REGULAR DOMAINS IN HOMOGENEOUS GROUPS

ROBERTO MONTI AND DANIELE MORBIDELLI

ABSTRACT. We study John, uniform and non-tangentially accessible domains in homogeneous groups of steps 2 and 3. We show that $C^{1,1}$ domains in groups of step 2 are non-tangentially accessible and we give an explicit condition which ensures the John property in groups of step 3.

1. INTRODUCTION

This paper deals with the study of John, uniform and non-tangentially accessible domains in homogeneous groups (Carnot groups) of steps 2 and 3 endowed with the Carnot–Carathéodory distance induced by a system of generators of their Lie algebra. Such regular domains are strongly related to the global embedding properties of Sobolev spaces and to the study of the boundary behavior of harmonic functions with respect to sub-elliptic Laplacians.

John domains have been introduced by John [Joh]. It is known from the classical theory of Sobolev spaces that if $\Omega \subset \mathbb{R}^n$ is a John domain and $\nabla u \in L^p(\Omega)$, then $u \in L^{pn/(n-p)}(\Omega)$ (see the references [Be] and [Bo]). In the setting of homogeneous groups it is known that if Ω is a John domain with respect to the Carnot–Carathéodory distance, the Sobolev-Poincaré inequality

$$(1.1) \qquad \Big(\int_\Omega |u - u_\Omega|^{p^*} \, dx \Big)^{1/p^*} \leq C \Big(\int_\Omega |Xu|^p \, dx \Big)^{1/p}$$

holds for all $u \in C^1(\mathbb{R}^n)$, where u_Ω is the average of u over Ω, $Xu = (X_1 u, ..., X_m u)$ is the sub-elliptic gradient of u with respect to the vector fields $X_1, ..., X_m$ generating the Lie algebra of the group, the integer Q is the homogeneous dimension of the group, $1 \leq p < Q$ and $p^* = pQ/(Q-p)$. Different forms of inequality (1.1) have been proved in [FLW], [GN1] and [HK] and they are a consequence of the Poincaré inequality proved by Jerison in [J] and of a chaining argument (relations between John condition and chaining properties are studied in [BKL], [GN1] and [HK]).

Uniform domains (also known as (ϵ, δ) domains) are a sub-class of John domains. The definition of uniform domain is due to Martio and Sarvas [MS] and to Jones [Jon]. In the latter paper an extension theorem for Sobolev functions in uniform domains is proved, and the theorem is generalized in [VG] and [GN2] to the setting of Carnot–Carathéodory spaces.

A sub-class of uniform domains are non-tangentially accessible domains (briefly NTA domains) which, in the Euclidean case, were introduced by Jerison and Kenig [JK] in connection with the study of the boundary behavior of harmonic functions. The notion of NTA domain can be formulated in terms of the Carnot-Carathéodory metric associated with a sub-elliptic Laplacian $\mathcal{L} = \sum X_j^2$ and used to prove a Fatou type theorem for non-negative weak solutions u of the equation $\mathcal{L}u = 0$ (see [CG] and see also [W]). In connection with harmonic measures for sub-elliptic Laplacians, a class of "cuspidal Harnack domains" has been recently studied in [FF1] and [FF2].

It is known that inequality (1.1) may fail even for smooth domains in Carnot–Carathéodory spaces. See [J, Section 6], where a counterexample is given. The natural problem that arises is to find explicit examples of domains where inequality (1.1) holds. This problem becomes very intriguing for homogeneous groups of step larger than 2 (in the step-two case more refined properties will be established; see the discussion below). In this paper we give a first answer to this problem.

In groups of step 3 we have found a differential condition on the boundary near characteristic points that seems to control in a sharp way the flatness behavior of the surface in order to have regularity. Our result gives a sufficient condition for the John condition and thus for (1.1). Precisely, consider two vector fields X_1 and X_2 in $\mathbb{R}^4$ generating the Lie algebra of a homogeneous group of step 3 with non-trivial commutators $[X_1, X_2] = X_3$ and $[X_1, X_3] = [X_2, X_3] = X_4$. Here $X_j = \partial/\partial x_j$, $j = 1, \ldots, 4$, at the origin (a complete description of this structure is given in Section 4). In this setting we prove the following result:

Theorem 1.1. *If $\Omega \subset \mathbb{R}^4$ is an* admissible *domain, then it is a John domain (with respect to the Carnot–Carathéodory distance generated by X_1 and X_2).*

The definition of *admissible domain* will be given in a detailed way in Section 4. We observe here that the key property that an *admissible domain* Ω should enjoy is the following "flatness" condition. If $\Omega = \{x \in \mathbb{R}^4 : \Phi(x_1, x_2, x_3, x_4) > 0\}$, we require that for all points in $\partial\Omega$ the estimate

$$(1.2) \quad |X_1^2\Phi| + |X_2^2\Phi| + |(X_1X_2 + X_2X_1)\Phi| \le k(|X_1\Phi|^{1/2} + |X_2\Phi|^{1/2} + |X_3\Phi|)$$

holds for a suitable positive constant k (see Lemma 4.3 and Theorem 4.4). This condition implies that at characteristic points of second type, that is, characteristic points where also $X_3\Phi$ vanishes (see Section 4), the second derivatives of Φ along X_1 and X_2 must also vanish. In particular, if (1.2) holds and Ω agrees near the origin with the set $\{x \in \mathbb{R}^4 : x_4 > \varphi(x_1, x_2, x_3)\}$, $\varphi(0) = 0$, $\nabla\varphi(0) = 0$, then (see Lemma 4.3) it should be $|\varphi(x_1, x_2, x_3)| \le k(|x_1| + |x_2| + |x_3|^{1/2})^3$. This flatness condition is not ensured by Euclidean regularity. In Example 5.2 we also explicitly construct admissible domains.

On the other hand, we have the following necessary condition, whose proof is contained in Proposition 5.6:

Theorem 1.2. *If a domain Ω locally agrees near the origin with the set $\{x \in \mathbb{R}^4 : x_4 > \varphi(x_1, x_2, x_3)\}$ and $|\varphi(x_1, x_2, x_3)| \le k(|x_1| + |x_2| + |x_3|^{1/2})^\gamma$ for some γ strictly larger than 3, then the Sobolev-Poincaré inequality (1.1) does not hold in Ω.*

This shows two surprising facts: (i) if a domain Ω locally agrees near the origin with the half-space $x_4 > 0$, then (1.1) cannot hold; (ii) the Sobolev-Poincaré inequality is false in the homogeneous ball $(x_1^2 + x_2^2)^6 + x_3^6 + x_4^4 < 1$. The latter fact disproves a conjecture stated in [CG, p. 429].

In a forthcoming paper we shall show how conditions similar to (1.2) can be used to identify regular domains in the different context of diagonal vector fields of the form $X_j = \lambda_j(x)\frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$, where the λ_j's are suitable functions.

In the setting of step 2 groups the situation is different. It has been proved that Carnot-Carathéodory balls in the Heisenberg group are uniform domains (see [VG]) but not NTA domains (see [CG]); examples of uniform domains in the Heisenberg group can also be found by means of quasiconformal mappings (see [CT]); half spaces and cubes centered at the origin in the Heisenberg group are uniform domains (see [G]). It was conjectured in [GN1], [CG] and [CGN1] that in any group of step 2 a connected, bounded open set with boundary of class $C^{1,1}$ is an NTA domain. A partial positive answer to this conjecture has been given in [CG] for sets with cylindrical symmetry near each characteristic point. An improvement has been announced to the authors by Luca Capogna and Nicola Garofalo for sets with "strongly isolated" characteristic set in the Heisenberg group. In the present paper we prove the full statement:

Theorem 1.3. *Any $C^{1,1}$ domain in a step 2 homogeneous group is NTA.*

Theorem 1.3 is sharp in the following sense: in groups of step 2, for any $\alpha \in (0, 1)$ there are open sets of class $C^{1,\alpha}$ for which the Sobolev–Poincaré inequality (1.1) does not hold (see Example 5.1).

Theorem 1.3 was recently used in cooperation with the results of [DGN] to give a complete characterization (in C^2 domains) of the trace space for Sobolev functions in step 2 homogeneous groups (see [DGN, Thm. 13.5]).

Since the proofs in the paper are technical we would like to sketch here the key ideas in the simple situation of the Heisenberg group. We shall briefly discuss only the John property, requiring a deeper analysis for the uniform condition. Consider the vector fields $X_1 = \partial_{x_1} - x_2\partial_{x_3}$ and $X_2 = \partial_{x_2} + x_1\partial_{x_3}$ in $\mathbb{R}^3$. Here $\mathbb{R}^3$ is equipped with the group law $x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1y_2 - x_2y_1)$, with the homogeneous norm $\|x\| = \left((x_1^2 + x_2^2)^2 + x_3^2\right)^{1/4}$ and with the left invariant distance $d(x, y) = \|x^{-1} \cdot y\|$. We consider an open set $\Omega = \{x_3 > \varphi(x_1, x_2)\}$, where $\varphi(0, 0) = 0$ and φ is a smooth function. Since rotations around the x_3-axis are isometries of the group, there is no loss of generality in requiring $\partial_{x_1}\varphi(0, 0) = -\nu$ with $\nu \geq 0$, and $\partial_{x_2}\varphi(0, 0) = 0$. We shall explain how to construct a John curve, i.e. a curve satisfying condition (1.3) below, starting from the origin. By left translation, this will also produce John curves starting from any point of the boundary of Ω. Moreover, since the map $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + \alpha)$, $\alpha \in \mathbb{R}$, is a left translation, the natural John curve starting from any point $(x_1, x_2, \varphi(x_1, x_2) + \alpha)$, $\alpha \geq 0$, can be consequently obtained.

Our curve γ will be constructed by two pieces:

$$\gamma(t) = \begin{cases} (t, 0, 0) & \text{if } 0 < t \leq \sigma\nu := t_1, \\ (t_1, 0, t - t_1) & \text{if } t \geq t_1. \end{cases}$$

Here $\sigma > 0$ is a parameter depending on the given function φ (it must not depend on ν) and will be fixed during the argument. We note that if $\nu = 0$, i.e. $\nabla\varphi(0, 0) = 0$ (the origin is a characteristic point), then $t_1 = 0$, the first piece of the path disappears and we simply have $\gamma(t) = (0, 0, t)$; this curve is not rectifiable.

We show that γ satisfies the following "John property":

$$(1.3) \qquad \text{dist}(\gamma(t); \partial\Omega) \geq \lambda d(\gamma(t), 0) \quad \text{for all } t > 0$$

for some small constant $\lambda > 0$ depending on φ, but not on ν (in Section 2 (1.3) is compared with similar conditions involving rectifiable curves). Subtracting from φ its linear part we get $\varphi(x) = -\nu x_1 + \psi(x_1, x_2)$ where, in view of the smoothness of φ, the remainder ψ satisfies for some uniform constant $C_0 > 0$ the quadratic growth estimate

$$|\psi(x_1, x_2)| \leq C_0(x_1^2 + x_2^2)$$

(the constant C_0 is essentially the Lipschitz constant of $\nabla\varphi$). Points of the ball $B(\gamma(t), \lambda d(\gamma(t), 0)) = \gamma(t) \cdot B(0, \lambda\|\gamma(t)\|)$ are of the form $\gamma(t) \cdot h$, with $\|h\| < \lambda\|\gamma(t)\|$. Thus (1.3) is equivalent to $\gamma(t) \cdot h \in \Omega$, that is,

$$(1.4) \qquad\qquad (\gamma(t) \cdot h)_3 \geq \varphi(\gamma(t) \cdot h)$$

for all $h \in \mathbb{R}^3$, $\|h\| < \lambda\|\gamma(t)\|$. By the quadratic estimate for ψ, (1.4) is ensured by

$$(1.5) \qquad (\gamma(t) \cdot h)_3 + \nu(\gamma(t) \cdot h)_1 \geq C_0 \left[(\gamma(t) \cdot h)_1^2 + (\gamma(t) \cdot h)_2^2\right].$$

Inequality (1.5) will be first checked for $t \leq t_1 = \sigma\nu$. Note that

$$\gamma(t) \cdot h = (t, 0, 0) \cdot (h_1, h_2, h_3) = (t + h_1, h_2, h_3 + th_2).$$

Then (1.5) becomes $h_3 + th_2 + \nu(t + h_1) \geq C_0\left[(t + h_1)^2 + h_2^2\right]$, that is,

$$\nu(t + h_1) \geq -h_3 - th_2 + C_0\left[(t + h_1)^2 + h_2^2\right].$$

Since $\|h\| \leq \lambda t$, the terms in the right-hand side can be estimated as: $-h_3 \leq |h_3| \leq (\lambda t)^2$, $-th_2 \leq |th_2| \leq \lambda t^2$, and the terms in square brackets can be estimated by Ct^2, with C an absolute constant. Moreover, in the left-hand side we have

$$\nu(t + h_1) \geq \nu(t - \|h\|) \geq \frac{1}{2}\nu t,$$

as soon as $\lambda \leq 1/2$. We ultimately get the inequality

$$(1.6) \qquad\qquad \nu t \geq C_0 t^2,$$

where C_0 is an absolute constant possibly larger than the ones written above. This inequality holds as soon as $t \leq \sigma\nu$, where σ is any positive fixed constant satisfying $\sigma \leq 1/C_0$.

We now check (1.5) for $t \geq t_1 = \sigma\nu$. Note that

$$\gamma(t) \cdot h = (t_1, 0, t - t_1) \cdot h = (t_1 + h_1, h_2, t - t_1 + h_3 + t_1 h_2).$$

Then (1.5) becomes $t - t_1 + h_3 + t_1 h_2 + \nu(t_1 + h_1) \geq C_0\left[(t_1 + h_1)^2 + h_2^2\right]$, that is,

$$(1.7) \qquad t - t_1 + \nu(t_1 + h_1) \geq -h_3 - t_1 h_2 + C_0\left[(t_1 + h_1)^2 + h_2^2\right].$$

This inequality has to be checked for $\|h\| \leq \lambda\|\gamma(t)\| = \lambda(t_1^4 + (t - t_1)^2)^{1/4}$, or, letting $a = (t - t_1)^{1/2}$, for $\|h\| \leq \lambda(t_1^4 + a^4)^{1/4}$. The terms in the right-hand side of (1.7) can be estimated as follows: $-h_3 \leq |h_3| \leq C\lambda(t_1^2 + a^2)$ and $-t_1 h_2 \leq |t_1 h_2| \leq t_1\|h\| \leq C\lambda(t_1^2 + t_1 a)$. Moreover,

$$(t_1 + h_1)^2 \leq (t_1 + \lambda(t_1^4 + a^4)^{1/4})^2 \leq (Ct_1 + C\lambda a)^2,$$

while $h_2^2 \leq \|h\|^2 \leq \lambda^2(t_1^4 + a^4)^{1/2} \leq C(t_1^2 + \lambda^2 a^2)$. The left-hand side can be estimated as follows:

$$\nu(t_1 + h_1) \geq \nu t_1 - \nu\|h\| \geq \nu t_1 - \nu\lambda C(t_1 + a) \geq \frac{1}{2}\nu t_1 - \lambda C\nu a,$$

as soon as $\lambda \leq \frac{1}{2C}$.

Then (1.7) will be proved if the following inequality holds:

$$a^2 + \frac{1}{2}\nu t_1 - \lambda C\nu a \geq C\lambda t_1^2 + C\lambda t_1 a + C\lambda a^2 + Ct_1^2 + C\lambda^2 a^2.$$

Choosing λ small enough the terms $C\lambda a^2$ and $C\lambda^2 a^2$ in the right-hand side can be absorbed in the left-hand side. Note also that $C\lambda t_1 a \leq \frac{C}{2}\lambda t_1^2 + \frac{C}{2}\lambda a^2$, so that $\frac{C}{2}\lambda a^2$ can be absorbed in the left-hand side, also. Then we find the stronger inequality $a^2 + \nu t_1 \geq C(t_1^2 + \lambda\nu a + \lambda t_1 a)$ which is ensured by $a^2 + \nu t_1 \geq C(t_1^2 + \lambda\nu a)$, and recalling now that $t_1 = \sigma\nu$ we finally get

(1.8) $$a^2 + \sigma\nu^2 \geq C_1(\sigma^2\nu^2 + \lambda\nu a).$$

We slightly modify the choice of σ requiring $C_1\sigma^2 \leq \sigma/2$. Then, in order to prove (1.8) it is enough to verify that

$$a^2 + \frac{1}{2}\sigma\nu^2 \geq C_1\lambda\Big(\frac{1}{2}\nu^2 + \frac{1}{2}a^2\Big).$$

This inequality is satisfied for any $a \geq 0$ provided that $C_1\lambda/2 \leq 1$ and $C_1\lambda \leq \sigma$. Indeed, such a λ can be found and it does not depend on ν, as required.

The described argument proves that smooth domains in the Heisenberg group enjoy the John property. They also enjoy the uniform property; this can be proved taking suitable cones having as core the John curves constructed above and showing that these cones meet appropriately (see Theorem 3.2).

Before closing this introduction we briefly explain our notation. In the paper we denote by C (or k) absolute positive constants (they may depend on the surface we are considering). By $a \lesssim b$ we mean $a \leq Cb$, and by $a \simeq b$ we mean $C^{-1}a \leq b \leq Ca$. By ε_0 and C_0 we denote, as well, absolute positive constants which are respectively smaller and larger than 1. The small parameter λ is used to denote the "aperture" of cones and in Sections 3 and 4 we shall several times write λ instead of $o(1)$, as $\lambda \to 0$. Finally, if $x, y \in \mathbb{R}^n$, then $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ denotes the usual Euclidean inner product.

2. John, uniform and NTA domains. Some general facts

In this section we recall the basic definitions and some general known results concerning John, uniform and NTA domains (see, for example, [MS], [V], [CT], [CG], [HK]) We consider a metric space (M, d). If $\gamma : [0, 1] \to M$ is a curve, we denote by $\text{length}(\gamma)$ the length of γ and by $\gamma_{[a,b]} : [0, b-a] \to M$ the curve $\gamma_{[a,b]}(t) = \gamma(t+a)$, $0 \leq a \leq b \leq 1$. A curve is rectifiable if $\text{length}(\gamma) < +\infty$.

Definition 2.1. Let (M, d) be a metric space. A bounded open set $\Omega \subset M$ is a *John domain* if there exist $x_0 \in \Omega$ and $\lambda > 0$ such that for every $x \in \Omega$ there exists a continuous rectifiable curve $\gamma : [0, 1] \to \Omega$, such that $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$\text{dist}(\gamma(t); \partial\Omega) \geq \lambda \, \text{length}(\gamma_{[0,t]})$$

for all $t \in [0, 1]$.

Definition 2.2. Let (M, d) be a metric space. An open set $\Omega \subset M$ is a *uniform domain* if there exists $\lambda > 0$ such that for every $x, y \in \Omega$ there exists a continuous rectifiable curve $\gamma : [0, 1] \to \Omega$ connecting them and such that

(2.1) $$\text{length}(\gamma) \leq \frac{1}{\lambda}d(x, y)$$

and

$$(2.2) \qquad \operatorname{dist}(\gamma(t); \partial\Omega) \geq \lambda \min\{\operatorname{length}(\gamma_{[0,t]}), \operatorname{length}(\gamma_{[t,1]})\}$$

for all $t \in [0, 1]$.

Let Ω be an open set in the metric space M and let $k \geq 1$. A ball $B(x, r) \subset \Omega$ is *k-non-tangential* if $k^{-1}r \leq \operatorname{dist}(B; \partial\Omega) \leq kr$. Given $x, y \in \Omega$, a *Harnack chain* joining x to y is a family $B_1, ..., B_n$ of k-non-tangential balls in Ω (for some $k \geq 1$) such that $x \in B_1$, $y \in B_n$ and $B_j \cap B_{j+1} \neq \varnothing$. The integer n is the length of the chain.

Definition 2.3. Let (M, d) be a metric space. A bounded domain $\Omega \subset M$ is *non-tangentially accessible* if there exist $r_0 > 0$ and $k \geq 1$ such that:

  (i) *Corkskrew condition.* For all $x \in \partial\Omega$ and $r \in (0, r_0)$ there exists $y \in \Omega$ such that $k^{-1}r \leq d(x, y) \leq r$ and $\operatorname{dist}(y; \partial\Omega) \geq k^{-1}r$.
  (ii) The set $M \setminus \bar{\Omega}$ satisfies the corkscrew condition.
  (iii) *Harnack chain condition.* If $\epsilon > 0$ and $x, y \in \Omega$ with $\operatorname{dist}(x; \partial\Omega) > \epsilon$, $\operatorname{dist}(y; \partial\Omega) > \epsilon$ and $d(x, y) < C\epsilon$, then there exists a Harnack chain of k-non-tangential balls joining x to y with length depending on C but not on ϵ.

This paper deals with homogeneous groups. Endowed with their Carnot-Carathéodory distance homogeneous groups are metric spaces with geodesics, and moreover, if μ denotes their Haar measure (which is Lebesgue measure), then there are positive constants C and $Q \in \mathbb{N}$ such that $\mu(B(x, r)) = Cr^Q$ for all x belonging to the group and for all $r > 0$ (see [FS]). More generally, a metric space (M, d) endowed with a Borel measure μ such that $0 < \mu(B(x, 2r)) \leq \delta\mu(B(x, r)) < +\infty$ for all $x \in M$ and $r > 0$ is called a *doubling metric space*, and $\delta > 0$ is its doubling constant.

We establish some propositions that will be needed in the next sections.

Proposition 2.4. *Let (M, d) be a doubling metric space with geodesics. Let $\Omega \subset M$ be a bounded open set and for any $r > 0$ define $\Omega_r = \{y \in \Omega : \operatorname{dist}(y; \partial\Omega) > r\}$. Assume that there exist $r > 0$ and $\lambda > 0$ such that Ω_r is arcwise connected and such that for any $x \in \Omega$ there exists a continuous curve $\gamma_x : [0, 1] \to \Omega$ such that $\gamma_x(0) = x$, $\gamma_x(1) \in \Omega_r$ and*

$$(2.3) \qquad \operatorname{dist}(\gamma_x(t); \partial\Omega) \geq \lambda d(\gamma_x(t), x)$$

for all $t \in [0, 1]$. Then Ω is a John domain.

Proof. We show that there exist $x_0 \in \Omega$ and $C > 0$ such that for all $x \in \Omega$ there exists a *continuous* curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and (2.3) holds. In the terminology of [HK] this means that Ω is a weak John domain. Since (M, d) is a doubling metric space with geodesics from [HK, Proposition 9.6], it follows that Ω is a John domain.

Fix $x_0 \in \Omega_r$ and for any $y \in \Omega_r$ denote by $\gamma_{y,x_0} : [0, 1] \to \Omega_r$ a continuous path such that $\gamma_{y,x_0}(0) = y$ and $\gamma_{y,x_0}(1) = x_0$. Then $\operatorname{dist}(\gamma_{y,x_0}(t); \partial\Omega) \geq r$ and $d(\gamma_{y,x_0}(t), x) \leq \operatorname{diam}(\Omega)$ for all $t \in [0, 1]$. If $x \in \Omega$, let γ_x be as in the statement of the proposition, let $\bar{y} = \gamma_x(1)$ and consider $\gamma_{\bar{y},x_0}$. The path γ sum of γ_x and $\gamma_{\bar{y},x_0}$ enjoys property (2.3) (possibly with a new λ). $\square$

Proposition 2.5. *Let (M, d) be a doubling metric space with geodesics. Let $\Omega \subset M$ be a bounded open set. If there exist $\lambda > 0$ and $r > 0$ such that for any $z \in \partial\Omega$ and for all $x, y \in B(z, r) \cap \Omega$ there exists a continuous curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$,*

$$(2.4) \qquad \operatorname{diam}(\gamma) \leq \frac{1}{\lambda} d(x, y)$$

and

$$(2.5) \qquad \operatorname{dist}(\gamma(t); \partial\Omega) \geq \lambda \min\{\operatorname{diam}(\gamma_{[0,t]}), \operatorname{diam}(\gamma_{[t,1]})\}$$

for all $t \in [0, 1]$, then Ω is an uniform domain.

Proof. The proof relies on two facts. First, using the argument in [V, Theorem 4.1], which holds in any metric space, it can be proved that for all $x, y \in \Omega$ there is a continuous curve γ such that (2.4) and (2.5) hold (the constant λ may be different).

Second, curves can be taken rectifiable and satisfying (2.1) and (2.2). We sketch the proof. Let $x, y \in \Omega$ and fix a continuous curve γ such that $\gamma(0) = x$ and $\gamma(1) = y$, and such that (2.4) and (2.5) hold. Let $\bar{t} \in (0, 1)$ be such that $\operatorname{diam}(\gamma_{[0,\bar{t}]}) = \operatorname{diam}(\gamma_{[\bar{t},1]})$. Consider the path $\tilde{\gamma}_x = \gamma_{[0,\bar{t}]}$. By the choice of $\bar{t}$ and by (2.5) $\tilde{\gamma}_x$ satisfies $\operatorname{dist}(\tilde{\gamma}_x(t); \partial\Omega) \geq \lambda \operatorname{diam}((\tilde{\gamma}_x)_{[0,t]})$. By Lemma 2.7 in [MS], which can be proved in any doubling metric space with geodesics, there exists a rectifiable path γ_x such that $\gamma_x(0) = x$, $\gamma_x(1) = \gamma(\bar{t})$ and $\operatorname{dist}(\gamma_x(t); \partial\Omega) \geq \lambda' \operatorname{length}(\gamma_{[0,t]})$ (here λ' depends on λ and on the doubling constant). A rectifiable path γ_y can be analogously constructed. The sum of the paths γ_x and γ_y gives the required path and proves that Ω is actually a uniform domain in the sense of Definition 2.2. $\square$

A continuous curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$ and $\operatorname{dist}(\gamma(t); \partial\Omega) \geq \lambda d(\gamma(t), x)$ will be called a *John curve* starting from x and with constant $\lambda > 0$.

All the John curves constructed in Sections 3 and 4 are such that the function $t \mapsto d(\gamma(t), \gamma(0))$ is (equivalent to) a monotonic increasing function and satisfies

$$(2.6) \qquad d(\gamma(0), \gamma(t)) \simeq \operatorname{diam}(\gamma_{[0,t]}).$$

In Section 3 we shall refer to the following proposition.

Proposition 2.6. *Let (M, d) be a doubling metric space with geodesics and let $\Omega \subset M$ be a bounded open set. Assume that there exists $\lambda > 0$ and $r > 0$ such that for any $z \in \partial\Omega$ and for all $x, y \in B(z, r) \cap \Omega$ there exist two John curves $\gamma_x, \gamma_y : [0, 1] \to \Omega$ starting respectively from x and y, with John constant λ, such that $\gamma_x(1) = \gamma_y(1)$ and*

$$\max\{\operatorname{diam}(\gamma_x), \operatorname{diam}(\gamma_y)\} \leq \frac{1}{\lambda} d(x, y).$$

Assume also that γ_x and γ_y verify (2.6). Then Ω is a uniform domain.

Proof. Let γ be the curve sum of γ_x and γ_y parameterized over $[0, 2]$. First of all

$$\operatorname{diam}(\gamma) \leq \operatorname{diam}(\gamma_x) + \operatorname{diam}(\gamma_y) \leq \frac{2}{\lambda} d(x, y).$$

Consider now a point $\gamma(t)$ and assume that $\gamma(t) = \gamma_x(t)$. Then

$$\operatorname{dist}(\gamma(t); \partial\Omega) = \operatorname{dist}(\gamma_x(t); \partial\Omega) \geq \lambda d(\gamma_x(t), x)$$

$$\simeq \lambda \operatorname{diam}((\gamma_x)_{[0,t]}) \geq \lambda \min\{\operatorname{diam}(\gamma_{[0,t]}), \operatorname{diam}(\gamma_{[t,2]})\}.$$

If $\gamma(t)$ is in γ_y, the estimate is the same. The claim follows from Proposition 2.5. $\square$

It is easy to see that if both Ω and $M\backslash\Omega$ satisfy the hypotheses of Proposition 2.6, then the interior and the exterior corkskrew conditions are satisfied. The following result contains another useful connection between uniform and NTA domains.

Proposition 2.7. *Let (M, d) be a metric space. If $\Omega \subset M$ is a uniform domain, then Ω satisfies the Harnack chain condition* (iii) *in Definition* 2.3.

Proof. See [CT, Proposition 4.2] $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

3. Uniform and NTA domains in groups of step 2

In this section we study uniform and NTA domains in homogeneous groups of step 2. We shall work in $\mathbb{R}^n$ endowed a left invariant metric induced by a system of vector fields $X = (X_1, ..., X_m)$ which generates a stratified Lie algebra of step 2. In $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^q$ we denote $x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^q$ and by abuse of notation we shall write $x' = (x', 0)$ and $x'' = (0, x'')$. We say that x' are the variables of the first slice and that x'' are the variables of the second slice.

The vector fields can be assumed to be of the form

$$X_j = \partial_j + \sum_{k=m+1}^{n} q_{jk}\partial_k, \quad j = 1, ..., m,$$

where $q_{jk} = q_{jk}(x')$ are homogeneous polynomials of degree 1 in the variables x' (see [FS]). Introduce the group law

$$(3.1) \quad x \cdot y = x + y + Q(x, y) = \big(x_1 + y_1, \ldots, x_m + y_m, x_{m+1} + y_{m+1}$$
$$+ Q_{m+1}(x, y), \ldots, x_n + y_n + Q_n(x, y)\big),$$

where $Q = (Q_1, ..., Q_n)$ with $Q_1 = ... = Q_m = 0$, and $Q_j = Q_j(x', y')$, $j = m+1, ..., n$, are homogeneous polynomials of degree 2 (the fact that the Q_j's do not depend on the variables of the second slice will be used several times). Moreover, they can be assumed to satisfy

$$(3.2) \qquad\qquad\qquad\qquad |Q_j(x', y')| \leq C|x'||y'|.$$

The vector fields $X_1, \ldots, X_m$ are by assumption left invariant with respect to the introduced law.

We denote by d the Carnot-Carathéodory distance induced on $\mathbb{R}^n$ by $X_1, ..., X_m$ and by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^n$ with radius $r \geq 0$. We also introduce in $\mathbb{R}^n$ the following continuous homogeneous norm

$$(3.3) \qquad\qquad\qquad\qquad \|x\| = |x'| + |x''|^{1/2}.$$

By a standard argument it can be proved that $d(x, y) \simeq \|y^{-1} \cdot x\|$. Thus, letting $\text{Box}(x, r) = \{x \cdot y \in \mathbb{R}^n : \|y\| \leq r\}$ there exists $c > 1$ such that for all $x \in \mathbb{R}^n$ and $r \geq 0$,

$$\text{Box}(x, c^{-1}r) \subset B(x, r) \subset \text{Box}(x, cr).$$

Definition 3.1. Let $S \subset \mathbb{R}^n$ be a hypersurface of class C^1 given in a neighborhood $\mathcal{U}$ of $x_0 \in S$ by the local equation $\Phi = 0$ where $\Phi \in C^1(\mathcal{U})$. The point x_0 is *characteristic* if $X_1\Phi(x_0) = ... = X_m\Phi(x_0) = 0$.

We denote by e_j the jth coordinate versor and if $x = \sum_{i=1}^{n} x_i e_i \in \mathbb{R}^n$ and $j \in \{1, \ldots, n\}$ we let

$$\hat{x}_j = \sum_{1 \leq i \leq n, i \neq j} x_i e_i.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, $x_0 \in \partial\Omega$ and let $\mathcal{U} \subset \mathbb{R}^n$ be a neighborhood of x_0 such that $\partial\Omega \cap \mathcal{U} = \{x \in \mathcal{U} : \Phi(x) = 0\}$ for some defining function $\Phi \in C^1(\mathcal{U})$. If $x \in \partial\Omega \cap \mathcal{U}$, then, possibly shrinking $\mathcal{U}$, the translated surface $x^{-1} \cdot (\partial\Omega \cap \mathcal{U})$ can be expressed in parametric form by an equation of the type $y_j = \varphi(\hat{y}_j)$ for $\hat{y}_j$ belonging to a neighborhood of the origin in $\mathbb{R}^{n-1}$ and φ of class C^1. If x_0 is a characteristic point we must choose $j \in \{m+1, ..., n\}$, otherwise we can choose $j \in \{1, ..., m\}$. In the next theorem we shall assume $\Phi \in C^{1,1}$ and so $\varphi \in C^{1,1}$. Actually, in the proof we shall need the Lipschitz continuity only of the derivatives $\partial_1\varphi, ..., \partial_m\varphi$.

Theorem 3.2. *Any connected, bounded open set $\Omega \subset \mathbb{R}^n$ of class $C^{1,1}$ is an NTA domain in the metric space $(\mathbb{R}^n, d)$.*

Proof. The proof will be split into several numbered small steps.

1. We claim that for all $x_0 \in \partial\Omega$ there exists a neighborhood $\mathcal{U}$ of x_0 such that for all $x, y \in \mathcal{U} \cap \Omega$ there exist continuous curves γ_x and $\gamma_y : [0,1] \to \Omega$ satisfying the hypotheses of Proposition 2.6. The proof will show that conditions (i) and (ii) in Definition 2.3 are verified, and by Proposition 2.7 Ω will be an NTA domain.

2. Let $\mathcal{U}$ be a neighborhood of x_0 and let $\Phi \in C^1(\mathcal{U})$ be a defining function such that $\partial\Omega \cap \mathcal{U} = \{x \in \mathcal{U} : \Phi(x) = 0\}$. We shall distinguish two cases:

(C1) $|X_1\Phi(x_0)| = ... = |X_m\Phi(x_0)| = 0$ (x_0 is a characteristic point of $\partial\Omega$);
(C2) $|X_1\Phi(x_0)| + ... + |X_m\Phi(x_0)| > 0$ (x_0 is a non-characteristic point of $\partial\Omega$).

Let $x_j = \varphi(\hat{x}_j)$ be a local parameterization of $\partial\Omega \cap \mathcal{U}$ around x_0. We assume without loss of generality that $x_0 = 0$.

3. **Case 1.** We consider an open set $\{y \in \mathbb{R}^n : y_j > \varphi(\hat{y}_j)\}$ where $j > m$ and $\varphi \in C^1(\mathbb{R}^{n-1})$ is a function such that $\varphi(0) = 0$, $|\nabla\varphi| \le k$ and $\partial_1\varphi, ..., \partial_m\varphi$ are k-Lipschitz continuous functions, where $k > 0$ is a fixed given constant. Define

$$\nu_i = -\partial_i\varphi(0), \quad \text{for } i = 1, ..., m, \quad \text{and} \quad \nu = (\nu_1, ..., \nu_m, 0, ..., 0).$$

Write also

$$\varphi(\hat{y}_j) = -\sum_{i=1}^{m} \nu_i y_i + \psi(\hat{y}_j)$$

where ψ can be written by Taylor formula in the form

$$\psi(\hat{y}_j) = \varphi(\hat{y}_j) - \sum_{i=1}^{m} \partial_i\varphi(0)y_j = \sum_{i>m, i\neq j} \partial_i\varphi(0)y_i + O(|\hat{y}_j|^2),$$

and satisfies the growth estimate

$$(3.4) \qquad\qquad |\psi(\hat{y}_j)| \lesssim \|\hat{y}_j\|^2.$$

Here we used the homogeneous norm introduced in (3.3) and the Lipschitz continuity of $\partial_1\varphi, ..., \partial_m\varphi$.

Our construction will take place in two steps. In the first step we define "canonical" John curves starting from points near the boundary. In the second step we join points near the boundary by curves satisfying the hypotheses of Proposition 2.6.

4. *First step.* Define

$$N_1 = \frac{\nu_1}{|\nu|}, ..., N_m = \frac{\nu_m}{|\nu|} \quad \text{and} \quad N = (N_1, ..., N_m, 0, ..., 0),$$

and if $\nu = 0$, simply set $N = 0$. For $\sigma > 0$ let $t_1 = \sigma|\nu|$. Fix $x = x_j e_j$ with $x_j \geq 0$ and define the continuous curve $\gamma : [0,1] \to \mathbb{R}^n$,

$$(3.5) \quad \gamma(t) = \begin{cases} x \cdot tN = tN + x_j e_j, & \text{if } 0 \leq t \leq t_1, \\ x \cdot \big((t_1 N) + (t - t_1)e_j\big) = t_1 N + (t - t_1 + x_j)e_j, & \text{if } t_1 \leq t \leq 1. \end{cases}$$

5. We claim that there exist $\sigma, \lambda \in (0,1)$ such that for all $t \in [0,1]$,

$$(3.6) \qquad\qquad \operatorname{dist}(\gamma(t); \partial\Omega) \geq \lambda d(\gamma(t), x).$$

If $0 \leq t \leq t_1$, then $d(\gamma(t), x) \simeq \|x^{-1} \cdot \gamma(t)\| = \|tN\| = t$, and (3.6) is equivalent to

$$(3.7) \qquad\qquad \operatorname{Box}(\gamma(t), \lambda t) \cap \{y_j = \varphi(\hat{y}_j)\} = \varnothing,$$

which is implied by

$$(3.8) \qquad \langle \nu, y \rangle + y_j \geq |\psi(\hat{y}_j)|, \quad \text{for all } y \in \operatorname{Box}(\gamma(t), \lambda t).$$

Points in $\operatorname{Box}(\gamma(t), \lambda t)$ are of the form

$$(3.9) \qquad \gamma(t) \cdot h = (tN + x_j e_j) \cdot h = tN + x_j e_j + h + Q(tN, h)$$

with $\|h\| \leq \lambda t$ (Q does not depend on the variables on the second slice) and thus we have to check that

$$\langle \nu, tN + h \rangle + x_j + h_j + Q_j(tN, h) \geq \big|\psi\big((\widehat{\gamma(t) \cdot h})_j\big)\big|$$

which is guaranteed by

$$t|\nu| + \langle \nu, h \rangle + x_j \geq |h_j| + \big|Q_j(tN, h)\big| + \big|\psi\big((\widehat{\gamma(t) \cdot h})_j\big)\big|.$$

Now, since $|\langle \nu, h \rangle| \leq \lambda|\nu|t$, then $t|\nu| + \langle \nu, h \rangle \gtrsim t|\nu|$ as soon as $\lambda < 1/2$. Moreover, $|h_j| \leq t^2$ and by (3.4)

$$\big|\psi\big((\widehat{\gamma(t) \cdot h})_j\big)\big| \lesssim \big\|(\widehat{\gamma(t) \cdot h})_j\big\|^2 = \|tN + \hat{h}_j + \hat{Q}_j(tN, h)\|^2$$
$$\lesssim t^2 + \|h\|^2 + \|Q(tN, h)\|^2 \lesssim t^2.$$

Moreover, $|Q_j(tN, h)| \lesssim \lambda t^2 \lesssim t^2$. Thus (3.7) is implied by

$$(3.10) \qquad\qquad \epsilon_0(t|\nu| + x_j) \geq t^2,$$

where ϵ_0 is a small but absolute constant. Since $x_j \geq 0$, (3.10) holds for all $t \leq \sigma|\nu|$ as soon as $\sigma \leq \epsilon_0$. Our claim is proved if $0 \leq t \leq t_1$.

6. We study the case $t \geq t_1$. Notice that in this case

$$(3.11) \qquad d(\gamma(t), x) \simeq \|x^{-1} \cdot \gamma(t)\| \simeq t_1 + (t - t_1)^{1/2} =: \delta(t).$$

Let $a = (t - t_1)^{1/2}$ so that $\delta(t) = t_1 + a$. We shall sometimes write δ instead of $\delta(t)$. We claim that there exists $\lambda \in (0,1)$ such that the John property $\operatorname{Box}(\gamma(t), \lambda\delta(t)) \cap \{y_j = \varphi(\hat{y}_j)\} = \varnothing$ holds for all $t \geq t_1$.

Points in $\operatorname{Box}(\gamma(t), \lambda\delta)$ are of the form

$$(3.12) \qquad \begin{aligned} \gamma(t) \cdot h &= (t_1 N + (t - t_1 + x_j)e_j) \cdot h \\ &= t_1 N + (t - t_1 + x_j)e_j + h + Q(t_1 N, h), \end{aligned}$$

with $\|h\| \leq \lambda\delta$. Thus, the John property is ensured by

$$\langle \nu, t_1 N + h \rangle + (t - t_1) + x_j + Q_j(t_1 N, h) \geq \big|\psi\big(t_1 N + \hat{h}_j + \hat{Q}_j(t_1 N, h)\big)\big|,$$

which (let $t - t_1 = a^2$) is a consequence of the following stronger inequality:

$$(3.13) \qquad t_1|\nu| + a^2 + x_j \geq |\nu|\|h\| + |Q_j(t_1 N, h)| + |\psi(z)|,$$

where z denotes the argument of ψ in the previous inequality.

Now, $|\nu|\|h\| \lesssim \lambda|\nu|t_1 + \lambda|\nu|a$ and $\lambda|\nu|t_1$ can be absorbed in the left-hand side of (3.13) as soon as $\lambda \leq \frac{1}{2}$. We also note that $|Q_j(t_1N, h)| \leq t_1\|h\| \leq t_1\lambda\delta \leq t_1^2 + \lambda\delta^2$. Moreover,

$$\|z\| \lesssim t_1 + \|h\| + \|\hat{Q}_j(t_1N, h)\| \lesssim t_1 + \lambda\delta + (t_1\lambda\delta)^{1/2} \simeq t_1 + \lambda\delta,$$

and by (3.4)

$$|\psi(z)| \lesssim \|z\|^2 \lesssim t_1^2 + \lambda\delta^2 + \lambda t_1\delta \simeq t_1^2 + \lambda\delta^2 \simeq t_1^2 + \lambda a^2.$$

Since the term λa^2 can be absorbed in the left-hand side and $x_j \geq 0$, then (3.13) will follow if we prove that for all $a \geq 0$ it is $\epsilon_0 \left(t_1|\nu| + a^2\right) \geq t_1^2 + \lambda|\nu|a$ where $\epsilon_0 > 0$ is a small but absolute constant. Replacing $t_1 = \sigma|\nu|$ we get

$$(3.14) \qquad \epsilon_0 \left(\sigma|\nu|^2 + a^2\right) \geq \sigma^2|\nu|^2 + \lambda|\nu|a.$$

Now, since $\sigma^2|\nu|^2 + \lambda|\nu|a \leq (\sigma^2 + \lambda/2)|\nu|^2 + \lambda a^2/2$, (3.14) holds for all $a \geq 0$ provided $\sigma^2 + \lambda/2 < \epsilon_0\sigma$ and $\lambda/2 \leq \epsilon_0$.

7. *Second step.* We prove that, given x and y in the open set $\{z_j > \varphi(\hat{z}_j)\}$ there exists a continuous curve connecting them and satisfying (2.4) and (2.5). Without loss of generality, we can assume that $x = x_je_j$ with $x_j \geq 0$ and $y = y_je_j + \hat{y}_j$ with $y_j > \varphi(\hat{y}_j)$. In the first step (see (3.5)) the "canonical" John curve starting from x has been defined. The parameters ν, N and $t_1 = \sigma|\nu|$ are defined as in the first step and are relative to x. The constant σ does not depend on x.

8. Our next task is to define the curve starting from y. We first explain the "intrinsic" argument which provides the parameters ν_j's relative to x. Letting $\Phi(\xi) = \xi_j - \varphi(\hat{\xi}_j)$, we have for $i = 1, ..., m$,

$$X_i\Phi(\xi) = -\partial_i\varphi(\hat{\xi}_j) + \sum_{k>m} q_{ik}(\xi)\partial_k\Phi(\xi),$$

and hence

$$(3.15) \qquad \nu_i = -\partial_i\varphi(0) = X_i\Phi(0).$$

Now let $w = \hat{y}_j + \varphi(\hat{y}_j)e_j$. We look for the parameters $\nu_i, i = 1, \ldots, m$, of the curve starting from $w^{-1} \cdot y = (y_j - \varphi(\hat{y}_j))e_j$ relatively to the translated boundary $w^{-1} \cdot \{z_j = \varphi(\hat{z}_j)\}$. Denote these parameters by $\bar{\nu}_1, ..., \bar{\nu}_m$. Then we find by left invariance

(3.16)

$$\bar{\nu}_i = (X_i\Phi)(\hat{y}_j + \varphi(\hat{y}_j)e_j) = -\partial_i\varphi(\hat{y}_j) + \sum_{k>m} q_{ik}(y)\frac{\partial}{\partial\xi_k}\left(\xi_j - \varphi(\hat{\xi}_j)\right)\big|_{\hat{\xi}_j=\hat{y}_j,\xi_j=\varphi(\hat{y}_j)}$$

$$= -\partial_i\varphi(\hat{y}_j) + q_{ij}(y') - \sum_{k>m,k\neq j} q_{ik}(y)\partial_k\varphi(\hat{y}_j).$$

Define

$$\bar{N} = \left(\frac{\bar{\nu}_1}{|\bar{\nu}|}, ..., \frac{\bar{\nu}_m}{|\bar{\nu}|}, 0, ..., 0\right) \quad \text{and} \quad \bar{t}_1 = \sigma|\bar{\nu}|.$$

The "canonical" John curve γ_y starting from y can be defined (by left translation of (3.5)) in the following way. If $0 \leq t \leq \bar{t}_1$, let

$$(3.17) \qquad \begin{aligned} \gamma_y(t) &= \left(\hat{y}_j + \varphi(\hat{y}_j)e_j\right) \cdot \left(t\bar{N} + (y_j - \varphi(\hat{y}_j))e_j\right) \\ &= \hat{y}_j + t\bar{N} + y_je_j + Q(y, t\bar{N}), \end{aligned}$$

and if $t \geq \bar{t}_1$, let

$$(3.18) \quad \begin{aligned} \gamma_y(t) &= \big(\hat{y}_j + \varphi(\hat{y}_j)e_j\big) \cdot \big(\bar{t}_1 \bar{N} + (t - \bar{t}_1 + (y_j - \varphi(\hat{y}_j))e_j\big) \\ &= \hat{y}_j + \bar{t}_1 \bar{N} + (t - \bar{t}_1 + y_j)e_j + Q(y, \bar{t}_1 \bar{N}). \end{aligned}$$

9. Denote by γ_x and γ_y the curves starting from x and y. The curves γ_x and γ_y cannot be expected to meet as Proposition 2.6 requires. Thus we enlarge the curve γ_x by constructing a curvilinear cone around it. Define

$$\delta(t) = \begin{cases} t & \text{if } 0 \leq t \leq t_1, \\ t_1 + (t - t_1)^{1/2} & \text{if } t \geq t_1, \end{cases}$$

and recall that $\delta(t) \simeq d(\gamma_x(t), x)$. For $\lambda > 0$ let $\mathcal{U}(\lambda) = \{h \in \mathbb{R}^n : \|h\| \leq \lambda\}$, and if $h = (h', h'') \in \mathcal{U}(\lambda)$, define $h_t = (\delta(t)h', \delta(t)^2 h'')$. As $h \in \mathcal{U}(\lambda)$, the family of curves

$$\gamma_x^h(t) = \gamma_x(t) \cdot h_t = \begin{cases} tN + x_j e_j + h_t + Q(tN, h_t) & \text{if } 0 \leq t \leq t_1, \\ t_1 N + (t - t_1 + x_j)e_j + h_t + Q(t_1 N, h_t) & \text{if } t \geq t_1 \end{cases}$$

forms a curvilinear cone with core γ_x. By the triangle inequality, if λ is small enough, then for any $h \in \mathcal{U}(\lambda)$, the curve $t \mapsto \gamma_x^h(t)$ is a John curve starting from x. From now on we assume that λ has been fixed small enough in order to ensure this property.

10. Two cases must be distinguished:

(A) $d(x, y) \leq \eta|\nu|$;
(B) $d(x, y) > \eta|\nu|$.

The parameter $\eta \in (0, 1)$ will be fixed later. Note that if 0 is a characteristic point, then Case A is empty. In Case A the curves γ_x^h and γ_y will meet in their first (rectifiable) piece. In Case B they will meet in their second (non-rectifiable) piece.

11. *Study of Case A.* We claim that there exist $\eta > 0$ and $M > 1$ such that for all x and y there exists $h \in \mathcal{U}(\lambda)$ such that $\gamma_y(Md(x, y)) = \gamma_x^h(Md(x, y))$. A correct choice of $\eta \in (0, 1)$ and $M > 1$ will show that the two curves meet in their first tract (see condition (3.27)).

Without loss of generality, we can assume $|\nu| \leq |\bar{\nu}|$ (otherwise the roles of x and y should be interchanged). If $t \leq t_1 = \sigma|\nu|$, then $t \leq \bar{t}_1 = \sigma|\bar{\nu}|$ and $\gamma_y(t) = \gamma_x^h(t)$ reads

$$(3.19) \quad \hat{y}_j + t\bar{N} + y_j e_j + Q(y, t\bar{N}) = tN + x_j e_j + h_t + Q(tN, h_t).$$

We have to show that the solution $h = (h', h'')$ of this equation belongs to $\mathcal{U}(\lambda)$ if $t = Md(x, y)$ and M is large enough.

As $t \leq t_1$, then $\delta(t) = t$ and $h_t = (th', t^2 h'')$. Projecting (3.19) along the first m components we get the equation $y' + t\bar{N} = tN + h'_t$, that is,

$$(3.20) \quad th' = y' + t(\bar{N} - N).$$

Replacing $t = Md(x, y)$ we find that the solution h' satisfies

$$(3.21) \quad |h'| \leq \frac{|y'|}{Md(x, y)} + |N - \bar{N}|.$$

First, notice that $d(x, y) \simeq \|(-x_j e_j) \cdot (y_j e_j + \hat{y}_j)\| \geq |y'|$, which gives $\frac{|y'|}{Md(x,y)} \lesssim \frac{1}{M}$. Moreover, using the inequality $\left|\frac{v}{|v|} - \frac{w}{|w|}\right| \leq 2\frac{|v-w|}{|v|}$ if $v, w \in \mathbb{R}^n \setminus \{0\}$, and the

explicit form (3.15) and (3.16) of ν and $\bar{\nu}$, we get

$$
|N - \bar{N}| \leq 2 \frac{|\nu - \bar{\nu}|}{|\nu|}
$$

$$
(3.22) \qquad \leq \frac{2}{|\nu|} \sum_{i=1}^{m} \left| \partial_i \varphi(0) - \partial_i \varphi(\hat{y}_j) + q_{ij}(y) - \sum_{k>m,\, k\neq j} q_{ik}(y) \partial_k \varphi(\hat{y}_j) \right|
$$

$$
\lesssim \frac{1}{|\nu|} \left(|\hat{y}_j| + |y'| \right) \lesssim \frac{d(x,y)}{|\nu|}.
$$

The last string of estimates follows from the boundedness of $\partial_i \varphi$, $i = m+1, ..., n$ and $i \neq j$, from the Lipschitz continuity of $\partial_i \varphi$, $i = 1, ..., m$, and from the inequality

$$
(3.23) \qquad d(x,y) \simeq \|(-x_j e_j) \cdot (\hat{y}_j + y_j e_j)\| = \|\hat{y}_j + (y_j - x_j)e_j\| \geq \|\hat{y}_j\| \gtrsim |\hat{y}_j|,
$$

which holds because y lies in a bounded set.

Putting (3.22) into (3.21) and using Case A we get

$$
(3.24) \qquad |h'| \lesssim \frac{1}{M} + \frac{d(x,y)}{|\nu|} \leq \frac{1}{M} + \eta.
$$

This shows that $|h'| \leq \lambda$ as soon as M is large enough and η is small enough.

We project now (3.19) along the components of the second slice obtaining

$$
\hat{y}_j'' + y_j e_j + t Q(y, \bar{N}) = x_j e_j + h_t'' + t Q(N, h_t').
$$

Here $h_t'' = t^2 h''$ and $h_t' = t h'$ where h' is the vector determined in (3.20) and satisfies the estimate (3.24). The last equation has a unique solution h'' which satisfies

$$
|h''| \leq \frac{|\hat{y}_j''| + |y_j - x_j|}{t^2} + \frac{1}{t} |Q(y, \bar{N})| + |Q(N, h')|.
$$

Here we have to replace $t = M d(x,y)$ but first we notice that, as in (3.23),

$$
(3.25) \qquad d(x,y) \simeq \|\hat{y}_j + (y_j - x_j)e_j\| \geq |\hat{y}_j''|^{1/2} + |y_j - x_j|^{1/2}.
$$

Moreover, $|Q(y, \bar{N})| \lesssim |y'| \lesssim d(x,y)$ and, by (3.24), $|Q(N, h')| \lesssim |h'| \lesssim \frac{1}{M} + \eta$. Putting all these estimates together we find

$$
(3.26) \qquad |h''| \lesssim \frac{1}{M^2} + \frac{1}{M} + \eta.
$$

Thus $|h''| \leq \lambda$ as soon as M is large enough and η is small enough.

Our claim will be proved if we show that the choice of M and η is compatible with the condition $M d(x,y) \leq t_1 = \sigma |\nu|$. As we are in Case A, then $d(x,y) \leq \eta |\nu|$ and we find the stronger condition

$$
(3.27) \qquad M\eta \leq \sigma,
$$

which can be satisfied, taking if necessary a smaller η.

12. In view of Proposition 2.6 we have to estimate the diameter of the curves γ_x^h and γ_y. First, by (3.5) we have $\operatorname{diam}(\gamma_x) \simeq M d(x,y)$. Moreover, if $0 \leq s, t \leq M d(x,y)$ and $\|h\| \leq 1$, then

$$
d(\gamma_x^h(s), \gamma_x^h(t)) \leq d(\gamma_x^h(s), \gamma_x(s)) + d(\gamma_x(s), \gamma_x(t)) + d(\gamma_x(t), \gamma_x^h(t))
$$

$$
\lesssim \|h_s\| + \operatorname{diam}(\gamma_x) + \|h_t\| \lesssim d(x,y),
$$

and thus $\operatorname{diam}(\gamma_x^h) \lesssim d(x,y)$.

13. *Study of Case B.* In this case the points x and y satisfy $d(x,y) \geq \eta|\nu|$ where $\eta > 0$ is from now on a fixed constant. Recall that $t_1 = \sigma|\nu|$ and $\bar{t}_1 = \sigma|\bar{\nu}|$, and for $R > 0$ let

$$t_x = t_1 + R^2 d(x,y)^2 \quad \text{and} \quad t_y = \bar{t}_1 + R^2 d(x,y)^2.$$

As above let $\mathcal{U}(\lambda) = \{h \in \mathbb{R}^n : \|h\| \leq \lambda\}$ and $h_t = (\delta(t)h', \delta(t)^2 h'')$ where now $\delta(t) = t_1 + (t - t_1)^{1/2} \simeq d(\gamma(t), x)$ for $t \geq t_1$.

14. We claim that there exists $R > 0$ such that for all x, y there exists $h \in \mathcal{U}(\lambda)$ (λ is the parameter fixed at the end of 9) such that $\gamma_y(t_y) = \gamma_x^h(t_x)$ (the times t_x and t_y depend on R).

This equation gives

$$\hat{y}_j + \bar{t}_1 \bar{N} + (t_y - \bar{t}_1 + y_j)e_j + Q(y, \bar{t}_1 \bar{N}) = t_1 N + (t_x - t_1 + x_j)e_j$$
$$+ h_{t_x} + Q(t_1 N, h_{t_x}).$$

Replacing $\bar{t}_1 = \sigma|\bar{\nu}|$, $t_1 = \sigma|\nu|$, $t_y - \bar{t}_1 = R^2 d(x,y)^2$ and $t_x - t_1 = R^2 d(x,y)^2$ we find

(3.28)
$$\hat{y}_j + \sigma\bar{\nu} + (R^2 d(x,y)^2 + y_j)e_j + \sigma|\bar{\nu}|Q(y, \bar{N})$$
$$= \sigma\nu + (R^2 d(x,y)^2 + x_j)e_j + h_{t_x} + \sigma|\nu|Q(N, h_{t_x}).$$

Projecting this equation along the coordinates of the first slice we get

(3.29)
$$y' + \sigma\bar{\nu} = \sigma\nu + h'_{t_x},$$

and the solution h'_{t_x} satisfies

$$|h'_{t_x}| \leq |y'| + \sigma|\nu| + \sigma|\bar{\nu}|.$$

We use $|y'| \leq d(x,y)$ and $\sigma|\nu| \leq \sigma d(x,y)/\eta$ (this is Case B). By (3.16)

(3.30)
$$|\bar{\nu}_i| \leq |\partial_i\varphi(\hat{y}_j)| + |q_{ij}(y)| + \sum_{k>m, k\neq j} |q_{ik}(y)\partial_k\varphi(\hat{y}_j)|$$
$$\lesssim |\partial_i\varphi(0)| + |\partial_i\varphi(0) - \partial_i\varphi(\hat{y}_j)| + |y'|$$
$$\lesssim |\nu| + d(x,y) \lesssim \frac{d(x,y)}{\eta},$$

because $\partial_k\varphi$, $k > m$ and $k \neq j$, are bounded functions, $\partial_i\varphi$, $i = 1, ..., m$, are Lipschitz continuous, and $|\partial_i\varphi(\hat{y}_j) - \partial_i\varphi(0)| \lesssim |\hat{y}_j| \lesssim d(x,y)$ by (3.23). Ultimately, we obtain for some large but absolute constant C_0,

(3.31)
$$|h'_{t_x}| \leq C_0 \frac{d(x,y)}{\eta} = C_0 d(x,y)$$

(the parameter η has been fixed in 11 and can be considered from now on an absolute constant).

Projecting (3.28) along the coordinates of the second slice we have

$$\hat{y}''_j + y_j e_j + \sigma|\bar{\nu}|Q(y, \bar{N}) = x_j e_j + h''_{t_x} + \sigma|\nu|Q(N, h_{t_x}).$$

Thus

$$h''_{t_x} = \hat{y}''_j + (y_j - x_j)e_j + \sigma|\bar{\nu}|Q(y, \bar{N}) - \sigma|\nu|Q(N, h'_{t_x}),$$

where h'_{t_x} satisfies (3.31). Notice that by (3.25) $|\hat{y}''_j| + |y_j - x_j| \lesssim d(x,y)^2$ and moreover, taking into account (3.31) and Case B,

$$\sigma|\nu||Q(N, h'_{t_x})| \lesssim |\nu||h'_{t_x}| \lesssim d(x,y)^2.$$

By (3.30) $\sigma|\bar{\nu}||Q(y', \bar{N})| \lesssim |\bar{\nu}||y'| \lesssim d(x, y)^2$, and hence $|h''_{t_x}| \lesssim d(x, y)^2$. Finally,

$$\|h\| = \frac{\|h_{t_x}\|}{\delta(t_x)} = \frac{|h'_{t_x}| + |h''_{t_x}|^{1/2}}{\delta(t_x)} \lesssim \frac{d(x, y)}{\delta(t_x)} = \frac{d(x, y)}{t_1 + (t_x - t_1)^{1/2}} \leq \frac{1}{R},$$

and $\|h\| \leq \lambda$ as soon as $R \geq C_0/\lambda$ where C_0 is a large but absolute constant.

15. The estimates for $\text{diam}(\gamma_x^h)$ and $\text{diam}(\gamma_y)$ can be obtained as in 12. Our claims are proved and the proof of the theorem in the characteristic case is ended.

16. **Case 2.** We now study the non-characteristic case. Assume without loss of generality that $\Omega = \{y \in \mathbb{R}^n : y_j > \varphi(\hat{y}_j)\}$ where $j \in \{1, ..., m\}$ and $\varphi \in C^1(\mathbb{R}^{n-1})$ is a function such that $\varphi(0) = 0$, $|\nabla \varphi| \leq k$ and $\partial_i \varphi$, $i = 1, ..., m$ and $i \neq j$, are k-Lipschitz continuous functions. Let $\nu_i = -\partial_i \varphi(0)$ if $i = 1, ..., m$ with $i \neq j$, and $\nu_j = 1$. Finally, let $\nu = (\nu_1, ..., \nu_m, 0, ..., 0)$.

17. *First step.* We construct John curves starting from near the boundary. The function ψ defined by

$$(3.32) \qquad \psi(\hat{y}_j) = \varphi(\hat{y}_j) + \sum_{i=1,...,m,i\neq j} \nu_i y_i$$

satisfies

$$(3.33) \qquad \begin{aligned} |\psi(\hat{y}_j)| &= \left| \varphi(\hat{y}_j) - \sum_{i=1,...,m,i\neq j} \partial_i \varphi(0) y_i \right| \\ &= \left| \sum_{i>m} \partial_i \varphi(0) y_i + O(|\hat{y}_j|^2) \right| \lesssim \|\hat{y}_j\|^2, \end{aligned}$$

because $\partial_i \varphi$, $i > m$, are bounded and $\partial_i \varphi$, $i = 1, ..., m$ and $i \neq j$, are Lipschitz continuous.

Fix a point $x \in \Omega$ of the form $x = x_j e_j$ with $x_j > 0$. For $t \geq 0$ define the curve starting from x,

$$(3.34) \qquad \gamma_x(t) = x \cdot t\nu = x \cdot \left(te_j + t \sum_{i=1,...m,i\neq j} \nu_i e_i \right).$$

Note first that $d(\gamma(t), x) \simeq \|t\nu\| = t|\nu| \simeq t$.

18. We claim that there exist $t_0 > 0$ and $\lambda \in (0, 1)$ such that for all $t \leq t_0$,

$$(3.35) \qquad \text{dist}(\gamma(t); \partial\Omega) \geq \lambda t.$$

The John condition (3.35) is equivalent to $\text{Box}(\gamma(t), \lambda t) \cap \partial\Omega = \varnothing$.

Points in $\text{Box}(\gamma(t), \lambda t)$ are of the form

$$\begin{aligned} x \cdot t\nu \cdot h &= x_j e_j \cdot (t\nu + h + tQ(\nu, h')) \\ &= x_j e_j + t\nu + h + tQ(\nu, h') + Q(x_j e_j, t\nu + h') =: z, \end{aligned}$$

where $h \in \mathbb{R}^n$ and $\|h\| \leq \lambda t$. We have to check that (z is defined in the last equation)

$$x_j + t + h_j > \varphi(\hat{z}_j) = -\sum_{i\leq m,i\neq j} \nu_i(t\nu_i + h_i) + \psi(\hat{z}_j),$$

by (3.32). Since $|h_k| < \lambda t$, $k = 1, \ldots, m$, if $\lambda > 0$ is small enough, the last inequality is ensured by

$$x_j + (1-\lambda)t + t\sum_{i\leq m,i\neq j} (\nu_i^2 - \lambda|\nu_i|) \geq |\psi(\hat{z}_j)|,$$

which is implied by

(3.36) $$\varepsilon_0(x_j + t) \geq |\psi(\hat{z}_j)|.$$

The right-hand side of (3.36) can be estimated by (3.33)

$$\begin{aligned}
|\psi(\hat{z}_j)| &\lesssim \|\hat{z}_j\|^2 = \left\| t\hat{\nu}_j + \hat{h}_j + tQ(\nu, h') + Q(x_j e_j, t\nu + h') \right\|^2 \\
&\lesssim t^2 + \lambda t^2 + \|tQ(\nu, h')\|^2 + \|Q(x_j e_j, t\nu + h')\|^2 \\
&\lesssim t^2 + (t|\nu|\lambda t) + (x_j|t\nu + h'|) \lesssim t^2 + x_j t,
\end{aligned}$$

where we used $|\nu| \lesssim 1$. Then (3.36) is ensured by

$$\epsilon_0(x_j + t) \geq t^2 + x_j t,$$

where $\epsilon_0 > 0$ is a small but absolute constant. This inequality is trivially satisfied as soon as $t \leq \epsilon_0$.

19. *Second step.* We prove the uniform condition. Given two points $x, y \in \Omega$ we have to connect them by curves γ_x and γ_y satisfying the hypotheses of Proposition 2.6. Assume that $x = x_j e_j$ with $x_j > 0$ and let $y = \hat{y}_j + y_j e_j$ with $y_j > \varphi(\hat{y}_j)$.

We first notice that if $d(x, y) < \text{dist}(x; \partial\Omega)$, then x and y can be connected simply by a geodesic. Therefore, without loss of generality we can assume that

(3.37) $$d(x, y) \geq \text{dist}(x; \partial\Omega).$$

20. We claim that there exists a constant $C_0 > 0$ such that

(3.38) $$x_j \leq C_0 d(x, y)$$

for all $x = x_j e_j, y \in \Omega$ satisfying (3.37) and lying in a bounded set (say the unit Euclidean ball centered at the origin). Indeed, if $\xi = \hat{\xi}_j + \varphi(\hat{\xi}_j)e_j \in \partial\Omega$, then

$$\begin{aligned}
d(x, \xi) &\simeq \left\| (-x_j e_j) \cdot (\hat{\xi}_j + \varphi(\hat{\xi}_j)e_j) \right\| \\
&\simeq |\varphi(\hat{\xi}_j) - x_j| + |\hat{\xi}_j'| + \left| \xi'' + Q(-x_j e_j, \hat{\xi}_j + \varphi(\hat{\xi}_j)e_j) \right|^{1/2} \\
&= |\varphi(\hat{\xi}_j) - x_j| + |\hat{\xi}_j'| + \left| \xi'' + Q(-x_j e_j, \hat{\xi}_j') \right|^{1/2}.
\end{aligned}$$

We used here the bilinearity of Q and the property $0 = (-e_j) \cdot e_j = -Q(e_j, e_j)$. In order to prove (3.38) it will be enough to show that

(3.39) $$x_j \leq C_0 \left(|\varphi(\hat{\xi}_j) - x_j| + |\hat{\xi}_j'| + \left| \xi'' + Q(-x_j e_j, \hat{\xi}_j') \right|^{1/2} \right).$$

By the Lipschitz continuity of φ we find

$$\begin{aligned}
x_j &\leq |x_j - \varphi(\hat{\xi}_j)| + |\varphi(\hat{\xi}_j)| \lesssim |x_j - \varphi(\hat{\xi}_j)| + |\hat{\xi}_j| \\
&\simeq |x_j - \varphi(\hat{\xi}_j)| + |\hat{\xi}_j'| + |\xi''| \\
&\lesssim |x_j - \varphi(\hat{\xi}_j)| + |\hat{\xi}_j'| + |\xi'' + Q(-x_j e_j, \hat{\xi}_j')| + |Q(-x_j e_j, \hat{\xi}_j')| \\
&\lesssim |x_j - \varphi(\hat{\xi}_j)| + |\hat{\xi}_j'| + |\xi'' + Q(-x_j e_j, \hat{\xi}_j')|^{1/2} + x_j|\hat{\xi}_j'| \\
&\lesssim |x_j - \varphi(\hat{\xi}_j)| + |\hat{\xi}_j'| + |\xi'' + Q(-x_j e_j, \hat{\xi}_j')|^{1/2}.
\end{aligned}$$

We used here the fact that all the involved vectors lie in a bounded set. Our claim (3.38) is proved.

21. Our next step is to compute the "canonical" John curve starting from a generic point $y \in \Omega$. The point y and the boundary of Ω will be translated by a suitable vector $\eta \in \mathbb{R}^n$ in such a way that $\eta \cdot y$ lies in the half axis $\{\alpha e_j : \alpha > 0\}$. Using the equation of the translated surface the correct vector of parameters $\bar{\nu}$ can

be computed and the curve starting from y will be defined as $\gamma_y(t) = \eta^{-1} \cdot (\eta \cdot y) \cdot (t\bar{\nu}) = y \cdot (t\bar{\nu})$ for $t \geq 0$.

22. We claim that there exist $\varrho > 0$ and $C_0 > 1$ such that for all $y \in \Omega \cap \{|y| \leq \varrho\}$ there exists $\eta \in \mathbb{R}^n$ such that:

(i) $\eta \cdot \partial\Omega$ contains the origin;
(ii) $\eta \cdot y$ belongs to $\{\alpha e_j : \alpha > 0\}$;
(iii) $|\eta| \leq C_0 |\hat{y}_j|$.

We look for $\eta = (\eta', \eta'')$. If η'' is given, and we define η' by the equation

$$(3.40) \qquad \eta' = -\hat{y}_j' - \varphi(\hat{y}_j' - \eta'')e_j,$$

then (i) is satisfied. Indeed, $z := \hat{y}_j' - \eta'' + \varphi(\hat{y}_j' - \eta'')e_j \in \partial\Omega$ and

$$\eta \cdot z = \eta' + \eta'' + \hat{y}_j' - \eta'' + \varphi(\hat{y}_j' - \eta'')e_j + Q(\eta', \hat{y}_j' + \varphi(\hat{y}_j' - \eta'')e_j) = 0,$$

by (3.40).

We shall soon prove that the implicit equation

$$(3.41) \qquad \eta'' + y'' + Q(-\hat{y}_j' - \varphi(\hat{y}_j' - \eta'')e_j, \hat{y}_j' + y_j e_j) = 0,$$

has a solution η''. Then, the choice of η'' solution of (3.41) and of η' as in (3.40) ensures that the vector $\eta = (\eta', \eta'')$ satisfies (ii). Indeed,

$$\begin{aligned}
\eta \cdot y &= \left(-\hat{y}_j' + \eta'' - \varphi(\hat{y}_j' - \eta'')e_j \right) \cdot \left(\hat{y}_j' + y'' + y_j e_j \right) \\
&= y'' + \eta'' + \left(y_j - \varphi(\hat{y}_j' - \eta'') \right)e_j + Q\left(-\hat{y}_j' - \varphi(\hat{y}_j' - \eta'')e_j, \hat{y}_j' + y_j e_j \right),
\end{aligned}$$

which belongs to the jth axis if and only if (3.41) holds.

We prove the existence of a solution η''. First notice that by the bilinearity of Q,

$$\begin{aligned}
Q\left(-\hat{y}_j' - \varphi(\hat{y}_j' - \eta'')e_j, y' \right) &= Q\left(-\hat{y}_j' - y_j e_j + \left(y_j - \varphi(\hat{y}_j' - \eta'') \right)e_j, y' \right) \\
&= Q\left(\left(y_j - \varphi(\hat{y}_j' - \eta'') \right)e_j, y' \right) \\
&= \left(y_j - \varphi(\hat{y}_j' - \eta'') \right)Q(e_j, y').
\end{aligned}$$

The map $y' \mapsto Q(e_j, y')$ is linear and does not depend on y_j. Thus (3.41) is equivalent to

$$(3.42) \qquad \eta'' + y'' + \left(y_j - \varphi(\hat{y}_j' - \eta'') \right)Q(e_j, \hat{y}_j') = 0.$$

We show that there exists $\varrho > 0$ such that if $y \in \Omega$ and $|y| \leq \varrho$, then (3.42) has a solution η'' satisfying

$$(3.43) \qquad |\eta''| \leq 2|\hat{y}_j|.$$

We use a fixed point argument. Letting $F(\eta'') = -y'' - \left(y_j - \varphi(\hat{y}_j' - \eta'') \right)Q(e_j, \hat{y}_j')$ equation (3.42) becomes $F(\eta'') = \eta''$. Let $D = \{\eta'' : |\eta''| \leq 2|\hat{y}_j|\}$. If we show that $F(D) \subset D$, then the continuous map F has a fixed point by Brouwer theorem. Indeed,

$$\begin{aligned}
|F(\eta'')| &\leq |y''| + |Q(e_j, \hat{y}_j')|\left| \left(y_j - \varphi(\hat{y}_j' - \eta'') \right) \right| \\
&\leq |\hat{y}_j| + C|\hat{y}_j'|\left(|y_j| + |\hat{y}_j'| + |\eta''| \right) \\
&\leq |\hat{y}_j|(1 + 4C|y|) \leq 2|\hat{y}_j|,
\end{aligned}$$

as soon as $|y| \leq \varrho = 1/(4C)$ (here the constant C depends only on the surface).

Moreover, by (3.40) and by (3.43),

$$|\eta'| = |\hat{y}_j'| + |\varphi(\hat{y}_j' - \eta'')| \lesssim |\hat{y}_j'| + |\eta''| \lesssim |\hat{y}_j|.$$

This proves claim (iii).

23. We compute $\bar{\nu}$ by a left translation argument. Let $\Phi(y) = y_j - \varphi(\hat{y}_j)$. The parameters ν at the point $y = 0$ are given by $\nu_i = X_i\Phi(0)$, $i = 1, ..., m$. Then for any point $\xi = \hat{\xi}_j + \varphi(\hat{\xi}_j)e_j$ belonging to the surface $\{\Phi = 0\}$ the parameters $\nu_i = \nu_i(\xi)$ are given by

$$\nu_i(\xi) = (X_i\Phi)(\xi)$$
$$= \begin{cases} \left(\partial_j + \sum_{k>m} q_{jk}(\xi')\partial_k\right)\Phi(\xi) = 1 - \sum_{k>m} q_{jk}(\xi')\partial_k\varphi(\hat{\xi}_j) & \text{if } i = j, \\ \left(\partial_i + \sum_{k>m} q_{ik}(\xi')\partial_k\right)\Phi(\xi) = -\partial_i\varphi(\hat{\xi}) - \sum_{k>m} q_{ik}(\xi')\partial_k\varphi(\hat{\xi}_j) & \text{if } i \neq j. \end{cases}$$

Let $\eta \in \mathbb{R}^n$ be a vector relative to y as in claims (i), (ii) and (iii) of 22. The correct value of the parameters is given by the evaluation of the previous equation at the point $-\eta$ (this is because the point $-\eta$ is taken to the origin by the left translation τ_η). Define $\bar{\nu}_i = \nu_i(-\eta)$. Set $\bar{\nu} = (\bar{\nu}_1, ..., \bar{\nu}_m, 0, ..., 0)$. We claim that

(3.44) $$|\nu - \bar{\nu}| \lesssim |\hat{y}_j|.$$

If $i \neq j$, by the Lipschitz continuity of φ and by claim (iii),

$$|\bar{\nu}_i - \nu_i| = \left| -\partial_i\varphi(-\hat{\eta}_j) - \sum_{k=m+1}^n q_{ik}(-\eta')\partial_k\varphi(-\hat{\eta}_j) + \partial_i\varphi(0) \right|$$
$$\lesssim |\partial_i\varphi(0) - \partial_i\varphi(-\hat{\eta}_j)| + \sum_{k=m+1}^n |q_{ik}(-\eta')||\partial_k\varphi(-\hat{\eta}_j)|$$
$$\lesssim |\hat{\eta}_j| + |\eta'| \simeq |\eta| \lesssim |\hat{y}_j|.$$

The estimate of the jth component of $\nu - \bar{\nu}$ is easier and we do not prove it.

24. Let γ_x be the curve starting from $x = x_j e_j$ defined in (3.34) and let γ_y be the curve starting from $y \in \Omega$ defined for $t \geq 0$ by

$$\gamma_y(t) = y \cdot (t\bar{\nu}) = \hat{y}_j' + t\bar{\nu} + y_j e_j + y'' + Q(y', t\bar{\nu}),$$

where $\bar{\nu}$ is the vector of parameters discussed above. We now construct a cone with core γ_x. For $\lambda > 0$ let $\mathcal{U}(\lambda) = \{h \in \mathbb{R}^n : \|h\| \leq \lambda\}$ and for $t \geq 0$ define $h_t = (th', t^2h'')$. Note that $\|h_t\| = t\|h\| \simeq d(\gamma_x(t), \gamma_x(0))\|h\|$. Finally, let

$$\gamma_x^h(t) = x_j e_j \cdot (t\nu) \cdot h_t = x_j e_j \cdot (t\nu + h_t + Q(t\nu, h_t'))$$
$$= x_j e_j + t\nu + h_t + tQ(\nu, h_t') + Q(x_j e_j, t\nu + h_t').$$

25. We claim that there exist $M > 0$ and $\varrho > 0$ such that for all $x = x_j e_j \in \Omega$ and for all $y \in \Omega$ such that $|\hat{y}_j| \leq \varrho$ there exists $h \in \mathcal{U}(\lambda)$ such that $\gamma_x^h(Md(x, y)) = \gamma_y(Md(x, y))$. Here λ is a parameter small enough to ensure that for all $h \in \mathcal{U}(\lambda)$, γ_x^h is a John curve with constant λ.

Equality $\gamma_y(t) = \gamma_x^h(t)$ reads

(3.45)
$$\hat{y}_j' + t\bar{\nu} + y_j e_j + y'' + tQ(y', \bar{\nu}) = x_j e_j + t\nu + h_t + tQ(\nu, h_t') + x_j Q(e_j, t\nu + h_t').$$

Projecting this equation along the coordinates of the first slice we get

$$\hat{y}'_j + t\bar{\nu} + y_j e_j = x_j e_j + t\nu + th' \tag{3.46}$$

and the solution h' satisfies $|h'| \leq \frac{1}{t}\{|\hat{y}'_j| + |y_j - x_j| + t|\nu - \bar{\nu}|\}$. Replacing $t = Md(x,y)$ we find

$$|h'| \leq \frac{|\hat{y}'_j| + |y_j - x_j|}{Md(x,y)} + |\nu - \bar{\nu}|.$$

By the equivalence

$$d(x,y) \simeq \|(-x) \cdot y\| \simeq |y_j - x_j| + |\hat{y}'_j| + \left|y'' + Q(-x_j e_j, \hat{y}'_j)\right|^{1/2}, \tag{3.47}$$

and by (3.44) we obtain for some absolute constant C_0,

$$|h'| \leq C_0\Big(\frac{1}{M} + \varrho\Big) \tag{3.48}$$

as soon as $|\hat{y}_j| \leq \varrho$.

We project now (3.45) along the coordinates of the second slice obtaining

$$y'' + tQ(y', \bar{\nu}) = h''_t + tQ(\nu, h'_t) + x_j Q(e_j, t\nu + h'_t),$$

where $h'_t = th'$ and h' satisfies (3.48). We deduce that

$$|h''_t| \leq |y''| + t|Q(y', \bar{\nu})| + t|Q(\nu, h'_t)| + x_j|Q(e_j, t\nu + h'_t)|.$$

We estimate separately each term in the right-hand side. By (3.47) and (3.38)

$$|y''| \leq |y'' + Q(-x_j e_j, \hat{y}'_j)| + |Q(-x_j e_j, \hat{y}'_j)| \lesssim d(x,y)^2 + x_j|\hat{y}'_j| \lesssim d(x,y)^2.$$

Moreover, $|Q(y', \bar{\nu})| \lesssim |y'| \lesssim d(x,y)$ and by (3.48) $|Q(\nu, h'_t)| \lesssim |h'_t| \lesssim t\Big(\frac{1}{M}+\varrho\Big)$. The vectors ν and $\bar{\nu}$ are bounded. Finally, again by (3.38), $x_j|Q(e_j, t\nu + h'_t)| \lesssim td(x,y)$. Then

$$|h''_t| \lesssim d(x,y)^2 + td(x,y) + t^2\Big(\frac{1}{M} + \varrho\Big),$$

and replacing $t = Md(x,y)$ we finally get

$$|h''| \lesssim \frac{1}{M^2} + \frac{1}{M} + \varrho,$$

which shows that $\|h\| \leq \lambda$ if M is large and ϱ is small enough. $\qquad \square$

4. John domains in a group of step 3

In this section we study John domains in groups of step 3. In order to make explicit computations we shall study the simplest Carnot group of step 3 whose Lie algebra has the lowest dimension, which is 4.

Consider in $\mathbb{R}^4$ the vector fields

$$X_1 = \partial_1 - \frac{1}{2}x_2\partial_3 - \Big\{\frac{1}{12}(x_1 x_2 + \alpha x_2^2) + \frac{1}{2}x_3\Big\}\partial_4,$$

$$X_2 = \partial_2 + \frac{1}{2}x_1\partial_3 + \Big\{\frac{1}{12}(x_1^2 + \alpha x_1 x_2) - \frac{\alpha}{2}x_3\Big\}\partial_4,$$

$$X_3 = \partial_3 + \frac{1}{2}(x_1 + \alpha x_2)\partial_4$$

$$X_4 = \partial_4,$$

where $\alpha \in \mathbb{R}$ is a real parameter. The commutation relations are

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = \alpha X_4,$$

and all other commutators vanish. Thus, for any $\alpha \in \mathbb{R}$ the vector fields X_1, X_2 are generators of a Lie algebra of differential operators in $\mathbb{R}^4$ of step 3. It can be checked that the following group law on $\mathbb{R}^4$ makes X_1, X_2, X_3 and X_4 left invariant:

$$\begin{aligned}
x \cdot y = \Big(& x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1), \\
& x_4 + y_4 + \frac{1}{12}\{(y_1 + \alpha y_2)(x_2 y_1 - x_1 y_2) + (x_1 + \alpha x_2)(x_1 y_2 - x_2 y_1)\} \\
& + \frac{1}{2}\{(x_1 y_3 - x_3 y_1) + \alpha(x_2 y_3 - y_2 x_3)\}\Big).
\end{aligned}$$

(4.1)

Notice that $x^{-1} = -x$. Introduce the abbreviations

$$\begin{aligned}
q_1(x_1, x_2, x_3) &= -\Big\{\frac{1}{12}(x_1 x_2 + \alpha x_2^2) + \frac{1}{2}x_3\Big\}, \\
q_2(x_1, x_2, x_3) &= \Big\{\frac{1}{12}(x_1^2 + \alpha x_1 x_2) - \frac{\alpha}{2}x_3\Big\}, \\
q_3(x_1, x_2) &= \frac{1}{2}(x_1 + \alpha x_2),
\end{aligned}$$

(4.2)

and

(4.3)

$$Q_3(x_1, x_2, y_1, y_2) = \frac{1}{2}(x_1 y_2 - x_2 y_1)$$

$$\begin{aligned}
Q_4(x_1, x_2, x_3, y_1, y_2, y_3) = \frac{1}{2}\Big\{&(x_1 y_3 - x_3 y_1) + \alpha(x_2 y_3 - y_2 x_3)\Big\} \\
+ \frac{1}{12}\Big\{&(y_1 + \alpha y_2)(x_2 y_1 - x_1 y_2) + (x_1 + \alpha x_2)(x_1 y_2 - x_2 y_1)\Big\},
\end{aligned}$$

in such a way that

$$\begin{aligned}
x \cdot y = (&x_1 + y_1, x_2 + y_2, x_3 + y_3 + Q_3(x_1, x_2, y_1, y_2), \\
&x_4 + y_4 + Q_4(x_1, x_2, x_3, y_1, y_2, y_3)).
\end{aligned}$$

We denote by d the Carnot-Carathéodory distance induced on $\mathbb{R}^4$ by X_1 and X_2 and by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^4$ with radius $r \geq 0$. Define also the following homogeneous norm in $\mathbb{R}^4$,

$$\|x\| = |x_1| + |x_2| + |x_3|^{1/2} + |x_4|^{1/3}.$$

By a standard argument it can be proved that $d(x, y) \simeq \|y^{-1} \cdot x\|$. Define the Box

(4.4) $$\text{Box}(x, r) = \{x \cdot y \in \mathbb{R}^4 : \|y\| \leq r\}.$$

Then it follows that there exists $c > 1$ such that for all $x \in \mathbb{R}^n$ and $r \geq 0$,

$$\text{Box}(x, c^{-1}r) \subset B(x, r) \subset \text{Box}(x, cr).$$

Let $S \subset \mathbb{R}^4$ be a 3-dimensional surface of class C^1. If $x_0 \in S$, there exists a neighborhood $\mathcal{U}$ of x_0 in $\mathbb{R}^4$ and there exists $\Phi \in C^1(\mathcal{U}; \mathbb{R})$ such that $S \cap \mathcal{U} = \{x \in \mathcal{U} : \Phi(x) = 0\}$ and $\nabla \Phi \neq 0$ on $S \cap \mathcal{U}$. A point $x \in S \cap \mathcal{U}$ is said to be *characteristic* if and only if $X_1 \Phi(x) = X_2 \Phi(x) = 0$. From a geometric point of view this means that X_1 and X_2 belong to the tangent spaces to S at x.

Definition 4.1. A characteristic point $x \in S \cap \mathcal{U}$ is *of first type* if $X_3 \Phi(x) \neq 0$. If $X_3 \Phi(x) = 0$, then it is *of second type*.

If $x \in S \cap \mathcal{U}$ is a characteristic point of second type, then $X_4 \Phi(x) = \partial_4 \Phi(x)$ can not be 0. Otherwise, it would be $X_1 \Phi = \cdots = X_4 \Phi = 0$ at x and this is impossible because $\nabla \Phi \neq 0$ and $X_1, \dots, X_4$ are independent at each point.

We are interested in expressing S as a graph in a neighborhood of a point $\bar{x} \in S$ after a translation that takes $\bar{x}$ to the origin. Notice that $\bar{x}$ is a characteristic point (of first, second type) of S if and only if 0 is a characteristic point (of first, second type) of the translated surface $\bar{x}^{-1} \cdot S$. This is an easy consequence of the left invariance of the X_j's.

Now, let $S = \partial \Omega$ be the boundary of an open set of class C^1 and let $\bar{x} \in \partial \Omega$. Assume that $\bar{x}$ is a characteristic point of first type. Then, for any point $\tilde{x}$ in a neighborhood of $\bar{x}$, the surface $(\tilde{x}^{-1} \cdot \partial \Omega) \cap \mathcal{V}$, where $\mathcal{V} \subset \mathbb{R}^4$ is a suitable neighborhood of the origin, can be parameterized by a function $x_3 = \varphi(x_1, x_2, x_4)$, where $\varphi \in C^1(D)$ and D is a neighborhood of the origin in $\mathbb{R}^3$.

If $\bar{x}$ is a characteristic point of second type, then the variable x_4 must be given in terms of the variables x_1, x_2, x_3. Then, for any $\tilde{x}$ near $\bar{x}$, the surface $(\tilde{x}^{-1} \cdot \partial \Omega) \cap \mathcal{V}$, can be parameterized by a function $x_4 = \varphi(x_1, x_2, x_3)$, where $\varphi \in C^1(D)$ and D is a neighborhood of the origin in $\mathbb{R}^3$.

Definition 4.2. Let Ω be a C^1 bounded connected open set. Denote by $\Sigma_2 \subset \partial \Omega$ the characteristic set of second type. We say that Ω is *admissible* if

(i) there are a neighborhood A in $\partial \Omega$ of Σ_2, a neighborhood D of the origin in $\mathbb{R}^3$ and constants $\epsilon_0, k > 0$ such that for any $z \in A$,

$$z^{-1} \cdot \partial \Omega \cap (D \times]-\epsilon_0, \epsilon_0[) = \{x \in \mathbb{R}^4 : x_4 = \varphi_z(x'), \quad x' = (x_1, x_2, x_3) \in D\},$$

where the function φ_z satisfies

$$(4.5) \quad \begin{aligned} |\varphi_z(x') &- \langle \nabla \varphi_z(0), x' \rangle| \\ &\leq k \big(\|x'\|^3 + (|\partial_1 \varphi_z(0)|^{1/2} + |\partial_2 \varphi_z(0)|^{1/2} + |\partial_3 \varphi_z(0)|)(x_1^2 + x_2^2) \big) \end{aligned}$$

(here $\|x'\| = |x_1| + |x_2| + |x_3|^{1/2}$ and ∇ denotes the Euclidean gradient);

(ii) $\partial \Omega$ is of class $C^{1,1}$ away from Σ_2.

The requirement (4.5) near Σ_2 is the key point. The natural question now is how to check it for a given surface. An answer is contained in Lemma 4.3.

Write $x = (x_1, x_2, x_3, x_4) = (x', x_4) \in \mathbb{R}^3 \times \mathbb{R}$. Thus we can split the group law (4.1) as follows:

$$x \cdot y = ((x' \circ y'), x_4 + y_4 + Q_4(x', y')),$$

where $\circ$ denotes the composition law in the Heisenberg group. Denote by ϱ the control distance in the Heisenberg group.

Lemma 4.3. *Let $D_0 \subset \mathbb{R}^3$ be a neighborhood of the origin. Consider a C^1 function $\varphi : D_0 \to \mathbb{R}$ and its graph $\Gamma = \{x_4 = \varphi(x_1, x_2, x_3) = \varphi(x')\}$. Assume that $\varphi(0, 0, 0) = 0$ and that there exists $k > 0$ such that*

(i) *the second derivatives $X_i X_j \varphi(x')$, $i, j = 1, 2$, exist at any point $x' \in D$ and the functions $X_i X_j \varphi$, $i, j = 1, 2$, satisfy*

$$(4.6) \quad |X_i X_j \varphi(x') - X_i X_j \varphi(y')| \leq k \varrho(x', y'), \quad x', y' \in D_0, \quad i, j = 1, 2;$$

(ii) *writing* $\Phi(x) = \Phi(x_1, x_2, x_3, x_4) = \varphi(x_1, x_2, x_3) - x_4$, *for all* $x = (x', \varphi(x'))$
$\in \Gamma$, *the following estimate holds:*

(4.7)
$$|X_1^2 \Phi(x)| + |X_2^2 \Phi(x)| + |(X_1 X_2 + X_2 X_1)\Phi(x)|$$
$$\leq k(|X_1\Phi(x)|^{1/2} + |X_2\Phi(x)|^{1/2} + |X_3\Phi(x)|).$$

Then there is $D \subset D_0$ *such that, for any fixed point* $z' \in D$, *the surface* $(z', \varphi(z'))^{-1} \cdot$
Γ *is the graph of a function* φ_z *defined on* $(z')^{-1} \circ D_0$ *which satisfies* (4.5) *for all*
$x' \in D$.

In Example 5.2 we will give an example of admissible set by means of this lemma.

Proof. Fix a neighborhood of the origin $D \subset D_0$ such that $D \subset (z')^{-1} \circ D_0$ for all
$z' \in D$ (i.e. $D \circ D \subset D_0$). Take a point $z' \in D$. It is easy to check that the set
$(z', \varphi(z'))^{-1} \cdot \Gamma$ is defined by

$$y_4 = \varphi(z' \circ y') - \varphi(z') + Q_4(-z', z' \circ y')$$
$$= \varphi(z' \circ y') - \varphi(z') - Q_4(z', y') := \varphi_z(y), \qquad y \in (z')^{-1} \circ D_0.$$

The equality $Q_4(-z', z' \circ y') = -Q_4(z', y')$ is an immediate consequence of the
associative property $(y', y_4) = ((z', z_4)^{-1} \cdot (z', z_4)) \cdot (y', y_4) = (z', z_4)^{-1} \cdot ((z', z_4) \cdot$
$(y', y_4))$, $y, z \in \mathbb{R}^4$.

By (4.2) and (4.3) the function $Q_4(z, y)$ can be written as

(4.8)
$$Q_4(z', y') = q_1(z')y_1 + q_2(z')y_2 + q_3(z')y_3$$
$$+ X_1 q_1(z')\frac{y_1^2}{2} + X_2 q_2(z')\frac{y_2^2}{2} + \left(X_1 q_2(z') + X_2 q_1(z')\right)\frac{y_1 y_2}{2}.$$

We now claim that the function $y' \mapsto \varphi(z' \circ y') - \varphi(z')$ admits the following
"homogeneous Taylor expansion":

(4.9)
$$\varphi(z' \circ y') - \varphi(z') = X_1 \varphi(z')y_1 + X_2 \varphi(z')y_2 + X_3 \varphi(z')y_3 + \frac{1}{2}X_1^2 \varphi(z')y_1^2$$
$$+ \frac{1}{2}X_2^2 \varphi(z')y_2^2 + \frac{1}{2}(X_1 X_2 + X_2 X_1)\varphi(z')y_1 y_2 + O(\|y'\|^3),$$

where $O(\|y'\|^3) \leq C\|y'\|^3$ for all $y \in (z')^{-1} \circ D_0$ (and thus for all $y' \in D$). The
constant C does not depend on $z' \in D$. Now write

$$\varphi_z(y') = \varphi_z(y_1, y_2, y_3) - \varphi_z(y_1, y_2, 0) + \varphi_z(y_1, y_2, 0) - \varphi_z(0, 0, 0).$$

We examine the first term. By the mean value theorem

$$\varphi_z(y_1, y_2, y_3) - \varphi_z(y_1, y_2, 0) = \varphi(z' \circ (y_1, y_2, y_3)) - \varphi(z' \circ (y_1, y_2, 0))$$
$$= (\partial_3 \varphi)(z' \circ (y_1, y_2, \vartheta y_3))y_3$$
$$= (X_3 \varphi)(z' \circ (y_1, y_2, \vartheta y_3))y_3$$
$$= X_3 \varphi(z')y_3 + \{X_3 \varphi(z' \circ (y_1, y_2, \vartheta y_3)) - X_3 \varphi(z')\}y_3,$$

where $\vartheta \in]0, 1[$. By the Lipschitz continuity of $X_3 \varphi$ (recall that $X_3 = X_1 X_2 -$
$X_2 X_1$) we get $|\{X_3 \varphi(z' \circ (y_1, y_2, \vartheta y_3)) - X_3 \varphi(z')\}| \leq k\|y'\|$, which multiplied by y_3
can be estimated by $\|y'\|^3$.

We now look at the second piece. Let

$$g(t) = \varphi(z' \circ (ty_1, ty_2, 0)) = \varphi(\exp(t(y_1 X_1 + y_2 X_2))(z)), \quad t \in [0, 1].$$

Condition (4.6) ensures that g is of class C^2 and that

$$g'(t) = \sum_{i=1}^{2} y_i X_i \varphi(z' \circ (ty_1, ty_2, 0)), \quad \text{and}$$

$$g''(t) = \sum_{i,j=1}^{2} y_i y_j X_i X_j \varphi(z' \circ (ty_1, ty_2, 0)).$$

Thus, by the Taylor formula,

$$\varphi(z' \circ (ty_1, ty_2, 0)) - \varphi(z') = g(1) - g(0) = g'(0) + \frac{1}{2} g''(\vartheta)$$

$$= \sum_{i=1}^{2} X_i \varphi(z') y_i + \frac{1}{2} \sum_{i,j=1}^{2} X_i X_j \varphi(z') y_i y_j$$

$$+ \frac{1}{2} \sum_{i,j=1}^{2} \{X_i X_j \varphi(z' \circ (\vartheta y_1, \vartheta y_2, 0)) - X_i X_j \varphi(z')\} y_i y_j.$$

In view of the Lipschiz continuity of $X_i X_j \varphi$ the last line can be easily estimated with $\|y'\|^3$. Thus we have proved that (4.9) holds.

Subtracting (4.8) from (4.9) we find the Taylor expansion of φ_z:

$$\varphi_z(y) = \sum_{j=1}^{3} \{X_j \varphi(z') - q_j(z')\} y_j + \frac{1}{2} \sum_{i,j=1}^{2} \{X_i X_j \varphi(z') - X_i q_j(z')\} y_i y_j + O(\|y'\|^3).$$

Then

$$\partial_i \varphi_z(0) = X_i \varphi(z') - q_i(z') = X_i \Phi(z', \varphi(z')), \quad i = 1, 2, 3,$$

$$\partial_{ij}^2 \varphi_z(0) = \frac{1}{2} \big(X_i X_j \varphi(z') - X_i q_j(z') + X_j X_i \varphi(z') - X_j q_i(z') \big)$$

$$= \frac{1}{2} (X_i X_j + X_j X_i) \Phi(z', \varphi(z')), \quad i, j = 1, 2.$$

Thus, assuming (4.7) we immediately see that (4.5) holds. This ends the proof of the lemma. □

We are now ready to prove our main result.

Theorem 4.4. *If $\Omega \subset \mathbb{R}^4$ is an admissible domain, then it is a John domain in $(\mathbb{R}^4, d)$.*

Proof. We prove the theorem using Proposition 2.4. We shall construct "canonical" John curves starting from points near the boundary $\partial\Omega$. The proof will be split into several numbered small steps.

1. For a fixed point $\bar{x} \in \partial\Omega$, let $\mathcal{U} \subset \mathbb{R}^4$ be a neighborhood of $\bar{x}$ and let $\Phi \in C^{1,1}(\mathcal{U}; \mathbb{R})$ be a local equation for $\partial\Omega \cap \mathcal{U}$. We shall distinguish three cases:

(C1) $X_1 \Phi(\bar{x}) = X_2 \Phi(\bar{x}) = 0$, and $X_3 \Phi(\bar{x}) \neq 0$ ($\bar{x}$ is a characteristic point of first type);

(C2) $X_1 \Phi(\bar{x}) = X_2 \Phi(\bar{x}) = X_3 \Phi(\bar{x}) = 0$ ($\bar{x}$ is a characteristic point of second type);

(C3) $|X_1 \Phi(\bar{x})| + |X_2 \Phi(\bar{x})| > 0$ ($\bar{x}$ is a non-characteristic point of $\partial\Omega$).

2. **Case 1.** After a translation $0 \in \partial\Omega$ can be assumed to be near $\bar{x}$. Thus, in a neighborhood of 0, $\partial\Omega$ can be written in the form $y_3 = \varphi(y_1, y_2, y_4)$, where the function $\varphi = \varphi(y_1, y_2, y_4)$ is of class $C^{1,1}$ and $\varphi(0) = 0$. Define

$$\nu_1 = -\partial_1\varphi(0), \ \nu_2 = -\partial_2\varphi(0), \ \nu = (\nu_1, \nu_2), \ N_1 = \frac{\nu_1}{|\nu|}, \ N_2 = \frac{\nu_2}{|\nu|},$$

and if $\nu = 0$, simply set $N_1 = N_2 = 0$. Moreover, let $\psi(y) = \varphi(y) + \nu_1 y_1 + \nu_2 y_2$. Since φ is $C^{1,1}$, we have

(4.10) $$|\psi(y)| = |\varphi(y) + \nu_1 y_1 + \nu_2 y_2| \lesssim y_1^2 + y_2^2 + |y_4|.$$

Consider now a point $x = (0, 0, x_3, 0) \in \Omega$ with $0 < x_3 \leq 1$. We shall define a continuous path $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$ and the John property $\mathrm{dist}(\gamma(t); \partial\Omega) \geq \lambda d(\gamma(t), x)$ holds for all $t \in [0, 1]$ and for some $\lambda > 0$ depending only on Ω. The path will be made by two pieces.

3. *First piece.* Let $\sigma > 0$ and define $t_1 = \sigma|\nu|$. For $t \in [0, t_1]$ we define

$$\gamma(t) = (0, 0, x_3, 0) \cdot (tN_1, tN_2, 0, 0).$$

Notice that $d(\gamma(t), x) \simeq t$ and that the first piece degenerates if $\nu = 0$.

4. We claim that there exist $\sigma, \lambda \in (0, 1)$ absolute constants such that for all $t \leq t_1$,

(4.11) $$\mathrm{Box}(\gamma(t), \lambda t) \subset \Omega.$$

Condition (4.11) is equivalent to the John property for γ in this first piece. Points in $\mathrm{Box}(\gamma(t), \lambda t)$ are of the form

$$\begin{aligned}
\gamma(t) \cdot h &= (0, 0, x_3, 0) \cdot (tN_1, tN_2, 0, 0) \cdot (h_1, h_2, h_3, h_4) \\
&= (0, 0, x_3, 0) \cdot (tN_1 + h_1, tN_2 + h_2, h_3 + Q_3(tN_1, tN_2, h_1, h_2), \\
&\qquad\qquad h_4 + Q_4(tN_1, tN_2, 0, h_1, h_2, h_3)) \\
&= \Big(tN_1 + h_1, tN_2 + h_2, x_3 + h_3 + Q_3(tN_1, tN_2, h_1, h_2), \\
&\qquad h_4 + Q_4(tN_1, tN_2, 0, h_1, h_2, h_3) \\
&\qquad + Q_4\big(0, 0, x_3, tN_1 + h_1, tN_2 + h_2, h_3 + Q_3(tN_1, tN_2, h_1, h_2)\big)\Big)
\end{aligned}$$

with $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \leq \lambda t$.

Now, $\gamma(t) \cdot h \in \Omega$ provided that (recall that $\varphi(z) = -\nu_1 z_1 - \nu_2 z_2 + \psi(z)$)

$$\begin{aligned}
x_3 + h_3 &+ Q_3(tN_1, tN_2, h_1, h_2) \geq -\nu_1(tN_1 + h_1) - \nu_2(tN_2 + h_2) \\
(4.12) &\quad + \psi\Big(tN_1 + h_1, tN_2 + h_2, h_4 + Q_4(tN_1, tN_2, 0, h_1, h_2, h_3) \\
&\quad + Q_4\big(0, 0, x_3, tN_1 + h_1, tN_2 + h_2, h_3 + Q_3(tN_1, tN_2, h_1, h_2)\big)\Big).
\end{aligned}$$

Since $\nu_1 N_1 + \nu_2 N_2 = |\nu|$, this inequality is guaranteed by

$$x_3 + |\nu|t \geq |h_1||\nu_1| + |h_2||\nu_2| + |h_3| + |Q_3(tN_1, tN_2, h_1, h_2)| + |\psi(z)|,$$

where $z = (z_1, z_2, z_4)$ denotes the argument of ψ in (4.12). Note that $|h_1||\nu_1| + |h_2||\nu_2| \leq \lambda|\nu|t$ and this term can be absorbed in the left-hand side if λ is small. Moreover, $|h_3| \leq \lambda t^2$ and $|Q_3(tN_1, tN_2, h_1, h_2)| \lesssim \lambda t^2$. Then, in order to prove inclusion (4.11) it will be enough to show that

$$\epsilon_0(x_3 + |\nu|t) \geq \lambda t^2 + |\psi(z)|$$

for some $\epsilon_0 > 0$ small but absolute. We estimate z_1, z_2 and z_4. Clearly, $|z_1| = |tN_1 + h_1| \lesssim t$ and $|z_2| = |tN_2 + h_2| \lesssim t$. Moreover,

$$
\begin{aligned}
|z_4| = \big| h_4 &+ Q_4(tN_1, tN_2, 0, h_1, h_2, h_3) \\
&+ Q_4\big(0, 0, x_3, tN_1 + h_1, tN_2 + h_2, h_3 + Q_3(tN_1, tN_2, h_1, h_2)\big)\big| \\
&\lesssim \lambda t^3 + x_3 t
\end{aligned}
$$

because $Q_4(0, 0, x_3, \xi_1, \xi_2, \xi_3) = 1/2\{(-x_3\xi_1) + \alpha(-\xi_2 x_3)\}$.

Thus by (4.10)

$$
|\psi(z)| \lesssim z_1^2 + z_2^2 + |z_4| \lesssim t^2 + \lambda t^3 + x_3 t \simeq t^2 + x_3 t,
$$

because $\lambda t^3 \lesssim t^2$ (we assume $t \leq 1$).

We finally have to prove the inequality

$$
(4.13) \qquad \epsilon_0(x_3 + |\nu| t) \geq t^2 + x_3 t,
$$

which holds if $t \leq \sigma |\nu|$ with $\sigma > 0$ small depending only on Ω.

5. *Second piece.* From now up to the end of Case 1 $t_1 = \sigma |\nu|$ will be fixed. For $t \geq t_1$ define

$$
\gamma(t) = (0, 0, x_3, 0) \cdot (t_1 N_1, t_1 N_2, t - t_1, 0),
$$

and note that $d(\gamma(t), x) \simeq t_1 + (t - t_1)^{1/2}$. Write $b = (t - t_1)^{1/2}$ and $\delta(t) = t_1 + b$.

6. We claim that there exists a positive $\lambda < 1$ such that for all $t_1 \leq t \leq 1$,

$$
(4.14) \qquad \mathrm{Box}(\gamma(t), \lambda\delta(t)) \subset \Omega.
$$

Condition (4.14) is equivalent to the John property for γ in its second piece.

Points in $\mathrm{Box}(\gamma(t), \lambda\delta(t))$ have the form

$$
\begin{aligned}
\gamma(t) \cdot h &= (0, 0, x_3, 0) \cdot (t_1 N_1, t_1 N_2, b^2, 0) \cdot (h_1, h_1, h_3, h_4) \\
&= \big(t_1 N_1, t_1 N_2, x_3 + b^2, Q_4(0, 0, x_3, t_1 N_1, t_1 N_2, b^2)\big) \cdot (h_1, h_2, h_3, h_4) \\
&= \Big(t_1 N_1 + h_1, t_1 N_2 + h_2, x_3 + b^2 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2), \\
&\qquad Q_4(0, 0, x_3, t_1 N_1, t_1 N_2, b^2) + h_4 + Q_4(t_1 N_1, t_1 N_2, x_3 + b^2, h_1, h_2, h_3)\Big),
\end{aligned}
$$

with $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \leq \lambda\delta(t)$. Now, $\gamma(t) \cdot h \in \Omega$ provided that

$$
\begin{aligned}
(4.15) \qquad x_3 + b^2 &+ h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2) \geq -\nu_1(t_1 N_1 + h_1) - \nu_2(t_1 N_2 + h_2) \\
&+ \psi\Big(t_1 N_1 + h_1, t_1 N_2 + h_2, Q_4(0, 0, x_3, t_1 N_1, t_1 N_2, b^2) + h_4 \\
&\qquad + Q_4(t_1 N_1, t_1 N_2, x_3 + b^2, h_1, h_2, h_3)\Big),
\end{aligned}
$$

which is implied by

$$
(4.16) \quad t_1|\nu| + x_3 + b^2 \geq |\nu_1||h_1| + |\nu_2||h_2| + |h_3| + |Q_3(t_1 N_1, t_1 N_2, h_1, h_2)| + |\psi(z)|,
$$

where $z = (z_1, z_2, z_4)$ is the argument of ψ in (4.15). In order to prove (4.16) note that $|\nu_1||h_1| + |\nu_2||h_2| \lesssim \lambda|\nu|\delta(t) \simeq \lambda|\nu| t_1 + \lambda|\nu| b$. The term $\lambda|\nu| t_1$ can be put in the left-hand side. Moreover, $|h_3| \leq \lambda\delta^2(t) \lesssim \lambda t_1^2 + \lambda b^2$ and $|Q_3(t_1 N_1, t_1 N_2, h_1, h_2)| \leq \lambda t_1\delta(t) \lesssim \lambda t_1^2 + \lambda b^2$. The term λb^2 can also be absorbed in the left-hand side.

Claim (4.14) will be proved if we show that for $t_1 \leq t \leq 1$,

$$
(4.17) \qquad \varepsilon_0(t_1|\nu| + x_3 + b^2) \geq \lambda|\nu| b + \lambda t_1^2 + |\psi(z)|.
$$

We estimate z_1, z_2 and z_4. First, $|z_1| = |t_1 N_1 + h_1| \lesssim t_1 + \lambda \delta(t) \lesssim t_1 + \lambda b$ and the same estimate holds for $|z_2|$. Moreover, writing δ instead of $\delta(t)$ and recalling (4.3) we get

$$
\begin{aligned}
|z_4| &= \left| Q_4(0, 0, x_3, t_1 N_1, t_1 N_2, b^2) + h_4 + Q_4(t_1 N_1, t_1 N_2, x_3 + b^2, h_1, h_2, h_3) \right| \\
&\lesssim x_3 t_1 + \lambda \delta^3 + \lambda t_1^2 \delta + \lambda t_1 \delta^2 + \lambda(x_3 + b^2)\delta \\
&\simeq x_3 t_1 + \lambda(t_1 + b)^3 + \lambda t_1^2(t_1 + b) + \lambda t_1(t_1 + b)^2 + \lambda(x_3 + b^2)(t_1 + b) \\
&\simeq x_3 t_1 + \lambda t_1^3 + \lambda b^3 + \lambda x_3 t_1 + \lambda x_3 b \simeq x_3 t_1 + \lambda t_1^3 + \lambda b^3 + \lambda x_3 b.
\end{aligned}
$$

Then by (4.10)

$$
|\psi(z)| \lesssim |z_1|^2 + |z_2|^2 + |z_4| \leq t_1^2 + \lambda b^2 + x_3 t_1 + \lambda t_1^3 + \lambda b^3 + \lambda x_3 b.
$$

Thus (4.17) is implied by

(4.18)
$$
\begin{aligned}
\epsilon_0(t_1|\nu| + x_3 + b^2) &\geq \lambda|\nu|b + t_1^2 + \lambda b^2 + x_3 t_1 + \lambda t_1^3 + \lambda b^3 + \lambda x_3 b \\
&\simeq \lambda|\nu|b + t_1^2 + \lambda b^2 + x_3 t_1 + \lambda b^3 + \lambda x_3 b.
\end{aligned}
$$

Inequality (4.18) holds for $b = 0$. This has been proved in (4.13) with $t = t_1$.

Taking a smaller constant in the left-hand side of (4.13) we can assert that (4.18) is guaranteed by

(4.19) $$\epsilon_0(t_1|\nu| + x_3 + b^2) \geq \lambda|\nu|b + \lambda b^2 + \lambda b^3 + \lambda x_3 b.$$

We can estimate the right-hand side using $|\nu| \leq 1$ and $b \leq 1$ getting

$$
\lambda(|\nu|b + b^2 + b^3 + x_3 b) \leq \lambda(|\nu|^2 + b^2 + x_3).
$$

Recalling now that $t_1 = \sigma|\nu|$ inequality (4.19) is proved for all $b \in (0, 1)$ if λ is small enough.

7. **Case 2.** Let $\bar{x} \in \partial\Omega$ be a characteristic point of second type. Take a point near $\bar{x}$. Translate it to the origin and write locally $\partial\Omega$ in the form $y_4 = \varphi(y_1, y_2, y_3)$. Since the domain is admissible, (4.5) holds. Write

$$
\nu_1 = -\partial_1\varphi(0), \ \nu_2 = -\partial_2\varphi(0), \ \nu_3 = -\partial_3\varphi(0), \ \nu = (\nu_1, \nu_2),
$$
$$
N_1 = \frac{\nu_1}{|\nu|}, \ N_2 = \frac{\nu_2}{|\nu|}, \ N_3 = \frac{\nu_3}{|\nu_3|} = \mathrm{sgn}(\nu_3).
$$

If $\nu = 0$, simply set $N_1 = N_2 = 0$. If $\nu_3 = 0$, set $N_3 = 0$. Moreover, let $\psi(y) = \varphi(y) + \nu_1 y_1 + \nu_2 y_2 + \nu_3 y_3$. By (4.5) ψ satisfies the following growth condition

(4.20) $$|\psi(y)| \lesssim \|y\|^3 + (|\nu|^{1/2} + |\nu_3|)(y_1^2 + y_2^2).$$

We shall now construct the John curve starting from $x = x_4 e_4$, $x_4 \geq 0$. Without loss of generality (the map $z \mapsto z + \mu e_4$, $\mu \in \mathbb{R}$, is a left translation), assume that $x = 0 \in \partial\Omega$. We have to define a continuous path $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = 0$ and $\mathrm{dist}(\gamma(t); \partial\Omega) \geq \lambda d(\gamma(t), 0)$ for all $t \in [0, 1]$ and for some $\lambda > 0$ depending only on Ω. We split the path into three pieces.

8. *First piece.* For $\sigma > 0$ let

(4.21) $$
t_1 = \begin{cases} \sigma \min\left\{ |\nu|^{1/2}, |\nu|/|\nu_3| \right\} & \text{if } \nu_3 \neq 0, \\ \sigma|\nu|^{1/2} & \text{if } \nu_3 = 0, \end{cases}
$$

and if $t \in [0, t_1]$, define

$$
\gamma(t) = (N_1 t, N_2 t, 0, 0).
$$

Note that $d(\gamma(t), 0) = t$.

9. We claim that there exist positive constants $\sigma, \lambda \in (0,1)$ such that for all $t \in [0, t_1]$ the following John property holds:

$$(4.22) \qquad \text{Box}(\gamma(t), \lambda t) \subset \Omega.$$

Points in $\text{Box}(\gamma(t), \lambda t)$ are of the form

$$\gamma(t) \cdot h = (N_1 t, N_2 t, 0, 0) \cdot (h_1, h_2, h_3, h_4)$$
$$= \big(N_1 t + h_1, N_2 t + h_2, h_3 + Q_3(N_1 t, N_2 t, h_1, h_2),$$
$$h_4 + Q_4(N_1 t, N_2 t, 0, h_1, h_2, h_3)\big),$$

with $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \leq \lambda t$. Now, $\gamma(t) \cdot h \in \Omega$ if

$$h_4 + Q_4(N_1 t, N_2 t, 0, h_1, h_2, h_3) > -\nu_1(N_1 t + h_1) - \nu_2(N_2 t + h_2)$$
$$- \nu_3(h_3 + Q_3(N_1 t, N_2 t, h_1, h_2))$$
$$+ \psi\big(N_1 t + h_1, N_2 t + h_2, h_3 + Q_3(N_1 t, N_2 t, h_1, h_2)\big),$$

which is implied by

$$\begin{aligned}(4.23) \quad |\nu| t \geq{}& |\nu_1||h_1| + |\nu_2||h_2| + |\nu_3||h_3| + |\nu_3||Q_3(N_1 t, N_2 t, h_1, h_2)| \\ &+ \big|\psi\big(N_1 t + h_1, N_2 t + h_2, h_3 + Q_3(N_1 t, N_2 t, h_1, h_2)\big)\big| \\ &+ |h_4| + |Q_4(N_1 t, N_2 t, 0, h_1, h_2, h_3)|.\end{aligned}$$

Recall that $|\nu_1||h_1| + |\nu_2||h_2| \leq \lambda |\nu| t$, $|h_3| \leq \lambda t^2$, $|Q_3(N_1 t, N_2 t, h_1, h_2)| \leq \lambda t^2$, $|h_4| \leq \lambda t^3$ and $|Q_4(N_1 t, N_2 t, 0, h_1, h_2, h_3)| \leq \lambda t^3$. If $z = (z_1, z_2, z_3)$ is the argument of ψ in (4.23), then we get

$$\|z\| = \|(N_1 t + h_1, N_2 t + h_2, h_3 + Q_3(N_1 t, N_2 t, h_1, h_2))\| \lesssim t + \lambda t \simeq t,$$

and by (4.20)

$$|\psi(z)| \lesssim \|z\|^3 + (|\nu|^{1/2} + |\nu_3|)(z_1^2 + z_2^2) \leq t^3 + (|\nu|^{1/2} + |\nu_3|)t^2.$$

We finally get the following inequality which is stronger than (4.23)

$$\epsilon_0 |\nu| t \geq \lambda |\nu| t + \lambda |\nu_3| t^2 + t^3 + (|\nu|^{1/2} + |\nu_3|)t^2,$$

where $\epsilon_0 < 1$ is an absolute constant. Dividing by t we have to show that

$$(4.24) \qquad \epsilon_0 |\nu| \geq t^2 + (|\nu|^{1/2} + |\nu_3|)t.$$

($\lambda |\nu|$ has been absorbed in the left-hand side). It will be enough to determine all t that solve the following two inequalities:

$$t^2 < \epsilon_0 |\nu| \quad \text{and} \quad t(|\nu|^{1/2} + |\nu_3|) < \epsilon_0 |\nu|.$$

The first one gives $t \leq \epsilon_0 |\nu|^{1/2}$ and the second one is consequently solved by $t|\nu_3| \leq \epsilon_0 |\nu|$. Claim (4.22) is proved if t_1 is as in (4.21) for a small absolute constant $\sigma > 0$.

10. *Second piece.* From now on t_1 is fixed as in (4.21). For $\eta > 0$ let

$$(4.25) \qquad t_2 = \begin{cases} \eta \max\{|\nu|, |\nu_3|^2\} & \text{if } \nu_3 \neq 0, \\ 0 & \text{if } \nu_3 = 0, \end{cases}$$

and if $t \in [t_1, t_1 + t_2]$ define

$$\gamma(t) = (t_1 N_1, t_1 N_2, (t - t_1)N_3, 0).$$

Notice that

$$\delta(t) := t_1 + (t - t_1)^{1/2} \simeq d(\gamma(t), 0).$$

In the sequel we shall sometimes write δ instead of $\delta(t)$. Moreover, let $b = (t-t_1)^{1/2}$.

11. We claim that there exist positive constants $\eta, \lambda < 1$ such that for all $t \in [t_1, t_1 + t_2]$ the following John property for γ holds:

$$(4.26) \qquad \qquad \mathrm{Box}(\gamma(t), \lambda\delta(t)) \subset \Omega.$$

Points in $\mathrm{Box}(\gamma(t), \lambda\delta)$ are of the form

$$\gamma(t) \cdot h = (t_1 N_1, t_1 N_2, (t - t_1) N_3, 0) \cdot (h_1, h_2, h_3, h_4)$$
$$= (t_1 N_1 + h_1, t_1 N_2 + h_2, (t - t_1) N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2),$$
$$h_4 + Q_4(t_1 N_1, t_1 N_2, (t - t_1) N_3, h_1, h_2, h_3)),$$

with $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \leq \lambda\delta$. Now, $\gamma(t) \cdot h \in \Omega$ if

$$h_4 + Q_4(t_1 N_1, t_1 N_2, b^2 N_3, h_1, h_2, h_3) > -\nu_1(t_1 N_1 + h_1) - \nu_2(t_1 N_2 + h_2)$$
$$- \nu_3\big(b^2 N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2)\big)$$
$$+ \psi(t_1 N_1 + h_1, t_1 N_2 + h_2, b^2 N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2)),$$

which is implied by

$$(4.27)$$
$$|\nu| t_1 + |\nu_3| b^2 > |h_4| + |Q_4(t_1 N_1, t_1 N_2, b^2 N_3, h_1, h_2, h_3)|$$
$$+ |\nu_1||h_1| + |\nu_2||h_2| + |\nu_3||h_3| + |\nu_3|\big|Q_3(t_1 N_1, t_1 N_2, h_1, h_2)\big|$$
$$+ \big|\psi\big(t_1 N_1 + h_1, t_1 N_2 + h_2, b^2 N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2)\big)\big|.$$

We estimate the right-hand side: $|h_4| \leq \lambda\delta^3$, $|Q_4(t_1 N_1, t_1 N_2, b^2, h_1, h_2, h_3)| \leq \lambda\delta^3 + b^2\lambda\delta \simeq \lambda\delta^3$, $|Q_3(t_1 N_1, t_1 N_2, h_1, h_2)| \leq \lambda t_1\delta \leq \lambda\delta^2$ and finally $|\nu_1||h_1| + |\nu_2||h_2| \leq \lambda|\nu|\delta$, $|h_3| \leq \lambda\delta^2$.

Let $z = (z_1, z_2, z_3)$ be the argument of ψ in (4.27). Then $|z_1| = |t_1 N_1 + h_1| \lesssim t_1 + \lambda\delta$ and analogously $|z_2| \lesssim t_1 + \lambda\delta$. Moreover, as $b \leq \delta$ and $t_1 \leq \delta$,

$$\|(z_1, z_2, z_3)\| = \|(t_1 N_1 + h_1, t_1 N_2 + h_2, b^2 N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2)\|$$
$$\simeq t_1 + \lambda\delta + b + \lambda\delta + (t_1\lambda\delta)^{1/2} \lesssim \delta.$$

By (4.20) this furnishes

$$|\psi(z)| \lesssim \delta^3 + (|\nu|^{1/2} + |\nu_3|)(t_1 + \lambda\delta)^2$$
$$\lesssim \delta^3 + |\nu|^{1/2} t_1^2 + \lambda|\nu|^{1/2}\delta^2 + |\nu_3| t_1^2 + \lambda|\nu_3|\delta^2,$$

and (4.27) is guaranteed by

$$(4.28)$$
$$|\nu| t_1 + |\nu_3| b^2 \geq \lambda\delta^3 + \lambda|\nu|\delta + \lambda|\nu_3|\delta^2 + \delta^3 + |\nu|^{1/2} t_1^2 + \lambda|\nu|^{1/2}\delta^2 + |\nu_3| t_1^2 + \lambda|\nu_3|\delta^2.$$

Replacing $\delta = t_1 + b$ we get

$$\epsilon_0(|\nu| t_1 + |\nu_3| b^2) \geq (t_1 + b)^3 + \lambda|\nu|(t_1 + b) + \lambda|\nu_3|(t_1 + b)^2 + |\nu|^{1/2} t_1^2$$
$$+ \lambda|\nu|^{1/2}(t_1 + b)^2 + |\nu_3| t_1^2,$$

where ϵ_0 is a small but absolute constant. Possibly changing ϵ_0 it will be enough to show that

$$\epsilon_0(|\nu| t_1 + |\nu_3| b^2) \geq t_1^3 + b^3 + \lambda|\nu| t_1 + \lambda|\nu| b + \lambda|\nu_3| t_1^2 + \lambda|\nu_3| b^2$$
$$+ |\nu|^{1/2} t_1^2 + \lambda|\nu|^{1/2} t_1^2 + \lambda|\nu|^{1/2} b^2 + |\nu_3| t_1^2.$$

Now, $\lambda|\nu|t_1$ and $\lambda|\nu_3|b^2$ can be absorbed in the left-hand side, and $\lambda|\nu_3|t_1^2 + |\nu_3|t_1^2 \simeq |\nu_3|t_1^2$. Then

$$(4.29) \qquad \epsilon_0(|\nu|t_1 + |\nu_3|b^2) \geq t_1^3 + b^3 + \lambda|\nu|b + |\nu_3|t_1^2 + |\nu|^{1/2}t_1^2 + \lambda|\nu|^{1/2}b^2.$$

Inequality (4.29) holds with $b = 0$ by (4.24) with $t = t_1$. It will be enough to show that

$$(4.30) \qquad \epsilon_0(|\nu|t_1 + |\nu_3|b^2) \geq b^3 + \lambda|\nu|b + \lambda|\nu|^{1/2}b^2.$$

12. In order to prove (4.30) the following two cases must be distinguished:

(2A) $|\nu_3| \leq |\nu|^{1/2}$;
(2B) $|\nu_3| > |\nu|^{1/2}$.

13. Case 2A. In this case $t_1 = \sigma|\nu|^{1/2}$ and (4.30) becomes (with a smaller ϵ_0)

$$\epsilon_0(|\nu|^{3/2} + |\nu_3|b^2) \geq b^3 + \lambda|\nu|b + \lambda|\nu|^{1/2}b^2.$$

By the trivial estimate $|\nu_3|b^2 \geq 0$ and letting $\lambda = 1$ in the right-hand side we get the stronger inequality $\epsilon_0|\nu|^{3/2} \geq b^3 + |\nu|b + |\nu|^{1/2}b^2$. Setting $b = |\nu|^{1/2}a$ (this can be done because in Case 2A it should be $\nu \neq 0$) we find $\epsilon_0 \geq a^3 + a^2 + a$ which holds for all $0 \leq a < a_0$. Then (4.30) holds for all $0 \leq b \leq a_0|\nu|^{1/2}$ and consequently our claim (4.26) holds for all $t \leq t_1 + a_0^2|\nu|$.

14. Case 2B. Here $t_1 = \sigma|\nu|/|\nu_3|$. The term $\lambda|\nu|^{1/2}b^2$ in the right-hand side of (4.30) is less than $\epsilon_0|\nu_3|b^2$ and can be absorbed in the left-hand side. Then we get the inequality (with a possibly smaller ϵ_0)

$$\epsilon_0\left(\frac{|\nu|^2}{|\nu_3|} + |\nu_3|b^2\right) \geq b^3 + \lambda|\nu|b,$$

that is,

$$\epsilon_0(|\nu|^2 + |\nu_3|^2b^2) \geq b^3|\nu_3| + \lambda|\nu||\nu_3|b.$$

Now, $\lambda|\nu||\nu_3|b \leq \frac{\lambda}{2}|\nu|^2 + \frac{\lambda}{2}|\nu_3|^2b^2$ and both these terms can be absorbed in the left-hand side if λ is suitable. Thus it suffices to solve

$$\epsilon_0(|\nu|^2 + |\nu_3|^2b^2) \geq b^3|\nu_3|.$$

Setting $|\nu| = 0$ we find $b \leq \epsilon_0|\nu_3|$ which gives the correct choice $t_2 = \epsilon_0^2|\nu_3|^2$, as declared in (4.25). Claim (4.26) is proved in Case 2B also.

15. *Third piece.* From now on t_2 is fixed as in (4.25). If $t \geq t_1 + t_2$, define

$$\gamma(t) = (t_1N_1, t_1N_2, t_2N_3, t - (t_1 + t_2)),$$

and notice that

$$\delta(t) := t_1 + t_2^{1/2} + (t - (t_1 + t_2))^{1/3} \simeq d(\gamma(t), 0).$$

As before we shall sometimes write δ instead of $\delta(t)$. Moreover, let

$$a = (t - (t_1 + t_2))^{1/3}.$$

16. We claim that there exists $\lambda < 1$ such that the following John property for γ holds for all $t_1 + t_2 \leq t \leq 1$:

$$(4.31) \qquad \text{Box}(\gamma(t), \lambda\delta(t)) \subset \Omega.$$

Points in $\mathrm{Box}(\gamma(t), \lambda\delta(t))$ are of the form

$$\gamma(t) \cdot h = (t_1 N_1, t_1 N_2, t_2 N_3, a^3) \cdot (h_1, h_2, h_3, h_4)$$
$$= \Big(t_1 N_1 + h_1, t_1 N_2 + h_2, t_2 N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2),$$
$$a^3 + h_4 + Q_4(t_1 N_1, t_1 N_2, t_2 N_3, h_1, h_2, h_3)\Big),$$

where $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \leq \lambda\delta$. Now, $\gamma(t) \cdot h \in \Omega$ if

$$a^3 + h_4 + Q_4(t_1 N_1, t_1 N_2, t_2 N_3, h_1, h_2, h_3) \geq -\nu_1(t_1 N_1 + h_1) - \nu_2(t_1 N_2 + h_2)$$
$$- \nu_3\big(t_2 N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2)\big)$$
$$+ \big|\psi\big(t_1 N_1 + h_1, t_1 N_2 + h_2, t_2 N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2)\big)\big|.$$

As usual, we find the stronger inequality

$$|\nu|t_1 + |\nu_3|t_2 + a^3 \geq |h_4| + |Q_4(t_1 N_1, t_1 N_2, t_2 N_3, h_1, h_2, h_3)|$$
$$+ |\nu_1||h_1| + |\nu_2||h_2| + |\nu_3||h_3| + |\nu_3|\big|Q_3(t_1 N_1, t_1 N_2, h_1, h_2)\big|$$
$$+ \big|\psi(t_1 N_1 + h_1, t_1 N_2 + h_2, t_2 N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2))\big|.$$

In the right-hand side we can estimate $|h_4|, |Q_4| \lesssim \lambda\delta^3$, $|\nu_1||h_1| + |\nu_2||h_2| \leq \lambda|\nu|\delta$, $|h_3| \leq \lambda\delta^2$ and $|Q_3(t_1 N_1, t_1 N_2, h_1, h_2)| \lesssim \lambda t_1 \delta \lesssim \lambda\delta^2$.

Let $z = (z_1, z_2, z_3)$ be the argument of ψ. Then $|z_1| = |t_1 N_1 + h_1| \leq t_1 + \lambda\delta$ and $|z_2| \leq t_1 + \lambda\delta$. Moreover,

$$\|z\| = \|(t_1 N_1 + h_1, t_2 N_2 + h_2, t_2 N_3 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2))\|$$
$$\lesssim t_1 + \lambda\delta + t_2^{1/2} + \lambda\delta + (t_1 \lambda\delta)^{1/2} \simeq t_1 + \lambda\delta + t_2^{1/2}.$$

By (4.20)

$$|\psi(z)| \leq \|z\|^3 + (|\nu|^{1/2} + |\nu_3|)(z_1^2 + z_2^2)$$
$$\leq (t_1 + \lambda\delta + t_2^{1/2})^3 + (|\nu|^{1/2} + |\nu_3|)(t_1 + \lambda\delta)^2$$
$$\simeq t_1^3 + t_2^{3/2} + \lambda\delta^3 + t_1^2|\nu|^{1/2} + t_1^2|\nu_3| + \lambda|\nu|^{1/2}\delta^2 + \lambda|\nu_3|\delta^2.$$

Ultimately, we have to show that

$$\epsilon_0(t_1|\nu| + t_2|\nu_3| + a^3) \geq \lambda\delta^3 + \lambda|\nu|\delta + \lambda|\nu_3|\delta^2 + t_1^3 + t_2^{3/2}$$
$$+ t_1^2|\nu|^{1/2} + t_1^2|\nu_3| + \lambda|\nu|^{1/2}\delta^2.$$

Notice that $|\nu|^{1/2}\delta^2 \leq \frac{1}{2}(|\nu|\delta + \delta^3)$ and thus the term $\lambda|\nu|^{1/2}\delta^2$ in the right-hand side can be deleted. Now, writing $\delta = t_1 + t_2^{1/2} + a$ we get

$$\epsilon_0(t_1|\nu| + t_2|\nu_3| + a^3) \geq \lambda t_1^3 + \lambda t_2^{3/2} + \lambda a^3 + \lambda|\nu|t_1 + \lambda|\nu|t_2^{1/2} + \lambda|\nu|a + \lambda|\nu_3|t_1^2$$
$$+ \lambda|\nu_3|t_2 + \lambda|\nu_3|a^2 + t_1^3 + t_2^{3/2} + t_1^2|\nu|^{1/2} + t_1^2|\nu_3|,$$

and letting $\lambda|\nu|t_1$, $\lambda|\nu_3|t_2$ and λa^3 be absorbed by the left-hand side we find the stronger inequality

$$\epsilon_0(t_1|\nu| + t_2|\nu_3| + a^3) \geq t_1^3 + t_2^{3/2} + \lambda|\nu|t_2^{1/2} + \lambda|\nu|a$$
$$+ \lambda|\nu_3|a^2 + t_1^2|\nu|^{1/2} + t_1^2|\nu_3|.$$

Such inequality holds if $a = 0$ (let $b^2 = t_2$ in (4.29)). Thus, it will be enough to prove that for a small but absolute constant ϵ_0,

$$(4.32) \qquad \epsilon_0(t_1|\nu| + t_2|\nu_3| + a^3) \geq \lambda|\nu|a + \lambda|\nu_3|a^2,$$

for all $a \geq 0$.

We distinguish Case 2A and Case 2B.

17. **Case 2A.** In this case $|\nu_3| \leq |\nu|^{1/2}$, $t_1 = \sigma|\nu|^{1/2}$ and $t_2 = \eta|\nu|$. Using $t_2|\nu_3| \geq 0$ in the left-hand side of (4.32), replacing $t_1 = \sigma|\nu|^{1/2}$ and using also $|\nu_3| \leq |\nu|^{1/2}$ in the right-hand side we get the stronger inequality

$$\epsilon_0(|\nu|^{3/2} + a^3) \geq \lambda(|\nu|a + |\nu|^{1/2}a^2)$$

which holds for all $a \geq 0$ if λ is small enough (3 and 3/2 are Hölder conjugate exponents).

18. Case 2B. Here $|\nu_3| > |\nu|^{1/2}$, $t_1 = \sigma|\nu|/|\nu_3|$ and $t_2 = \eta|\nu_3|^2$. In the left-hand side of (4.32) we use $t_1|\nu| \geq 0$ and put $t_2 = \eta|\nu_3|^2$. In the right-hand side we estimate $|\nu| \leq |\nu_3|^2$. Thus we find the stronger inequality

$$\epsilon_0(|\nu_3|^3 + a^3) \geq \lambda|\nu_3|^2a + \lambda|\nu_3|a^2,$$

which holds for all $a \geq 0$ if λ is small enough.

19. **Case 3.** This is the non-characteristic case and can be analyzed as in Theorem 3.2. $\qquad\square$

5. Examples

Example 5.1. In the setting of the Heisenberg group we give an example of domain of class $C^{1,\alpha}$ with $\alpha \in (0,1)$ which is not a John domain. To this aim we construct a counterexample to the Sobolev-Poincaré inequality (1.1).

We consider $(x,y,t) = (z,t) \in \mathbb{R}^2 \times \mathbb{R} = \mathbb{H}^1$ and the vector fields $X_1 = \partial_x + y\partial_t$ and $X_2 = \partial_y - x\partial_t$, $(x,y,t) = (z,t) \in \mathbb{R}^3$. We shall write $X = (X_1, X_2)$. It is well known that these vector fields are associated with a homogeneous group with dimension $Q = 4$.

Let $\Omega = \{(z,t) \in \mathbb{H}^1 : |z|^{\alpha+1} < t < 1\}$ where $\alpha \geq 0$ is a real parameter. The domain $\partial\Omega$ is not smooth when $|z| = t = 1$ but this does not matter as we are interested in the characteristic point $0 \in \partial\Omega$. If $\alpha \geq 1$, then Ω is of class C^2 in a neighborhood of $0 \in \partial\Omega$ and it belongs to the regular class studied in Section 3.

We consider the case $0 \leq \alpha < 1$. We let $u(z,t) = t^{-\gamma}$ and look for an exponent $\gamma > 0$ such that

$$(5.1) \qquad \int_\Omega |Xu|^p \, dzdt < +\infty \quad \text{but} \quad \int_\Omega |u|^q \, dzdt = +\infty,$$

where $1 \leq p < Q = 4$, for some $q \geq 1$ which should be less than $p^* = 4p/(4-p)$. We have

$$\int_\Omega |Xu|^p \, dzdt \simeq \int_0^1 t^{-p(\gamma+1)} \int_{|z|<t^{1/(\alpha+1)}} |z|^p \, dz \, dt$$

$$\simeq \int_0^1 t^{-p(\gamma+1)+(2+p)/(\alpha+1)} \, dt < +\infty \quad \Leftrightarrow \quad \gamma < \frac{3+\alpha-p\alpha}{p(\alpha+1)}.$$

On the other hand,

$$\int_\Omega |u|^q \, dzdt \simeq \int_0^1 t^{-q\gamma+4/(\alpha+1)} \, dt = +\infty \quad \Leftrightarrow \quad \gamma \geq \frac{3+\alpha}{q(\alpha+1)}.$$

An exponent γ ensuring (5.1) can be found if the following condition holds:

$$(5.2) \qquad \frac{3+\alpha}{q(\alpha+1)} < \frac{3+\alpha-p\alpha}{p(\alpha+1)} \qquad \Leftrightarrow \qquad q > \frac{p(3+\alpha)}{3+\alpha-p\alpha}.$$

If $\alpha = 1$ (5.2) becomes $q > p^*$ which is exactly what one should expect. If $\alpha < 1$, we can find $q < p^*$ such that (5.2) holds (the function $\alpha \mapsto (3+\alpha)p/(3+\alpha-p\alpha)$ is increasing near $\alpha = 1$). Thus if $\alpha < 1$, the Sobolev–Poincaré inequality does not hold and Ω is not a John domain.

Example 5.2. Using Theorem 4.4 we construct an example of John domain $\Omega \subset \mathbb{R}^4$ with respect to the metric structure of the group of step 3 considered in Section 4. Let $g \in C^2(0,1) \cap C([0,1])$ be a function such that

$$g(t) = \begin{cases} 1 - t^{1/4} & \text{if } 0 \le t \le 1/4, \\ (1-t)^{1/4} & \text{if } 3/4 \le t \le 1. \end{cases}$$

Such a function can be chosen with the additional property $g'(t) < 0$ for all $t \in (0,1)$.

Let

$$(5.3) \qquad N(x_1, x_2, x_3) = (x_1^2 + x_2^2)^6 + x_3^6,$$

and define the open set

$$(5.4) \qquad \Omega = \{x \in \mathbb{R}^4 : |x_4| < g(N(x_1, x_2, x_3))\}.$$

Notice that if $N(x_1, x_2, x_3) \ge 3/4$, then $\partial\Omega$ has local equation $(x_1^2+x_2^2)^6+x_3^6+x_4^4 = 1$. If $N(x_1, x_2, x_3) \le 1/4$, then $\partial\Omega$ has equation $|x_4| + [(x_1^2+x_2^2)^6 + x_3^6]^{1/4} = 1$.

We first show that the points $(0,0,0,\pm 1) \in \partial\Omega$ are the unique characteristic points of second type of $\partial\Omega$. Indeed, let $\Phi(x_1, x_2, x_3, x_4) = g(N(x)) - x_4$ and compute

$$X_1\Phi(x) = g'(N(x))X_1N(x) - q_1(x),$$
$$X_2\Phi(x) = g'(N(x))X_2N(x) - q_2(x),$$
$$X_3\Phi(x) = g'(N(x))X_3N(x) - q_3(x),$$

where q_1, q_2 and q_3 are defined in (4.2). Note that $x_1q_1(x) + x_2q_2(x) = -x_3q_3(x)$. Moreover,

$$X_1N(x) = 12x_1(x_1^2 + x_2^2)^5 - 3x_2x_3^5,$$
$$X_2N(x) = 12x_2(x_1^2 + x_2^2)^5 + 3x_1x_3^5,$$

and thus $x_1X_1N(x) + x_2X_2N(x) = 12(x_1^2 + x_2^2)^6$. Then

$$x_1X_1\Phi + x_2X_2\Phi = g'(N(x))12(x_1^2 + x_2^2)^6 + x_3q_3 = 0,$$

if x is characteristic. Moreover, if x is of second type, we have

$$x_3X_3\Phi = g'(N(x))6x_3^6 - x_3q_3(x) = 0.$$

Summing up the last two equations we finally get

$$g'(N(x))\big(12(x_1^2 + x_2^2)^6 + 6x_3^6\big) = 0,$$

which implies $x_1 = x_2 = x_3 = 0$, as $g'(N(x)) \neq 0$.

We now check that hypotheses of Lemma 4.3 are satisfied. We start with (i). Consider the graph $x_4 = \varphi(x_1, x_2, x_3) = N(x)^{1/4}$, $x \in D_0 = \{(x_1, x_2, x_3) : N(x) < 1\}$. We need to show that the derivatives $X_i X_j \varphi$, $i, j = 1, 2$, are Lipschitz continuous on D_0 (with respect to the distance ϱ in the Heisenberg group). This is equivalent to showing that the derivatives $X_i X_j X_k \varphi$ $(i, j, k = 1, 2)$ are bounded. This is trivially true because these functions are smooth away from the origin and homogeneous of degree zero with respect to the dilations $(x_1, x_2, x_3) \mapsto (\lambda x_1, \lambda x_2, \lambda^2 x_3)$ (we are taking derivatives of order 3 of the function $\varphi = N^{1/4}$ which is homogeneous of degree 3).

In order to check hypothesis (ii) of Lemma 4.3 we have to prove that, letting $\Phi(x) = N(x)^{1/4} - x_4$, there exists a constant $k > 0$ such that

$$|X_1^2 \Phi| + |X_2^2 \Phi| + |(X_1 X_2 + X_2 X_1)\Phi| \leq k(|X_1 \Phi|^{1/2} + |X_2 \Phi|^{1/2} + |X_3 \Phi|)$$

for all $x \in \partial\Omega$ such that $0 < N(x) \leq 1/4$. We note that away from the origin the function $\Phi(x) = N(x)^{1/4} - x_4$ is smooth, and moreover, it is homogeneous of degree 3 with respect to the dilations $(x_1, x_2, x_3, x_4) \to (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4)$. Then the derivatives $X_1 \Phi$ and $X_2 \Phi$ are homogeneous of degree 2 and their square roots $|X_1 \Phi|^{1/2}$ and $|X_2 \Phi|^{1/2}$ are homogeneous of degree 1. Analogously, $X_1^2 \Phi$, $X_2^2 \Phi$, $X_3 \Phi$ and $(X_1 X_2 + X_2 X_1)\Phi$, being derivatives of degree 2, are homogeneous of degree 1. Then the function $H = H(x_1, x_2, x_3)$ defined by

$$H = \frac{|X_1^2 \Phi| + |X_2^2 \Phi| + |(X_1 X_2 + X_2 X_1)\Phi|}{|X_1 \Phi|^{1/2} + |X_2 \Phi|^{1/2} + |X_3 \Phi|}$$

is homogeneous of degree 0. We showed above that $|X_1 \Phi(x)|^{1/2} + |X_2 \Phi(x)|^{1/2} + |X_3 \Phi(x)| > 0$ for all $N(x) > 0$, and thus by 0-homogeneity

$$\sup_{0 < N(x) \leq 1/4} H(x) = \max_{N(x) = 1/4} H(x) = k < +\infty.$$

Thus Lemma 4.3 can be applied and the set Ω is admissible.

In the following examples we show that in groups of step 3 there are domains of class C^∞ which are not John domains and for which the Sobolev-Poincaré inequality (1.1) does not hold.

We begin with a well-known lemma. Here $X = (X_1, ..., X_m)$ is a system of generators of a homogeneous group on $\mathbb{R}^n$ with homogeneous dimension Q. Let d be the left invariant metric induced by these vector fields.

Lemma 5.3. *If $\Omega \subset \mathbb{R}^n$ is a bounded open set such that the Sobolev-Poincaré inequality (1.1) holds, then there exist $r_0 > 0$ and $C > 0$ such that*

$$|\Omega \cap B(x, r)| \geq C r^Q$$

for all $x \in \overline{\Omega}$ and $0 \leq r \leq r_0$.

Proof. If $p = 1$, (1.1) reads

$$(5.5) \qquad \left(\int_\Omega |u - u_\Omega|^{Q/(Q-1)} \, dx \right)^{(Q-1)/Q} \leq C \int_\Omega |Xu| \, dx.$$

Fix $\bar{x} \in \overline{\Omega}$ and $0 < r < R$. The function

$$u(x) = \begin{cases} 1 & \text{if } d(x, \bar{x}) < r, \\ 1 - \frac{d(x,\bar{x}) - r}{R - r} & \text{if } r \leq d(x, \bar{x}) \leq R, \\ 0 & \text{if } d(x, \bar{x}) > R \end{cases}$$

can be used in (5.5). Moreover, taking R small enough to ensure $|\Omega \setminus B(\bar{x}, R)| \geq \frac{1}{2}|\Omega|$ we can let $u_\Omega = 0$ to find (for a possibly different constant)

$$\left(\int_\Omega |u|^{Q/(Q-1)}\, dx \right)^{(Q-1)/Q} \leq C \int_\Omega |Xu|\, dx.$$

Define $g(r) = |B(\bar{x}, r) \cap \Omega|$. First of all

$$\left(\int_\Omega |u|^{Q/(Q-1)}\, dx \right)^{(Q-1)/Q} \geq g(r)^{(Q-1)/Q},$$

and on the other hand, since $|Xd(x, \bar{x})| = 1$ for almost every x,

$$\int_\Omega |Xu|\, dx = \frac{1}{R-r}|(B(\bar{x}, R) \setminus B(\bar{x}, r)) \cap \Omega| = \frac{g(R) - g(r)}{R - r}.$$

Thus we find

(5.6) $$g(r)^{(Q-1)/Q} \leq C\frac{g(R) - g(r)}{R - r}, \qquad 0 < r < R \leq r_0.$$

The function g is differentiable almost everywhere, being monotonic, and letting $R \to r$ in (5.6) we get $g(r)^{(Q-1)/Q} \leq Cg'(r)$ at every point of differentiability, or equivalently

$$\frac{d}{dr}g(r)^{1/Q} \geq \frac{1}{C}.$$

Thus

$$g(r)^{1/Q} \geq \int_0^r \frac{d}{ds}g(s)^{1/Q}\, ds \geq \frac{1}{C}r,$$

and the claim is proved. $\square$

Example 5.4. Consider now $\mathbb{R}^4$ with the homogeneous group structure introduced in Section 4 (we choose $\alpha = 0$). The homogeneous dimension of the group is $Q = 7$. Let $\Omega \subset \mathbb{R}^4$ be a bounded open set of class C^∞ such that for some open neighborhood $\mathcal{U}$ of the origin $\Omega \cap \mathcal{U} = \{x \in \mathbb{R}^4 : x_4 > 0\} \cap \mathcal{U}$. We show that the Sobolev$-$Poincaré inequality (1.1) fails in Ω.

We localize our analysis in a neighborhood of 0. Take a point $\bar{x} = (0, -b, 0, 0) \in \partial\Omega$, $b \neq 0$, and translate it to the origin. The translated boundary is

(5.7)
$$\{\bar{x}^{-1} \cdot (y_1, y_2, y_3, 0)\} = \{(y_1, y_2 + b, y_3 + Q_3(0, b, y_1, y_2), Q_4(0, b, 0, y_1, y_2, y_3))\}$$

$$= \left\{ \left(y_1, y_2 + b, y_3 - \frac{1}{2}by_1, \frac{b}{12}y_1^2 \right) \right\} = \left\{ y_4 = \frac{b}{12}y_1^2 \right\}.$$

We write $\Omega_0 = \bar{x}^{-1} \cdot \Omega$ and assume that $b = 12$ (any $b > 0$ gives the same result). It will be enough to show that Ω_0 does not support the Sobolev$-$Poincaré

inequality. We can use the boxes (4.4) instead of balls to estimate for $0 < r < 1$,

$$|\text{Box}(0,r) \cap \Omega_0| = \int_{\|y\|<r,\, y_4 > y_1^2} dy$$

$$= \int_{|y_1|<r} dy_1 \int_{|y_2|<r} dy_2 \int_{|y_3|<r^2} dy_3 \int_{|y_4|<r^3,\, y_1^2 < y_4} dy_4$$

$$= 4r^3 \int_{|y_1|<r,\, |y_4|<r^3,\, y_1^2 < y_4} dy_1 dy_4.$$

On the integration domain we have $|y_1| \leq |y_4|^{1/2} \leq r^{3/2}$. Therefore,

$$|\text{Box}(0,r) \cap \Omega_0| \leq 4r^3 \int_{-r^{3/2}}^{r^{3/2}} dy_1 \int_{y_1^2}^{r^3} dy_4 = 8r^3 \int_0^{r^{3/2}} (r^3 - y_1^2) dy_1 = \frac{16}{3} r^{15/2}.$$

If the Sobolev-Poincaré inequality holds, the estimate $|\text{Box}(0,r) \cap \Omega_0| \geq Cr^7$ should be true, but this is not possible.

We note that $0 \in \partial\Omega$ is a characteristic point of second type which is "flat", but near 0 there are characteristic points of the same type $\bar{x} = (0, -b, 0, 0)$, $b \neq 0$, which are not "flat": by (5.7) the parametric equation for the translated boundary $\bar{x}^{-1} \cdot \partial\Omega$ is $y_4 = \frac{b}{12} y_1^2$ and the flatness condition $|y_4| \leq C\|(y_1, y_2, y_3)\|^3$ does not hold. This makes Ω a bad domain.

The following example is a refined version of the previous one.

Example 5.5. In this example we disprove Conjecture 1 stated in [CG, p. 429]. We consider again the group of step 3 studied in Section 4 (with $\alpha = 0$). We prove that in the "gauge ball"

$$\Omega = \{x \in \mathbb{R}^4 : (x_1^2 + x_2^2)^6 + x_3^6 + x_4^4 < 1\}$$

the Sobolev-Poincaré inequality does not hold. We actually show the following stronger fact.

Proposition 5.6. *Assume that $\Omega \cap \mathcal{U} = \{x_4 > \varphi(x_1, x_2, x_3)\} \cap \mathcal{U}$ for some neighborhood $\mathcal{U}$ of the point $(0, 0, 0, \varphi(0, 0, 0))$. Assume also that for some $\gamma > 3$,*

$$(5.8) \qquad |\varphi(x_1, x_2, x_3) - \varphi(0, 0, 0)| \leq C\|x\|^\gamma,$$

for all x near the origin. Then the Sobolev-Poincaré inequality does not hold in Ω.

We note that in the particular case of the homogeneous ball $\Omega = \{(x_1^2 + x_2^2)^6 + x_3^6 + x_4^4 < 1\}$ the function φ is $\varphi(x_1, x_2, x_3) = -\{1 - [(x_1^2 + x_2^2)^6 + x_3^6]\}^{1/4}$. Thus (5.8) is satisfied.

Proof. As in Example 5.4 we show that the estimate of Lemma 5.3 is violated, but here the choice of the center and of the radius of the balls which give the counterexample is more delicate.

Fix $\alpha \in \,]1, \frac{\gamma}{3}[$ (this open interval is non-empty as soon as $\gamma > 3$). We claim that there exist $\epsilon \in (0, 1)$ and $C > 0$ such that

$$(5.9) \quad \left|\text{Box}\big((0, -b, 0, \varphi(0, -b, 0)), b^\alpha\big) \cap \Omega\right| \leq Cb^{7\alpha + (\alpha - 1)/2} \quad \text{for all} \quad b \in (0, \epsilon).$$

Since α is strictly larger than 1, this is not compatible as $b \to 0^+$ with the estimate

$$\left|\text{Box}\big((0, -b, 0, \varphi(0, -b, 0)), b^\alpha\big) \cap \Omega\right| \gtrsim b^{7\alpha},$$

which by Lemma 5.3 should be satisfied if the Sobolev-Poincaré inequality would hold.

If $\bar{x} = (0, -b, 0, \varphi(0, -b, 0))$, $b > 0$, let $\Omega_b = \bar{x}^{-1} \cdot \Omega$. It is easy to check that the surface $\partial\Omega_b := \bar{x}^{-1} \cdot \partial\Omega$ contains the origin and can be locally parameterized by

$$x_4 = \varphi\Big(x_1, x_2 - b, x_3 + \frac{1}{2}bx_1\Big) - \varphi(0, -b, 0) + \frac{1}{12}bx_1^2 := \varphi_b(x_1, x_2, x_3).$$

If x belongs to Box$(0, b^\alpha)$, then $|x_1|, |x_2| \leq b^\alpha$ and $|x_3| < b^{2\alpha}$. Thus we have the estimate

$$\Big|\varphi\Big(x_1, x_2 - b, x_3 + \frac{1}{2}bx_1\Big)\Big| \lesssim |x_1|^\gamma + |x_2 - b|^\gamma + \Big|x_3 + \frac{1}{2}bx_1\Big|^{\gamma/2} \lesssim b^\gamma,$$

because $\alpha > 1$. Moreover, $|\varphi(0, -b, 0)| \leq b^\gamma$.

This implies that if $x \in$ Box$(0, b^\alpha) \cap \Omega_b$, then $x_4 > \varphi_b(x_1, x_2, x_3)$ and

$$\frac{1}{12}bx_1^2 \leq |x_4| + \Big|\varphi(\dots) - \varphi(\dots)\Big|$$
$$\lesssim b^{3\alpha} + b^\gamma \simeq b^{3\alpha},$$

because $3\alpha < \gamma$. As a consequence $|x_1| \lesssim b^{(3\alpha-1)/2}$ and

$$\text{Box}(0, b^\alpha) \cap \Omega_b \subset \{|x_1| < C_0 b^{(3\alpha-1)/2}, \, |x_2| < b^\alpha, \, |x_3| < b^{2\alpha}, \, |x_4| < b^{3\alpha}\}$$

for some C_0 which stays bounded as b approaches 0. This proves (5.9).   □

Acknowledgments

We would like to thank Nicola Garofalo who has drawn our attention to the problems studied in Section 3, Olli Martio who has discussed with us some general properties of John and uniform domains and Nicoletta Ripamonti for her help with the computer. Finally, we wish to thank Bruno Franchi and Luca Capogna for pointing out some references to us.

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Dipartimento di Matematica Pura ed Applicata, Università di Padova, via Belzoni 7,
35131 Padova, Italy
  E-mail address: `monti@math.unipd.it`

Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato, 5,
40127 Bologna, Italy
  E-mail address: `morbidel@dm.unibo.it`