LIPSCHITZ APPROXIMATION OF $\mathbb{H}$-PERIMETER MINIMIZING BOUNDARIES

ROBERTO MONTI

Abstract. We prove that the boundary of $\mathbb{H}$-perimeter minimizing sets in the Heisenberg group can be approximated by graphs that are intrinsic Lipschitz in the sense of [10]. The Hausdorff measure of the symmetric difference in a ball of graph and boundary is estimated by excess in a larger concentric ball. This result is motivated by a research program on the regularity of $\mathbb{H}$-perimeter minimizing sets.

1. Introduction

The Heisenberg group is the set $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \geq 1$, with group law $* : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}^n$, $(z,t) * (\zeta,\tau) = \big(z+\zeta, t+\tau+Q(z,\zeta)\big)$, where $Q : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ is the bilinear form

$$Q(z,\zeta) = 2\mathrm{Im}\Big(\sum_{j=1}^n z_j\bar{\zeta}_j\Big), \quad z,\zeta \in \mathbb{C}^n.$$

We identify the element $z = x+iy \in \mathbb{C}^n$ with the pair $(x,y) \in \mathbb{R}^{2n}$. The Lie algebra of $\mathbb{H}^n$ is generated by the *horizontal* left invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad j = 1,\dots,n. \tag{1.1}$$

These vector fields span a distribution of $2n$-dimensional planes that is called *horizontal distribution*.

In Section 2, we recall the definition of several objects depending on the horizontal distribution. The $\mathbb{H}$-*perimeter* of a measurable set in $\mathbb{H}^n$ is the total variation of its characteristic function in horizontal directions. We also introduce a distance with its balls $B_r(p)$ and $B_r = B_r(0)$, $p \in \mathbb{H}^n$ and $r > 0$, that is equivalent to the Carnot-Carathéodory distance. Using this distance one can define a family of Hausdorff measures $\mathcal{S}^s$, $s \geq 0$. In particular, the measure $\mathcal{S}^{2n+1}$ is strictly related to $\mathbb{H}$-perimeter, see (2.4). Finally, in Section 4 we recall the central notion of *intrinsic Lipschitz graph*, see Definition 4.6.

Aim of this paper is to prove the following theorem.

Date: 30-09-2012.

Theorem 1.1. *Let $L > 0$ be a constant, that is suitably large when $n = 1$. For any set $E \subset \mathbb{H}^n$ that is $\mathbb{H}$-perimeter minimizing in B_{kr} with $0 \in \partial E$ and $r > 0$, there exists an intrinsic L-Lipschitz graph $\Gamma \subset \mathbb{H}^n$ such that*

$$\mathcal{S}^{2n+1}\big((\Gamma \Delta \partial E) \cap B_r\big) \leq c\,(kr)^{2n+1}\mathrm{Exc}(E, B_{kr}), \tag{1.2}$$

where $k > 1$ is a large geometric constant, $c > 0$ is a constant depending on n and L, and Exc *denotes the horizontal excess, see Definition 3.1.*

The reader is invited to consider the precise formulation given in Theorem 5.1. When $n = 1$, the theorem holds only for Lipschitz constants L that are large enough.

Geometric measure theory in the Heisenberg group started from the pioneering work [9] and now there is a wide literature in the area. One of the most important open problems is the regularity of sets that are $\mathbb{H}$-perimeter minimizing. The issue of regularity is also relevant in the resolution of the Heisenberg isoperimetric problem. All known regularity results assume some strong apriori regularity and/or some restrictive geometric structure of the minimizer, see [7], [6], [5], and [19]. On the other hand, there are examples of minimal surfaces in $\mathbb{H}^n$ with $n = 1$ that are only Lipschitz continuous in the standard sense, see for instance [16] and [17].

The first step in the regularity theory is a good approximation of the boundary of minimizing sets. In De Giorgi's original approach, the approximation is made by convolution and the techniques rely on the monotonicity formula (see, e.g., [12]). The validity of a monotonicity formula for $\mathbb{H}$-perimeter, however, is not clear. A more flexible approximation scheme is via Lipschitz graphs (see, e.g., [18]). The approximation of area minimizing integral currents of general codimension by means of multiple valued Lipschitz functions is one of the central results of Almgren's book [1, Chapter 3]. These results have been recently improved in [8].

In this paper, we follow the Lipschitz approximation scheme. The boundary of sets with finite $\mathbb{H}$-perimeter is not rectifiable, and, in fact, it may have fractional Hausdorff dimension (see [13]). Nevertheless, the notion of intrinsic graph turns out to be effective in the approximation. The starting point is the analysis of pair of points with small horizontal excess and then we use some ideas from [2, Sections 4.3, 4.4] to prove Theorem 5.1.

The second step in the regularity theory is the harmonic approximation. Elliptic estimates imply the decay estimate for excess and this, in turn, yields the interior partial regularity of the support of the current (the regularity of the reduced boundary for perimeter minimizing sets, respectively). In the Heisenberg group, the harmonic approximation with its various adaptations cannot be easily implemented and, so far, the regularity problem remains open.

Section 2 fixes notation and definitions. In Section 3, we prove some auxiliary results on sets with zero or small horizontal excess in a ball, see in particular Propositions 3.7, 3.9, and 3.10. In Section 4, we start the construction of the intrinsic Lipschitz approximation (see Propositions 4.1 and 4.2) and we prove a covering lemma (see Theorem 4.14). Finally, in Section 5 we prove Theorem 5.1.

Acknowledgements. It is a pleasure to acknowledge with gratitude L. Ambrosio and D. Vittone for their careful reading of an early version of this paper.

2. Preliminary definitions and results

For any $p = (z,t) \in \mathbb{C}^n \times \mathbb{R} = \mathbb{H}^n$ we let $\|p\| = \max\{|z|, |t|^{1/2}\}$. The homogeneous norm $\|\cdot\|$ satisfies the triangle inequality

$$\|p * q\| \leq \|p\| + \|q\|, \quad p, q \in \mathbb{H}^n. \tag{2.1}$$

Moreover, the function $d : \mathbb{H}^n \times \mathbb{H}^n \to [0,\infty)$, $d(p,q) = \|p^{-1} * q\|$, is a left invariant distance on $\mathbb{H}^n$ equivalent to the Carnot-Carathéodory distance. Using this distance, we define the ball centered at $p \in \mathbb{H}^n$ and with radius $r > 0$

$$B_r(p) = \big\{q \in \mathbb{H}^n : d(p,q) < r\big\} = p * \big\{q \in \mathbb{H}^n : \|q\| < r\big\}. \tag{2.2}$$

In the case $p = 0$, we let $B_r = B_r(0)$.

The horizontal divergence of a vector valued function $\varphi \in C^1(\mathbb{H}^n; \mathbb{R}^{2n})$ is

$$\mathrm{div}_{\mathbb{H}}\varphi = \sum_{j=1}^n X_j\varphi_j + Y_j\varphi_{j+n}.$$

This is the standard divergence of the vector field $\sum_{j=1}^n \varphi_j X_j + \varphi_{j+n} Y_j$ with respect to the Lebesgue measure $\mathcal{L}^{2n+1}$, that is the Haar measure of $\mathbb{H}^n$. A measurable set $E \subset \mathbb{H}^n$ is of *locally finite* $\mathbb{H}$*-perimeter* in an open set $\Omega \subset \mathbb{H}^n$ if there exists a $\mathbb{R}^{2n}$-vector valued Radon measure μ_E on Ω such that

$$\int_E \mathrm{div}_{\mathbb{H}}\varphi \, dp = -\int_\Omega \langle \varphi, d\mu_E\rangle$$

for all $\varphi \in C^1_c(\Omega; \mathbb{R}^{2n})$. Here, $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^{2n}$. We denote by $|\mu_E|$ the total variation measure of μ_E. If $|\mu_E|(\Omega) < \infty$ we say that E has finite perimeter in Ω. We also use the notation

$$P(E; B) = |\mu_E|(B),$$

for any Borel set $B \subset \Omega$, to denote the $\mathbb{H}$*-perimeter* of E in B. When $B = \mathbb{H}^n$ we write $P(E) = P(E; \mathbb{H}^n)$. By Radon-Nykodim theorem (equivalently, by Riesz representation theorem), there exists a Borel function $\nu_E : \Omega \to \mathbb{R}^{2n}$ such that $\mu_E = \nu_E |\mu_E|$. Moreover, we have $|\nu_E| = 1$ $|\mu_E|$-a.e. in Ω. The function ν_E is called *measure theoretic inner normal.*

When E is a set with Lipschitz boundary ∂E, the Heisenberg perimeter of E can be represented by the following area formula

$$P(E;\Omega) = \int_{\partial E\cap\Omega} \Big(\sum_{j=1}^{n} \langle N, X_j\rangle^2 + \langle N, Y_j\rangle^2\Big)^{1/2} d\mathcal{H}^{2n}, \tag{2.3}$$

where N is the standard unit normal to ∂E, $\langle\cdot,\cdot\rangle$ is the standard scalar product of vectors of $\mathbb{R}^{2n+1}$, and $\mathcal{H}^{2n}$ is the standard $2n$-dimensional Hausdorff measure of $\mathbb{R}^{2n+1}$.

The *measure theoretic boundary* of a measurable set $E \subset \mathbb{H}^n$ is

$$\partial E = \big\{ p \in \mathbb{H}^n : |E \cap B_r(p)| > 0 \text{ and } |B_r(p) \setminus E| > 0 \text{ for all } r > 0 \big\}.$$

Here and in the sequel, we denote by $|E| = \mathcal{L}^{2n+1}(E)$ the Lebesgue measure of E.

Assume that E has locally finite $\mathbb{H}$-perimeter in $\mathbb{H}^n$. The *reduced boundary* of E is the set $\partial^* E$ of all points $p \in \mathbb{H}^n$ such that the following three conditions hold:

(1) $|\mu_E|(B_r(p)) > 0$ for all $r > 0$;
(2) We have
$$\lim_{r\to 0} \frac{1}{|\mu_E|(B_r(p))} \int_{B_r(p)} \nu_E \, d|\mu_E| = \nu_E(p);$$
(3) There holds $|\nu_E(p)| = 1$.

This definition is introduced and studied in [9].

The Heisenberg perimeter has the following representation in terms of Hausdorff measures. For any $s \geq 0$ we denote by $\mathcal{S}^s$ the spherical Hausdorff measure in $\mathbb{H}^n$ constructed with the left invariant metric d. Namely, for any set $E \subset \mathbb{H}^n$ we let

$$\mathcal{S}^s(E) = \sup_{\delta>0} \mathcal{S}^s_\delta(E)$$

where

$$\mathcal{S}^s_\delta(E) = \inf\Big\{ \sum_{i\in\mathbb{N}} (\mathrm{diam} B_i)^s : E \subset \bigcup_{i\in\mathbb{N}} B_i, B_i \text{ balls as in (2.2)}, \mathrm{diam}(B_i) < \delta \Big\},$$

and diam is the diameter in the distance d. Then there exists a constant $\delta(n) > 0$ depending on $n \in \mathbb{N}$ such that for any set $E \subset \mathbb{H}^n$ with locally finite $\mathbb{H}$-perimeter and for any Borel set $B \subset \mathbb{H}^n$ we have

$$P(E, B) = \delta(n) \mathcal{S}^{Q-1}\big(\partial^* E \cap B\big), \tag{2.4}$$

where the integer $Q = 2n + 2$ is the homogeneous and metric dimension of $\mathbb{H}^n$. Formula (2.4) is proved in [9].

Definition 2.1. Let $\Omega \subset \mathbb{H}^n$ be an open set. A set $E \subset \mathbb{H}^n$ with locally finite $\mathbb{H}$-perimeter in Ω is $\mathbb{H}$-*perimeter minimizing in* Ω if for all $p \in \mathbb{H}^n$ and $r > 0$ and for any $F \subset \mathbb{H}^n$ such that $E \Delta F \subset\subset B_r(p) \subset \Omega$ we have

$$P(E, B_r(p)) \leq P(F, B_r(p)). \tag{2.5}$$

Sets that are $\mathbb{H}$-perimeter minimizing admit lower and upper density estimates with geometric constants. The proof of the following lemma is well known in the case of the standard perimeter.

Lemma 2.2. *Let $E \subset \mathbb{H}^n$ be an $\mathbb{H}$-perimeter minimizing set in B_r for some $r > 0$. Then we have*

$$P(E, B_r) \le c_1 r^{Q-1}, \tag{2.6}$$

where $c_1 = P(B_1)$.

Proof. Let $0 < s < r$. Since the sets E and $E \setminus B_s$ agree inside $B_r \setminus \bar{B}_s$, we have

$$P(E, B_r \setminus \bar{B}_s) = P(E \setminus B_s, B_r \setminus \bar{B}_s) = P(E \setminus B_s, B_r) - P(E \setminus B_s, \bar{B}_s).$$

On the other hand, using $P(E \setminus B_s, B_s) = 0$ and (2.4) we obtain

$$\begin{aligned} P(E \setminus B_s, \bar{B}_s) &= P(E \setminus B_s, \partial B_s) = \delta(n)\mathcal{S}^{Q-1}(\partial^*(E \setminus B_s) \cap \partial B_s) \\ &\le \delta(n)\mathcal{S}^{Q-1}(\partial B_s) = P(B_s) = c_1 s^{Q-1}. \end{aligned}$$

The formula $P(B_s) = s^{Q-1}P(B_1)$ follows by an elementary homogeneity argument. Then we obtain the inequality $P(E \setminus B_s, B_r) \le P(E, B_r \setminus \bar{B}_s) + c_1 s^{Q-1}$. For E is $\mathbb{H}$-perimeter minimizing in B_r, by (2.5) we get

$$P(E, B_r) \le P(E \setminus B_s, B_r) \le P(E, B_r \setminus \bar{B}_s) + c_1 s^{Q-1},$$

and, letting $s \uparrow r$, we obtain (2.6). □

The proof of the following lemma is in [19], Proposition 2.14 (see also Theorem 2.4).

Lemma 2.3. *There exist constants $c_2, c_3 > 0$ depending on $n \ge 1$ such that for any set $E \subset \mathbb{H}^n$ that is $\mathbb{H}$-perimeter minimizing in $B_{2\varrho}$, $\varrho > 0$, we have, for all $p \in \partial E \cap B_\varrho$ and for all $0 < r < \varrho$,*

$$\min\big\{|E \cap B_r(p)|, |B_r(p) \setminus E|\big\} \ge c_2 r^Q, \tag{2.7}$$

and

$$P(E, B_r(p)) \ge c_3 r^{Q-1}. \tag{2.8}$$

We always have the inclusion $\partial^* E \subset \partial E$. This follows from the structure theorem for sets with locally finite $\mathbb{H}$-perimeter of [9]. If we also have the uniform density estimate (2.8), then the difference $\partial E \setminus \partial^* E$ is negligible.

Lemma 2.4. *Let $E \subset \mathbb{H}^n$ be an $\mathbb{H}$-perimeter minimizing set in an open set $\Omega \subset \mathbb{H}^n$. Then we have*

$$\mathcal{S}^{Q-1}\big((\partial E \setminus \partial^* E) \cap \Omega\big) = 0. \tag{2.9}$$

Proof. Let $K = (\partial E \setminus \partial^* E) \cap \Omega$, let $A \subset \Omega$ be an open set containing K, and fix $\delta > 0$. For any $p \in K$ there is an $0 < r_p < \delta/10$ such that $B_{5r_p}(p) \subset A$. Then $\{B_{r_p}(p) : p \in p \in K\}$ is a covering of K and by the 5-covering lemma, there exists a sequence $p_i \in K$, $i \in \mathbb{N}$, such that the balls $B_i = B_{r_i}(p_i)$, with $r_i = r_{p_i}$, are pairwise disjoint and

$$K \subset \bigcup_{i \in \mathbb{N}} B_{5r_i}(p_i).$$

It follows that

$$\begin{aligned}\mathcal{S}^{Q-1}_\delta(K \cap A) &\leq \sum_{i\in\mathbb{N}} \operatorname{diam}(B_{5r_i}(p_i))^{Q-1} = 10^{Q-1} \sum_{i\in\mathbb{N}} r_i^{Q-1} \\ &\leq 10^{Q-1} c_3^{-1} \sum_{i\in\mathbb{N}} P(E, B_{r_i}(p_i)) \leq 10^{Q-1} c_3^{-1} P(E, A).\end{aligned}$$

Because $\delta > 0$ is arbitrary, we deduce that $\mathcal{S}^{Q-1}(K) \leq 10^{Q-1} c_3^{-1} P(E, A)$. As A is arbitrary and, by (2.4), $P(E, K) = 0$, we conclude that $\mathcal{S}^{Q-1}(K) = 0$. □

The density estimates (2.6), (2.7), and (2.8) are the unique facts concerning $\mathbb{H}$-perimeter minimizing sets that are used in the rest of the paper.

3. Sets with locally constant measure theoretic normal

In this section, we discuss some properties of sets in $\mathbb{H}^n$ having zero or small excess in a ball. In the following, we let $m = 2n - 1$ and $\mathbb{S}^m = \{\nu \in \mathbb{R}^{2n} : |\nu| = 1\}$. With abuse of notation, we identify $\nu \in \mathbb{S}^m$ with $(\nu, 0) \in \mathbb{R}^{2n} \times \mathbb{R} = \mathbb{H}^n$.

Definition 3.1 (Excess)**.** Let $E \subset \mathbb{H}^n$ be a set of locally finite $\mathbb{H}$-perimeter, $p \in \mathbb{H}^n$, and $r > 0$. For any $\nu \in \mathbb{S}^m$, we define the *ν-directional (horizontal) excess of E in $B_r(p)$*

$$\operatorname{Exc}(E, B_r(p), \nu) = \frac{1}{r^{Q-1}} \int_{B_r(p)} |\nu_E(q) - \nu|^2 d|\mu_E|.$$

The *(horizontal) excess of E in $B_r(p)$* is

$$\operatorname{Exc}(E, B_r(p)) = \min_{\nu\in\mathbb{S}^m} \operatorname{Exc}(E, B_r(p), \nu).$$

Remark 3.2. (i) Using $|\nu| = 1$ and $|\nu_E| = 1$ $|\mu_E|$–a.e., we obtain the equivalent definition

$$\operatorname{Exc}(E, B_r(p)) = \frac{2}{r^{Q-1}} \min_{\nu\in\mathbb{S}^m} \big\{ |\mu_E|(B_r(p)) - \langle \nu, \mu_E(B_r(p)) \rangle \big\}. \tag{3.1}$$

The minimum is achieved at the vector $\nu = \mu_E(B_r(p))/|\mu_E(B_r(p))|$, when $\mu_E(B_r(p)) \neq 0$, and we have

$$\operatorname{Exc}(E, B_r(p)) = \frac{2}{r^{Q-1}} \big\{ |\mu_E|(B_r(p)) - |\mu_E(B_r(p))| \big\}. \tag{3.2}$$

(ii) By the continuity of measures on increasing sequences of sets, we have

$$\lim_{s\uparrow r} \operatorname{Exc}(E, B_s(p)) = \operatorname{Exc}(E, B_r(p)). \tag{3.3}$$

(iii) If $B_r(p) \subset B_\varrho(q)$ we also have the monotonicity

$$r^{2n+1}\operatorname{Exc}(E, B_r(p)) \leq \varrho^{2n+1}\operatorname{Exc}(E, B_\varrho(q)). \tag{3.4}$$

Lemma 3.3. *Let $E, E_h \subset \mathbb{H}^n$, $h \in \mathbb{N}$, be sets of locally finite $\mathbb{H}$-perimeter such that $\mu_{E_h} \rightharpoonup \mu_E$ in the weak sense of Radon measures, as $h \to \infty$. Then for any $\nu \in \mathbb{S}^m$, $p \in \mathbb{H}^n$, and $r > 0$ we have*

$$\int_{B_r(p)} |\nu_E(q) - \nu|^2 d|\mu_E|(q) \leq \liminf_{h\to\infty} \int_{B_r(p)} |\nu_{E_h}(q) - \nu|^2 d|\mu_{E_h}|(q). \tag{3.5}$$

In particular, we have the lower semicontinuity of excess

$$\operatorname{Exc}(E, B_r(p)) \leq \liminf_{h\to\infty} \operatorname{Exc}(E_h, B_r(p)). \tag{3.6}$$

Proof. For a.e. $s > 0$ we have $|\mu_E|(\partial B_s(p)) = 0$. By the weak convergence of Radon measures, we have for any such s

$$\int_{B_s(p)} |\nu_E(q) - \nu|^2 d|\mu_E|(q) = \lim_{h\to\infty} \int_{B_s(p)} |\nu_{E_h}(q) - \nu|^2 d|\mu_{E_h}|(q). \tag{3.7}$$

Approximating $r > 0$ by an increasing sequence of s such that (3.7) holds, we obtain (3.5).

The same argument starting from (3.2) proves (3.6).

□

The automorphisms $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$, of the form

$$\delta_\lambda(z, t) = (\lambda z, \lambda^2 t), \quad (z, t) \in \mathbb{H}^n,$$

are called *dilations*. We use the abbreviations $\lambda p = \delta_\lambda(p)$ and $\lambda E = \delta_\lambda(E)$, for $p \in \mathbb{H}^n$ and $E \subset \mathbb{H}^n$. Left translations $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$

$$\tau_q(p) = q * p, \quad p, q \in \mathbb{H}^n, \tag{3.8}$$

and rotations of the form

$$(z, t) \mapsto (Tz, t), \quad (z, t) \in \mathbb{H}^n, \quad \text{with } T \in U(n) \tag{3.9}$$

are isometries of $\mathbb{H}^n$ with the distance d.

The proof of the following lemma follows elementarily from the invariance properties of $\mathbb{H}$-perimeter and we omit it.

Lemma 3.4. *Let $E \subset \mathbb{H}^n$ be a set with locally finite $\mathbb{H}$-perimeter, $p \in \mathbb{H}^n$, and $r > 0$.*

(i) *For any $\lambda > 0$*

$$\operatorname{Exc}(\lambda E, B_{\lambda r}(\lambda p)) = \operatorname{Exc}(E, B_r(p)). \tag{3.10}$$

(ii) *For any isometry* $T : \mathbb{H}^n \to \mathbb{H}^n$

$$\mathrm{Exc}(T(E), T(B_r(p))) = \mathrm{Exc}(E, B_r(p)). \tag{3.11}$$

The proof of the following lemma is in [15, Lemma 2.1]. Recall that a horizontal left invariant vector field Z in $\mathbb{H}^n$ is a linear combination with constant coefficients of the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$.

Lemma 3.5. *Let $E \subset \mathbb{H}^n$ be a set with finite $\mathbb{H}$-perimeter in B_r, $r > 0$, and let Z be a horizontal left invariant vector field such that*

$$\int_E Z\varphi(p)\, dp \le 0 \quad \textit{for all } \varphi \in C_c^1(B_r) \textit{ with } \varphi \ge 0. \tag{3.12}$$

Then for any $\mathcal{L}^{2n+1}$-measurable set $A \subset B_r$ we have $|E \cap A| \le |E \cap \exp(sZ)(A)|$ for all $s \ge 0$ such that $\exp(sZ)(A) \subset B_r$.

In the following proposition, we require $n \ge 2$. This is a localized version of an important lemma in the theory of sets with finite horizontal perimeter, see [9]. See also [14] for the problem of characterizing sets with constant horizontal normal in the Engel group.

Proposition 3.6. *Let $E \subset \mathbb{H}^n$, $n \ge 2$, be a set with finite $\mathbb{H}$-perimeter in $B_r(q)$, $q \in \partial E$ and $r > 0$, and let $\nu \in \mathbb{S}^m$. If $\nu_E(p) = \nu$ for $|\mu_E|$-a.e. $p \in B_r(q)$ then, up to a $\mathcal{L}^{2n+1}$-negligible set, we have*

$$E \cap B_r(q) = \big\{p \in B_r(q) : \langle \nu, q^{-1} * p\rangle > 0\big\}. \tag{3.13}$$

Proof. Possibly modifying E in a $\mathcal{L}^{2n+1}$-negligible set, we can assume that E coincides with the set of its Lebesgue points:

$$E = \Big\{p \in \mathbb{H}^n : \lim_{r\downarrow 0} \frac{|E \cap B_r(p)|}{|B_r(p)|} = 1\Big\}. \tag{3.14}$$

By (3.11) and (3.8)-(3.9), we can assume that $\nu = (1, 0, \dots, 0)$ and $q = 0$. Thus, for any $\varphi \in C_c^1(B_r; \mathbb{R}^{2n})$ we have

$$\int_E \mathrm{div}_{\mathbb{H}}\varphi(p)\, dp = -\int_{\mathbb{H}^n} \langle \varphi, \nu_E\rangle d|\mu_E| = -\int_{\mathbb{H}^n} \varphi_1 d|\mu_E|.$$

Then (3.12) holds with $Z = X_1$ and it follows that

$$p \in E \cap B_r \quad \Rightarrow \quad \exp(sX_1)(p) \in E \quad \text{for } s > 0, \tag{3.15}$$

as long as $\exp(sX_1)(p) \in B_r$.

Condition (3.12) holds also with $Z \in \big\{\pm X_2, \dots, \pm X_n, \pm Y_1, \dots, \pm Y_n\big\}$. In particular, the positive and negative flow along Z preserves Lebesgue measure. Thus

$$p \in E \cap B_r \quad \Rightarrow \quad \exp(sZ)(p) \in E \quad \text{for } s \in \mathbb{R}, \tag{3.16}$$

as long as $\exp(sZ)(p) \in B_r$. Finally, for $s \in \mathbb{R}$ consider the mappings $\Phi_s, \Psi_s : \mathbb{H}^n \to \mathbb{H}^n$

$$\Phi_s(p) = \exp(-sY_2)\exp(-sX_2)\exp(sY_2)\exp(sX_2)(p) = p * (0, -4s^2),$$
$$\Psi_s(p) = \exp(-sX_2)\exp(-sY_2)\exp(sX_2)\exp(sY_2)(p) = p * (0, 4s^2).$$

Then we have

$$p \in E \cap B_r \quad \Rightarrow \quad \Phi_s(p), \Psi_s(p) \in E \text{ for } s > 0 \text{ small enough.} \tag{3.17}$$

If $0 \in \partial E \cap B_r$, from (3.15)–(3.17) it follows that $E \cap B_r = \{p \in B_r : p_1 > 0\}$. $\square$

When $n = 1$, the situation is different. Let Z be a horizontal left invariant vector field. We say that a set $E \subset \mathbb{H}^1$ is *Z-ruled in a set $A \subset \mathbb{H}^1$* if

$$p \in E \cap A \quad \Rightarrow \quad \exp(sZ)(p) \in E$$

for all $s \in \mathbb{R}$ such that $\exp(sZ)(p) \in A$.

In the following proposition, it will be useful to work with suitable boxes in $\mathbb{H}^1$ replacing the balls $B_r(p)$. For $r > 0$ and $p \in \mathbb{H}^1$, we let

$$\begin{aligned} &Q_r = \{(x, y, t) \in \mathbb{H}^1 : |x| < r,\ |y| < r,\ |t| < r^2\}, \\ &Q_r(p) = p * Q_r. \end{aligned} \tag{3.18}$$

For $r > 0$ and $(y_0, t_0) \in \mathbb{R}^2$, we also define

$$\begin{aligned} &D_r = \{(y, t) \in \mathbb{R}^2 : |y| < r,\ |t| < r^2\}, \\ &D_r(y_0, t_0) = \{(y, t) \in \mathbb{R}^2 : |y - y_0| < r,\ |t - t_0| < r^2\}. \end{aligned} \tag{3.19}$$

Proposition 3.7. *Let $E \subset \mathbb{H}^1$ be a set with finite $\mathbb{H}$-perimeter in Q_{4r}, $r > 0$, with $0 \in \partial E$. Assume that $\nu_E(p) = (1, 0) \in \mathbb{S}^1$ for $|\mu_E|$-a.e. $p \in Q_{4r}$. Then there exists a function $g : D_r \to (-r/4, r/4)$ such that:*

(i) *Up to an $\mathcal{L}^3$-negligible set we have*

$$E \cap Q_r = \{(x, y, t) \in Q_r : x > g(y, t)\}.$$

(ii) *$g(0) = 0$ and for all $(y, t), (y', t') \in D_r$*

$$|g(y, t) - g(y', t')| \le |y - y'| + \frac{1}{2r}|t - t'|. \tag{3.20}$$

(iii) *The graph of g is Y_1-ruled in Q_r and namely*

$$g(y, t) = g(0, t + 2yg(y, t)), \quad (y, t) \in D_r. \tag{3.21}$$

Proof. As in the proof of Proposition 3.6, we can assume that

$$E = \Big\{p \in \mathbb{H}^1 : \lim_{s \downarrow 0} \frac{|E \cap B_s(p)|}{|B_s(p)|} = 1\Big\}.$$

For any $\alpha, \beta \in \mathbb{R}$ with $\alpha \geq 0$, let $Z = \alpha X_1 + \beta Y_1$. Then, for any $\varphi \in C^1_c(Q_{4r})$ with $\varphi \geq 0$, we have

$$\int_E Z\varphi \, dp = -\alpha \int_{Q_{4r}} \varphi \, d|\mu_E| \leq 0.$$

By Lemma 3.5, it follows that

$$p \in E \cap Q_{4r} \quad \Rightarrow \quad \exp(sZ)(p) \in E, \tag{3.22}$$

for all $s \in \mathbb{R}$ such that $\exp(sZ)(p) \in Q_{4r}$.

For any point $q \in E \cap Q_{2r}$ consider the set $E_q = q^{-1} * E$. The set E_q has constant measure theoretic normal $(1,0) \in \mathbb{S}^1$ in Q_{2r}. We apply (3.22) to the set E_q starting from the point $p = 0 \in E_q$. We deduce that

$$\big\{(x,y,0) \in Q_{2r} : x > 0\big\} \subset E_q.$$

Then we apply apply (3.22) to the set E_q starting from a generic point $p = (x,y,0) \in Q_{2r} \cap E_q$ with $|y| < 2r$ and $x > 0$, and we let $x \to 0$. We deduce that

$$\big\{(x,y,t) \in Q_{2r} : x > 0,\ |t| < 4rx\big\} \subset E_q.$$

In other words, we have

$$q \in E \cap Q_{2r} \quad \Rightarrow \quad q * \big\{(x,y,t) \in Q_{2r} : x > 0,\ |t| < 4rx\big\} \subset E. \tag{3.23}$$

From (3.23), it follows that $E \cap Q_{2r} \cap \{y = 0\}$ is a planar set with the cone property, the cones having all axis parallel to the x-axis and aperture $4r$. We deduce that there exists a Lipschitz function $h : (-r^2, r^2) \to \mathbb{R}$ such that:

(a) $\big\{(x,t) \in \mathbb{R}^2 : (x,0,t) \in E\big\} = \big\{(x,t) \in D_{2r} : x > h(t)\big\}$;

(b) $|h(t) - h(t')| \leq \dfrac{1}{4r}|t - t'|$ for all $t, t' \in (-r^2, r^2)$.

Because $0 \in \partial E$, we infer that $h(0) = 0$. From (3.23), we also deduce that ∂E is Y_1-ruled in Q_{2r}. Then we have

$$\partial E \cap Q_{2r} = \big\{(h(\tau), \sigma, \tau - 2\sigma h(\tau)) \in \mathbb{H}^1 : (\sigma, \tau) \in D_{2r}\big\}. \tag{3.24}$$

For any $(y,t) \in D_r$, the system of equations

$$\sigma = y, \quad \tau - 2\sigma h(\tau) = t$$

has a unique solution $(\sigma, \tau) \in D_{2r}$. This is an easy consequence of the Banach fixed point theorem and we omit the details. We claim that the solution $\tau = \tau(y,t)$ of the equation $\tau - 2yh(\tau) = t$ is Lipschitz continuous. Namely, by (b), we have for $(y,t), (y',t') \in D_r$

$$\begin{aligned} |\tau(y,t) - \tau(y',t')| &= |t - 2yh(\tau(y,t)) - t' + 2y'h(\tau(y',t'))| \\ &\leq |t - t'| + 2|y||h(\tau(y,t)) - h(\tau(y',t'))| + 2|h(\tau(y',t'))||y - y'| \\ &\leq |t - t'| + \frac{1}{r}|\tau(y,t) - \tau(y',t')| + \frac{1}{2r}|\tau(y',t')||y - y'|, \end{aligned}$$

and this implies

$$|\tau(y,t)-\tau(y',t')| \le 4r|y-y'| + 2|t-t'|. \tag{3.25}$$

The function $g = h\circ\tau$ satisfies (i), (ii), and (iii). In particular, (3.20) follows from (3.25), and (3.21) follows from (3.24). Finally, $|g(y,t)| < r/4$ follows from (b). ☐

Remark 3.8. If in Proposition 3.7 the radius r can be taken arbitrarly large, then from (3.20) we deduce that the function g does not depend on t. Then from (3.21), we deduce that g does not depend on y, either. Thus E is a vertical half-space.

Proposition 3.9. *Let $n \ge 2$. For any $0<\tau<1$ there exists an $\eta>0$ such that for all $p\in\mathbb{H}^n$, $r>0$, $\nu\in\mathbb{S}^m$, and for any set $E\subset\mathbb{H}^n$ that is $\mathbb{H}$-perimeter minimizing in $B_{3r}(p)$ with $p\in\partial E$ we have*

$$\mathrm{Exc}(E, B_{3r}(p),\nu) < \eta \quad \Rightarrow \quad P\big(E, B_r(p*q)\big) > 4\tau\omega_{2n-2}r^{Q-1}$$

for all $q\in B_{2r}$ such that $\langle q,\nu\rangle = 0$. Above, ω_{2n-2} is the Lebesgue measure of the standard unit ball in $\mathbb{R}^{2n-2}$.

Proof. By contradiction, there exists $0<\tau<1$ such that for all $h\in\mathbb{N}$ there are $p_h\in\mathbb{H}^n$, $r_h>0$, $\nu_h\in\mathbb{S}^m$ and $q_h\in B_{2r_h}$ such that $\langle q_h,\nu_h\rangle = 0$, and sets $E_h\subset\mathbb{H}^n$ that are $\mathbb{H}$-perimeter minimizing in $B_{3r_h}(p_h)$ with $p_h\in\partial E_h$ such that

$$\mathrm{Exc}(E, B_{3r_h}(p_h),\nu_h) < \frac{1}{h} \quad \text{and} \quad P\big(E_h, B_{r_h}(p_h*q_h)\big) \le 4\tau\omega_{2n-2}r_h^{Q-1}.$$

By (3.10) and (3.11), we can assume that $r_h = 1$, $p_h = 0$, and $q_h\in B_2$. Moreover, by compactness we can assume that $\nu_h\to\nu$, $q_h\to q\in\bar{B}_2$ with $\langle q,\nu\rangle = 0$.

By the compactness theorem for sets with finite $\mathbb{H}$-perimeter, [11, Sec. 4], we can assume that, up to a subsequence, $E_h\to E$ as $h\to\infty$ in the $L^1(B_3)$-convergence of characteristic functions, for some set $E\subset\mathbb{H}^n$ with finite $\mathbb{H}$-perimeter in B_3.

As $0\in\partial E_h$, by (2.7), we have for $0<r<1$

$$\min\big\{|E_h\cap B_r|, |B_r\setminus E_h|\big\} \ge c_3 r^Q. \tag{3.26}$$

Passing to the limit as $h\to\infty$ in (3.26), we obtain the same estimate for the set E and thus $0\in\partial E$.

Passing to the limit as $h\to\infty$ in

$$\mathrm{Exc}(E_h, B_3,\nu_h) < \frac{1}{h},$$

using (3.5) and $\nu_h\to\nu$, we deduce that $\mathrm{Exc}(E,B_3,\nu) = 0$ and thus $\nu_E(p) = \nu$ for $|\mu_E|$-a.e. $p\in B_3$. By Proposition 3.6, E coincides in B_3 with a vertical half-space having boundary ∂E orthogonal to ν. As $q\in\partial E$ and $B_1(q)\subset B_3$, we have

$$P(E, B_1(q)) = P(E, B_1) = 4\omega_{2n-2}.$$

The exact formula for $P(E,B_1)$ follows from the area formula (2.3).

On the other hand, for any $0 < r < 1$ we have

$$P(E, B_r(q)) \le \liminf_{h\to\infty} P(E_h, B_r(q)) \le \liminf_{h\to\infty} P(E_h, B_1(q_h)) \le 4\tau\omega_{2n-2},$$

and thus $P(E, B_1(q)) < 4\omega_{2n-2}$. This is a contradiction. $\square$

Next, we prove a version of Proposition 3.9 in the case $n = 1$.

Proposition 3.10. *There are constants $k_0, \eta > 0$ such that for all $p \in \mathbb{H}^1$, $r > 0$, $\nu \in \mathbb{S}^1$, $k \ge k_0$, and for any set $E \subset \mathbb{H}^1$ that is $\mathbb{H}$-perimeter minimizing in $B_{kr}(p)$ with $p \in \partial E$, we have*

$$\operatorname{Exc}(E, B_{kr}(p), \nu) < \eta \quad \Rightarrow \quad P(E, B_r(p * q)) > \frac{r^3}{32}, \tag{3.27}$$

for all $q \in B_{2r}$ such that $\langle q, \nu\rangle = 0$. In fact, we can choose $k_0 = 370$.

Proof. By (3.10) and (3.11), it is enough to prove the proposition for $p = 0$, $\nu = (1, 0) \in \mathbb{S}^1$, and $r = 1$. We replace the balls B_r with the boxes Q_r introduced in (3.18). Notice that $Q_{r/2} \subset B_r \subset Q_r$.

Assume by contradiction that there exist a sequence of sets E_h, $h \in \mathbb{N}$, that are $\mathbb{H}$-perimeter minimizing in $Q_{k/2}$ with $0 \in \partial E_h$, and a sequence of points $q_h \in B_2 \subset Q_2$ such that $\langle q_h, \nu\rangle = 0$, and moreover

$$\operatorname{Exc}(E_h, Q_{k/2}, \nu) < \frac{1}{h}, \quad P(E_h, Q_{1/2}(q_h)) \le \frac{1}{32}.$$

By compactness, we can assume that $q_h \to q \in \bar{Q}_2$ with $\langle q, \nu\rangle = 0$. Moreover, by the compactness theorem for sets with finite $\mathbb{H}$-perimeter we can assume that, up to a subsequence, $E_h \to E$ as $h \to \infty$ for some set $E \subset \mathbb{H}^n$ such that

$$\operatorname{Exc}(E, Q_{k/2}, \nu) = 0 \quad \text{and} \quad P(E, Q_{1/2}(q)) \le \frac{1}{32}. \tag{3.28}$$

By Proposition 3.7, there exists a function $g : D_{k/8} \to \mathbb{R}$ such that:

(i) $\partial E \cap Q_{k/8} = \big\{(g(y,t), y, t) \in \mathbb{H}^1 : (y,t) \in D_{k/8}\big\}$;

(ii) $g(0) = 0$ and $|g(y,t) - g(y',t')| \le |y - y'| + \frac{4}{k}|t - t'|$ for all $(y,t), (y',t') \in D_{k/8}$;

(iii) The graph of g is Y_1-ruled in $Q_{k/8}$, and namely for any $(y,t) \in D_{k/8}$ we have

$$g(y,t) = g\big(0, t + 2yg(y,t)\big). \tag{3.29}$$

We claim that for any $q = (y_0, t_0) \in \bar{D}_2$, i.e.,

$$\max\{|y_0|, |t_0|^{1/2}\} \le 2, \tag{3.30}$$

and for any $(y,t) \in D_{1/4}(q)$, i.e.,

$$\max\{|y - y_0|, |t - t_0|^{1/2}\} < \frac{1}{4}, \tag{3.31}$$

we have

$$|g(y,t)| < \frac{1}{2}, \quad |y - y_0| < \frac{1}{2}, \quad |t - t_0 - 2y_0 g(y,t)| < \frac{1}{4}. \tag{3.32}$$

The inequality $|y-y_0|<1/2$ is trivially satisfied by (3.31). Moreover, by (3.29), (ii), (3.30), and (3.31), we have

$$\begin{aligned}|g(y,t)| &= |g(0,t+2yg(y,t))|\\ &\le \frac{4}{k}|t+2yg(y,t)|\\ &\le \frac{4}{k}\Big[|t-t_0|+|t_0|+2(|y-y_0|+|y_0|)|g(y,t)|\Big]\\ &\le \frac{1}{k}\Big[\frac{65}{4}+18|g(y,t)|\Big].\end{aligned}$$

Then we have for $k>18$

$$|g(y,t)| \le \frac{65}{4(k-18)}. \tag{3.33}$$

This implies $|g(y,t)|<1/2$ for $k>51$. Moreover, by (3.30), (3.31), and (3.33)

$$|t-t_0-2y_0g(y,t)| \le |t-t_0|+2|y_0||g(y,t)| \le \frac{1}{16}+\frac{65}{k-18}, \tag{3.34}$$

and this implies $|t-t_0-2y_0g(y,t)|<1/4$ for $k\ge 370$.

Then, by (3.31)–(3.34), we have

$$G=\big\{\big(g(y,t),y,t\big)\in\mathbb{R}^3 : (y,t)\in D_{1/4}(q)\big\}\subset \partial E\cap Q_{1/2}(q),$$

and, by the area formula (2.3), it follows that

$$P(E,Q_{1/2}(q)) \ge \int_G \sqrt{\langle X_1,N\rangle^2+\langle Y_1,N\rangle^2}\,d\mathcal{H}^2,$$

where N is the standard unit normal to G and $\mathcal{H}^2$ is the standard 2-dimensional Hausdorff measure in $\mathbb{R}^3$. We deduce that

$$P(E,Q_{1/2}(q)) \ge \int_{D_{1/4}(q)} \sqrt{\big(1-2y\partial_t g\big)^2+\big(\partial_y g-2g\partial_t g\big)^2}\,dydt.$$

By (ii) we have $|\partial_t g|\le 4/k$ a.e. on $D_{k/8}$, and thus

$$\big|1-2y\partial_t g\big| \ge 1-2|y|\big|\partial_t g\big| \ge 1-\frac{18}{k} > \frac{1}{2},$$

for $k>36$. We conclude that for $k\ge 370$, we have

$$P(E,Q_{1/2}(q)) \ge \int_{D_{1/4}(q)} \big|1-2y\partial_t g\big|dydt > \frac{1}{2}|D_{1/4}(q)| = \frac{1}{32}.$$

This contradicts (3.28). □

4. Intrinsic Lipschitz graphs

Let $\nu \in \mathbb{S}^m$. With abuse of notation, we identify ν with $(\nu, 0) \in \mathbb{H}^n$. For any $p \in \mathbb{H}^n$, we let $\nu(p) = \langle p, \nu\rangle \nu \in \mathbb{H}^n$ and we define $\nu^\perp(p) \in \mathbb{H}^n$ as the unique point such that

$$p = \nu^\perp(p) * \nu(p). \tag{4.1}$$

We use the coordinates $p = (z, t) \in \mathbb{C}^n \times \mathbb{R}$ and we let

$$z^\top = \langle z, \nu\rangle \nu \quad \text{and} \quad z^\perp = z - z^\top = z - \langle z, \nu\rangle \nu. \tag{4.2}$$

In the notation for $z^\perp$ and $z^\top$, we omit reference to ν. Then we have

$$\nu^\perp(p) = \big(z^\perp, t - Q(z^\perp, z^\top)\big). \tag{4.3}$$

Recall that the homogeneous norm of $p = (z, t) \in \mathbb{C}^n \times \mathbb{R}$ is $\|p\| = \max\{|z|, |t|^{1/2}\}$. By the triangle inequality (2.1), there holds

$$\|\nu(p)\| = \|\langle \nu, p\rangle \nu\| = |\langle \nu, z\rangle| \leq |z| \leq \|p\|, \tag{4.4}$$

$$\|\nu^\perp(p)\| = \|p * \nu(p)^{-1}\| \leq \|p\| + \|\nu(p)^{-1}\| = \|p\| + \|\nu(p)\| \leq 2\|p\|. \tag{4.5}$$

Proposition 4.1. *Let $n \geq 2$ and $\nu \in \mathbb{S}^m$. For any $L > 0$ there exists a $\sigma > 0$ such that for all $r > 0$ and for any set $E \subset \mathbb{H}^n$ that is $\mathbb{H}$-perimeter minimizing in B_{3r} the condition*

$$\mathrm{Exc}(E, B_s(u), \nu) < \sigma \quad \text{for all } 0 < s < 2r \text{ and } u \in \{p, q\} \subset \partial E \cap B_r \tag{4.6}$$

implies

$$\|\nu(q^{-1} * p)\| \leq L\|\nu^\perp(q^{-1} * p)\|. \tag{4.7}$$

Proof. By contradiction, assume that there exists an $L > 0$ such that for all $h \in \mathbb{N}$ there are $r_h > 0$, sets $E_h \subset \mathbb{H}^n$ that are $\mathbb{H}$-perimeter minimizing in B_{3r_h}, and points $p_h, q_h \in \partial E_h \cap B_{r_h}$ such that

$$\mathrm{Exc}(E_h, B_s(u), \nu) < \frac{1}{h} \quad \text{for all } 0 < s < 2r_h \text{ and } u \in \{p_h, q_h\} \tag{4.8}$$

and

$$\|\nu(q_h^{-1} * p_h)\| > L\|\nu^\perp(q_h^{-1} * p_h)\|. \tag{4.9}$$

By (3.10), we can without loss of generality assume that $r_h = 1$ for all $h \in \mathbb{N}$. Possibly taking a subsequence, we can assume that as $h \to \infty$ we have:

i) $p_h \to p$ and $q_h \to q$ with $p, q \in \bar{B}_1$. In particular, passing to the limit in (4.9), we have

$$\|\nu(q^{-1} * p)\| \geq L\|\nu^\perp(q^{-1} * p)\|. \tag{4.10}$$

ii) $E_h \to E$ in $L^1(B_3)$, for some set $E \subset \mathbb{H}^n$ that is $\mathbb{H}$-perimeter minimizing in B_3. This can be proved by the compactness theorem of [11, Sec. 4]. In particular, by the lower semicontinuity (3.5), we have for any $0 < s < 2$, and in fact for $s = 2$,

$$\mathrm{Exc}(E, B_s(u), \nu) = 0 \quad \text{for } u \in \{p, q\}. \tag{4.11}$$

As in Lemma 3.9, we have $p, q \in \partial E$.

Now there are two cases:

a) $\|\nu(q^{-1} * p)\| > 0$;
b) $\|\nu(q^{-1} * p)\| = 0$.

By (4.11) and Proposition 3.6, the set E coincides in $B_2(p) \cap B_2(q)$ with a vertical half-space H having boundary orthogonal to ν, with $p, q \in \partial H$. In the case a), this is a contradiction.

In the case b), we have $\lambda_h = \|\nu(q_h^{-1} * p_h)\| \to 0$ as $h \to \infty$, and $\lambda_h > 0$ for all $h \in \mathbb{N}$, by (4.9). Let us define

$$F_h = \frac{1}{\lambda_h}\big(q_h^{-1} * E_h\big) \quad \text{and} \quad v_h = \frac{1}{\lambda_h}(q_h^{-1} * p_h), \quad h \in \mathbb{N}.$$

Then we have $0, v_h \in \partial F_h$ and, by (4.9), $1 = \|\nu(v_h)\| > L\|\nu^\perp(v_h)\|$. By (4.1) and (2.1), we get $\|v_h\| \le \|\nu(v_h)\| + \|\nu^\perp(v_h)\| \le 1 + 1/L$.

Let $R > 0$ be any large number, e.g., $R > 1 + 1/L$. Possibly taking a subsequence, we can assume that, as $h \to \infty$, we have:

1) $v_h \to v$ with $\|v\| \le 1 + 1/L$ and $\|\nu(v)\| = 1$.
2) $F_h \to F$ in $L^1(B_{3R})$, for some set $F \subset \mathbb{H}^n$ that is $\mathbb{H}$-perimeter minimizing in B_{3R}. In particular, we have $\mathrm{Exc}(F, B_{2R}(u), \nu) = 0$ for $u \in \{0, v\}$.

Then F coincides in $B_{2R}(v) \cap B_{2R}$ with a halfspace H having boundary orthogonal to ν with $0, v \in \partial H$. This contradicts $\|\nu(v)\| = 1$. □

Proposition 4.2. *For any $L > 2$ there exists a $\sigma > 0$ such that for all $r > 0$, for all $\nu \in \mathbb{S}^1$, and for any set $E \subset \mathbb{H}^1$ that is $\mathbb{H}$-perimeter minimizing in B_{128r}, the condition*

$$\mathrm{Exc}(E, B_s(u), \nu) < \sigma \quad \textit{for all } 0 < s < 64r \textit{ and } u \in \{p, q\} \subset \partial E \cap B_r \tag{4.12}$$

implies

$$\|\nu(q^{-1} * p)\| \le L\|\nu^\perp(q^{-1} * p)\|. \tag{4.13}$$

Proof. The first part of the argument is the same as in the proof of Proposition 4.1. We arrive at the following statement: by contradiction, there are points $p, q \in \partial E \cap \bar{B}_r$ such that

$$\mathrm{Exc}(E, B_s(u), \nu) = 0 \quad \text{for all } 0 < s < 64r \text{ and } u \in \{p, q\} \tag{4.14}$$

and

$$\|\nu(q^{-1} * p)\| \geq L\|\nu^{\perp}(q^{-1} * p)\|. \tag{4.15}$$

There are two cases:

a) $\|\nu(q^{-1} * p)\| > 0$;
b) $\|\nu(q^{-1} * p)\| = 0$.

The analysis of case a) is more complicated than in Proposition 4.1. From (4.15) and $L > 4$ we have in this case the strict inequality

$$\|\nu(q^{-1} * p)\| > 4\|\nu^{\perp}(q^{-1} * p)\|. \tag{4.16}$$

By (3.11), we can assume that $\nu = (1, 0) \in \mathbb{S}^1$. By (4.14), we have $\mathrm{Exc}(E, Q_s(u), \nu) = 0$ for all $0 < s < 32r$ and $u \in \{p, q\}$. By (3.11) again, we can assume that $q = 0$ and $\mathrm{Exc}(E, Q_s(u), \nu) = 0$ for all $0 < s < 16r$ and $u \in \{p, 0\}$ with $p \in \bar{B}_{2r} \subset \bar{Q}_{2r}$.

By Proposition 3.7, there exists a function $g : D_{4r} \to (-r, r)$ such that:

(i) Up to an $\mathcal{L}^3$-negligible set we have

$$E \cap Q_{4r} = \big\{(x, y, t) \in Q_{4r} : x > g(y, t)\big\}.$$

(ii) $g(0) = 0$ and for all $(y, t), (y', t') \in D_{4r}$

$$|g(y, t) - g(y', t')| \leq |y - y'| + \frac{1}{2r}|t - t'|. \tag{4.17}$$

(iii) The graph of g is Y_1 ruled in Q_{4r}, i.e., $g(y, t) = g(0, 2yg(y, t))$ for $(y, t) \in Q_{4r}$.

Then we have $p = (g(y, t), y, t)$ for some $(y, t) \in \bar{D}_{2r}$. By the formula $\nu(p) = \langle p, \nu\rangle\nu$ and (4.3), we have

$$\nu(p) = g(y, t)\nu \quad \text{and} \quad \nu(p)^{\perp} = \big(0, y, t - 2yg(y, t)\big).$$

By $|g(y, t)| \leq r$, (ii), and (iii), we obtain

$$\begin{aligned}
\|\nu(p)\| &= |g(y, t)| = |g(0, t + 2yg(y, t))| \\
&\leq \frac{1}{2r}|t + 2yg(y, t)| \\
&\leq \frac{1}{2r}\big\{|t - 2yg(y, t)| + 4|y||g(y, t)|\big\} \\
&\leq \frac{1}{2r}\big\{\sqrt{8}r|t - 2yg(y, t)|^{1/2} + 4r|y|\big\} \\
&\leq 2\big\{|t - 2yg(y, t)|^{1/2} + |y|\big\} = 2\|\nu(p)^{\perp}\|.
\end{aligned}$$

This contradicts (4.16) with $q = 0$.

The analysis of the case b) is the same as in the proof of Proposition 4.1. In this case, we use Remark 3.8. Details are omitted. □

Definition 4.3. The (open) cone with vertex $0 \in \mathbb{H}^n$, axis $\nu \in \mathbb{S}^m$, and aperture $\alpha \in (0, \infty]$ is the set

$$C(0, \nu, \alpha) = \big\{p \in \mathbb{H}^n : \|\nu^\perp(p)\| < \alpha \|\nu(p)\|\big\}. \tag{4.18}$$

The negative and positive cones are, respectively,

$$C^-(0, \nu, \alpha) = \big\{p = (z,t) \in \mathbb{H}^n : \|\nu^\perp(p)\| < \alpha \|\nu(p)\|,\ \langle z, \nu\rangle < 0\big\},$$
$$C^+(0, \nu, \alpha) = \big\{p = (z,t) \in \mathbb{H}^n : \|\nu^\perp(p)\| < \alpha \|\nu(p)\|,\ \langle z, \nu\rangle > 0\big\}.$$

The cone with vertex $p \in \mathbb{H}^n$, axis $\nu \in \mathbb{S}^m$, and aperture $\alpha \in (0, \infty]$ is the set $C(p, \nu, \alpha) = p * C(0, \nu, \alpha)$. Analogously, we let $C^-(p, \nu, \alpha) = p * C^-(0, \nu, \alpha)$ and $C^+(p, \nu, \alpha) = p * C^+(0, \nu, \alpha)$.

Remark 4.4. By (4.3) and (4.2), the inequality $\|\nu^\perp(p)\| < \alpha \|\nu(p)\|$, with $p = (z, t)$, defining the cone in (4.18) is

$$\max\big\{|z^\perp|, |t - Q(z^\perp, z^\top)|^{1/2}\big\} < \alpha |z^\top|. \tag{4.19}$$

In the next lemma, we collect some elementary facts about cones in the Heisenberg group.

Lemma 4.5. *Let $\nu \in \mathbb{S}^m$.*

i) *For any $\alpha > 0$, $p \in \mathbb{H}^n$, and $s_0 \in \mathbb{R}$ we have*

$$\bigcup_{s<s_0} C^+(p * s\nu, \nu, \alpha) = \mathbb{H}^n. \tag{4.20}$$

ii) *Let $\iota : \mathbb{H}^n \to \mathbb{H}^n$ be the mapping $\iota(p) = p^{-1} = -p$. For any $\alpha > 0$ we have*

$$C^-(0, \nu, \alpha) \subset \iota C^+(0, \nu, \alpha + 2\sqrt{\alpha}). \tag{4.21}$$

iii) *For all $\alpha, \beta \geq 0$ and $p \in C^+(0, \nu, \alpha)$, we have $C^+(p, \nu, \beta) \subset C^+(0, \nu, \gamma)$ with*

$$\gamma = \max\big\{\alpha, \beta, \sqrt{\alpha\beta + 2\beta}\big\}. \tag{4.22}$$

Analogously, for any $p \in C^-(0, \nu, \alpha)$ we have $C^-(p, \nu, \beta) \subset C^-(0, \nu, \gamma)$.

Proof. When $p = 0$, by (4.19) and the estimate $|Q(z, \zeta)| \leq 2|z||\zeta|$ we have

$$\begin{aligned} C^+(s\nu, \nu, \alpha) &= s\nu * C^+(0, \nu, \alpha) \\ &= s\nu * \Big\{(z,t) \in \mathbb{H}^n : \max\{|z^\perp|, |t - Q(z^\perp, z^\top)|^{1/2}\} < \alpha \langle \nu, z\rangle\Big\} \\ &= \Big\{(z,t) \in \mathbb{H}^n : \max\{|z^\perp|, |t - Q(z^\perp, z^\top - 2s\nu)|^{1/2}\} < \alpha \langle \nu, z - s\nu\rangle\Big\}. \end{aligned}$$

Claim (4.20) with $p = 0$ then follows from the fact that for any $(z, t) \in \mathbb{H}^n$ there exists $\sigma \in \mathbb{R}$ such that for all $s < \sigma$ there holds

$$|t - Q(z^\perp, z^\top - 2s\nu)|^{1/2} < \alpha \langle \nu, z - s\nu\rangle.$$

By left translation, the claim holds for any $p \in \mathbb{H}^n$.

We prove (4.21). Notice that, for any $\beta > 0$, we have

$$\iota C^+(0,\nu,\beta) = \Big\{(z,t) \in \mathbb{H}^n : \max\big\{|z^\perp|, |t + Q(z^\perp, z^\top)|^{1/2}\big\} < -\beta\langle z,\nu\rangle\Big\}.$$

For any $(z,t) \in C^-(0,\nu,\alpha)$, there holds $|Q(z^\perp, z^\top)| \le 2|z^\perp||z^\top| \le 2\alpha|z^\top|^2$ and

$$|t + Q(z^\perp, z^\top)|^{1/2} \le |t - Q(z^\perp, z^\top)|^{1/2} + |2Q(z^\perp, z^\top)|^{1/2} \le -(\alpha + 2\sqrt{\alpha})\langle z,\nu\rangle.$$

The claim follows.

We prove iii). A point $p = (z,t) \in \mathbb{H}^n$ belongs to $C^+(0,\nu,\alpha)$ if

$$\max\big\{|z^\perp|, |t - Q(z^\perp, z^\top)|^{1/2}\big\} \le \alpha\langle z,\nu\rangle. \tag{4.23}$$

A point $q \in C^+(p,\nu,\beta) = p{*}C^+(0,\nu,\beta)$ is of the form $q = p{*}w$ where $w = (\zeta,\tau) \in \mathbb{H}^n$ is such that

$$\max\big\{|\zeta^\perp|, |\tau - Q(\zeta^\perp, \zeta^\top)|^{1/2}\big\} \le \beta\langle \zeta,\nu\rangle. \tag{4.24}$$

As $q = \big(z+\zeta, t+\tau+Q(z,\zeta)\big)$, the claim $\|\nu^\perp(q)\| \le \gamma\|\nu(q)\|$ reads

$$\max\big\{|z^\perp + \zeta^\perp|, |t+\tau+Q(z,\zeta) - Q(z^\perp+\zeta^\perp, z^\top+\zeta^\top)|^{1/2}\big\} \le \gamma\langle z+\zeta,\nu\rangle. \tag{4.25}$$

On the one hand, by (4.23), (4.24), and (4.22) we have

$$|z^\perp + \zeta^\perp| \le |z^\perp| + |\zeta^\perp| \le \alpha\langle z,\nu\rangle + \beta\langle\zeta,\nu\rangle \le \gamma\langle z+\zeta,\nu\rangle.$$

Then, to prove (4.25) it is sufficient to show that

$$|t+\tau+Q(z,\zeta) - Q(z^\perp+\zeta^\perp, z^\top+\zeta^\top)| \le \gamma^2\{|z^\top|^2 + 2\langle z,\nu\rangle\langle\zeta,\nu\rangle + |\zeta^\top|^2\}. \tag{4.26}$$

By the triangle inequality, we have

$$\begin{aligned}|t+\tau+Q(z,\zeta) - Q(z^\perp+\zeta^\perp, z^\top+\zeta^\top)| \le & |t - Q(z^\perp, z^\top)| + |\tau - Q(\zeta^\perp,\zeta^\top)| \\ & + |Q(z,\zeta) - Q(\zeta^\perp, z^\top) - Q(z^\perp,\zeta^\top)|.\end{aligned} \tag{4.27}$$

The first and second term in the right hand side of (4.27) are estimated by (4.23) and (4.24). Moreover, we have

$$\begin{aligned}|Q(z,\zeta) - Q(\zeta^\perp, z^\top) - Q(z^\perp,\zeta^\top)| &= |Q(z^\perp,\zeta^\perp) + 2Q(z^\top,\zeta^\perp)| \\ &\le 2\big\{|z^\perp||\zeta^\perp| + 2|z^\top||\zeta^\perp|\big\} \\ &\le 2\{\alpha\beta + 2\beta\}|z^\top||\zeta^\top| \\ &\le 2\gamma^2\langle z,\nu\rangle\langle\zeta,\nu\rangle.\end{aligned} \tag{4.28}$$

Claim (4.26) follows. $\square$

For $\nu \in \mathbb{S}^m$ we denote by $H_\nu = \{p \in \mathbb{H}^n : \langle p,\nu\rangle = 0\}$ the vertical hyperplane in $\mathbb{H}^n$ orthogonal to ν. We endow $\mathrm{span}\{\nu\} = \{(s\nu, 0) \in \mathbb{H}^n : s \in \mathbb{R}\}$ with its natural total ordering.

Definition 4.6. (i) The *intrinsic graph* of a function $\varphi : A \to \mathrm{span}\{\nu\}$, $A \subset H_\nu$, is the set

$$\mathrm{gr}(\varphi) = \big\{ p * \varphi(p) \in \mathbb{H}^n : p \in A \big\}. \tag{4.29}$$

In gr we omit reference to ν.

(ii) A function $\varphi : A \to \mathrm{span}\{\nu\}$, $A \subset H_\nu$, is *L-intrinsic Lipschitz*, $L \in [0, \infty)$, if for any $p \in \mathrm{gr}(\varphi)$ there holds

$$\mathrm{gr}(\varphi) \cap C(p, \nu, 1/L) = \emptyset. \tag{4.30}$$

Remark 4.7. The notion of intrinsic Lipschitz function of Definition 4.6 is introduced in [10]. The cones (4.18) are relevant in the theory of H-convex sets [3].

The following extension theorem is proved in [10]. Here, we give a self-contained proof with an estimate of the Lipschitz constant of the extension.

Proposition 4.8. *Let $\varphi : A \to \mathrm{span}\{\nu\}$, $\nu \in \mathbb{S}^m$ and $A \subset H_\nu$ nonempty set, be an L-intrinsic Lipschitz function, $L \geq 0$. Then there exists an M-intrinsic Lipschitz function $\psi : H_\nu \to \mathrm{span}\{\nu\}$ with*

$$M = \Big(\sqrt{1 + \frac{1}{L + 2L^2}} - 1 \Big)^{-2}, \tag{4.31}$$

such that $\psi(p) = \varphi(p)$ for all $p \in A$.

Proof. Let $\alpha = 1/L$ and define the set

$$E = \bigcup_{p \in A} C^+\big(p * \varphi(p), \nu, \alpha\big).$$

The set E is open, as it is the union of open half-cones, and, moreover, $E \neq \emptyset$ because $\mathrm{gr}(\varphi) \cap E = \emptyset$, by (4.30).

Let

$$\beta = \frac{\alpha^2}{\alpha + 2}. \tag{4.32}$$

In view of Lemma 4.5, part iii), notice that $\sqrt{\alpha\beta + 2\beta} \leq \alpha$ and $\beta \leq \alpha$. Then for any $q \in E$ we have

$$C^+(q, \nu, \beta) \subset E. \tag{4.33}$$

By an elementary continuity argument, the inclusion (4.33) also holds for any $q \in \partial E$, the topological boundary of E. Then we have the implication

$$p, q \in \partial E \quad \Rightarrow \quad p \notin C^+(q, \nu, \beta). \tag{4.34}$$

For any $p \in H_\nu$, the set $L_p = \big\{ s \in \mathbb{R} : p * s\nu \in E \big\}$ is open, because E is open; it is nonempty (we skip the elementary proof); it is bounded from below by (4.20). In fact, L_p is an open half-line. Then we can define the function $\psi : H_\nu \to \mathrm{span}\{\nu\}$

$$\psi(p) = s_p \nu \quad \text{where} \quad s_p = \inf L_p \in \mathbb{R}.$$

We claim that $\psi(p) = \varphi(p)$ for any $p \in A$. If $p \in A$ we have $p * s\nu \in E$ for all $s > \langle \varphi(p), \nu\rangle$ and this implies that $\psi(p) \le \varphi(p)$. By contradiction, assume that $\psi(p) < \varphi(p)$. Then there exists $s < \langle \varphi(p), \nu\rangle$ such that $p * s\nu \in E$ and (4.33) implies that

$$p * \varphi(p) \in C^+(p * s\nu, \nu, \beta) \subset E.$$

Then there exists $q \in A$ such that $p * \varphi(p) \in C^+(q * \varphi(q), \nu, \alpha)$, and this contradicts (4.30), because

$$\mathrm{gr}(\varphi) \cap C^+(q * \varphi(q), \nu, \alpha) \neq \emptyset.$$

Finally, we prove that ψ is M-intrinsic Lipschitz. Let $p, q \in \mathrm{gr}(\psi) \subset \partial E$. The inclusion $\mathrm{gr}(\psi) \subset \partial E$ follows easily from the construction of ψ. By (4.34), each of the following equivalences holds true

$$p \notin C^+(q, \nu, \beta) \quad \Leftrightarrow \quad q^{-1} * p \notin C^+(0, \nu, \beta) \quad \Leftrightarrow \quad p^{-1} * q \notin \iota C^+(0, \nu, \beta), \tag{4.35}$$

where $\iota : \mathbb{H}^n \to \mathbb{H}^n$ is the map $\iota(p) = p^{-1}$. Let $\gamma > 0$ be such that $\gamma + 2\sqrt{\gamma} = \beta$, and nameley

$$\gamma = \big(\sqrt{1+\beta} - 1\big)^2. \tag{4.36}$$

By Lemma 4.5, (4.21), we have $C^-(0, \nu, \gamma) \subset \iota C^+(0, \nu, \beta)$, and thus, by (4.34), each of the (4.35) implies that $q \notin C^-(p, \nu, \gamma)$. After all, we proved that

$$p \notin C^+(q, \nu, \beta) \quad \Rightarrow \quad q \notin C^-(p, \nu, \gamma).$$

Then for any $p \in \mathrm{gr}(\psi)$ it holds $\mathrm{gr}(\psi) \cap C(0, \nu, \gamma) = \emptyset$, i.e., ψ is M-intrinsic Lipschitz with $M = 1/\gamma$. By (4.32) and (4.36), M satisfies (4.31). □

For any $r > 0$ and $p \in \mathbb{H}^n$, we let $B_r^\perp = \big\{q \in B_r : \langle q, \nu\rangle = 0\big\}$ and $B_r^\perp(p) = p * B_r^\perp$. $B_r^\perp(p)$ is the section of $B_r(p)$ with the hyperplane through p orthogonal to ν. Let $\psi : H_\nu \to \mathrm{span}\{\nu\}$ be an M-intrinsic Lipschitz function. We denote by $\pi : \mathbb{H}^n \to H_\nu$ the projection $\pi(p) = \nu^\perp(p)$, see (4.3).

Definition 4.9 (ψ-balls and ψ-cylinders)**.** The ψ-*ball* with radius $r > 0$ and center $p \in H_\nu$ is the subset of H_ν

$$D_r^\psi(p) = \pi\Big(B_r^\perp(p * \psi(p))\Big).$$

The ψ-*cylinder* with base $D_r^\psi(p)$ is the subset of $\mathbb{H}^n$

$$C_r^\psi(p) = \big\{q * s\nu \in \mathbb{H}^n : q \in D_r^\psi(p),\ s \in \mathbb{R}\big\}.$$

Remark 4.10. If $p = (z, t) \in H_\nu$, i.e. $z^\top = 0$, we have the formula for ψ-balls

$$D_r^\psi(p) = \big\{\big(z + \zeta, t + \tau + Q(z + 2\psi(p), \zeta)\big) \in \mathbb{H}^n : (\zeta, \tau) \in B_r^\perp\big\}. \tag{4.37}$$

Formula (4.37) can be checked after a short computation.

Notice that, by Fubini-Tonelli Theorem, the measure of ψ-balls is given by the formula

$$\mathcal{L}^{2n}(D_r^{\psi}(p)) = 2\omega_{2n-1} r^{Q-1}, \tag{4.38}$$

where ω_{2n-1} is the Lebesgue measure of the standard unit ball in $\mathbb{R}^{2n-1}$.

A well known problem concerning sets as $D_r^{\psi}(p)$ is that they origin from a projection, the mapping π, that is not Lipschitz continuous for the Carnot-Carathéodory metric. In the case of intrinsic Lipschitz functions, however, we have the following fact.

Lemma 4.11. *Let $\psi : H_\nu \to \mathrm{span}\{\nu\}$ be M-intrinsic Lipschitz, $M \geq 0$. For all points $p, q \in H_\nu$ such that $q \in D_r^{\psi}(p)$, $r > 0$, there holds*

$$|\psi(p) - \psi(q)| \leq C_M r, \tag{4.39}$$

with

$$C_M = (M + \sqrt{M^2 + M})^2. \tag{4.40}$$

Proof. Let $p = (z, t) \in H_\nu$ and $q = \big(z + \zeta, t + \tau + Q(z + 2\psi(p), \zeta)\big) \in D_r^{\psi}(p)$ for some $(\zeta, \tau) \in B_r^{\perp}$, i.e.,

$$|\zeta| < r \quad \text{and} \quad |\tau| < r^2. \tag{4.41}$$

As ψ is M-instrinsic Lipschitz, the point $w = \psi(q)^{-1} * q^{-1} * p * \psi(p)$ satisfies $w \notin C(0, \nu, 1/M)$, i.e.,

$$\|\nu(w)\| \leq M \|\nu^{\perp}(w)\|. \tag{4.42}$$

A short computation provides

$$\nu^{\perp}(w) = \big(-\zeta, -\tau - 2Q(\zeta, \psi(q) - \psi(p))\big) \quad \text{and} \quad \nu(w) = \psi(p) - \psi(q),$$

and thus, by (4.19), (4.42) is equivalent with the inequality

$$|\psi(p) - \psi(q)| \leq M \max\big\{|\zeta|, |\tau - 2Q(\zeta, \psi(p) - \psi(q)|^{1/2}\big\}.$$

Then by the estimate $|Q(z, \zeta)| \leq 2|z||\zeta|$ and (4.41), we have

$$\begin{aligned}|\psi(p) - \psi(q)| &\leq M \max\big\{|\zeta|, |\tau|^{1/2} + 2|\zeta|^{1/2}|\psi(p) - \psi(q)|^{1/2}\big\} \\ &\leq M\big(r + 2r^{1/2}|\psi(p) - \psi(q)|^{1/2}\big).\end{aligned}$$

This implies $|\psi(p) - \psi(q)| \leq C_M r$ with C_M as in (4.40).

$\square$

Lemma 4.12. *Let $\psi : H_\nu \to \mathrm{span}\{\nu\}$ be M-intrinsic Lipschitz, $M > 0$.*

i) *There exists a constant $\lambda = \lambda(M) > 0$ such that for all $p, q \in H_\nu$ and for all $r > 0$ we have*

$$p \in D_r^{\psi}(q) \quad \Rightarrow \quad D_r^{\psi}(q) \subset D_{\lambda r}^{\psi}(p). \tag{4.43}$$

ii) *For any* $\lambda > 0$ *there exists a constant* $\mu = \mu(M, \lambda)$ *such that for all* $r > 0$ *and for all* $p, q \in H_\nu$ *with* $p \in D_r^\psi(q)$ *we have* $D_{\lambda r}^\psi(p) \subset D_{\mu r}^\psi(q)$.

Proof. i) Let $p = (z_p, t_p)$ and $q = (z_q, t_q)$. Pick a point $(z, t) \in D_r^\psi(q)$, i.e., by (4.37) we have

$$z = z_q + \zeta_q, \quad t = t_q + \tau_q + Q(z_q + 2\psi(q), \zeta_q), \quad \text{with } (\zeta_q, \tau_q) \in B_r^\perp.$$

Analogously, we have $z_p = z_q + \zeta_p$, $t_p = t_q + \tau_p + Q(z_q + 2\psi(q), \zeta_p)$, with $(\zeta_p, \tau_p) \in B_r^\perp$, and thus $z = z_p + \zeta$ and $t = t_p + \tau + Q(z_p + 2\psi(p), \zeta)$ with

$$\zeta = \zeta_q - \zeta_p, \quad \tau = \tau_q - \tau_p + Q(z_q - z_p + 2(\psi(q) - \psi(p)), \zeta_q - \zeta_p).$$

We thus have $|\zeta| \le 2r$, and using $z_p - z_q = \zeta_p$ and (4.39), we obtain

$$|\tau| \le |\tau_q| + |\tau_p| + 2(|\zeta_p| + 2|\psi(q) - \psi(p)|)|\zeta_q - \zeta_p| \le r^2(6 + 8C_M),$$

where C_M is the constant (4.40). Claim (4.43) follows.

ii) As above, let $z_p = z_q + \zeta_p$, $t_p = t_q + \tau_p + Q(z_q + 2\psi(q), \zeta_p)$, with $(\zeta_p, \tau_p) \in B_r^\perp$. Pick a point $(z, t) \in D_{\lambda r}^\psi(p)$, i.e., by (4.37),

$$z = z_p + \zeta, \quad t = t_p + \tau + Q(z_p + 2\psi(p), \zeta), \quad \text{with } (\zeta, \tau) \in B_{\lambda r}^\perp.$$

Thus we have $z = z_q + \zeta_q$ and $t = t_q + \tau_q + Q(z_q + 2\psi(q), \zeta_q)$ with

$$\zeta_q = \zeta_p + \zeta \quad \text{and} \quad \tau_q = \tau_p + \tau - Q(\zeta - \zeta_p + 2(\psi(q) - \psi(p)), \zeta).$$

Then, by (4.39),

$$|\zeta_q| \le (1 + \lambda) r \quad \text{and} \quad |\tau_q| \le (1 + 3\lambda + 2\lambda^2 + 4\lambda C_M) r^2,$$

and the claim ii) follows with $\mu = \sqrt{1 + 3\lambda + 2\lambda^2 + 4\lambda C_M}$. □

Corollary 4.13. *For any* $M > 0$ *there exists a* $\mu > 0$ *with the following property. Let* $\psi : H_\nu \to \mathrm{span}\{\nu\}$ *be* M*-intrinsic Lipschitz. Then for all* $r, s > 0$ *such that* $s \le 2r$ *and for all* $p, q \in H_\nu$ *we have*

$$D_r^\psi(p) \cap D_s^\psi(q) \neq \emptyset \quad \Rightarrow \quad D_s^\psi(q) \subset D_{\mu r}^\psi(p). \tag{4.44}$$

Proof. Let $w \in D_r^\psi(p) \cap D_s^\psi(q)$. Let $\lambda = \lambda(M) > 0$ be the constant given by Lemma 4.12 part i), and let $\mu = \mu(M, 2\lambda)$ be given by part ii). Then we have

$$D_s^\psi(q) \subset D_{\lambda s}^\psi(w) \subset D_{2\lambda r}^\psi(w) \subset D_{\mu r}^\psi(p).$$

□

Theorem 4.14. *For any* $M > 0$ *there exists a constant* $\mu > 0$ *with the following property. Let* $R \subset \mathbb{R}^+$ *and* $P \subset H_\nu$ *be nonempty bounded sets,* $\psi : H_\nu \to \mathrm{span}\{\nu\}$

be M-intrinsic Lipschitz, and let $\mathcal{D} = \{D_r^\psi(p) : r \in R,\, p \in P\}$. Then there exists a sequence $D_{r_i}^\psi(p_i) \in \mathcal{D}$, $i \in \mathbb{N}$, such that

$$P \subset \bigcup_{i\in\mathbb{N}} D_{\mu r_i}^\psi(p_i) \quad \textit{and} \quad D_{r_i}^\psi(p_i) \cap D_{r_j}^\psi(p_j) = \emptyset \textit{ for } i \neq j.$$

Proof. The proof is a variation, based on Corollary 4.13, of the proof of the $5r$-covering lemma in metric spaces. Let $\mathcal{D}_1 = \mathcal{D}$, $R_1 = \sup\{r > 0 : \text{there exists } D_r^\psi(p) \in \mathcal{D}_1\}$, and pick $D_1 = D_{r_1}^\psi(p_1) \in \mathcal{D}_1$ such that $2r_1 \geq R_1$. Then we let $\mathcal{D}_2 = \{D_r^\psi(p) \in \mathcal{D} : D_r^\psi(p) \cap D_1 = \emptyset\}$ and $R_2 = \sup\{r > 0 : \text{there exists } D_r^\psi(p) \in \mathcal{D}_2\}$. Then we pick $D_2 = D_{r_2}^\psi(p_2) \in \mathcal{D}_2$ such that $2r_2 \geq R_2$.

By induction, once $D_1, \ldots, D_{i-1}$ are chosen, we let $\mathcal{D}_i = \{D_r^\psi(p) \in \mathcal{D} : D_r^\psi(p) \cap (D_1 \cup \ldots \cup D_{i-1}) = \emptyset\}$ and $R_i = \sup\{r > 0 : \text{there exists } D_r^\psi(p) \in \mathcal{D}_i\}$. Then we pick $D_i = D_{r_i}^\psi(p_i) \in \mathcal{D}_i$ such that $2r_i \geq R_i$.

If $\mathcal{D}_i = \emptyset$ for some $i \in \mathbb{N}$ then the selection process ends after a finite number of steps. In this case, each ψ-ball $D_r^\psi(p) \in \mathcal{D}$ intersects at least one of the $D_1, \ldots, D_{i-1}$.

Otherwise, we obtain a disjoint sequence of sets $(D_i)_{i\in\mathbb{N}}$. Because R and P are bounded the union $\bigcup_{i\in\mathbb{N}} D_i$ is also bounded and has thus finite Lebesgue measure in H_ν. By (4.38), we obtain

$$\mathcal{L}^{2n}\Big(\bigcup_{i\in\mathbb{N}} D_i\Big) = \sum_{i\in\mathbb{N}} \mathcal{L}^{2n}(D_i) = 2\omega_{2n-1} \sum_{i\in\mathbb{N}} r_i^{Q-1} < \infty,$$

and it follows that $r_i \to 0$ as $i \to \infty$.

Pick any $D_r^\psi(p) \in \mathcal{D}$, where $r > 0$. There exists $i_0 \in \mathbb{N}$ such that $2r_i < r$ for all $i > i_0$. Then there exists $j \in \{1, \ldots, i_0\}$ such that $D_r^\psi(p) \cap D_j \neq \emptyset$, because otherwise we would contradict the choice of r_i for $i = i_0 + 1$. Let $j \in \{1, \ldots, i_0\}$ be the smallest integer such that $D_r^\psi(p) \cap D_j \neq \emptyset$. Then we have $r \leq R_j \leq 2r_j$ and thus, by (4.44), $D_r^\psi(p) \subset D_{\mu r_j}^\psi(p_j)$, where $\mu > 0$ is given by Corollary 4.13. □

5. Lipschitz approximation

In this section, we prove our main theorem, Theorem 5.1 below. In the proof we need some further properties of instrinsic Lipschitz functions. Let $\psi : H_\nu \to \mathbb{R}$ be intrinsic Lipschitz. Here, we identify the target space with $\mathbb{R}$ and assume without loss of generality that $\nu = (1, 0, \ldots, 0) \in \mathbb{S}^m$. Then $H_\nu = \{p \in \mathbb{H}^n : p_1 = 0\}$ and we can identify H_ν and $\mathbb{R}^{2n}$ via the coordinates $(x_2, \ldots, x_n, y_1, \ldots, y_n, t) \in \mathbb{R}^{2n}$. With abuse of notation, we denote by $X_2, ..., X_n$ and $Y_2, ..., Y_n$ the restrictions of the vector fields in (1.1) to $\mathbb{R}^{2n}$.

Let $W^\psi\psi$ be the distribution acting on $\vartheta \in C_c^1(\mathbb{R}^{2n})$ as

$$\langle W^\psi\psi, \vartheta\rangle = -\int_{\mathbb{R}^{2n}} \Big(\psi \frac{\partial\vartheta}{\partial y_1} - 2\psi^2 \frac{\partial\vartheta}{\partial t}\Big) d\widehat{p},$$

where $d\widehat{p} = dx_2 \dots dx_n dy_1 \dots dy_n dt$ is the Lebesgue measure on $\mathbb{R}^{2n}$. The intrinsic gradient of ψ is then defined as the $2n-1$-vector

$$\nabla^\psi \psi = (X_2\psi, \dots, X_n\psi, W^\psi\psi, Y_2\psi, \dots, Y_n\psi). \tag{5.1}$$

This gradient has to be understood in distributional sense. By [4, Theorem 1.2], $\nabla^\psi\psi$ is in fact a vector of L^∞-functions. Moreover, we have the estimate

$$\|\nabla^\psi\psi\|_{L^\infty(\mathbb{R}^{2n})} \le M. \tag{5.2}$$

This follows from [10, Prop. 3.18 and 4.8].

Before, stating our main result we recall that in Proposition 4.8 we have the relation (4.31), and namely

$$M(L) = \Big(\sqrt{1+\frac{1}{L+2L^2}} - 1\Big)^{-2}, \quad L > 0.$$

Moreover, in Proposition 4.2 there is the restriction $L > 2$, for the case $n = 1$ only.

Theorem 5.1. *Let $L > 0$ if $n \ge 2$ and $L > M(2)$ if $n = 1$. There are constants $k > 1$ and $c(L, n) > 0$ with the following property. For any set $E \subset \mathbb{H}^n$ that is $\mathbb{H}$-perimeter minimizing in B_{kr} with $0 \in \partial E$ and $r > 0$, there exist $\nu \in \mathbb{S}^m$ and an L-intrinsic Lipschitz function $\varphi : H_\nu \to \operatorname{span}\{\nu\}$ such that*

$$\mathcal{S}^{Q-1}\big((\mathrm{gr}(\varphi)\Delta\partial E) \cap B_r\big) \le c(L,n)(kr)^{Q-1}\mathrm{Exc}(E, B_{kr}). \tag{5.3}$$

Proof. Step 1. Let $\nu \in \mathbb{S}^m$ be such that

$$\mathrm{Exc}(E, B_{kr}) = \mathrm{Exc}(E, B_{kr}, \nu) = \frac{1}{(kr)^{Q-1}} \int_{B_{kr}} |\nu_E(w) - \nu|^2 d|\mu_E|(w). \tag{5.4}$$

When $n \ge 2$, let $\eta > 0$ be the constant depending on $\tau = (128\omega_{2n-2})^{-1}$ of Proposition 3.9. When $n = 1$, let $\eta > 0$ and k_0 be the constants given by Proposition 3.10. When $s > 0$, p and q are as in Proposition 3.9 (as in Proposition 3.10 when $n = 1$, respectively), then we have

$$P(E, B_s(p * q)) > \frac{s^{Q-1}}{32}, \tag{5.5}$$

as long as $B_{k_0 s}(p) \subset B_{kr}$. Finally, let $\sigma = \sigma(L) > 0$ be the constant depending on $L > 0$ of Propositions 4.1 and 4.2, and set

$$\varepsilon = \frac{1}{2}\min\{\eta, \sigma\}. \tag{5.6}$$

Let $\alpha > 0$ be a parameter that will be fixed later, in (5.31). In view of Propositions 4.1 and 4.2, we require $\alpha \ge 32$. The set

$$G = \big\{p \in \bar{B}_{\alpha r} \cap \partial E : \mathrm{Exc}(E, B_s(p), \nu) \le \varepsilon \text{ for all } s \in (0, 2\alpha r)\big\} \tag{5.7}$$

is compact. The proof of this fact starts from the closure of the condition $\operatorname{Exc}(E, B_s(p), \nu) \leq \varepsilon$ for any $s < 2\alpha r$ such that $|\mu_E|(\partial B_s(p)) = 0$. The compactness of G then easily follows.

Let $\pi : \mathbb{H}^n \to H_\nu$, $\pi(p) = \nu^\perp(p)$, be the projection onto H_ν defined in (4.3). By Propositions 4.1 and 4.2, we have

$$\|\nu(q^{-1} * p)\| \leq L\|\nu^\perp(q^{-1} * p)\| \quad \text{for all } p, q \in G. \tag{5.8}$$

An elementary computation, that is omitted, shows that $\nu^\perp(p) = \nu^\perp(q)$ implies $\nu^\perp(q^{-1} * p) = 0$. Then, by (5.8), we have

$$\nu^\perp(p) = \nu^\perp(q), \quad p, q \in G \quad \Rightarrow \quad p = q.$$

In other words, the projection π is injective on G and thus there exists a function $\varphi : \pi(G) \to \operatorname{span}\{\nu\}$ such that $p \mapsto p * \varphi(p)$ is the inverse of π restricted to G. By (5.8), the mapping φ is L-intrinsic Lipschitz. By Proposition 4.8, there exists $\psi : H_\nu \to \operatorname{span}\{\nu\}$ that is M-intrinsic Lipschitz with M given by (4.31) and such that $\psi = \varphi$ on $\pi(G)$. We denote by $\Gamma = \operatorname{gr}(\psi)$ the intrinsic graph of ψ, as in (4.29). Notice that when $n \geq 2$, M can be as small as we wish, provided that we start from a small L. When $n = 1$, we can assume that M stays bounded near $M(2)$, provided that we choose L close to 2. From now on, we assume without loss of generality that

$$M \leq M(2) + 1. \tag{5.9}$$

Step 2. Let $U = (\partial E \setminus G) \cap B_{\alpha r}$. For any $p \in U$ there exists $s_p \in (0, 2\alpha r)$ such that

$$\int_{B_{s_p}(p)} |\nu_E(w) - \nu|^2 d|\mu_E|(w) > \varepsilon s_p^{Q-1}. \tag{5.10}$$

The family of balls $\big\{B_{s_p}(p) : p \in U\big\}$ is a covering of U. By the $5r$-covering Lemma, there exists a sequence of points $p_i \in U$ and radii $s_i = s_{p_i}$, $i \in \mathbb{N}$, such that the balls $B_{s_i}(p_i)$ are pairwise disjoint and $\big\{B_{5s_i}(p_i) : i \in \mathbb{N}\big\}$ is still a covering of U. To have the inclusion $B_{5s_i}(p_i) \subset B_{kr}$, we require

$$11\alpha \leq k. \tag{5.11}$$

By (2.9), (2.4) and (2.6), we obtain

$$\begin{aligned}
\mathcal{S}^{Q-1}(U) &\leq \sum_{i=1}^{\infty} \mathcal{S}^{Q-1}(\partial E \cap B_{5s_i}(p_i)) = \sum_{i=1}^{\infty} \mathcal{S}^{Q-1}(\partial E \cap B_{5s_i}(p_i)) \\
&= \delta(n)^{-1} \sum_{i=1}^{\infty} P(E; B_{5s_i}(p_i)) \leq c_1 \delta(n)^{-1} 5^{Q-1} \sum_{i=1}^{\infty} s_i^{Q-1}.
\end{aligned}$$

Now, by (5.4), (5.10), and (5.11)

$$
\begin{aligned}
\mathcal{S}^{Q-1}(U) &\le c_1\delta(n)^{-1}5^{Q-1}\varepsilon^{-1}\sum_{i=1}^{\infty}\int_{B_{s_i}(p_i)}|\nu_E(w)-\nu|^2 d|\mu_E|(w)\\
&\le c_1\delta(n)^{-1}5^{Q-1}\varepsilon^{-1}\int_{B_{kr}}|\nu_E(w)-\nu|^2 d|\mu_E|(w)\\
&= (kr)^{Q-1}c_4\mathrm{Exc}(E,B_{kr}),
\end{aligned}
\tag{5.12}
$$

where we set $c_4 = c_1\delta(n)^{-1}5^{Q-1}\varepsilon^{-1}$. Now, (5.12) implies the first half of (5.3); namely,

$$
\mathcal{S}^{Q-1}((\partial E\setminus\Gamma)\cap B_r) \le \mathcal{S}^{Q-1}((\partial E\setminus G)\cap B_r) \le (kr)^{Q-1}c_4\mathrm{Exc}(E,B_{kr}). \tag{5.13}
$$

Step 3. We claim we can assume that $G\cap B_{r/3}\neq\emptyset$. If we had $G\cap B_{r/3}=\emptyset$, then by (2.9), (2.4), and (2.8)

$$
\begin{aligned}
\mathcal{S}^{Q-1}(U) &\ge \mathcal{S}^{Q-1}((\partial E\setminus G)\cap B_{r/3})\\
&= \mathcal{S}^{Q-1}(\partial E\cap B_{r/3}) = \delta(n)^{-1}P(E,B_{r/3})\\
&\ge \delta(n)^{-1}c_3(r/3)^{Q-1}.
\end{aligned}
\tag{5.14}
$$

From (5.12) and (5.14), we would obtain

$$
\mathrm{Exc}(E,B_{kr}) \ge c_3c_4^{-1}\delta(n)^{-1}(3k)^{1-Q}.
$$

Then, with $\Gamma = H_\nu$ we would have

$$
\begin{aligned}
\mathcal{S}^{Q-1}((\Gamma\setminus\partial E)\cap B_r) &\le \mathcal{S}^{Q-1}(\Gamma\cap B_r)\\
&= \delta(n)^{-1}P(\{p_1>0\},B_r) = \delta(n)^{-1}2\omega_{2n-1}r^{Q-1}\\
&\le 2\omega_{2n-1}c_3^{-1}c_4 3^{Q-1}(rk)^{Q-1}\mathrm{Exc}(E,B_{kr}),
\end{aligned}
$$

and the claim (5.3) would follow.

Step 4. We claim that for all $s,r>0$ and for all $w\in\pi(\Gamma\cap B_r)$ there holds

$$
r\le s/5 \quad\Rightarrow\quad B_{r/3}\subset C_s^{\psi}(w). \tag{5.15}
$$

It is sufficient to prove the inclusion $\pi(B_{r/3})\subset D_s^{\psi}(\pi(p))$. Let $w*\psi(w)\in B_r$ with $w=(z,t)$ and $q=(\zeta,\tau)\in B_{r/3}$. Then we have

$$
|\psi(w)|<r,\quad \max\{|z|,|t|^{1/2}\}<2r,\quad \max\{|\zeta|,|\tau|^{1/2}\}<r/3. \tag{5.16}
$$

By (4.3), we have $\pi(q) = \big(\zeta^\perp, \tau - Q(\zeta^\perp,\zeta^\top)\big)$, and thus, by (4.37), $\zeta^\perp = z+\zeta'$ and $\tau - Q(\zeta^\perp,\zeta^\top) = t+\tau'+Q(z+2\psi(w),\zeta')$ with

$$
\zeta' = z-\zeta^\perp \quad\text{and}\quad \tau' = \tau - t - Q(z+2\psi(w),\zeta') - Q(\zeta^\perp,\zeta^\top). \tag{5.17}
$$

From (5.16), we obtain $|\zeta'|\le|z|+|\zeta|\le 7r/3$ and

$$
|\tau'|\le|\tau|+|t|+2(|z|+2|\psi(w)|)|\zeta'|+2|\zeta^\perp||\zeta^\top|\le 23r^2,
$$

and the claim $\max\{|\zeta'|,|\tau'|^{1/2}\}<s$ follows.

Step 5. Let $V = (\Gamma \setminus \partial E) \cap B_r$. We claim there exists a constant $c_5 > 0$ such that

$$\mathcal{S}^{Q-1}(V) \leq c_5 \mathcal{S}^{Q-1}(U). \tag{5.18}$$

The inequalities (5.18) and (5.12) complete the proof of the second half of (5.3). This along with (5.13) proves the theorem. The proof of (5.18) finishes with *Step 8*.

Pick a point $p = (z, t) \in \pi(V)$. Because $V \subset B_r$, then we have

$$\|p\| = \max\{|z|, |t|^{1/2}\} < 2r. \tag{5.19}$$

We also fix the number $\beta = \beta(M) > 0$ such that

$$3 + 4C_M\beta = \beta^2, \tag{5.20}$$

where C_M is the constant (4.40). The choice (5.20) of β will ensure the inclusion (5.29). Then we set

$$s_p = \frac{1}{\beta} \max\{s > 0 : C_s^\psi(p) \cap G = \emptyset\}.$$

For a moment, we omit reference to p and write $s = s_p$. We have $C_{\beta s}^\psi(p) \cap G = \emptyset$ and, by *Step 3*, $G \cap B_{r/3} \neq \emptyset$. Thus $B_{r/3}$ is not contained in $C_{\beta s}^\psi(p)$, and by *Step 4*, (5.15), we have

$$\beta s < 5r. \tag{5.21}$$

By the maximality of $s > 0$, there exists $q \in \partial D_{\beta s}^\psi(p)$ such that $q * \psi(q) \in G$. By the formula (4.37), q is of the form

$$q = \big(z + \zeta, t + \tau + Q(z + 2\psi(p), \zeta)\big), \quad \text{with } \max\{|\zeta|, |\tau|^{1/2}\| = \beta s,$$

where, by (5.19) and (5.21),

$$|z + \zeta| < 2r + \beta s < 7r,$$

and, also using $|\psi(p)| < r$,

$$|t + \tau + Q(z + 2\psi(p), \zeta)| \leq (2r)^2 + (\beta s)^2 + 8r\beta s < 69r^2.$$

We deduce that $\|q\| < 9r$. By (4.39) and (5.21), there also holds

$$|\psi(q)| \leq |\psi(q) - \psi(p)| + |\psi(p)| \leq C_M\beta s + r < (1 + 5C_M)r. \tag{5.22}$$

We conclude that, by (2.1) , $\|q\| < 9r$, and (5.22)

$$\|q * \psi(q)\| \leq \|q\| + \|\psi(q)\| \leq (10 + 5C_M)r. \tag{5.23}$$

At this point, we require the condition on k

$$10 + 5C_M \leq \frac{k}{2}, \tag{5.24}$$

and so we have $q * \psi(q) \in B_{kr/2}$.

We are going to use Lemmas 3.9 and 3.10, see (5.5), to estimate from below $P(E, B_s(p * \psi(q))$. To this aim, notice that we have $p * \psi(q) = q * \psi(q) * u$ where

$u = \psi(q)^{-1} * q^{-1} * p * \psi(q)$ satisfies $\langle u, \nu\rangle = 0$. By the structure (4.37) of the box $D^\psi_{\beta s}(p)$, after a short computation one can check that

$$u = \big(-\zeta, -\tau - 2Q(\psi(p) - \psi(q), \zeta)\big), \quad \text{with } \max\{|\zeta|, |\tau|^{1/2}\} = \beta s. \tag{5.25}$$

Using (5.25) and (4.39), we obtain

$$|\tau + 2Q(\psi(p) - \psi(q), \zeta)| \le (\beta s)^2(1 + 4C_M),$$

and so, by $\beta \le 2$ and (5.9), there exists a constant $\gamma > 1$ independent of M such that

$$\|u\| \le \beta\sqrt{1 + 4C_M} s \le \gamma s. \tag{5.26}$$

Let $k_0 = 370$ be the constant given by Lemma 3.10. With the condition

$$k \ge 2k_0, \tag{5.27}$$

by (5.24) and (5.23), we have the inclusion $B_{k_0 s}(q * \psi(q)) \subset B_{kr}$. In particular, we can use Lemma 3.10. Then we have

$$P(E, B_{s/\gamma}(p * \psi(q)) \ge \frac{1}{32}(s/\gamma)^{Q-1}. \tag{5.28}$$

Step 6. We claim that with the choice (5.20) we have

$$B_s(p * \psi(q)) \subset C^\psi_{\beta s}(p). \tag{5.29}$$

To prove this inclusion, it is sufficient to show that $\pi(B_s(p * \psi(q)) \subset D^\psi_{\beta s}(p)$. In fact, a point $p * \psi(q) * w$ with $p = (z, t)$ and $w = (\zeta, \tau) \in B_s$ is of the form $p * \psi(q) * w = (z + \zeta', t + \tau' + Q(z, +2\psi(p), \zeta'))$, where

$$\zeta' = \zeta^\perp \quad \text{and} \quad \tau' = \tau + 2Q(\psi(q) - \psi(p), \zeta^\perp) - Q(\zeta^\perp, \zeta^\top),$$

and thus, by (4.39) and (5.20),

$$|\tau'| \le |\tau| + 4|\psi(q) - \psi(p)||\zeta^\perp| + 2|\zeta^\perp||\zeta^\top| \le s^2(3 + 4C_M\beta) \le \beta^2 s^2.$$

This proves (5.29). Moreover, we also have

$$B_s(p * \psi(q)) \subset B_{\delta r}, \quad \text{with } \delta = 3 + 5C_M + \frac{5}{\beta}. \tag{5.30}$$

In fact, by (5.19) and (5.22),

$$\|p * \psi(q)\| \le \|p\| + \|\psi(q)\| \le (3 + 5C_M)r,$$

an (5.30) follows from (5.21).

Step 7. We fix the constants α and k. Recalling (5.30), the final choice for α is

$$\alpha = \max\{32, \delta\}. \tag{5.31}$$

Now, (5.29) and (5.30) imply that

$$B_s(p * \psi(q)) \subset C^\psi_{\beta s}(p) \cap B_{\alpha r}. \tag{5.32}$$

The constant k has to satisfy the conditions (5.11), (5.24), and (5.27). Then we can choose

$$k = \max\{11\alpha, 20 + 10C_M, 2k_0\}. \tag{5.33}$$

Step 8. We finish the proof of (5.18). Since we have $G \cap C^{\psi}_{\beta s}(p) = \emptyset$, by (5.32), (2.9), (2.4), and (5.28) we obtain

$$\begin{aligned} \mathcal{S}^{Q-1}(U \cap C^{\psi}_{\beta s}(p)) &= \mathcal{S}^{Q-1}((\partial E \setminus G) \cap B_{\alpha r} \cap C^{\psi}_{\beta s}(p)) \\ &\geq \mathcal{S}^{Q-1}(\partial E \cap B_s(p * \psi(q))) \\ &\geq \delta(n)^{-1} P(E, B_s(p * \psi(q))) \\ &\geq \frac{1}{32\delta(n)}(s/\gamma)^{Q-1}. \end{aligned} \tag{5.34}$$

Recall that $s = s_p$. The family of sets $\{D^{\psi}_{\beta s_p}(p) : p \in \pi(V)\}$ is a converging of $\pi(V)$. By Theorem 4.14, there exists a sequence $p_i \in \pi(V)$, $i \in \mathbb{N}$, such that, with $s_i = \beta s_{p_i}$, we have

$$D^{\psi}_{s_i}(p_i) \cap D^{\psi}_{s_j}(p_j) = \emptyset \text{ for } i \neq j \text{ and } \pi(V) \subset \bigcup_{i \in \mathbb{N}} D^{\psi}_{\mu s_i}(p_i), \tag{5.35}$$

where $\mu = \mu(M)$ is the constant given by Theorem 4.14.

By the area formula for intrisic Lipschitz functions (see Theorem [10, Lemma 4.30]),(5.2), (5.35), (4.38), (5.34), and $s_i = \beta s_{p_i}$ we have

$$\begin{aligned} \mathcal{S}^{Q-1}(V) &= \delta(n)^{-1} \int_{\pi(V)} \sqrt{1 + |\nabla^{\psi}\psi(\widehat{p})|^2}\, d\widehat{p} \leq \delta(n)^{-1}\sqrt{1+M^2}\mathcal{L}^{2n}(\pi(V)) \\ &\leq \delta(n)^{-1}\sqrt{1+M^2} \sum_{i=1}^{\infty} \mathcal{L}^{2n}(D^{\psi}_{\mu s_i}(p_i)) \\ &= 2\omega_{2n-1}\delta(n)^{-1}\sqrt{1+M^2}(\mu\beta)^{Q-1} \sum_{i=1}^{\infty} s_{p_i}^{Q-1} \\ &\leq c_5 \sum_{i=1}^{\infty} \mathcal{S}^{Q-1}(U \cap C^{\psi}_{\beta s_{p_i}}(p_i)) \leq c_5 \mathcal{S}^{Q-1}(U), \end{aligned}$$

with $c_5 = 64\omega_{2n-1}\sqrt{1+M^2}(\mu\beta/\gamma)^{Q-1}$. This ends the proof of (5.18) and so of the Theorem.

□

References

[1] Almgren, Jr., F. J. *Almgren's big regularity paper*, vol. 1 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000. Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2, With a preface by Jean E. Taylor and Vladimir Scheffer.

[2] Ambrosio, L. *Corso introduttivo alla teoria geometrica della misura ed alle superfici minime*. Scuola Normale Superiore, Pisa, 1997.

[3] Arena, G., Caruso, A. O., and Monti, R. Regularity properties of H-convex sets. *J. Geom. Anal. 22*, 2 (2012), 583–602.

[4] Bigolin, F., Caravenna, L., and Serra Cassano, F. Intrinsic lipschitz graphs in Heisenberg groups and continuous solutions of balance equations. *Preprint* (2012).

[5] Capogna, L., Citti, G., and Manfredini, M. Regularity of non-characteristic minimal graphs in the Heisenberg group $\mathbb{H}^1$. *Indiana Univ. Math. J. 58*, 5 (2009), 2115–2160.

[6] Capogna, L., Citti, G., and Manfredini, M. Smoothness of Lipschitz minimal intrinsic graphs in Heisenberg groups $\mathbb{H}^n$, $n > 1$. *J. Reine Angew. Math. 648* (2010), 75–110.

[7] Cheng, J.-H., Hwang, J.-F., and Yang, P. Regularity of C^1 smooth surfaces with prescribed p-mean curvature in the Heisenberg group. *Math. Ann. 344*, 1 (2009), 1–35.

[8] De Lellis, C., and Spadaro, E. N. Higher integrability and approximation of minimal currents. *Preprint* (2011).

[9] Franchi, B., Serapioni, R., and Serra Cassano, F. Rectifiability and perimeter in the Heisenberg group. *Math. Ann. 321*, 3 (2001), 479–531.

[10] Franchi, B., Serapioni, R., and Serra Cassano, F. Intrinsic Lipschitz graphs in Heisenberg groups. *J. Nonlinear Convex Anal. 7*, 3 (2006), 423–441.

[11] Garofalo, N., and Nhieu, D.-M. Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces. *Comm. Pure Appl. Math. 49*, 10 (1996), 1081–1144.

[12] Giusti, E. *Minimal surfaces and functions of bounded variation*, vol. 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[13] Kirchheim, B., and Serra Cassano, F. Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 3*, 4 (2004), 871–896.

[14] Le Donne, E., and Bellettini, C. Regularity of sets with constant horizontal normal in the Engel group. *Preprint* (2012).

[15] Monti, R., and Vittone, D. Sets with finite $\mathbb{H}$-perimeter and controlled normal. *Math. Z. 270*, 1-2 (2012), 351–367.

[16] Pauls, S. D. H-minimal graphs of low regularity in $\mathbb{H}^1$. *Comment. Math. Helv. 81*, 2 (2006), 337–381.

[17] Ritoré, M. Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ with low regularity. *Calc. Var. Partial Differential Equations 34*, 2 (2009), 179–192.

[18] Schoen, R., and Simon, L. A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals. *Indiana Univ. Math. J. 31*, 3 (1982), 415–434.

[19] Serra Cassano, F., and Vittone, D. Graphs of bounded variation, existence and local boundedness of non-parametric minimal surfaces in Heisenberg groups. *Preprint* (2010).