) = 0$$

because $p(x) = O(1 - x)$, $u(x) - \eta v(x) = O(1 - x)^2$ by Lemma 4.3, and $2 - \alpha/(\alpha + 1) > 0$. Note also that $z(1) = \zeta(1) = 0$, which gives $(2/2^*)(z(1)^{2^*} - \eta^2\zeta(1)^{2^*}) = 0$. Moreover, it is also

$$\lim_{x\to 1} p(x)^{2/(\alpha+1)}\big(z'(x)^2 - \eta^2\zeta'(x)^2\big) = 0. \quad (4.18)$$

In order to prove (4.18), write $z' = h'u + hu'$, where $h(x) = p(x)^{\alpha/(2(\alpha+1))}$, so that $h = O(1-x)^{\alpha/(2(\alpha+1))}$ and $h' = O(1-x)^{-(\alpha+2)/(2(\alpha+1))}$, as $x \to 1$. The same computation can be done for ζ. Inserting these estimates in the left-hand side of (4.18) along with Lemma 4.3 proves the claim. Thus we have shown that $\lim_{x\to 1} E(z(x))-\eta^2 E(\zeta(x)) = 0$.

Now we analyze the right-hand side in (4.17). Again, write $z' = (p^{\alpha/(2(\alpha+1))})'u + p^{\alpha/(2(\alpha+1))}u'$. We have $z'(0) = u'(0)$ and $\zeta'(0) = v'(0) = -u'(0)$, by $v(x) = u(-x)$. The energies for z and ζ are equal at 0; $E(z(0)) = E(\zeta(0)) = (2/2^*)u(0)^{2^*} + u'(0)^2$. Thus, letting $x \to 1$ in (4.17) yields

$$0 = (1 - \eta^2)\Big(u'(0)^2 + \frac{2}{2^*}u(0)^{2^*}\Big) + \int_0^1 G'(x)\big(z(x)^2 - \eta^2\zeta(x)^2\big)\,dx. \quad (4.19)$$

This is a contradiction because the right-hand side is strictly positive. Indeed, $\eta \in (0, 1)$ and $z/\zeta = u/v > \eta$ on $(0, 1)$. The proof of Theorem 4.4 is concluded. □

Thanks to Theorem 4.4, the proof of Theorem 5 or, equivalently, the uniqueness for problem (1.9), is reduced to the uniqueness for the problem

$$\begin{cases} (pu')' - qu + pu^{2^*-1} = 0 & \text{in } (0, 1),\\ u > 0 & \text{in } [0, 1],\\ u'(0) = 0, & \\ \alpha u(1) + 2u'(1) = 0. & \end{cases} \quad (4.20)$$

Thus the proofs of both Theorems 5 and 7 are a consequence of the following.

THEOREM 4.5
Let $m \neq 2$, and let $k = 1$. There exists a unique solution $u \in C^2(0,1) \cap C^1([0,1])$ to the problem

$$\begin{cases} (psu')' - qsu + psu^{2^*-1} = 0 & \text{in } (0,1), \\ u > 0 & \text{in } [0,1], \\ u'(0) = 0, \\ u'(1) + \left(\frac{Q}{2} - 1\right)u(1) = 0. \end{cases} \tag{4.21}$$

Proof
Assume that there exist two different solutions u, v of problem (4.21). By the uniqueness for the Cauchy problem for our equation with data at 0 or at 1, we can assume that $u(0) \neq v(0)$ and that $u(1) \neq v(1)$. We show that this assumption gives a contradiction. The argument of the proof is based on a Sturm-type comparison of u and v.

First, we prove that u and v intersect at least once in $(0,1)$. Assume by contradiction that the solutions u and v satisfy $u < v$ on $(0,1)$. Here, we use only superlinearity. The Wronskian $w = uv' - vu'$ satisfies the equation

$$(psw)' = (u^{2^*-2} - v^{2^*-2})psuv. \tag{4.22}$$

Integrating (4.22) over $(0,1)$ and using $(psw)(0) = (psw)(1) = 0$, we get $\int_0^1 (v^{2^*-2} - u^{2^*-2})psuv\,dx = 0$. But this is not possible because $u < v$ on $(0,1)$.

Next, we prove that the solutions u and v must intersect at least twice in $(0,1)$. This is the most delicate part of the proof, and it involves the fact that the function G is increasing. Assume by contradiction that u and v intersect only once in $(0,1)$. For example, assume that $u(b) = v(b)$, $u < v$ on $(0,b)$, and $u > v$ on $(b,1)$ for some $b \in (0,1)$. The function u/v is strictly increasing on $(0,1)$. Indeed, since $(u/v)' = -w/v^2$, in order to prove this statement, we show that $w(x) < 0$ for all $x \in (0,1)$. Take $x \in (0,1)$. If $x \leq b$, then integrating (4.22) yields

$$(psw)(x) - (psw)(0) = (psw)(x) = \int_0^x (u^{2^*-2} - v^{2^*-2})psuv\,dt < 0,$$

and thus $w(x) < 0$ because $u < v$ on $(0,b)$. If $x > b$, integrate (4.22) on $(x,1)$, and use the fact that $u > v$ on $(b,1)$. We have proved that u/v is increasing.

Now, let $\eta = u(1)/v(1)$, and recall that G is increasing on $(0,1)$. Let $z = s^\vartheta p^\vartheta u$, and let $\zeta = s^\vartheta p^\vartheta v$, where $\vartheta = (Q-2)/(2(Q-1))$, as in (4.3) and (4.6). Fix $0 < \varepsilon < \sigma < 1$, and integrate (4.8) on (ε,σ) to get

$$E\big(z(\sigma)\big) - \eta^2 E\big(\zeta(\sigma)\big) = E\big(z(\varepsilon)\big) - \eta^2 E\big(\zeta(\varepsilon)\big) + \int_\varepsilon^\sigma G'(t)[z(t)^2 - \eta^2\zeta(t)^2]\,dt. \tag{4.23}$$

We now prove the following.

CLAIM 1
We have

$$\lim_{\varepsilon\to 0} E\big(z(\varepsilon)\big) - \eta^2 E\big(\zeta(\varepsilon)\big) \le 0.$$

CLAIM 2
We have

$$\lim_{\sigma\to 1} E\big(z(\sigma)\big) - \eta^2 E\big(\zeta(\sigma)\big) = 0.$$

As soon as the claims are proved, letting $\varepsilon \to 0$ and $\sigma \to 1$ in (4.23) gives a contradiction, and we may conclude, as desired, that u and v intersect at least twice. Indeed, the integral in the right-hand side of (4.23) is negative because $z(x)-\eta\zeta(x) < 0$ and $G'(x) > 0$ for all $x \in (0, 1)$.

Proof of Claim 1
Write (4.7) explicitly for $k = 1$. A short computation gives

$$E\big(z(x)\big) = x^{2(m-1)/(Q-1)} p(x)^{2/(Q-1)} \big(z'(x)\big)^2 + \frac{2}{2^*} z(x)^{2^*} + G(x) z(x)^2. \tag{4.24}$$

Now by (4.4) for $k = 1$, we have

$$G(x) = p(x)^{2/(Q-1)} x^{2(m-1)/(Q-1)} \big(c_1 x^{2\alpha} p(x)^{-2} + c_2 x^{-2}\big) \tag{4.25}$$

with c_1 and c_2 as in (4.5). Therefore, since $c_2 = 0$ if $m = 1$, we have, as $\varepsilon \to 0$,

$$G(\varepsilon) = \begin{cases} O(\varepsilon^{2\alpha}) & \text{if } m = 1, \\ O(\varepsilon^{2(m-1)/(Q-1)-2}) & \text{if } m \ge 2. \end{cases}$$

Moreover, since $z(\varepsilon) = u(\varepsilon)\varepsilon^{(m-1)(Q-2)/(2(Q-1))} p(\varepsilon)^{(Q-2)/(2(Q-1))}$, we have $z'(\varepsilon) = k_1(\varepsilon)u(\varepsilon) + k_2(\varepsilon)u'(\varepsilon)$ for functions k_1 and k_2 such that for $\varepsilon \to 0$,

$$k_1 = \begin{cases} O(\varepsilon^{2\alpha+1}) & \text{if } m = 1, \\ O(\varepsilon^{(m-1)(Q-2)/(2(Q-1))-1}) & \text{if } m \ge 2, \end{cases}$$

and

$$k_2 = O(\varepsilon^{(m-1)(Q-2)/(2(Q-1))}).$$

We used $p'(\varepsilon) = O(\varepsilon^{2\alpha+1})$.

We prove Claim 1 for $m \ge 3$. In this case, inserting into (4.24) the asymptotic behavior of the terms appearing in it,

$$\begin{aligned} E\big(z(\varepsilon)\big) = \varepsilon^{2(m-1)/(Q-1)}\Big(&O(\varepsilon^{(m-1)(Q-2)/(2(Q-1))-1})u(\varepsilon) \\ &+ O(\varepsilon^{(m-1)(Q-2)/(2(Q-1))})u'(\varepsilon)\Big)^2 \\ &+ \frac{2}{2^*}O(\varepsilon^{(m-1)(Q-2)/(2(Q-1))2^*})u(\varepsilon)^{2^*} \\ &+ O(\varepsilon^{2(m-1)/(Q-1)-2})O(\varepsilon^{(m-1)(Q-2)/(Q-1)})u(\varepsilon)^2 \\ = \varepsilon^{Q(m-1)/(Q-1)-2}O(1) \to 0 \end{aligned}$$

as $\varepsilon \to 0$ because $Q(m-1)/(Q-1)-2 > 0$ if $m \geq 3$. The same holds for $E(\zeta(\varepsilon))$.

We prove Claim 1 for $m = 1$. In this case,

$$\begin{aligned} E\big(z(\varepsilon)\big) &= \big(O(\varepsilon^{1+2\alpha})u(\varepsilon) + O(1)u'(\varepsilon)\big)^2 + \frac{2}{2^*}p(\varepsilon)^{2^*(Q-2)/(2(Q-1))}u(\varepsilon)^{2^*} \\ &\quad + O(\varepsilon^{2\alpha})u(\varepsilon)^2 \to \frac{2}{2^*}u(0)^{2^*}, \end{aligned}$$

as $\varepsilon \to 0$ (recall that $u'(0) = 0$). Therefore, if $m = 1$, we have

$$\lim_{\varepsilon\to 0} E\big(z(\varepsilon)\big) - \eta^2 E\big(\zeta(\varepsilon)\big) = \frac{2}{2^*}[u(0)^{2^*} - \eta^2 v(0)^{2^*}] \leq 0$$

because $u(0) \leq v(0)$, and then $u(0)^{2^*}/v(0)^{2^*} \leq (u(0)/v(0))^2 \leq (u(1)/v(1))^2 = \eta^2$. This finishes the proof of Claim 1. □

Proof of Claim 2
By (4.24),

$$\begin{aligned} &E\big(z(x)\big) - \eta^2 E\big(\zeta(x)\big) \\ &\quad = x^{2(m-1)/(Q-1)}p(x)^{2/(Q-1)}[z'(x)^2 - \eta^2\zeta'(x)^2] + \frac{2}{2^*}[z(x)^{2^*} - \eta^2\zeta(x)^{2^*}] \\ &\qquad + G(x)[z(x)^2 - \eta^2\zeta(x)^2] := A_1 + A_2 + A_3. \end{aligned}$$

We first show that $A_3 \to 0$ as $x \to 1$. By (4.25), $G(x) = O(1)p(x)^{2/(Q-1)-2}$ as $x \to 1$. Thus

$$\begin{aligned} G(x)[z(x)^2 - \eta^2\zeta(x)^2] &= O(1-x)^{2/(Q-1)-2}[z(x) - \eta\zeta(x)][z(x) + \eta\zeta(x)] \\ &= O(1-x)^{-(Q-2)/(Q-1)}[u(x) - \eta v(x)][u(x) + \eta v(x)] \end{aligned}$$

because $z(x) = x^{(m-1)(Q-2)/(2(Q-1))}p(x)^{(Q-2)/(2(Q-1))}u(x) = O(1-x)^{(Q-2)/(2(Q-1))}u(x)$, and the same is true for ζ. Then the last line tends to zero as $x \to 1$ because $u(x) - \eta v(x) = O(1-x)^2$ by Lemma 4.3 and $2 - (Q-2)/(Q-1) > 0$.

The proof that A_2 tends to zero is trivial because both $z(x)$ and $\zeta(x)$ tend to zero.

In order to show that $\lim_{x\to 1-} A_1 = 0$, a more careful analysis is needed. Indeed, we have, as $x \to 1$,

$$A_1 = O(1-x)^{2/(Q-1)}[z'(x) - \eta\zeta'(x)][z'(x) + \eta\zeta'(x)].$$

Since $z(x) = x^{(m-1)(Q-2)/(2(Q-1))} p(x)^{(Q-2)/(2(Q-1))} u(x)$, we have $z'(x) = k_1(x)u(x) + k_2(x)u'(x)$ with $k_1(x) = O(1-x)^{(Q-2)/(2(Q-1))-1}$ and $k_2(x) = O(1-x)^{(Q-2)/(2(Q-1))}$ as $x \to 1$. Therefore

$$\begin{aligned} z'(x) - \eta\zeta'(x) = O(1-x)^{(Q-2)/(2(Q-1))-1}[u(x) - \eta v(x)] \\ + O(1-x)^{(Q-2)/(2(Q-1))}[u'(x) - \eta v'(x)], \end{aligned}$$

and $z'(x) + \eta\zeta'(x) = O(1-x)^{(Q-2)/(2(Q-1))-1}$. Hence

$$A_1 = O(1-x)^{Q/(Q-1)-2}[u(x) - \eta v(x)] + O(1-x)^{Q/(Q-1)-1}[u'(x) - \eta v'(x)].$$

By Lemma 4.3, $u(x) - \eta v(x) = O(1-x)^2$ and $u'(x) - \eta v'(x) = O(1-x)$ as $x \to 1$. This finishes the proof of Claim 2. $\square$

Thus far, we have proved that two solutions of problem (4.21) must intersect at least twice in $(0, 1)$. Now an essentially similar argument provides the conclusion of the proof of the theorem. Indeed, assume by contradiction that there exist two solutions u and v such that $v > u$ on $(0, a)$, $v < u$ on (a, b), and $u(b) = v(b)$ for some $0 < a < b < 1$. Integrating equation (4.22), we get

$$(psw)(x) = \int_0^x (u^{2^*-2} - v^{2^*-2}) psuv\, dt.$$

Since $v > u$ on $(0, a)$, then $w(x) < 0$ for $x \in (0, a]$. By continuity, $w < 0$ on $(0, a+\delta)$ for some positive δ. Since $(v/u)' = w/v^2$, the function v/u is strictly decreasing on $(0, a+\delta]$. Moreover, since $(v/u)(a) = (v/u)(b) = 1$, there exists $\xi \in (a, b)$ such that $(v/u)'(\xi) = 0$ and $(v/u)' < 0$ on $(0, \xi)$. Therefore $v'(\xi)u(\xi) = v(\xi)u'(\xi)$, so that, letting $\tau = v(\xi)/u(\xi)$, we also have $\tau = v'(\xi)/u'(\xi)$.

As in (4.23), for $\varepsilon \in (0, \xi)$, we have

$$E\big(\zeta(\xi)\big) - \tau^2 E\big(z(\xi)\big) = E\big(\zeta(\varepsilon)\big) - \tau^2 E\big(z(\varepsilon)\big) + \int_\varepsilon^\xi G'(x)[\zeta(x)^2 - \tau^2 z(x)^2]\, dx. \tag{4.26}$$

As in the proof of Claim 1,

$$\lim_{\varepsilon\to 0} E\big(\zeta(\varepsilon)\big) - \tau^2 E\big(z(\varepsilon)\big) = \delta_{m,1}\frac{2}{2^*}[v(0)^{2^*} - \tau^2 u(0)^{2^*}],$$

where $\delta_{m,1}$ is the Kronecker symbol. This term is nonnegative because $v(0) > u(0)$ and $\tau \in (0, 1)$. The integral in (4.26) is also positive because $G' > 0$ and $\zeta - \tau z > 0$ on $(0, \xi)$.

Thus, letting $\varepsilon \to 0$ in the right-hand side of (4.26), we get a positive limit. This is a contradiction because the left-hand side is negative. Indeed, by $\zeta(\xi) - \tau z(\xi) = 0$, and $\zeta'(\xi) - \tau z'(\xi) = 0$, we obtain

$$E\big(\zeta(\xi)\big) - \tau^2 E\big(z(\xi)\big) = \frac{2}{2^*}[\zeta(\xi)^{2^*} - \tau^2 z(\xi)^{2^*}] = \frac{2}{2^*} z(\xi)^{2^*}\left[\frac{\zeta(\xi)^{2^*}}{z(\xi)^{2^*}} - \frac{\zeta(\xi)^2}{z(\xi)^2}\right] < 0.$$

The functions u and v cannot exist. Problem (4.21) has a unique solution. The proof of Theorem 4.5 is concluded. □

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Monti
Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni 7, I-35100 Padova, Italy; monti@math.unipd.it

Morbidelli
Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, I-40127 Bologna, Italy; morbidel@dm.unibo.it