Distances, boundaries and surface measures in Carnot-Carathéodory spaces

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Introduction

Analysis in metric spaces is a field in strong development and touches different areas of classical analysis. The theory of Sobolev spaces in metric spaces has been deeply studied [**100**], a general theory of currents in metric spaces has been developed [**10**], the theory of quasiconformal maps has been generalized to metric spaces with controlled geometry [**105**], the analysis on fractals has been linked to that of metric spaces (see [**57**] and [**161**]). The first books on the subject begin to appear (see [**6**] and [**104**]), and the English edition [**95**] of [**93**] should also be mentioned along with [**160**].

A special class of metric spaces are Carnot-Carathéodory spaces. Even before the formal introduction of these spaces, the metric structures involved have been used in the study of hypoelliptic equations, degenerate elliptic equations, singular integrals and differentiability properties of functions, along the path [**108**], [**28**], [**52**], [**156**], [**64**], [**77**], [**151**], [**172**], [**164**] (and many others). A corresponding theory of Sobolev spaces in C-C spaces has been systematically worked out, including Poincaré inequalities, compactness theorems, embedding and extension theorems. We refer to chapter 4 and to the references in [**100**] for an up to date bibliography on the subject, which is considerably wide.

The study of C-C spaces from the point of view of Geometric Measure Theory is more recent, only few results are known and some more detail can here be explained. The first step was perhaps the proof of the isoperimetric inequality in the Heisenberg group [**154**]. The connection with geometric Sobolev embeddings was subsequently used to prove more general isoperimetric inequalities for C-C metrics in [**73**] and [**89**] (but also in [**26**] within the theory of Dirichlet forms). The notion of set of finite perimeter introduced by Caccioppoli ([**35**]) and De Giorgi ([**58**], [**59**]) has a natural formulation in C-C spaces (see [**80**] and [**89**]) and enjoys several nice properties that were used in [**89**] to prove the existence of minimal surfaces. This formulation is a special case of a general definition of function with bounded variation in metric spaces (see [**138**]). The problem of finding a good notion of rectifiable sets even in the simplest non Riemmannian C-C space, the Heisenberg group, has obtained only partial answers. The classical definition, which looks for sets that are Lipschitz images of open sets of Euclidean spaces, does not work [**9**] and different proposals have been put forward in [**82**] and [**155**]. The one proposed in the former paper seems to be the most promising because there can rely upon it a proof of a structure theorem for sets of finite perimeter in the Heisenberg group which is a counterpart of the Euclidean one. However, some fundamental results of Geometric Measure Theory in the Euclidean setting are no longer true in C-C spaces. For instance, the lack of a covering theorem of Besicovitch type in C-C spaces (see [**120**] and [**158**]) yields the difficult task of differentiating a general Radon measure. Moreover, the metric differentiability of Lipschitz functions may fail [**116**]. Nevertheless, some deep results in this direction have been obtained in [**7**] for the perimeter measure in the general setting of metric spaces. The study of surface measures in C-C spaces is far from being complete. Perimeter and Minkowski content of a sufficiently regular surface agree [**148**] and, again in the Heisenberg group, perimeter equals spherical Hausdorff measure of the right dimension, at least on regular surfaces [**82**]. Finally, in the setting of Carnot groups some area and coarea formulas have been proved in [**175**], [**173**], [**129**], [**130**], and the study of the isoperimetric set in the Heisenberg group has begun and [**123**].

C-C spaces, and specifically Carnot groups, are also of great interest in the theory of quasiconformal maps in metric spaces. Several characterizations, properties and examples of such maps in the Heisenberg group have been given in [**119**] and [**120**] (see also [**44**]), the connection with quasilinear equations has been explored in [**39**], and the problem of regularity in Carnot groups has been studied in [**18**].

In Differential Geometry C-C spaces are also known under the name of sub-Riemannian manifolds (see for example [**31**], [**165**] and the book [**21**]). The study of geodesics in these manifolds has a controversial history (see [**143**] and [**124**]) and seems still to be at its beginning. In [**137**] Carnot groups have been proved to be the natural tangent space to a sub-Riemannian manifold with equiregular distribution (see also [**20**] and [**134**]). Finally, in spite of their geometric flavour, the papers [**153**] and [**94**] have had a great influence in the analytic literature, so as to impose the expressions "Carnot group" and "Carnot-Carathéodory space".

It is now time to introduce the metric spaces we are talking about. Suppose that a family $X = (X_1, ..., X_m)$ of vector fields is given in $\mathbb{R}^n$ and that every couple of points $x, y \in \mathbb{R}^n$ can be connected by a Lipschitz curve $\gamma : [0, T] \to \mathbb{R}^n$ such that for a.e. $t \in [0, T]$

$$\dot\gamma(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)) \quad \text{and} \quad \sum_{j=1}^{m} h_j(t)^2 \leq 1.$$

Such a curve will be called $X-$subunit. The function $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ defined by

$$\begin{aligned} d(x, y) = \inf\{T \geq 0 : \ & \text{there exists a } X\text{−subunit curve } \gamma : [0, T] \to \mathbb{R}^n \\ & \text{such that } \gamma(0) = x \text{ and } \gamma(T) = y\} \end{aligned}$$

is a metric and the metric space $(\mathbb{R}^n, d)$ is called Carnot-Carathéodory space (C-C space). Typically, a C-C space is not bi-Lipschitz equivalent, not even locally, to any Euclidean space and it is not Ahlfors regular either, but in most cases it is locally of homogeneous type. C-C spaces are length spaces.

We underline the fact that the manifold considered will always be $\mathbb{R}^n$ or an open subset of $\mathbb{R}^n$ endowed with Lebesgue measure, and not a more general manifold. Similarly, we shall always have in mind vector fields rather than distributions (in the sense of Differential Geometry). Once the vector fields are fixed and connect the space a uniquely defined C-C metric is given. Our approach will be metric rather than differential geometric.

The general problem may be described as the study of the interplay between the analytical objects which can be defined directly by the vector fields $X_1, ..., X_m$ (such as anisotropic Sobolev spaces, functions of bounded variation, sets of finite perimeter and sub-elliptic differential operators) and the analytical objects that are instead defined using the metric (such as Lipschitz functions, rectifiable curves and geodesics, properties of domains, different integral kernels, Minkowski content and Hausdorff measures). Within such a general research program this thesis deals with distances, boundaries and surface measures. Such topics are deeply connected with several problems of Geometric Measure Theory and of Functional Analysis in metric spaces.

The first item appearing in the title of the thesis is "distances". The C-C distance d_K from a closed set K (typically a surface or a point) is a tool which must have nice intrinsic properties to be useful. Motivated by the application of an intrinsic coarea formula in C-C spaces (see chapter 5) we focus our attention on the eikonal equation $|Xd_K| = 1$ which is studied in detail in chapter 2. The problem of "boundaries", the second item, is very delicate. Here comes into play a feature of C-C spaces that does not appear in the Euclidean case: a boundary can be characteristic at some point, i.e. all the vector fields $X_1, ..., X_m$ are tangent to the boundary at that point. If this happens then the boundary and the open set it encloses are bad from all points of view. Roughly speaking, in order to be regular a domain must have "flat" boundary at characteristic points: this is the philosophy which inspired the regularity theorems of chapter 3, theorems that have several applications to global Sobolev-Poincaré and isoperimetric inequalities, embedding and extension of functions, regularity up to the boundary of solutions of hypoelliptic equations. "Surface measures" is the third item. Such measures are perimeter, Minkowski content and Hausdorff measures of suitable dimension. Our results mainly deal with the first two and in particular we shall prove that in a quite general C-C space perimeter and Minkowski content are the same.

The last chapter of the thesis describes an application to the Calculus of Variations that will be discussed later. This chapter can be seen as a final summary of the entire work: each of the previous chapters contains at least one theorem that here is needed and used.

Not all the results the reader will find in the thesis are due to the author. The original contributions are now going to be illustrated. All results in chapter 3 along with Theorem 1.6.10 are joint work with D. Morbidelli of University of Bologna and refer to the papers [**146**] and [**147**]. All results in chapter 6, in sections 1 and 2 of chapter 5, in sections 3 and 6 of chapter 2 are joint work with F. Serra Cassano, my thesis advisor, and refer to the papers [**148**] and [**149**]. Theorem 1.3.5 in chapter 1 and sections 4 and 5 in chapter 2 are due to the author and the results there proved are mostly unpublished.

The basic properties of C-C spaces are studied in chapter 1. We introduce different definitions of the metric that turn out to be equal, we prove the Riemannian approximation theorem, we study rectifiable curves and geodesics, we prove Chow theorem for systems of vector fields satisfying the Hörmander condition and we state the structure theorem of C-C balls of [**151**]. Then we introduce the main examples of C-C spaces that will be object of study: Carnot groups, the Heisenberg group and C-C spaces of Grushin type. Most of the results proved in this chapter are well known, but we would like to mention two theorems that seem to be new.

In section 3 we show how to compute the metric derivative of Lipschitz curves in C-C spaces. A Lipschitz curve $\gamma : [0, 1] \to (\mathbb{R}^n, d)$ is differentiable almost everywhere and its derivative lies in the horizontal space, i.e. $\dot{\gamma} = \mathcal{A}(\gamma)h$ almost everywhere, where $\mathcal{A}$ is the $(n \times m)-$matrix whose columns are the coefficients of the vector fields $X_1, ..., X_m$, and $h = (h_1, ..., h_m)$ is the vector of *canonical coordinates* of γ. In Theorem 1.3.5 we prove that

$$\lim_{\delta \to 0} \frac{d(\gamma(t+\delta), \gamma(t))}{|\delta|} = |h(t)|$$

for a.e. $t \in [0,1]$, thus obtaining the representation formula for the length of γ

$$\mathrm{Var}(\gamma) = \int_0^1 |h(t)|\, dt,$$

where $\mathrm{Var}(\gamma)$ is the variation of γ with respect to the C-C distance d.

In section 6 we give a variant of the structure theorem of C-C balls of [**151**] which is very useful in the study of problems involving non characteristic surfaces (see Theorem 1.6.10). This theorem will be used in chapter 3 to show that domains with non characteristic boundary are uniform and to prove a trace theorem on non characteristic boundaries.

Chapter 2 deals with differentiability of Lipschitz functions in C-C spaces and eikonal equations for C-C distance functions. The differentiability of Lipschitz functions in metric spaces is a topic that seems to be arousing an increasing interest (see for example [**48**]). There are two main results which are known in C-C spaces. Pansu differentiability Theorem established in [**153**] states that any Lipschitz map between two Carnot groups has almost everywhere a differential which is a homogeneous homomorphism (see Theorem 2.1.6). On the other hand, if we consider only real valued functions, but now in a general C-C space, then every Lipschitz function has weak derivatives along the vector fields inducing the metric and these are L^∞ functions. This result has been proved in [**81**] and then in [**90**] (see Theorem 2.2.1). We prove a differentiability Theorem of intermediate character: we consider real valued functions, we require more regularity on the C-C space but we get a strong differentiability result. More precisely, if $(\mathbb{R}^n, d, |\cdot|)$ is a doubling C-C space induced by the vector fields $X_1, ..., X_m$ which are "of Carnot type" (see (2.3.10)) and $f : (\mathbb{R}^n, d) \to \mathbb{R}$ is a Lipschitz map then for almost every $x \in \mathbb{R}^n$ there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - T(y-x)}{d(x,y)} = 0,$$

and this linear map actually is $T = (X_1 f(x), ..., X_m f(x), 0, ..., 0)$ (see Theorem 2.3.3). In order to prove the theorem we need a weak form of the Morrey inequality (see (2.3.12)) which holds in very general situations, for example if the vector fields are of Hörmander type and the C-C space is Ahlfors regular in a neighborhood of almost every point.

In the second part of chapter 2 we study the eikonal equation for C-C metrics. If K is a closed set in a C-C space $(\mathbb{R}^n, d)$ the distance from K is the function

$$d_K(x) = \inf\{d(x,y) : y \in K\}.$$

Since d_K is $1-$Lipschitz then $|Xd_K(x)| \leq 1$ for a.e. $x \in \mathbb{R}^n$. The problem is to prove that

$$|Xd_K(x)| = 1 \quad \text{for a.e. } x \in \mathbb{R}^n \setminus K.$$

In Theorem 2.6.1 we prove this eikonal equation essentially whenever the differentiation Theorem 2.3.3 is available.

In the special case of the Heisenberg group if K is one point we get a stronger result. In Theorem 2.4.1 we show that the Heisenberg distance from the origin is of class C^∞ outside the center of the group and that here the eikonal equation holds everywhere. Moreover, if K is a compact subset of a surface of class C^1 having the uniform ball property (see Definition 2.5.3) then the distance from the surface is of class C^1 and again the eikonal equation holds everywhere in a neighborhood of K (see Theorem 2.5.8). This result relies upon a kind of Gauss Lemma in the Heisenberg group which can be formulated as follows. If a hypersurface S of class C^1 having the uniform ball property is given in the Heisenberg group and its local equation is $f(z,t) = 0$, then the geodesic flow starting from S having horizontal velocity

$$\nu(z,t) = \frac{\nabla_{\mathbb{H}} f(z,t)}{|\nabla_{\mathbb{H}} f(z,t)|}$$

realizes the distance from the surface (see Lemma 2.5.6). Here $\nabla_{\mathbb{H}} f$ is the Heisenberg gradient of f and $\nu(z,t)$ is the normalized projection onto the horizontal space of the Euclidean normal to the surface at (z,t) (see chapter 2 section 5).

Chapter 3 is entirely devoted to regular domains in C-C spaces and to trace theorems. The domains studied are John and uniform domains (see Definitions 3.1.1 and 3.1.10). John domains, which in the Euclidean setting have been introduced by F. John [**112**], support the global Sobolev-Poincaré inequality (see [**78**], [**89**], [**100**] for our metric setting), the Rellich-Kondrachov compactness theorem (see [**89**] and [**100**]) and the relative isoperimetric inequality (see [**78**] and [**89**]). These results will be discussed in chapter 4. Uniform domains (also known as $(\varepsilon, \delta)-$domains) are a sub-class of John domains and have been introduced by Martio and Sarvas [**136**] and Jones [**113**]. In particular, in [**113**] an extension theorem for Sobolev functions in uniform domains was proved, theorem generalized in [**174**] and [**90**] to the setting of Carnot–Carathéodory spaces. In connection with the study of harmonic measures for sub-elliptic equations a class of regular domains ($\varphi-$Harnack domains) has also been introduced in [**65**] and [**66**].

In spite of all such results only few examples of John and uniform domains are known in Carnot-Carathéodory spaces (to our knowledge, at least), and precisely:

(i) Carnot-Carathéodory balls are John domains. This is a general fact which holds in any metric space with geodesics (see [**100**, Corollary 9.5] and see also [**74**]).
(ii) Carnot-Carathéodory balls in groups of step 2 are uniform domains (see [**92**] and also [**174**] for the special case of the Heisenberg group).
(iii) In groups of step 2 every connected, bounded open set of class $C^{1,1}$ having cylindrical symmetry in a neighborhood of each characteristic point (see [**42**] for precise definitions) is a non tangentially accessible (nta) domain (see [**42**] and [**43**]). This property is stronger than the uniform one.
(iv) In the Heisenberg group global quasiconformal maps preserve the uniform property [**44**].

We prove that:

1) If d is the Carnot-Carathéodory metric induced on $\mathbb{R}^n$ by a system of Hörmander vector fields and $\Omega \subset \mathbb{R}^n$ is a connected, bounded open set of class C^∞ without characteristic points on its boundary then Ω is a uniform domain in $(\mathbb{R}^n, d)$ (see Theorem 3.2.1).
2) In the setting of Carnot-Carathéodory spaces of Grushin type we introduce a class of admissible domains which are uniform (see Definition 3.3.1 and Theorem 3.3.3).
3) In any group of step 2 a connected bounded open set with boundary of class $C^{1,1}$ is a uniform domain, and actually a nta−domain (see Theorem 3.4.2).
4) In a group of step 3 we introduce a class of admissible domains (see Definition 3.5.2) that are John domains (see Theorem 3.5.5). We also produce examples of domains belonging to this class (see Example 3.5.6).

Result 3) proves a conjecture stated in [**89**], [**42**] and [**43**] and is sharp in the sense that in groups of step 2 there are open sets of class $C^{1,\alpha}$ for any $\alpha \in (0,1)$ which are not John domains (see Example 4.1.9). Result 2) is sharp, too. Moreover we show that the Grushin ball in the plane is not uniform. In groups of step 3 the C^∞ regularity does not ensure metric regularity. In Section 5 we give a sufficient condition for the John property expressed in terms of an inequality involving the local equation of the boundary and its derivatives along the horizontal vector fields. Precisely, consider two vector fields X_1 and X_2 in $\mathbb{R}^4$ generating a homogeneous group of step 3 with commutators $[X_1, X_2] = X_3$ and $[X_1, X_3] = [X_2, X_3] = X_4$ (all other commutators vanish). Let $\Omega = \{\Phi > 0\}$ be a connected, bounded open set with boundary $\partial\Omega = \{\Phi = 0\}$ where Φ is a function of class C^2, we require that for all points in $\partial\Omega$

$$|X_1^2\Phi| + |X_2^2\Phi| + |(X_1X_2 + X_2X_1)\Phi| \le k(|X_1\Phi|^{1/2} + |X_2\Phi|^{1/2} + |X_3\Phi|),$$

where $k > 0$ is a uniform constant (see Definition 3.5.2). This condition can be reformulated for the parametric representation of the surface (see formula (3.5.96)) and quantitatively describes the flatness behavior of the surface near characteristic points.

The second group of results of chapter 3 deals with the trace problem for Sobolev functions. Let us recall the following classical result of [**87**]. If $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set with regular boundary $\partial\Omega$, then there exists a constant $C > 0$ such that for any $u \in \mathrm{W}^{1,p}(\Omega)$

$$\int_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n-1+ps}}\, d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \le C \int_\Omega |\nabla u(x)|^p \, dx,$$

where $s = 1 - 1/p$ is the fractional order of differentiability of the trace $u = u_{|\partial\Omega}$, and $\mathcal{H}^{n-1}$ is the $(n-1)-$Hausdorff measure in $\mathbb{R}^n$. The problem of finding similar estimates for vector fields has deserved some attention in the last years (see [**71**], [**25**], [**13**], [**56**]) but even the choice of the fractional semi-norm to use in the left hand side of the above inequality is not clear. We prove trace estimates of the following type. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega$ of class C^1. There exists $C > 0$ such that

$$\int_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^p \, d\mu(x) d\mu(y)}{d(x,y)^{ps} \mu(B(x, d(x,y)))} \le C \int_\Omega |Xu(x)|^p \, dx \qquad (*)$$

for all $u \in C^1(\Omega) \cap C(\overline{\Omega})$. Here d is the C-C metric induced by the vector fields $X = (X_1, ..., X_m)$, B is a C-C ball and μ is the surface measure $\mu = |Xn| \llcorner \mathcal{H}^{n-1}$, $n(x)$ being the unit normal to $\partial\Omega$ at x. The measure μ, which is exactly the perimeter measure of Ω (see chapter 5), seems to take correctly into account characteristic points in $\partial\Omega$.

The main problem with $(*)$ is, again, the regularity of the boundary $\partial\Omega$. If $\partial\Omega$ is smooth and does not contain characteristic points then the trace estimate holds for a general system of Hörmander vector fields (see Theorem 3.6.4). The proof is based on a technique inspired by the original paper of Gagliardo [**87**] which relies upon the possibility of connecting points on the boundary $\partial\Omega$ by means of sub-unit curves lying in Ω. When the boundary contains characteristic points the analysis is much more difficult. But in the setting of the Grushin plane we introduce a class of admissible domains of class C^1, which is the same of 2) above (see Definition 3.7.3), that are "flat" at characteristic points in such a way that the trace estimates hold (see Theorem 3.7.5). By a non trivial counterexample this result will be shown to be sharp in the sense that there exist domains of class C^1 which are not admissible for which the theorem fails (see Proposition 3.7.6).

In the remarkable paper [**56**], assuming some regularity on the measure μ and the uniform property for Ω, the authors prove general trace theorems of the form $(*)$ for Hörmander vector fields with applications to Carnot groups of step 2. Our results on uniform domains proved in section 4 could help to give a very satisfactory answer to the trace theorem in this class of groups.

Chapter 4 is a brief survey on the basic properties of anisotropic Sobolev spaces and of functions with bounded $X-$variation. Here all results are well known, possibly except the counterexample to the Sobolev-Poincaré inequality in the Heisenberg group in Example 4.1.9.

Chapter 5 is entirely devoted to the study of surface measures in C-C spaces. A first natural measure that can be introduced is the perimeter variational measure induced by the vector fields $X = (X_1, ..., X_m)$. If $\varphi \in C^1(\mathbb{R}^n; \mathbb{R}^m)$ its $X-$divergence is

$$\operatorname{div}_X(\varphi) = -\sum_{j=1}^{m} X_j^* \varphi_j$$

where X_j^* is the operator formally adjoint to X_j in $\mathrm{L}^2(\mathbb{R}^n)$. Then, if $E \subset \mathbb{R}^n$ is a measurable set, its $X-$perimeter in an open set $\Omega \subset \mathbb{R}^n$ is

$$|\partial E|_X(\Omega) = \sup \left\{ \int_E \operatorname{div}_X(\varphi)\, dx : \varphi \in C_0^1(\Omega; \mathbb{R}^m),\ \|\varphi\|_\infty \leq 1 \right\}.$$

Using measures of this type to integrate over the boundary of the level sets of a function, a general coarea formula in C-C spaces can be obtained (see Theorem 5.1.6).

Let now $(\mathbb{R}^n, d)$ be the C-C space induced by the vector fields X and assume d continuous. If K is a closed set (for instance a hypersurface) and $r > 0$, its $r-$tubular neighborhood is

$$I_r(K) = \{x \in \mathbb{R}^n : \min_{y \in K} d(x, y) < r\},$$

and the Minkowski content of K in an open set Ω is, if the limit exists
$$M(K)(\Omega) = \lim_{r\downarrow 0} \frac{|I_r(K)\cap\Omega|}{2r}.$$
Our main Theorem states that if $K = \partial E$ and E is an open set of class C^2 such that $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$ then
$$M(\partial E)(\Omega) = |\partial E|_X(\Omega).$$
The proof is based on a Riemannian approximation technique (see Theorem 5.2.1).

Results concerning perimeter and Hausdorff measures are much less general and are mainly confined to the Heisenberg group. Here, however, there is a nice result to which, unfortunately, the author did not give any contribution.

The Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ endowed with its C-C metric has a well distinguished metric and homogeneous dimension that is $Q = 2n+2$. Therefore, the natural dimension of a hypersurface is $Q-1$. The $(Q-1)-$dimensional spherical Hausdorff measure of a set $K \subset \mathbb{R}^{2n+1}$ is
$$\mathcal{S}_d^{Q-1}(K) = \gamma(Q-1)\sup_{\delta>0}\inf\Big\{\sum_{j=1}^{+\infty}(\mathrm{diam}(B_j))^{Q-1} : K\subset\bigcup_{j=1}^{+\infty}B_j,\ \mathrm{diam}(B_j)\le\delta\Big\},$$
where B_j are balls in the C-C Heisenberg metric d, and $\gamma(Q-1)$ is a suitable normalization constant.

If $K = \partial E$ and E is an open set of class C^1 and Ω is an open set then (see Corollary 5.3.12)
$$|\partial E|_X(\Omega) = \mathcal{S}_d^{Q-1}(\partial E\cap\Omega),$$
where here X denotes the system of Heisenberg vector fields. This result was first proved in [**82**] for a metric equivalent to the C-C distance and then in [**131**] for the C-C distance itself. The proof relies on a structure theorem for sets of finite perimeter and on a differentiation of the perimeter measure made possible by the asymptotic doubling estimates for perimeter established in [**7**]. A role is also played by the fact that the set of characteristic points in a surface of class C^1 is $\mathcal{S}_d^{Q-1}-$negligible, fact proved in [**17**].

We finally come to chapter 6. Here the application of C-C spaces techniques to the study of the $\Gamma-$convergence of functionals involving degenerate energies plays a central role. Let $\Omega\subset\mathbb{R}^n$ be a regular bounded open set and let $A(x)$ be a non negative matrix such that $A(x) = C(x)C(x)^T$ for all $x\in\Omega$ and for some $(n\times m)-$matrix C with Lipschitz entries: the rows of C can be thought of as a system $X = (X_1, ..., X_m)$ of vector fields. Fix $0 < V < |\Omega|$ and for any $\varepsilon > 0$ define the functional $G_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$
$$G_\varepsilon(u) = \begin{cases} \varepsilon\displaystyle\int_\Omega \langle ADu, Du\rangle\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx & \text{if } u\in C^1(\Omega),\ u\ge 0 \text{ and } \displaystyle\int_\Omega u\,dv = V,\\ +\infty & \text{otherwise.}\end{cases}$$
where $W(u) = u^2(1-u)^2$. In [**139**] Modica proved that when $A = I_n$ is the identity matrix the functionals G_ε $\Gamma-$converge as $\varepsilon\downarrow 0$ to the perimeter functional. Afterwards, many other results of the same type have been proved (we refer to the introduction of chapter 6 for more detailed references and for the physical interpretation of the problem), but all assuming some kind of ellipticity or coercivity.

The perimeter $|\partial E|_A(\Omega)$ with respect to a non negative matrix can be defined (see the definitions (4.2.15)−(4.2.16) in chapter 4) in such a way that it coincides with $|\partial E|_X(\Omega)$ whenever $A = CC^T$ as above. Consider the functional $G : \mathrm{L}^1(\Omega) \to [0, +\infty]$

$$G(u) = \begin{cases} |\partial E|_A(\Omega) & \text{if } u = \chi_E \text{ and } |E| = V, \\ +\infty & \text{otherwise.} \end{cases}$$

The main theorem proved in chapter 6 states that the (relaxed of the) functionals G_ε $\Gamma(\mathrm{L}^1(\Omega))-$converge to the functional G (see Theorem 6.3.3). The proof is based on a Riemannian approximation of the vector fields X, which can be chosen monotonic in a precise sense. Moreover, many examples can be found where each functional G_ε has, after relaxation, minimum and the family of such minima is compact in $\mathrm{L}^1(\Omega)$ relatively to the parameter $\varepsilon > 0$.

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Basic notation

$\mathbb{R}^n$	$n-$dimensional Euclidean space
$\mathbb{H}^n$	$n-$Heisenberg group
$\mathbb{G}$	a Carnot group
Ω	open set in $\mathbb{R}^n$
d	C-C metric induced by a given family of vector fields
$\lvert x\rvert$	Euclidean norm of $x \in \mathbb{R}^n$
$\lVert x\rVert$	homogeneous norm of $x \in \mathbb{R}^n$ in a Carnot group
$B(x,r)$	open C-C ball centered at $x \in \mathbb{R}^n$ with radius $r \geq 0$
$U(x,r)$	open Euclidean ball centered at $x \in \mathbb{R}^n$ with radius $r \geq 0$
$\mathrm{Box}(x,r)$	Ball$-$Box in a Carnot group or in a Grushin space centered at $x \in \mathbb{R}^n$ with radius $r \geq 0$
$\mathrm{diam}(K)$	C-C diameter of a set $K \subset \mathbb{R}^n$
$d_K(x)$	$d_K(x) \equiv \mathrm{dist}(x;K)$ C-C distance of x from a set $K \subset \mathbb{R}^n$
$\mathrm{Var}(\gamma)$	total variation of a rectifiable curve in a C-C space
$\mathrm{length}_p(\gamma)$	$p-$length of an admissible curve in a C-C space
∇	Euclidean gradient
$\nabla_{\mathbb{H}}$	Heisenberg gradient
X	gradient with respect to the vector fields $X_1, ..., X_m$
$\mathcal{A}$	$(n \times m)-$matrix of the vector fields $X_1, ..., X_m$ disposed in columns
C	$(m \times n)-$matrix of the vector fields $X_1, ..., X_m$ disposed in rows
div	divergence
div_X	$X-$divergence
$C_0^1(\Omega)$	continuously differentiable functions with support compactly contained in Ω
$\mathrm{L}^p(\Omega)$	$p-$summable functions in Ω, $1 \leq p \leq +\infty$
$\mathrm{W}^{1,p}(\Omega)$	space of classical Sobolev functions in Ω
$\mathrm{H}^1(\Omega)$	$\mathrm{W}^{1,2}(\Omega)$
$\mathrm{BV}(\Omega)$	space of functions with bounded variation in Ω
$\mathrm{W}^{1,p}_X(\Omega)$	anisotropic Sobolev space associated with X
$\mathrm{H}^1_X(\Omega)$	$\mathrm{W}^{1,2}_X(\Omega)$
$\mathrm{BV}_X(\Omega)$	space of functions with bounded $X-$variation in Ω
$\mathrm{BV}_A(\Omega)$	space of functions with bounded variation in Ω with respect to a non negative matrix A
$\mathrm{Lip}(\Omega, d)$	real valued Lipschitz functions from the C-C space (Ω, d)
$\mathrm{Lip}(\Omega)$	real valued Lipschitz functions (Euclidean metric)
$\lvert Xf\rvert$	$X-$variation measure of a $\mathrm{L}^1_{\mathrm{loc}}$ function f
$\lvert\partial E\rvert_X$	$X-$perimeter measure of a measurable set $E \subset \mathbb{R}^n$
$\lvert Df\rvert_A$	total variation of a $\mathrm{L}^1_{\mathrm{loc}}$ function f with respect to a non negative matrix A

$\lvert\partial E\rvert_A$	perimeter of a measurable set E with respect to a non negative matrix A, not to be confused with $\lvert\partial E\rvert_X$
$M(K)(\Omega)$	C-C Minkowski content of a compact set $K \subset \mathbb{R}^n$ in an open set Ω
$\mathcal{H}^k$	$k-$dimensional Hausdorff measure in $\mathbb{R}^n$ in the Euclidean metric
$\mathcal{H}_d^k$	$k-$dimensional Hausdorff measure in $\mathbb{R}^n$ in a specified C-C metric d
$\mathcal{S}_d^k$	$k-$dimensional spherical Hausdorff measure in $\mathbb{R}^n$ in a specified C-C metric d
$\lVert\mathcal{A}\rVert$	operator norm of a matrix $\mathcal{A}$
$\lvert E\rvert$	Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$
$\langle x, y\rangle$	standard Euclidean inner product of $x, y \in \mathbb{R}^n$
$x \cdot y$	$x \cdot y \equiv P(x,y) \equiv x + y + Q(x,y)$, product of $x, y \in \mathbb{R}^n$ with respect to a Carnot group structure
δ_λ	dilations in a Carnot group
$\subset$	contained
$\Subset$	compactly contained
$\hookrightarrow$	embedding
$\mu \llcorner A$	measure μ restricted to the set A
e_j	$(0, ..., 0, 1, 0, ..., 0)$ with 1 in the $j-$th component
$u \lesssim v$	$u \leq Cv$ with $C > 0$ absolute constant
$u \simeq v$	$u \lesssim v$ and $v \lesssim u$
$\mathrm{spt}(\varphi)$	support of the function φ
a.e.	almost everywhere, always referred to Lebesgue measure

CHAPTER 1

Introduction to Carnot-Carathéodory spaces

1. Carnot-Carathéodory metrics

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $X = (X_1, ..., X_m)$ be a family a vector fields with locally Lipschitz continuous coefficients on Ω. Vector fields will be written and thought of indifferently as vectors and differential operators

$$X_j(x) = (a_{1j}(x), ..., a_{nj}(x)) = \sum_{i=1}^{n} a_{ij}(x)\partial_i, \quad j = 1, ..., m,$$

where $a_{ij} \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, $j = 1, ..., m$ and $i = 1, ..., n$. We shall write the coefficients a_{ij} in the $n \times m$ matrix $\mathcal{A} = \mathrm{col}\,[X_1, ..., X_m]$, i.e.

$$\mathcal{A}(x) = \begin{pmatrix} a_{11}(x) & \dots & a_{1m}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \dots & a_{nm}(x) \end{pmatrix}. \tag{1.1.1}$$

For every $x \in \Omega$ the vector fields span the vector space $\mathrm{span}\{X_1(x), ..., X_m(x)\}$ which has dimension less or equal than $\min\{m, n\}$.

Definition 1.1.1. A Lipschitz continuous curve $\gamma : [0, T] \to \Omega$, $T \geq 0$, is $X-$*admissible* if there exists a vector of measurable functions $h = (h_1, ..., h_m) : [0, T] \to \mathbb{R}^m$ such that

(i) $\dot\gamma(t) = \mathcal{A}(\gamma(t))h(t) = \sum_{j=1}^{m} h_j(t)X_j(\gamma(t))$ for a.e. $t \in [0, T]$;
(ii) $|h| \in \mathrm{L}^\infty(0, T)$.

The curve γ is $X-$*subunit*, if it is $X-$admissible and $\|h\|_\infty \leq 1$.

Let γ be a Lipschitz curve such that for a.e. $t \in [0, T]$ there exists $h(t) \in \mathbb{R}^m$ such that $\dot\gamma(t) = \mathcal{A}(\gamma(t))h(t)$ and $|h(t)| \leq M$ for some constant $M > 0$. By measurable selection theorems it follows that the function $t \mapsto h(t)$ can be assumed to be measurable (see [**106**] and [**46**]). In general, such function is not unique. But if $h(t)$ is required to be orthogonal to $\mathrm{Ker}(\mathcal{A}(\gamma(t)))$ for a.e. $t \in [0, T]$ – or equivalently $h(t) \in \mathrm{Im}(\mathcal{A}^T(\gamma(t)))$ – then h is also uniquely determined. We shall refer to such a h as to the vector of *canonical coordinates* of γ with respect to $X_1, ..., X_m$.

Introduce the Hamilton function $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$

$$H(x, \xi) = \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2 = \langle \mathcal{A}(x)\mathcal{A}(x)^T\xi, \xi\rangle. \tag{1.1.2}$$

The matrix $\mathcal{B}(x) = \mathcal{A}(x)\mathcal{A}(x)^T$ is semidefinite positive and with locally Lipschitz entries. The following proposition shows that subunit curves can be defined for a generic semidefinite positive quadratic form $\mathcal{B}$ on Ω even not admitting a factorization $\mathcal{B} = \mathcal{A}\mathcal{A}^T$. This was the original definition of subunit curve in [**64**].

PROPOSITION 1.1.2. *A Lipschitz continuous curve $\gamma : [0, T] \to \Omega$ is $X-$subunit if and only if*

$$\langle \dot{\gamma}(t), \xi \rangle^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 \tag{1.1.3}$$

for all $\xi \in \mathbb{R}^n$, and for a.e. $t \in [0, T]$.

Proof. Let γ be a subunit curve and fix $\xi \in \mathbb{R}^n$. By Schwarz inequality

$$\langle \dot{\gamma}(t), \xi \rangle^2 = \Big(\sum_{j=1}^{m} h_j(t) \langle X_j(\gamma(t)), \xi \rangle \Big)^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2$$

for a.e. $t \in [0, T]$.

Conversely, let $t \in [0, T]$ be a point of differentiability of γ and write

$$\dot{\gamma}(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)) + \sum_{i=1}^{n} b_i(t) \partial_i$$

for suitable vectors of coefficients $h(t) = (h_1(t), ..., h_m(t)) \in \mathbb{R}^m$ and $b(t) = (b_1(t), ..., b_n(t)) \in \mathbb{R}^n$. Choose $\xi \in \mathbb{R}^n$ such that $\langle X_j(\gamma(t)), \xi \rangle = 0$ for all $j = 1, ..., m$. By (1.1.3)

$$\langle b(t), \xi \rangle^2 = \langle \dot{\gamma}(t), \xi \rangle^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 = 0,$$

and thus $\langle b(t), \xi \rangle = 0$. This means that $\dot{\gamma}(t) \in \operatorname{span}\{X_1(\gamma(t)), ..., X_m(\gamma(t))\}$. We can write $\dot{\gamma}(t) = \mathcal{A}(\gamma(t))h(t)$ and assume that $h(t) = \mathcal{A}(\gamma(t))^T \xi$ for some $\xi = \xi(t) \in \mathbb{R}^n$. Thus

$$\begin{aligned} |h(t)|^4 &= \langle h(t), \mathcal{A}(\gamma(t))^T \xi \rangle^2 = \langle \mathcal{A}(\gamma(t)) h(t), \xi \rangle^2 \\ &= \langle \dot{\gamma}(t), \xi \rangle^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi \rangle^2 = |\mathcal{A}(\gamma(t))^T \xi|^2 = |h(t)|^2, \end{aligned}$$

and this proves that $|h(t)|^2 \leq 1$. $\square$

We introduce the function that will be the metric object of our study. Define $d : \Omega \times \Omega \to [0, +\infty]$ by

$$\begin{aligned} d(x, y) = \inf\{T \geq 0 : \ & \text{there exists a } X\text{−subunit path } \gamma : [0, T] \to \Omega \\ & \text{such that } \gamma(0) = x \text{ and } \gamma(T) = y\}. \end{aligned} \tag{1.1.4}$$

If the above set is empty put $d(x, y) = +\infty$. If $x \in \Omega$ is a fixed point the set of the points $y \in \Omega$ such that $d(x, y) < +\infty$ is the $X-$*reachable set* from x (or *orbit* of x). We are interested in the case when orbits are equal to Ω. In general, if $X_1, ..., X_m \in C^\infty(\Omega; \mathbb{R}^n)$ orbits are C^∞ submanifolds of Ω [**168**].

Our next task is to prove that if $d(x, y) < +\infty$ for all $x, y \in \Omega$ then d is a metric in Ω. We need the following propositions. If $\mathcal{A}$ is a $(n \times m)-$matrix its norm is by definition

$$\|\mathcal{A}\| := \sup_{h \in \mathbb{R}^m, |h| \leq 1} |\mathcal{A}h|. \tag{1.1.5}$$

LEMMA 1.1.3. *Let $x_0 \in \Omega$ and $r > 0$ be such that $U = U(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\} \Subset \Omega$. Let $M = \sup_{x \in U} \|\mathcal{A}(x)\|$ and $\gamma : [0, T] \to \Omega$ be a $X-$subunit curve such that $\gamma(0) = x_0$. If $MT < r$ then $\gamma(t) \in U$ for all $t \in [0, T]$.*

Proof. Assume by contradiction that

$$\bar{t} := \inf\{t \in [0, T] : \gamma(t) \notin U\} \leq T.$$

Then

$$\begin{aligned}
|\gamma(\bar{t}) - x_0| &= \Big| \int_0^{\bar{t}} \dot{\gamma}(\tau)\, d\tau \Big| = \Big| \int_0^{\bar{t}} \mathcal{A}(\gamma(\tau)) h(\tau)\, d\tau \Big| \\
&\leq \int_0^{\bar{t}} |\mathcal{A}(\gamma(\tau)) h(\tau)|\, d\tau \leq \int_0^{\bar{t}} \|\mathcal{A}(\gamma(\tau))\|\, |h(\tau)|\, d\tau \\
&\leq \bar{t} M \leq TM < r,
\end{aligned}$$

and hence $\gamma(\bar{t}) \in U$ which is open. This is in contradiction with the definition of $\bar{t}$. □

PROPOSITION 1.1.4. *Let $K \Subset \Omega$ be a compact set. There exists a constant $\beta > 0$ such that*

$$d(x, y) \geq \beta |x - y| \tag{1.1.6}$$

for all $x, y \in K$.

Proof. Let $\varepsilon > 0$ and $K_\varepsilon = \{x \in \Omega : \min_{y \in K} |x - y| \leq \varepsilon\}$. If ε is small enough then $K_\varepsilon \Subset \Omega$. Let $M = \sup_{x \in K_\varepsilon} \|\mathcal{A}(x)\|$, take $x, y \in K$ and set $r = \min\{\varepsilon, |x - y|\}$. Let $\gamma : [0, T] \to \Omega$ be a $X-$subunit curve such that $\gamma(0) = x$ and $\gamma(T) = y$. Since $|\gamma(T) - \gamma(0)| = |x - y| \geq r$, by Lemma 1.1.3 we have $TM \geq r$. If $r = \varepsilon$ then

$$T \geq \frac{\varepsilon}{M} \geq \frac{\varepsilon}{MD} |x - y|,$$

where $D := \sup_{x, y \in K} |x - y|$. If $r = |x - y|$ then $T \geq |x - y|/M$. Since the subunit curve γ is arbitrary, by the definition of d we get

$$d(x, y) \geq \min\Big\{\frac{1}{M}, \frac{\varepsilon}{MD}\Big\} |x - y|. \tag{1.1.7}$$

□

PROPOSITION 1.1.5. *If $d(x, y) < +\infty$ for all $x, y \in \Omega$ then (Ω, d) is a metric space.*

Proof. The symmetry property $d(x, y) = d(y, x)$ follows from the fact that if $\gamma : [0, T] \to \Omega$ is $X-$subunit then $\bar{\gamma}(t) = \gamma(T - t)$ is $X-$subunit too.

Moreover, if $\gamma_1 : [0, T_1] \to \Omega$ and $\gamma_2 : [0, T_2] \to \Omega$ are subunit curves such that $\gamma_1(0) = x$, $\gamma_1(T_1) = z$, $\gamma_2(0) = z$ and $\gamma_2(T_2) = y$ then

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, T_1] \\ \gamma_2(t - T_1) & \text{if } t \in [T_1, T_1 + T_2], \end{cases}$$

is a $X-$subunit curve such that $\gamma(0) = x$ and $\gamma(T_1 + T_2) = y$. Taking the infimum one finds the triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$.

Finally, $d(x, x) = 0$ and if $x \neq y$ by (1.1.6) it follows $d(x, y) > 0$. □

The metric space (Ω, d) is called *Carnot-Carathéodory (C-C) space.* If $x \in \Omega$, $r \geq 0$ and $K \subset \Omega$ we shall write

$$B(x,r) = \{y \in \Omega : d(x,y) < r\} \quad \text{and} \quad \operatorname{diam}(K) := \sup_{x,y \in K} d(x,y).$$

$B(x,r)$ is the C-C ball centered at x and with radius r. If the vector fields X define a C^∞ distribution on Ω (or more generally on a manifold) which by iterated brackets generates the tangent space at every point of Ω the resulting C-C space is also called *sub-Riemmaninan space* (see [**165**] and [**21**]).

Inequality (1.1.6) shows that the Euclidean metric is continuous with respect to the C-C metric d. The converse is in general not true. For example, consider in $\mathbb{R}^2$ the vector fields $X_1 = \partial_x$ and $X_2 = a(x)\partial_y$, where $a \in \operatorname{Lip}(\mathbb{R})$ is such that $a(x) = 0$ if $x \leq 0$ and $a(x) > 0$ if $x > 0$. Any couple of points in $\mathbb{R}^2$ can be connected by piecewise integral curves of X_1 and X_2, which therefore induce on $\mathbb{R}^2$ a finite C-C metric d. But if $x_1 < 0$

$$\lim_{y_1 \to 0} d((x_1, y_1), (x_1, 0)) = 2|x_1| \neq 0.$$

We now turn to a different definition of d which is useful in the study of the geodesic problem. Let $\gamma : [0,1] \to \Omega$ be an $X-$admissible curve with canonical vector of coordinates $h \in \mathrm{L}^\infty(0,1)^m$. For $1 \leq p \leq +\infty$ define

$$\operatorname{length}_p(\gamma) = \|h\|_p = \begin{cases} \left(\displaystyle\int_0^1 |h(t)|^p \, dt \right)^{1/p} & \text{if } 1 \leq p < +\infty \\ \operatorname{ess\,sup}_{t \in [0,1]} |h(t)| & \text{if } p = +\infty, \end{cases} \tag{1.1.8}$$

and

$$\begin{aligned} d_p(x,y) = \inf\{ \operatorname{length}_p(\gamma) : \gamma : [0,1] \to \Omega \text{ is an } X-\text{admissible curve} \\ \text{such that } \gamma(0) = x \text{ and } \gamma(1) = y\}. \end{aligned} \tag{1.1.9}$$

If the above set is empty put $d_p(x,y) = +\infty$.

Theorem 1.1.6. *For all $x, y \in \Omega$ and for all $1 \leq p \leq +\infty$ the equality $d(x,y) = d_p(x,y)$ holds.*

Proof. By Hölder inequality $\|h\|_1 \leq \|h\|_p \leq \|h\|_\infty$ for any $h \in \mathrm{L}^\infty(0,1)^m$ and for all $1 \leq p \leq \infty$. This yields $d_1(x,y) \leq d_p(x,y) \leq d_\infty(x,y)$.

We show that $d(x,y) = d_\infty(x,y)$. Let $\gamma : [0,T] \to \Omega$ be a $X-$subunit curve such that $\gamma(0) = x$, $\gamma(T) = y$ and $\dot\gamma(t) = \mathcal{A}(\gamma(t))h(t)$ for a.e. $t \in [0,T]$ with $\|h\|_\infty \leq 1$. The reparametrized curve $\widetilde\gamma : [0,1] \to \Omega$ defined by $\widetilde\gamma(t) = \gamma(Tt)$ is $X-$admissible and $\dot{\widetilde\gamma}(t) = \mathcal{A}(\widetilde\gamma(t))\widetilde h(t)$ for a.e. $t \in [0,1]$, where $\widetilde h(t) = Th(t)$. Because $\|\widetilde h\|_\infty \leq T$ and γ is arbitrary we get $d_\infty(x,y) \leq d(x,y)$. The converse inequality $d(x,y) \leq d_\infty(x,y)$ can be proved in the same way.

If we show that $d_\infty(x,y) \leq d_1(x,y)$ the theorem is proved. Let $\gamma : [0,1] \to \Omega$ be an $X-$admissible curve such that $\gamma(0) = x$, $\gamma(1) = y$ and $\dot\gamma(t) = \mathcal{A}(\gamma(t))h(t)$. We shall construct a new curve $\widetilde\gamma$ such that $\operatorname{length}_\infty(\widetilde\gamma) \leq \|h\|_1$. We may assume $\|h\|_1 > 0$. Let $\varphi : [0,1] \to [0,1]$ be the absolutely continuous function defined by

$$\varphi(t) = \frac{1}{\|h\|_1} \int_0^t |h(\tau)| d\tau, \quad t \in [0,1].$$

The function φ is non decreasing and its "inverse" function is $\psi : [0,1] \to [0,1]$ defined by $\psi(s) = \inf\{t \in [0,1] : \varphi(t) = s\}$, which - being monotonic - is differentiable for a.e. $s \in [0,1]$. We need to differentiate the identity $s = \varphi(\psi(s))$ by the chain rule. Let $B = \{t \in [0,1] : \varphi \text{ is not differentiable at } t\}$ and $D = \{s \in [0,1] : \varphi \text{ is not differentiable at } \psi(s)\}$. Since φ is absolutely continuous it transforms set with zero measure into set with zero measure, but $|B| = 0$ and as a consequence $|\varphi(B)| = 0$. From $D \subset \varphi(B)$ it follows that $|D| = 0$. This proves that for a.e. $s \in [0,1]$ we can write $\dot\varphi(\psi(s))\dot\psi(s) = 1$.

Define $\widetilde\gamma : [0,1] \to \Omega$ by $\widetilde\gamma(s) = \gamma(\psi(s))$ for $s \in [0,1]$. Let $E = \{s \in [0,1] : \gamma \text{ is not differentiable at } \psi(s)\}$. Since γ is a Lipschitz curve, arguing as above we deduce that $|E| = 0$. As a consequence for a.e. $s \in [0,1]$ we can compute

$$\dot{\widetilde\gamma}(s) = \dot\gamma(\psi(s))\dot\psi(s) = \mathcal{A}(\widetilde\gamma(s))h(\psi(s))\dot\psi(s).$$

Notice that if $|h(\psi(s))| \neq 0$ then

$$\dot\psi(s) = \frac{1}{\dot\varphi(\psi(s))} = \frac{\|h\|_1}{|h(\psi(s))|}.$$

If for $j = 1, ..., m$ we define

$$\widetilde h_j(s) = \begin{cases} \|h\|_1 \frac{h_j(\psi(s))}{|h(\psi(s))|} & \text{if } |h(\psi(s))| \neq 0 \\ 0 & \text{if } |h(\psi(s))| = 0, \end{cases}$$

then $\dot{\widetilde\gamma}(s) = \mathcal{A}(\widetilde\gamma(s))\widetilde h(s)$ for a.e. $s \in [0,1]$. As $\widetilde h \in \mathrm{L}^\infty(0,1)^m$, then $\widetilde\gamma$ is $X-$admissible and finally $\|\widetilde h\|_\infty \leq \|h\|_1 = \mathrm{length}_1(\gamma)$. $\square$

2. Riemannian approximation of the C-C distance

In this section we show that C-C spaces are "limit" of Riemannian manifolds (see [**93**] and [**70**]). Let d be the C-C metric induced on $\mathbb{R}^n$ by the family of vector fields $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$.

Let $J \in C_0^\infty(\mathbb{R}^n)$ be such that $J(x) \geq 0$ for all $x \in \mathbb{R}^n$, $\mathrm{spt}(J) \subset \{x \in \mathbb{R}^n : |x| < 1\}$ and $\int_{\mathbb{R}^n} J(x)\,dx = 1$, and introduce the mollifiers $J_\varepsilon(x) = \frac{1}{\varepsilon^n} J(x/\varepsilon)$, $\varepsilon > 0$. Define

$$X_j^\varepsilon(x) = X_j * J_\varepsilon(x) = \int_{\mathbb{R}^n} J_\varepsilon(x-y)X_j(y)\,dy, \quad j = 1, ..., m.$$

Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded open set, define

$$M := \max_{j=1,\dots,m} \sup_{x\in\Omega_0} |X_j(x)|, \tag{1.2.10}$$

and let $L > 0$ be a constant such that

$$|X_j(x) - X_j(y)| \leq L|x-y| \quad \text{for all } x, y \in \Omega_0,\ j = 1, ..., m. \tag{1.2.11}$$

Take $\Omega \Subset \Omega_0$ and $0 < \varepsilon < \min_{x\in\partial\Omega,\, y\in\partial\Omega_0} |x-y|$. If $x \in \Omega$

$$\begin{aligned} |X_j^\varepsilon(x) - X_j(x)| &= \Big|\int_{\mathbb{R}^n} (X_j(y) - X_j(x))J_\varepsilon(y)\,dy\Big| \leq \int_{\mathbb{R}^n} |X_j(y) - X_j(x)|J_\varepsilon(y)\,dy \\ &\leq L\int_{\mathbb{R}^n} |x-y|J_\varepsilon(y)\,dy \leq L\varepsilon, \end{aligned}$$

and $\sup_{x\in\Omega} |X_j^\varepsilon(x)| \leq \sup_{x\in\Omega_0} |X_j(x)| \leq M$.

If $x \in \Omega$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$

$$|\sum_{j=1}^{m} \langle X_j^\varepsilon(x), \xi\rangle^2 - \sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2| \leq 2M \sum_{j=1}^{m} |X_j^\varepsilon(x) - X_j(x)| \leq 2mML\varepsilon. \quad (1.2.12)$$

Consider the family of $m+n$ vector fields $X_\varepsilon^{(k)} = (X_1^\varepsilon, ..., X_m^\varepsilon, 1/k\partial_1, ..., 1/k\partial_n)$, with $k \in \mathbb{N}$. Thanks to (1.2.12) there exists a decreasing sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, $\varepsilon_k \to 0$, such that if $x \in \Omega$ and $|\xi| = 1$ then

$$\sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2 \leq \frac{1}{k^2}|\xi|^2 + \sum_{j=1}^{m} \langle X_j^{\varepsilon_k}(x), \xi\rangle^2 =: H_k(x, \xi), \quad (1.2.13)$$

and

$$H_{k+1}(x, \xi) \leq H_k(x, \xi). \quad (1.2.14)$$

By homogeneity (1.2.13) and (1.2.14) hold for $\xi \in \mathbb{R}^n$.

Let $d^{(k)}$ be the C-C metric induced on $\mathbb{R}^n$ by the vector fields

$$X^{(k)} := X_{\varepsilon_k}^{(k)} \quad (1.2.15)$$

and consider the $n \times (m+n)-$matrix

$$\mathcal{A}_k = \mathrm{col}[X_1^{\varepsilon_k}, ..., X_m^{\varepsilon_k}, 1/k\partial_1, ..., 1/k\partial_n]. \quad (1.2.16)$$

The matrix $\mathcal{A}_k\mathcal{A}_k^T$ is definite positive. Indeed, if $\mathcal{A}_k\mathcal{A}_k^T\xi = 0$ then $\langle \mathcal{A}_k^T\xi, \mathcal{A}_k^T\xi\rangle = 0$ and $\xi = 0$. The quadratic form on $\mathbb{R}^n$

$$g_k(x, \xi) = \langle(\mathcal{A}_k(x)\mathcal{A}_k^T(x))^{-1}\xi, \xi\rangle \quad (1.2.17)$$

is a Riemannian tensor which induces the metric $d^{(k)}$. To check this consider an $X^{(k)}-$admissible curve $\gamma : [0, 1] \to \mathbb{R}^n$ such that $\dot\gamma(t) = \mathcal{A}_k(\gamma(t))h(t)$ for a.e. $t \in [0, 1]$. The linear map $\mathcal{A}_k^T(\mathcal{A}_k\mathcal{A}_k^T)^{-1}\mathcal{A}_k$ is the identity on $\mathrm{Im}(\mathcal{A}_k^T)$. Then

$$g_k(\gamma, \dot\gamma) = \langle(\mathcal{A}_k(\gamma)\mathcal{A}_k^T(\gamma))^{-1}\mathcal{A}_k(\gamma)h, \mathcal{A}_k(\gamma)h\rangle = |h|^2,$$

a.e. on $[0, 1]$, and thus

$$\int_0^1 \sqrt{g_k(\gamma(t), \dot\gamma(t))}\, dt = \int_0^1 |h(t)|\, dt. \quad (1.2.18)$$

The Riemannian metric is the infimum of integrals as in the left hand side and by Theorem 1.1.6 the C-C metric is the infimum of integrals as in the right hand side. So the Riemmanian metric induced by the quadratic form (1.2.17) and the C-C metric $d^{(k)}$ are the same.

THEOREM 1.2.1. *Let $(\mathbb{R}^n, d)$ be the C-C space induced by the locally Lipschitz vector fields $X_1, ..., X_m$. Let Ω_0, Ω, M and $d^{(k)}$ be defined as above. If $K \subset \Omega$ is such that*

$$(M + 1)\mathrm{diam}(K) < \min_{x\in K,\, y\in\partial\Omega} |x - y| \quad (1.2.19)$$

then

(i) *$d^{(k)}(x, y) \leq d^{(k+1)}(x, y) \leq d(x, y)$ for all $k \in \mathbb{N}$ and*

$$\lim_{k\to+\infty} d^{(k)}(x, y) = d(x, y) \quad (1.2.20)$$

for all $x, y \in K$;

(ii) *if, in addition, d is continuous in the Euclidean topology then the convergence* (1.2.20) *is uniform on* $K \times K$.

Proof. Fix $\lambda > 1$ such that $\lambda(M+1)\mathrm{diam}(K) < \min_{x\in K,\, y\in\partial\Omega} |x-y|$. Let $x, y \in K$ and let $\gamma : [0,T] \to \mathbb{R}^n$ be a $X-$subunit curve such that $\gamma(0) = x$ and $\gamma(T) = y$. In view of (1.1.4) we can assume $T \leq \lambda\mathrm{diam}(K)$. Lemma 1.1.3 and (1.2.19) thus imply $\gamma(t) \in \Omega$ for all $t \in [0,T]$, and by (1.2.13) and Proposition 1.1.2 γ is also $X^{(k)}-$subunit. It follows that $d^{(k)}(x,y) \leq d(x,y)$. Moreover, by (1.2.14) $d^{(k)}(x,y) \leq d^{(k+1)}(x,y)$.

Write $d_k = d^{(k)}(x,y)$ and set $\delta_k = d_k + 1/k$. There exists a $X^{(k)}-$subunit curve $\widetilde{\gamma}_k : [0,\delta_k] \to \mathbb{R}^n$ such that $\widetilde{\gamma}_k(0) = x$, $\widetilde{\gamma}_k(\delta_k) = y$. Write $\dot{\widetilde{\gamma}}_k = \mathcal{A}_k(\widetilde{\gamma}_k)(\widetilde{h}^k, \widetilde{b}^k)$, with $\mathcal{A}_k$ as in (1.2.16), and $\widetilde{h}^k = (\widetilde{h}^k_1, ..., \widetilde{h}^k_m)$, $\widetilde{b}^k = (\widetilde{b}^k_1, ..., \widetilde{b}^k_n)$ measurable coefficients such that $|\widetilde{h}^k|^2 + |\widetilde{b}^k|^2 \leq 1$ a.e. on $[0,\delta_k]$. We can assume $\delta_k \leq \lambda\mathrm{diam}(K)$ for all $k \in \mathbb{N}$. Moreover

$$\sup_{x\in\Omega_0} \|\mathcal{A}_k(x)\| \leq M+1$$

so that Lemma 1.1.3 and (1.2.19) imply $\gamma_k(t) \in \Omega$ for all $k \in \mathbb{N}$ and $t \in [0,\delta_k]$.

Define $\gamma_k : [0,1] \to \Omega$ by $\gamma_k(t) = \widetilde{\gamma}_k(\delta_k t)$. Then $\dot{\gamma}_k = \mathcal{A}_k(\gamma_k)(h^k, b^k)$ with $h^k = \delta_k\widetilde{h}^k$ and $b^k = \delta_k\widetilde{b}^k$ and thus $|h^k|^2 + |b^k|^2 \leq \delta_k^2$ a.e. on $[0,1]$. Since $\gamma_k(t) \in \Omega$ for all $k \in \mathbb{N}$ and $t \in [0,1]$, being this set bounded we get $\|\gamma_k\|_\infty \leq C_1 < +\infty$ and consequently $\|\dot{\gamma}_k\|_\infty \leq C_2 < +\infty$ for all $k \in \mathbb{N}$. The sequence of curves $(\gamma_k)_{k\in\mathbb{N}}$ is uniformly bounded and uniformly Lipschitz continuous and by Ascoli-Arzelà Theorem there exists a subsequence that converges uniformly to a Lipschitz curve $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(1) = y$. On the other hand, the sequences $(h^k_j)_{k\in\mathbb{N}}$ and $(b^k_i)_{k\in\mathbb{N}}$ are uniformly bounded in $\mathrm{L}^\infty(0,1)$ and by the weak* compactness theorem there exist subsequences which weakly* converge to $h_j, b_i \in \mathrm{L}^\infty(0,1)$, $j = 1, ..., m$, $i = 1, ..., n$. Without loss of generality the sequences $(\gamma_k)_{k\in\mathbb{N}}$, $(h^k_j)_{k\in\mathbb{N}}$ and $(b^k_i)_{k\in\mathbb{N}}$ can be assumed to converge themselves. Now

$$\gamma_k(t) = x + \int_0^t \Big(\sum_{j=1}^m h^k_j(s) X^{\varepsilon_k}_j(\gamma_k(s)) + \frac{1}{k}\sum_{i=1}^n b^k_i(s)\mathrm{e}_i\Big) ds,$$

and taking the limit using the uniform convergence of $(\gamma_k)_{k\in\mathbb{N}}$, the weak convergence of $(h^k_j)_{k\in\mathbb{N}}$ and $(b^k_i)_{k\in\mathbb{N}}$, the uniform convergence (1.2.12) and $X_j \in \mathrm{Lip}(\Omega;\mathbb{R}^n)$ we obtain

$$\gamma(t) = x + \int_0^t \sum_{j=1}^m h_j(s) X_j(\gamma(s)) ds \quad \text{and thus} \quad \dot{\gamma}(t) = \mathcal{A}(\gamma(t))h(t),$$

where $h = (h_1, ..., h_m)$. Since $\|h^k\|_\infty \leq \delta_k$ for all $k \in \mathbb{N}$ the lower semicontinuity of $\|\cdot\|_\infty$ with respect to the weak* convergence implies

$$\|h\|_\infty \leq \liminf_{k\to\infty} \|h^k\|_\infty \leq \lim_{k\to\infty} \delta_k.$$

The curve $\widetilde{\gamma} : [0, \|h\|_\infty] \to \mathbb{R}^n$ defined by $\widetilde{\gamma}(t) = \gamma(t/\|h\|_\infty)$ is $X-$subunit and

$$d(x,y) \leq \|h\|_\infty \leq \lim_{k\to\infty} d^{(k)}(x,y) \leq d(x,y).$$

Equalities hold and the pointwise convergence of the metrics is proved.

Finally, suppose d continuous. The set K is bounded and without loss of generality it can also be assumed closed in the topology of d which - being d continuous - is the

Euclidean topology. Therefore K is compact. The functions $d^{(k)} : K \times K \to [0, +\infty)$, $k \in \mathbb{N}$, are continuous and converge monotonically to d. Since $K \times K$ is compact this implies the uniform convergence by Dini theorem. $\square$

REMARK 1.2.2. If $X = (X_1, ..., X_m)$ is a system of vector fields with $X_j \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ then Friedrichs regularization is not needed. For any $k \in \mathbb{N}$ let $d^{(k)}$ be the C-C metric induced on $\mathbb{R}^n$ by the vector fields $X^{(k)} = (X_1, ..., X_m, 1/k\partial_1, ..., 1/k\partial_n)$. Every $X^{(k)}-$subunit curve is $X^{(h)}-$subunit for all $h > k$ and also $X-$subunit. Then

$$d^{(k)}(x, y) \le d^{(k+1)}(x, y) \le d(x, y) \quad \text{for all } k \in \mathbb{N} \text{ and } x, y \in \mathbb{R}^n. \tag{1.2.21}$$

If, in addition, C-C balls in the metric $d^{(1)}$ are bounded in the Euclidean metric then the sequence of curves $(\gamma_k)_{k\in\mathbb{N}}$ constructed in the proof of Theorem 1.2.1 may be assumed to be equibounded and the Ascoli-Arzelà argument applies. Thus

$$\lim_{k\to\infty} d^{(k)}(x, y) = d(x, y) \tag{1.2.22}$$

for all $x, y \in \mathbb{R}^n$ and if the C-C metric metric d is continuous the convergence is uniform on compact sets.

REMARK 1.2.3. If $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\mathbb{R}^n)$ then (1.2.19) is satisfied for a compact set $K \subset \mathbb{R}^n$ as soon as Ω is a bounded open set containing it such that the Euclidean distance of $\partial\Omega$ from K is large enough. Under such assumptions all conclusions of Theorem 1.2.1 hold.

REMARK 1.2.4. It is worth noticing that the proof of Theorem 1.2.1 implicitly contains a proof of the *local* existence of geodesics in C-C spaces.

3. Rectifiable curves in C-C spaces

Let (M, d) be a metric space. The *total variation* of a curve $\gamma : [0, 1] \to M$ is by definition

$$\mathrm{Var}(\gamma) = \sup_{0\le t_1<...<t_k\le 1} \sum_{i=1}^{k-1} d(\gamma(t_{i+1}), \gamma(t_i)).$$

The supremum is taken over all finite partition of $[0, 1]$. If $\mathrm{Var}(\gamma) < +\infty$ the curve γ is said *rectifiable.*

A curve $\gamma : [0, 1] \to M$ is *$L-$Lipschitz*, $L \ge 0$, if $d(\gamma(t), \gamma(s)) \le L|t - s|$ for all $t, s \in [0, 1]$. Lipschitz curves are rectifiable and the total variation has an integral representation in terms of the *metric derivative*

$$|\dot\gamma|(t) := \lim_{\delta\to 0} \frac{d(\gamma(t + \delta), \gamma(t))}{|\delta|}. \tag{1.3.23}$$

The existence of the limit is a general fact that holds in any metric space as stated in the next theorem (see [**6**]).

THEOREM 1.3.1. *Let (M, d) be a metric space and $\gamma : [0, 1] \to M$ a Lipschitz curve. The metric derivative $|\dot\gamma|(t)$ exists for a.e. $t \in [0, 1]$, is a measurable function and*

$$\mathrm{Var}(\gamma) = \int_0^1 |\dot\gamma|(t)\, dt.$$

Aim of this section is to compute the metric derivative of Lipschitz curves in a C-C space $(\mathbb{R}^n, d)$.

PROPOSITION 1.3.2. *If $\gamma : [0,1] \to \mathbb{R}^n$ is $X-$admissible then it is Euclidean Lipschitz continuous.*

Proof. If $\dot\gamma = \mathcal{A}(\gamma)h$ and $h \in \mathrm{L}^\infty(0,1)^m$ then

$$|\gamma(t)-\gamma(s)| = \Big|\int_s^t \mathcal{A}(\gamma(\tau))h(\tau)d\tau\Big| \le \|h\|_\infty \sup_{x\in\gamma([0,1])} \|\mathcal{A}(x)\||t-s|.$$

□

PROPOSITION 1.3.3. *A curve $\gamma : [0,1] \to (\mathbb{R}^n, d)$ is $L-$Lipschitz if and only it is $X-$admissible and $\dot\gamma = \mathcal{A}(\gamma)h$ with $\|h\|_\infty \le L$.*

Proof. If γ is $X-$admissible then by definition (1.1.4) $d(\gamma(t),\gamma(s)) \le \|h\|_\infty|t-s|$ for all $s,t \in [0,1]$.

We assume now that γ is $1-$Lipschitz and prove that it is $X-$subunit. By Proposition 1.1.4 $\beta|\gamma(t)-\gamma(s)| \le d(\gamma(t),\gamma(s)) \le |t-s|$ for some $\beta > 0$ and γ is Euclidean Lipschitz continuous and thus differentiable a.e. on $[0,1]$. Suppose that $t = 0$ is a point of differentiability. For all $k \in \mathbb{N}$ let $\delta_k = (k+1)/k^2$. There exists a $X-$subunit curve $\gamma_k : [0,\delta_k] \to \mathbb{R}^n$ such that $\gamma_k(0) = \gamma(0)$ and $\gamma_k(\delta_k) = \gamma(1/k)$. Write $\dot\gamma_k = \mathcal{A}(\gamma_k)h_k$ a.e. on $[0,\delta_k]$ for some $h_k \in \mathrm{L}^\infty(0,\delta_k)^m$ with $\|h_k\|_\infty \le 1$, and consider

$$k(\gamma(1/k)-\gamma(0)) = k\int_0^{\delta_k} \mathcal{A}(\gamma(0))h_k(t)\,dt + k\int_0^{\delta_k} (\mathcal{A}(\gamma_k(t)) - \mathcal{A}(\gamma_k(0)))h_k(t)\,dt.$$

Since $\mathcal{A}$ has locally Lipschitz entries there exists a constant $C > 0$ such that

$$\int_0^{\delta_k} |(\mathcal{A}(\gamma_k(t)) - \mathcal{A}(\gamma_k(0)))h_k(t)|\,dt \le C\int_0^{\delta_k} |\gamma_k(t)-\gamma_k(0)|\,dt \le \frac{C}{\beta}\delta_k^2.$$

Indeed, $\beta|\gamma_k(t)-\gamma_k(0)| \le d(\gamma_k(t),\gamma_k(0)) \le \delta_k$. As $k\delta_k^2 \to 0$ we finally find

$$\dot\gamma(0) = \lim_{k\to+\infty} k(\gamma(1/k)-\gamma(0)) = \lim_{k\to+\infty} k\int_0^{\delta_k} \mathcal{A}(\gamma(0))h_k(t)\,dt.$$

The second limit exists, and in particular there exists

$$h(0) := \lim_{k\to+\infty} k\int_0^{\delta_k} h_k(t)\,dt, \quad \text{and} \quad |h(0)| \le \liminf_{k\to+\infty} k\int_0^{\delta_k} |h_k(t)|\,dt \le 1.$$

We have proved that $\dot\gamma(t) = \mathcal{A}(\gamma(t))h(t)$ and $|h(t)| \le 1$ for a.e. $t \in [0,1]$ and the claim follows. □

REMARK 1.3.4. Let $\Omega \subset \mathbb{R}^n$ be and open set and let $\Phi : \Omega \to \Phi(\Omega)$ be a C^1-diffeomorphism. Let X be a family of vector fields $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, $j = 1, ..., m$ and define the new family Ξ of vector fields on $\Phi(\Omega)$ by

$$\Xi_j(\xi) = d\Phi(x)X_j(x), \quad \xi = \Phi(x),$$

where $d\Phi(x)$ is the differential of Φ at $x \in \Omega$. Let $\mathcal{A}$ be the matrix of the vector fields X as in (1.1.1) and $\mathcal{B} = d\Phi\mathcal{A}$ the matrix of the transformed vector fields Ξ.

If $\gamma : [0,T] \to \Omega$ is a $X-$subunit curve such that $\dot{\gamma} = \mathcal{A}(\gamma)h$ define the transformed curve $\kappa : [0,T] \to \Phi(\Omega)$ by

$$\kappa(t) = \Phi(\gamma(t)) \tag{1.3.24}$$

and notice that

$$\begin{aligned}\dot{\kappa}(t) &= \frac{d}{dt}\Phi(\gamma(t)) = d\Phi(\gamma(t))\dot{\gamma}(t) = d\Phi(\gamma(t))\mathcal{A}(\gamma(t))h(t) \\ &= \sum_{j=1}^{m} h_j(t) d\Phi(\gamma(t))X_j(\gamma(t)) = \sum_{j=1}^{m} h_j(t)\Xi_j(\kappa(t)) = \mathcal{B}(\kappa(t))h(t).\end{aligned} \tag{1.3.25}$$

Thus $X-$subunit curves are transformed to $\Xi-$subunit ones. Moreover, the curves γ and $\kappa = \Phi(\gamma)$ have the same canonical vector of coordinates h.

If d and ϱ are the C-C metrics defined respectively on Ω by X and on $\Phi(\Omega)$ by Ξ it follows that

$$d(x,y) = \varrho(\Phi(x), \Phi(y)) \quad \text{for all } x, y \in \Omega, \tag{1.3.26}$$

and according to the definition of length of admissible curves (for any $1 \le p \le +\infty$ in (1.1.8))

$$\text{length}_d(\gamma) = \text{length}_\varrho(\kappa).$$

THEOREM 1.3.5. *Assume that $X_1, ..., X_m$ are pointwise linearly independent. Let $\gamma : [0,1] \to (\mathbb{R}^n, d)$ be a Lipschitz curve with canonical coordinates $h \in \mathrm{L}^\infty(0,1)^m$. Then*

$$\lim_{\delta\to 0} \frac{d(\gamma(t+\delta), \gamma(t))}{|\delta|} = |h(t)| \tag{1.3.27}$$

for a.e. $t \in [0,1]$, and therefore

$$\mathrm{Var}(\gamma) = \int_0^1 |h(t)|\, dt. \tag{1.3.28}$$

Proof. By Theorem 1.3.1 identity (1.3.28) will hold if we prove (1.3.27).

By Proposition 1.3.3 if γ is Lipschitz then it is $X-$admissible and we can write $\dot{\gamma} = \mathcal{A}(\gamma)h$ where $\mathcal{A}$ is the matrix of the vector fields (1.1.1). Define

$$E = \Big\{ t \in (0,1) : \dot{\gamma}(t) \text{ exists and is } \mathcal{A}(\gamma(t))h(t), \text{ and } \lim_{\delta\to 0} \frac{1}{\delta} \int_t^{t+\delta} |h(\tau)|\, d\tau = |h(t)| \Big\}.$$

By Proposition 1.3.2 γ is differentiable a.e. and $h \in \mathrm{L}^\infty(0,1)^m$. Therefore the set $[0,1] \setminus E$ is negligible.

It will be enough to consider the case $\delta > 0$. By Theorem 1.1.6

$$d(\gamma(t+\delta), \gamma(t)) \le \int_t^{t+\delta} |h(\tau)|\, d\tau,$$

and thus if $t \in E$

$$\limsup_{\delta\downarrow 0} \frac{d(\gamma(t+\delta), \gamma(t))}{\delta} \le \lim_{\delta\downarrow 0} \frac{1}{\delta} \int_t^{t+\delta} |h(\tau)|\, d\tau = |h(t)|. \tag{1.3.29}$$

Now fix $t \in E$, $\varepsilon, \eta > 0$ and set

$$K_\eta = \gamma([t, t+\eta]) \quad \text{and} \quad K_{\varepsilon,\eta} = \{x \in \mathbb{R}^n : \min_{y \in K_\eta} |x-y| \le \varepsilon\},$$

$$D_\eta = \sup\{|x-y| : x, y \in K_\eta\} \quad \text{and} \quad M_{\varepsilon,\eta} = \sup_{x \in K_{\varepsilon,\eta}} \|\mathcal{A}(x)\|,$$

$$\text{and} \quad M_\varepsilon = \sup_{|x-\gamma(t)| \le \varepsilon} \|\mathcal{A}(x)\|.$$

Here $\|\mathcal{A}\|$ is the norm (1.1.5).

The vectors $X_1(\gamma(t)), ..., X_m(\gamma(t))$ are linearly independent. Assume that $\mathcal{A}(\gamma(t))$, which is a $n \times m$ matrix, has the form

$$\mathcal{A}(\gamma(t)) = \begin{pmatrix} I_m \\ 0 \end{pmatrix}, \tag{1.3.30}$$

where I_m is the identity $m \times m$ matrix.

By (1.1.7), if $x, y \in K_\eta$ then

$$d(x,y) \ge \min\left\{\frac{1}{M_{\varepsilon,\eta}}, \frac{\varepsilon}{M_{\varepsilon,\eta} D_\eta}\right\} |x-y|,$$

and thus

$$\begin{aligned}\liminf_{\delta \downarrow 0} \frac{d(\gamma(t+\delta), \gamma(t))}{\delta} &\ge \min\left\{\frac{1}{M_{\varepsilon,\eta}}, \frac{\varepsilon}{M_{\varepsilon,\eta} D_\eta}\right\} \lim_{\delta \downarrow 0} \frac{|\gamma(t+\delta) - \gamma(t)|}{\delta} \\ &\ge \min\left\{\frac{1}{M_{\varepsilon,\eta}}, \frac{\varepsilon}{M_{\varepsilon,\eta} D_\eta}\right\} |\mathcal{A}(\gamma(t)) h(t)|.\end{aligned}$$

Notice that

$$\lim_{\eta \downarrow 0} \min\left\{\frac{1}{M_{\varepsilon,\eta}}, \frac{\varepsilon}{M_{\varepsilon,\eta} D_\eta}\right\} = \frac{1}{M_\varepsilon} \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \frac{1}{M_\varepsilon} = \frac{1}{\|\mathcal{A}(\gamma(t))\|}.$$

We first let $\eta \downarrow 0$ and then $\varepsilon \downarrow 0$ to find

$$\liminf_{\delta \downarrow 0} \frac{d(\gamma(t+\delta), \gamma(t))}{\delta} \ge \frac{|\mathcal{A}(\gamma(t)) h(t)|}{\|\mathcal{A}(\gamma(t))\|}. \tag{1.3.31}$$

By (1.3.30) we have $\|\mathcal{A}(\gamma(t))\| = 1$ and moreover $|\mathcal{A}(\gamma(t))h(t)| = |h(t)|$. Then (1.3.31) reads

$$\liminf_{\delta \downarrow 0} \frac{d(\gamma(t+\delta), \gamma(t))}{\delta} \ge |h(t)|,$$

which, along with (1.3.29), proves our thesis if $\mathcal{A}(\gamma(t))$ is of the form (1.3.30).

If $\mathcal{A}(\gamma(t))$ is not of the form (1.3.30) we argue in the following way. Since $X_1(\gamma(t)), ..., X_m(\gamma(t))$ are linearly independent there exists an invertible linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi X_j(\gamma(t)) = \mathrm{e}_j$ for $j = 1, ..., m$. Define the new family of vector fields $\Xi_j = \Phi X_j$, $j = 1, ..., m$, let ϱ be the C-C metric induced by them, and let $\kappa(t) = \Phi(\gamma(t))$ be the transformed curve. Now, if $\mathcal{B} = \Phi\mathcal{A}$ is the matrix of the vector fields $\Xi_1, ..., \Xi_m$ then $\mathcal{B}(\kappa(t))$ is of the form (1.3.30) and the above argument does apply. Then

$$\lim_{\delta \downarrow 0} \frac{\varrho(\kappa(t+\delta), \kappa(t))}{\delta} = |h(t)|.$$

Indeed, by (1.3.25) $\dot\kappa(t) = \mathcal{B}(\kappa(t))h(t)$. But by (1.3.26) we have $d(\gamma(t+\delta), \gamma(t)) = \varrho(\kappa(t+\delta), \kappa(t))$ and our thesis is proved. This ends the proof of (1.3.27). $\square$

REMARK 1.3.6. In the proof of Theorem 1.3.5 the assumption that $X_1, ..., X_m$ be pointwise linearly independent can be omitted and formula (1.3.27) holds in quite general C-C spaces. An alternative proof of Theorem 1.3.5 could be obtained using the Riemmanian approximation discussed in section 2. If $\mathrm{Var}_k(\gamma)$ denotes the variation of γ in the Riemannian metric $d^{(k)}$ induced by $X^{(k)}$ then $\mathrm{Var}_k(\gamma) \le \mathrm{Var}(\gamma)$, as $d^{(k)} \le d$. If $h^{(k)}$ is the vector of canonical coordinates of γ with respect to $X^{(k)}$ then, assuming formula (1.3.28) for the metric derivative in the Riemmanian case (recall also (1.2.18)), we have

$$\mathrm{Var}_k(\gamma) = \int_0^1 |h^{(k)}(t)|\, dt.$$

Moreover, a weak* compactness argument as in the proof of Theorem 1.2.1 and the fact that h is the vector of canonical coordinates of γ yield

$$\int_0^1 |h(t)|\, dt \le \liminf_{k\to\infty} \int_0^1 |h^{(k)}(t)|\, dt,$$

and thus

$$\int_0^1 |h(t)|\, dt \le \mathrm{Var}(\gamma).$$

The easier opposite inequality is a consequence of Theorem 1.1.6. This proves formula (1.3.28) and now (1.3.27) follows from Theorem 1.3.1.

4. Geodesics

In this section we study geodesics in C-C spaces and for a special class we shall write the differential equations they have to satisfy. The general framework in which study existence of geodesics is that of length metric spaces (see [**33**], [**34**] and [**95**]).

A metric space (M, d) is a *length space* (or space with *intrinsic* metric) if for each $x, y \in M$

$$d(x, y) = \inf\{\mathrm{Var}(\gamma) : \gamma : [0, 1] \to M \text{ continuous and rectifiable curve such that } \gamma(0) = x \text{ and } \gamma(1) = y\}.$$

A continuous rectifiable curve $\gamma : [0, 1] \to M$ is a *geodesic,* if $\mathrm{Var}(\gamma) = d(\gamma(0), \gamma(1))$. By an arclength reparametrization a geodesic γ can always be reparametrized on the interval $[0, \mathrm{Var}(\gamma)]$ in such a way that $d(\gamma(t), \gamma(s)) = |t - s|$ for all $s, t \in [0, \mathrm{Var}(\gamma)]$ (see [**33**]).

If $x \in M$ and $r > 0$ write $B(x, r) = \{y \in M : d(x, y) < r\}$ and denote by $\bar{B}(x, r) = \{y \in M : d(x, y) \le r\}$ the closed ball. In length metric spaces the closure of the open ball is the closed ball, i.e. $\overline{B(x, r)} = \bar{B}(x, r)$ (see, for instance, [**34**]).

PROPOSITION 1.4.1. *Let (M, d) be a length space. If $d(x, y) \le r$ and the closed ball $\bar{B}(x, r)$ is compact then there exists a geodesic $\gamma : [0, 1] \to M$ connecting x to y.*

We shall see the proof in the special case of C-C spaces. The following Hopf-Rinow theorem is due to Cohn-Vossen [**51**] and Busemann [**33**], [**34**].

THEOREM 1.4.2. *In a locally compact length space (M, d) the following three conditions are equivalent:*

(i) *closed balls are compact;*
(ii) *M is complete;*

(iii) *every geodesic* $\gamma : [0, \delta) \to M$, $\delta > 0$, *can be completed.*

We now turn our analysis to C-C spaces.

PROPOSITION 1.4.3. *A C-C space* $(\mathbb{R}^n, d)$ *is a length space. Moreover, if* d *is continuous the space is also locally compact.*

Proof. If $x, y \in \mathbb{R}^n$ by Theorem 1.1.6

$d(x, y) = \inf\{\text{length}_1(\gamma) : \gamma : [0, 1] \to \mathbb{R}^n \text{ admissible }, \gamma(0) = x \text{ and } \gamma(1) = y\}.$

and by Theorem 1.3.5 $\text{length}_1(\gamma) = \text{Var}(\gamma)$. Up to a reparameterization rectifiable curves are Lipschitz and thus admissible by Proposition 1.3.3. This proves that $(\mathbb{R}^n, d)$ is a length space.

Assume d continuous. Fix $x \in \mathbb{R}^n$, $r_0 > 0$, $K = \{y \in \mathbb{R}^n : |x - y| \le r_0\}$ and $M = \max_{x \in K} \|\mathcal{A}(x)\|$, where $\mathcal{A}$ is the matrix (1.1.1). By Proposition 1.1.3 if $0 < rM < r_0$ then $B(x, r) \subset K$. It follows that $\bar{B}(x, r)$ is compact in the Euclidean topology and consequently in the topology of d. □

THEOREM 1.4.4. *Let* $(\mathbb{R}^n, d)$ *be a C-C space.*

(i) *If metric balls are bounded then for all* $x, y \in \mathbb{R}^n$ *there exists a geodesic connecting them.*
(ii) *If* d *is continuous and* $K \subset \mathbb{R}^n$ *is compact there exists* $r > 0$ *such that if* $x \in K$ *and* $d(x, y) < r$ *there exists a geodesics connecting* x *to* y.

Proof. We prove statement (ii). Fix $\varepsilon > 0$, $K_\varepsilon = \{x \in \mathbb{R}^n : \min_{y \in K} |x - y| \le \varepsilon\}$ and $M = \sup_{x \in K_\varepsilon} \|\mathcal{A}(x)\|$. If $x \in K$ and $0 < rM < \varepsilon$ then $\bar{B}(x, r) \subset K_\varepsilon$ and thus $\bar{B}(x, r)$ is compact. Take $y \in B(x, r)$ and choose a sequence $(\gamma_k)_{k \in \mathbb{N}}$ of rectifiable curves $\gamma_k : [0, 1] \to \mathbb{R}^n$ such that $\gamma_k(0) = x$, $\gamma_k(1) = y$, and $\text{Var}(\gamma_k) \le d(x, y) + 1/k$. Such curves may be assumed to be Lipschitz in $(\mathbb{R}^n, d)$ with Lipschitz constant less or equal than $1 + d(x, y)$ and moreover $\gamma_k(t) \in \bar{B}(x, r)$ for all $t \in [0, 1]$ and $k \in \mathbb{N}$. By Ascoli-Arzelà Theorem there exists a subsequence - which may be assumed to be $(\gamma_k)_{k \in \mathbb{N}}$ itself - converging uniformly to a Lipschitz curve $\gamma : [0, 1] \to (\mathbb{R}^n, d)$. Since the total variation is lower semicontinuous with respect to the pointwise convergence

$$d(x, y) \le \text{Var}(\gamma) \le \liminf_{k \to +\infty} \text{Var}(\gamma_k) = d(x, y),$$

and thus $\text{Var}(\gamma) = d(x, y)$.

Statement (i) can be proved in the same way noticing that the sequence $(\gamma_k)_{k \in \mathbb{N}}$ is uniformly Lipschitz continuous also in the Euclidean metric by Propositions 1.3.3 and 1.3.2. □

REMARK 1.4.5. If the vector fields are globally Lipschitz continuous then it is easy to see using Gronwall Lemma that C-C balls are bounded.

EXAMPLE 1.4.6. Let $a \in \text{Lip}(\mathbb{R})$ be defined by $a(x) = x$ if $x \ge 0$ and $a(x) = 0$ if $x < 0$. In $\mathbb{R}^2$ consider the vector fields

$$X_1 = \partial_x \quad \text{and} \quad X_2 = a(x)\partial_y.$$

We show that $\mathbb{R}^2$ with the induced C-C metric d is not locally compact. A C-C ball $B(0, r)$, $r > 0$, is an Euclidean bounded open neighborhood of the origin $0 \in \mathbb{R}^2$. If $0 < \varepsilon < r$ the open sets $\Omega = \{(x, y) \in \mathbb{R}^2 : x > -\varepsilon\}$ and $\Omega_y = \{(x, \eta) \in \mathbb{R}^2 : x <$

0 and $\eta = y\}$, $y \in \mathbb{R}$, form an open covering of $B(0,r)$ which does not have any finite subcovering. Notice that every Ω_y is open in $(\mathbb{R}^2, d)$.

$(\mathbb{R}^2, d)$ is not locally compact. Nonetheless, metric balls are bounded and by Theorem 1.4.4 geodesics exist globally. They could be computed explicitly.

The most promising way to derive geodesics equations in C-C spaces is to reformulate the geodesic problem as an *optimal control theory* problem in order to apply Pontryagin Maximum Principle. This seems to have been first realized in [**31**], [**70**], [**165**].

Geodesics in C-C spaces are solution of the following *control problem.* We have the *state equation*

$$\dot{x}(t) = \mathcal{A}(x(t))h(t) \quad \text{for a.e. } t \in [0,1] \tag{1.4.32}$$

with constraints

$$x(0) = x_0 \quad \text{and} \quad x(1) = x_1, \quad x_0, x_1 \in \mathbb{R}^n. \tag{1.4.33}$$

The *cost functional* to minimize is $J : \mathrm{L}^\infty(0,1)^m \to \mathbb{R}$

$$J(h) = \frac{1}{2} \int_0^1 |h(t)|^2 \, dt. \tag{1.4.34}$$

By Theorem 1.1.6 the functional J is a "length" functional. Consider the minimum problem

$$\begin{aligned} \min\{J(h) : &\text{ there exists } x \in \mathrm{Lip}([0,1];\mathbb{R}^n) \text{ solving (1.4.32) relatively to } h \\ &\text{and satisfying the constraints (1.4.33)}\}. \end{aligned} \tag{1.4.35}$$

A pair (x,h) that solves this minimum problem is said to be *optimal.* By Theorem 1.4.4 problem (1.4.35) has a solution if $d(x_0, x_1)$ is small enough or more generally if C-C balls are bounded. Pontryagin Maximum Principle gives *necessary* conditions for a pair (x,h) to be optimal. Such conditions replace the Euler-Lagrange equations of the Calculus of Variations. In our context the Maximum Principle can be stated in the following way. We refer to [**114**] for a general introduction to it.

THEOREM 1.4.7 (Pontryagin Maximum Principle). *If the pair (x,h) is optimal then there exist a $\lambda \in \{0,1\}$ (the "multiplier") and $\xi \in \mathrm{Lip}([0,1];\mathbb{R}^n)$ (the "dual variable") such that:*

(i) $|\xi(t)| + \lambda \neq 0$ *for all* $t \in [0,1]$*;*
(ii) $\dot{\xi} = -\frac{\partial}{\partial x}\langle \mathcal{A}(x)h, \xi\rangle$ *a.e. on* $[0,1]$*;*
(iii) $\langle \mathcal{A}(x)h, \xi\rangle - \lambda\frac{1}{2}|h|^2 = \max_{u\in\mathbb{R}^m}\langle \mathcal{A}(x)u, \xi\rangle - \lambda\frac{1}{2}|u|^2$ *a.e. on* $[0,1]$.

DEFINITION 1.4.8. Geodesics corresponding to case $\lambda = 1$ are called *normal.* Geodesics corresponding to case $\lambda = 0$ are called *singular* (or *abnormal*).

In Riemannian spaces the case $\lambda = 0$ can not occur. That in C-C spaces singular geodesics may actually exist was shown by Montgomery [**142**]. The Hamiltonian formalism is particularly useful here. Introduce the Hamilton function

$$H(x,\xi) = \sum_{j=1}^m \langle X_j(x), \xi\rangle^2, \tag{1.4.36}$$

and the corresponding system of Hamilton equations

$$\begin{cases} \dot{x} = \dfrac{1}{2}\dfrac{\partial H(x,\xi)}{\partial \xi} \\ \dot{\xi} = -\dfrac{1}{2}\dfrac{\partial H(x,\xi)}{\partial x}. \end{cases} \tag{1.4.37}$$

If (x, ξ) solves (1.4.37) ("bicharacteristics") then $H(x(t), \xi(t))$ is constant.

PROPOSITION 1.4.9. *A normal geodesic x and its dual variable ξ solve equations (1.4.37). A singular geodesic x and its dual variable ξ solve $H(x(t), \xi(t)) \equiv 0$.*

Proof. If $\lambda = 1$ from (iii) in Theorem 1.4.7 we find the explicit expression for the optimal control

$$h(t) = \mathcal{A}^T(x(t))\xi(t), \tag{1.4.38}$$

which replaced in (ii) and in the state equation (1.4.32) gives (1.4.37).

If $\lambda = 0$ condition (iii) becomes $\langle \mathcal{A}^T\xi(t), h\rangle = \max_{u\in\mathbb{R}^m}\langle \mathcal{A}^T\xi(t), u\rangle$ and this forces $\mathcal{A}^T\xi = 0$, which means

$$\langle X_j(x(t)), \xi(t)\rangle \equiv 0, \quad j = 1, ..., m. \tag{1.4.39}$$

□

EXAMPLE 1.4.10. Singular geodesics do not satisfy the system of Hamilton equations. The following example, which we mention without proofs, is analyzed in detail in [**124**] section 2.3. In $\mathbb{R}^3$ consider the vector fields

$$X_1 = \partial_x \quad \text{and} \quad X_2 = (1-x)\partial_y + x^2\partial_z.$$

The C-C metric d is finite because X_1 and X_2 are bracket generating (see section 5). The curve $\gamma : [0, \varepsilon] \to \mathbb{R}^3$ defined by $\gamma(t) = (0, t, 0)$ does not solve equations (1.4.37) for any choice of ξ. But if $\varepsilon > 0$ is small enough γ is a geodesic.

5. Chow theorem

If the vector fields are smooth a general condition is known to imply connectivity, the "maximal rank" Chow-Hörmander condition. Such connectivity result was first proved by Chow [**49**], and named after Hörmander [**108**] that used the condition in the study of hypoelliptic equations. Here we shall follow the approach developed in [**121**].

If $X = \sum_{i=1}^n a_i(x)\partial_i$ and $Y = \sum_{j=1}^n b_j(x)\partial_i$ are smooth vector fields their commutator (bracket) is the vector field

$$[X, Y] = \sum_{i=1}^n \sum_{j=1}^n \big(a_j(x)\partial_j b_i(x) - b_j(x)\partial_j a_i(x)\big)\partial_i, \tag{1.5.40}$$

which amounts to write formally $[X, Y] = XY - YX$. Such product is skew-symmetric $[X, Y] = -[Y, X]$ and satisfies the Jacobi relation $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$. In the Lie algebra formalism $\mathrm{ad}X(Y) := [X, Y]$ is the adjoint representation.

By iterated brackets the vector fields $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ generate a Lie algebra which shall be denoted by $\mathcal{L}(X_1, ..., X_m)$ and for each $x \in \mathbb{R}^n$ this Lie algebra is a vector space $\mathcal{L}(X_1,...,X_m)(x)$. The Chow-Hörmander condition requires this vector space to have maximal rank

$$\mathrm{rank}\mathcal{L}(X_1,...,X_m)(x)=n, \quad \text{for all } \ x\in\mathbb{R}^n. \tag{1.5.41}$$

From a differential-geometric point of view condition (1.5.41) states that the vector fields and their iterated brackets generate the whole tangent space at every point.

Before stating and proving Chow Theorem we introduce some preliminary notions about exponential maps and about the Campbell-Hausdorff formula.

If $Y\in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $K\subset\mathbb{R}^n$ is a compact set consider the Cauchy problem

$$\begin{cases} \dot\gamma_x(t)=Y(\gamma_x(t)) \\ \gamma_x(0)=x\in K. \end{cases}$$

The solution γ_x is defined for $|t|\le\delta$ for some $\delta>0$ and we can define the exponential map

$$\mathrm{e}^{tY}(x)=\exp(tY)(x)=\gamma_x(t), \quad |t|\le\delta,\ x\in K. \tag{1.5.42}$$

The function $t\to\mathrm{e}^{tY}(x)$ is C^∞ and

$$\mathrm{e}^{tY}(x)=x+tY(x)+t^2O(1), \tag{1.5.43}$$

where $O(1)$ is a function bounded for $|t|\le\delta$ and $x\in K$.

If X and Y are two non commuting indeterminates in a Lie algebra the Campbell-Hausdorff formula links the composition of exponentials with a suitable exponential (see [**107**], [**108**], [**151**, Appendix] and [**170**] for Lie groups).

If $\alpha=(\alpha_1,\dots,\alpha_k)$ is a multi-index of non negative integers define $|\alpha|=\alpha_1+\dots+\alpha_k$ and $\alpha!=\alpha_1!\cdots\alpha_k!$. If α and β are multi-indeces set

$$D_{\alpha\beta}(X,Y)=\begin{cases} (\mathrm{ad}X)^{\alpha_1}(\mathrm{ad}Y)^{\beta_1}\cdots(\mathrm{ad}X)^{\alpha_k}(\mathrm{ad}Y)^{\beta_{k-1}}Y & \text{if } \beta_k\neq 0 \\ (\mathrm{ad}X)^{\alpha_1}(\mathrm{ad}Y)^{\beta_1}\cdots(\mathrm{ad}X)^{\alpha_{k-1}}X & \text{if } \beta_k=0, \end{cases} \tag{1.5.44}$$

and

$$c_{\alpha\beta}=\frac{1}{|\alpha+\beta|\alpha!\beta!}. \tag{1.5.45}$$

The Campbell-Hausdorff formula states that

$$\exp(X)\exp(Y)=\exp(P(X,Y)) \tag{1.5.46}$$

where $P(X,Y)$ is formally given by

$$P(X,Y)=\sum_{k=1}^\infty\frac{(-1)^{k+1}}{k}\sum_{\alpha_i+\beta_i\ge 1}c_{\alpha\beta}D_{\alpha\beta}(X,Y). \tag{1.5.47}$$

The inner sum ranges over all $\alpha=(\alpha_1,\dots,\alpha_k)$ and $\beta=(\beta_1,\dots,\beta_k)$ satisfying $\alpha_i+\beta_i\ge 1$. By direct computation it can be checked

$$P(X,Y)=X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]]+R(X,Y) \tag{1.5.48}$$

where $R(X,Y)$ is a formal series of commutators of length at least 4. Moreover, one can formally compute

$$\exp(-Y)\exp(-X)\exp(Y)\exp(X)=\exp([X,Y]+R(X,Y)), \tag{1.5.49}$$

where $R(X,Y)$ contains commutators of length at least 3. Formula (1.5.49) shows that the exponential of the commutator $[X,Y]$ can be represented up to lower order terms as a commutator of exponentials.

We now turn back to the Lie algebra generated by the vector fields $X_1, ..., X_m$. Let $J = (Y_1, ..., Y_r)$ be a $r-$tuple of vector fields $Y_i \in \{\pm X_1, ..., \pm X_m\}$, $i = 1, ..., r$. If $x \in K$ and $|t| \le \delta$ define

$$E(J,t)(x) = \mathrm{e}^{tY_r} ... \mathrm{e}^{tY_1}(x). \tag{1.5.50}$$

Notice that by Definition 1.1.4

$$d(x, E(J,t)(x)) \le |t| r. \tag{1.5.51}$$

If $I = (i_1, ..., i_k)$ is a vector of integer indeces $1 \le i_j \le m$, $j = 1, ..., k$, denote by X_I the iterated commutator

$$X_I = [X_{i_1}, [X_{i_2}, \cdots [X_{i_{k-1}}, X_{i_k}] \cdots]]. \tag{1.5.52}$$

Both I and X_I are said to have *length* k.

THEOREM 1.5.1. *Let* $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ *satisfy* (1.5.41). *Let* $K \subset \mathbb{R}^n$ *be a compact set and assume that for all* $x \in K$ *condition* (1.5.41) *is guaranteed by iterated commutators of length less than or equal to* k. *Then there exists* $C > 0$ *such that for all* $x, y \in K$

$$d(x,y) \le C|x-y|^{1/k}. \tag{1.5.53}$$

Consider now a vector of indeces $I = (i_1, ..., i_k)$ and let X_I be the iterated commutator defined in (1.5.52). By the Campbell-Hausdorff formula there exist $J = (Y_1, ..., Y_r)$ with $r \le 4^{k-1}$ and $C > 0$ such that

$$|E(J,t)(x) - \mathrm{e}^{tX_I}(x)| \le Ct^{(k+1)/k} \tag{1.5.54}$$

for all $x \in K$ and $|t| \le \delta$ (see Lemma 2.21 in [**151**] and [**150**]). For each commutator in X_I four terms "of smaller length" appear in the sequence J as in (1.5.49).

Define for $|\tau| \le \tau_0$

$$E_I(\tau)(x) = \begin{cases} E(J, \tau^{1/k})(x) & \text{if } \tau \ge 0 \\ E(J^-, (-\tau)^{1/k})(x) & \text{if } \tau < 0, \end{cases} \tag{1.5.55}$$

where J^- is a sequence of k vector fields that corresponds to $-X_I$ in such a way that (1.5.54) holds.

The function $\tau \to E_I(\tau)(x)$ is C^1 in a neighborhood of $\tau = 0$. Fix $\tau > 0$ and write $\tau + h = t^k$ e $\tau = t_0^k$ for some $t, t_0 > 0$. Then (we shall omit x)

$$\frac{\partial E_I(\tau)}{\partial \tau} = \lim_{t \to t_0} \frac{E(J,t) - E(J,t_0)}{t^k - t_0^k} = \frac{1}{k t_0^{k-1}} \frac{\partial E(J,t_0)}{\partial t},$$

and because of (1.5.54) and (1.5.43)

$$\begin{aligned} \frac{\partial E(J,t^k)}{\partial t} &= \frac{\partial}{\partial t} \mathrm{e}^{t^k X_I} + t^k O(1) \\ &= k t^{k-1} X_I(\mathrm{e}^{t^k X_I}) + t^k O(1) \\ &= k t^{k-1} \mathrm{e}^{t^k X_I} + t^k O(1), \end{aligned}$$

where here and in the sequel $O(1)$ is a bounded function for $|t| \leq \delta$ and $x \in K$. Finally

$$\frac{\partial E_I(\tau)}{\partial \tau} = X_I(x) + \tau^{1/k} O(1), \tag{1.5.56}$$

and, analogously, if $\tau < 0$ one can find

$$\frac{\partial E_I(\tau)}{\partial \tau} = X_I(x) + (-\tau)^{1/k} O(1).$$

This shows that $\dfrac{\partial E_I(\tau)(x)}{\partial \tau}$ is continuous at $\tau = 0$ and equals to $X_I(x)$.

Proof of Theorem 1.5.1. Let $x_0 \in K$ and fix n commutators $X_{I_1}, ..., X_{I_n}$ of length less than or equal to k such that $X_{I_1}(x_0), ..., X_{I_n}(x_0)$ are linearly independent. Let $E_{I_1}, ..., E_{I_n}$ be the approximated exponential maps defined in (1.5.55).

If $t \in \mathbb{R}^n$ belongs to a neighborhood of the origin define

$$F(t) = E_{I_n}(t_n) \cdots E_{I_1}(t_1)(x_0).$$

The map F is of class C^1. From (1.5.56)

$$\frac{\partial F(0)}{\partial t_i} = \frac{\partial E_{I_i}(0)}{\partial t_i}(x_0) = X_{I_i}(x_0),$$

and since

$$\det JF(0) = \det \operatorname{col}[X_{I_1}(x_0), ..., X_{I_n}(x_0)] \neq 0,$$

F is a local diffeomorphism. There exist $\varrho, \varepsilon, M > 0$ such that $\{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\} \subset F(\{t \in \mathbb{R}^n : |t| < \varrho\})$ and

$$|F(t) - F(t')| \geq M|t - t'| \tag{1.5.57}$$

for all $t, t' \in \{t \in \mathbb{R}^n : |t| < \varrho\}$.

Take $x \in \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}$. The C-C distance $d(x, x_0)$ can be estimated in the following way. There exists $t \in \mathbb{R}^n$ with $|t| < \varrho$ such that $F(t) = x$. Set $x_i = E_{I_i}(t_i)(x_{i-1})$, $i = 1, ...n$, and notice that $x = x_n$. By (1.5.51) with $r \leq 4^{k-1}$ and recalling (1.5.55) we have

$$d(x_i, x_{i-1}) \leq C|t_i|^{1/k}$$

where $C > 0$ depends only on K and k. Finally recalling (1.5.57)

$$\begin{aligned} d(x, x_0) &\leq \sum_{i=1}^n d(x_i, x_{i-1}) \leq C \sum_{i=1}^n |t_i|^{1/k} \leq nC|t|^{1/k} \leq \frac{nC}{M^{1/k}}|F(t) - F(0)|^{1/k} \\ &\leq \frac{nC}{M^{1/k}}|x - x_0|^{1/k}. \end{aligned}$$

□

6. Doubling metric spaces and structure theorems for C-C balls

6.1. Doubling metric spaces. Let (M, d) be a metric space endowed with a Borel measure μ positive and finite on balls.

DEFINITION 1.6.1. The space (M, d, μ) is said to be *doubling* (or *of homogeneous type*) if there exists $\delta > 1$ such that

$$\mu(B(x, 2r)) \le \delta\mu(B(x, r)) \quad \text{for all } x \in M \text{ and } r \ge 0. \tag{1.6.58}$$

The best constant δ in (1.6.58) is the *doubling constant* of M.

DEFINITION 1.6.2. The space (M, d, μ) is said to be *locally of homogeneous type* if for any compact set $K \subset M$ there exist $\delta > 1$ and $r_0 > 0$ such that

$$\mu(B(x, 2r)) \le \delta\mu(B(x, r)) \quad \text{for all } x \in K \text{ and } 0 \le r \le r_0. \tag{1.6.59}$$

PROPOSITION 1.6.3. *Let (M, d, μ) be a metric space locally of homogeneous type. For any compact set $K \subset M$ there exists $Q > 0$, such that if $x, x_0 \in K$, $B_0 = B(x_0, R)$ and $B = B(x, r)$ with $x \in B_0$ and $r \le R$ then*

$$\frac{\mu(B)}{\mu(B_0)} \ge \frac{1}{\delta^2}\Big(\frac{r}{R}\Big)^Q. \tag{1.6.60}$$

Proof. Let $t = r/R \le 1$ and fix $k \in \mathbb{N}$ such that $2^{k-1} \le t^{-1} < 2^k$. Then

$$\mu(B_0) \le \mu(B(x, 2R)) = \mu(B(x, 2r/t)) \le \mu(B(x, 2^{k+1}r)) \le \delta^{k+1}\mu(B(x, r)).$$

Now, $k \le 1 + \log_2(R/r)$ and

$$\delta^k \le \delta^{1+\log_2(R/r)} = \delta\Big(\frac{R}{r}\Big)^{\log_2(\delta)},$$

and the claim follows with $Q = \log_2(\delta)$. □

DEFINITION 1.6.4. The constant $Q = \log_2(\delta)$ is the *local homogeneous dimension* of (M, d, μ) relative to the compact set K.

Spaces of homogeneous type were introduced by Coifman and Weiss [**52**] in the study of maximal operators and singular integrals. Such spaces are of special interest because a Lebesgue differentiation theorem holds.

THEOREM 1.6.5. *Let (M, d, μ) be a doubling metric space. If $f \in \mathrm{L}^p_{\mathrm{loc}}(M, \mu)$, $1 \le p < +\infty$, then*

$$\lim_{r\downarrow 0} \fint_{B(x,r)} |f(x) - f(y)|^p \, d\mu(y) = 0$$

for $\mu-$a.e. $x \in M$.

The proof of Theorem 1.6.5 relies on the continuity of the maximal operator (see [**163**]).

6.2. Nagel-Stein-Wainger theorem. Carnot-Carathéodory spaces arising from vector fields satisfying the Chow-Hörmander condition are locally of homogeneous type. This is one of the main results of the basic paper [**151**] which will be briefly described.

Let $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be a family of vector fields satisfying the maximal rank condition (1.5.41). If $\Omega \subset \mathbb{R}^n$ is a bounded open set there exists an integer k such that condition (1.5.41) is verified at every $x \in \Omega$ by commutators of length equal or less than k. Let $\{Y_1, ..., Y_q\}$ be an enumeration of all the commutators with length equal or less than k, so that $\operatorname{rank}\{Y_1, ..., Y_q\} = n$ for all $x \in \Omega$. Denote by $d(Y_i)$ the length of the commutator Y_i. Let $\mathcal{I}$ be the family of all multi-indeces $I = (i_1, ..., i_n)$ such that $1 \le i_j \le q$, and for any $I \in \mathcal{I}$ let $(Y_{i_1}, ..., Y_{i_n})$ be the corresponding $n-$tuple of commutators. For $I \in \mathcal{I}$ and $h \in \mathbb{R}^n$ define

$$d(I) = d(Y_{i_1}) + ... + d(Y_{i_n}) \quad \text{ and } \quad \|h\|_I = \max_{j=1,...,n} |h_j|^{1/d(Y_{i_j})}.$$

In the homogeneous norm $\|h\|_I$ the $j-$th component is weighted by the length of the commutator Y_{i_j}.

Finally, if $I \in \mathcal{I}$ introduce the function

$$\lambda_I(x) = \det[Y_{i_1}(x)...Y_{i_n}(x)],$$

and the exponential map

$$\Phi_I(x,h) = \Phi_{I,x}(h) = \exp(h_1 Y_{i_1} + ... + h_n Y_{i_n})(x).$$

Nagel-Stein-Wainger Theorem can now be stated.

THEOREM 1.6.6. *Let $K \subset \Omega$ be a compact set and let $r_0 > 0$. There exist $0 < \eta_2 < \eta_1 < 1$ such that if $I = (i_1, ..., i_n) \in \mathcal{I}$, $x \in K$ and $0 < r < r_0$ satisfy*

$$|\lambda_I(x)| r^{d(I)} \ge \frac{1}{2} \max_{J \in \mathcal{I}} |\lambda_J(x)| r^{d(J)}, \tag{1.6.61}$$

then

(i) *if $\|h\|_I < \eta_1 r$ then*

$$\frac{1}{4} |\lambda_I(x)| \le |\det \frac{\partial \Phi_I}{\partial h}(x,h)| \le 4 |\lambda_I(x)|; \tag{1.6.62}$$

(ii) *the following inclusions hold*

$$B(x, \eta_2 r) \subset \Phi_{I,x}(\{h \in \mathbb{R}^n : \|h\|_I < \eta_1 r\}) \subset B(x, \eta_1 r); \tag{1.6.63}$$

(iii) *the function $\Phi_{I,x}$ is one-to-one on $\{h \in \mathbb{R}^n : \|h\|_I < \eta_1 r\}$.*

Thesis (ii) represents C-C balls by means of the image under the exponential map $\Phi_{I,x}$ of “homogeneous rectangles”. From (1.6.62) and (1.6.63) the following corollary easily follows, which proves the local doubling property for C-C balls measured by the Lebesgue measure. The size of the balls $B(x,r)$ is described by the functions

$$\Lambda(x,r) := \sum_{I \in \mathcal{I}} |\lambda_I(x)| r^{d(I)}.$$

COROLLARY 1.6.7. *Let $K \subset \Omega$ be a compact set and let $r_0 > 0$. There exists $C > 0$ such that*

$$\frac{1}{C}|B(x,r)| \le \Lambda(x,r) \le C|B(x,r)| \tag{1.6.64}$$

for all $x \in K$ and $0 < r < r_0$.

REMARK 1.6.8. Notice that $d(I) \le kn$ for all $I \in \mathcal{I}$. Then, if $x \in K$ and $0 < r < r_0$

$$\begin{aligned}|B(x,2r)| &\le C\Lambda(x,2r) = C\sum_{I\in\mathcal{I}}|\lambda_I(x)|(2r)^{d(I)}\\ &\le C2^{kn}\sum_{I\in\mathcal{I}}|\lambda_I(x)|r^{d(I)} \le C^2 2^{kn}|B(x,r)|.\end{aligned}$$

This is the local doubling property.

6.3. A variant of the structure theorem. Following the basic ideas contained in [**151**] and its generalization in [**150**], we shall represent C-C balls restricted to non characteristic surfaces by means of suitable exponential maps which are "small perturbations" of the exponential of the commutators of the vector fields.

Write $(x,t) \in \mathbb{R}^{n-1}\times\mathbb{R}$. Consider m vector fields $X_1,...,X_m \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ of the form

$$X_j = \sum_{i=1}^{n-1} a_{ij}(x,t)\partial_i,\ j = 1,...,m-1, \quad X_m = \partial_t. \tag{1.6.65}$$

and satisfying the Hörmander condition. We shall write $X_m = T$. For any multi-index $I = (i_1,\dots,i_k)$, $1 \le i_j \le m$ and $k \in \mathbb{N}$, let

$$X_{[I]} = [X_{i_1},[X_{i_2},\cdots[X_{i_{k-1}},X_{i_k}]\cdots]],$$

where $[X,Y]$ denotes the commutator of the vector fields X and Y. If $I = (i_1,\dots,i_k)$ we set $|I| = k$ and we say that the commutator $X_{[I]}$ has *length* or *degree* $d(X_{[I]}) = k$.

For any commutator $Y \ne T$ and for small $s \in \mathbb{R}$ we shall define a map $\exp_T(sY) : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$. We proceed by induction on $d(Y)$. If $d(Y) = 1$ and $Y = X_j$ with $j \in \{1,...,m-1\}$ define for $x \in \mathbb{R}^{n-1}$

$$\exp_T(sY)(x) = \begin{cases}\exp(-sT)\exp(s(X_j+T))(x) & \text{if } s \ge 0,\\ \exp(s(X_j+T))\exp(-sT)(x) & \\ \quad = \exp_T(|s|Y)^{-1}(x) & \text{if } s < 0.\end{cases} \tag{1.6.66}$$

The map is well defined provided x belongs to a compact set and s is small. We also set $\exp_T(sT) = \exp(sT)$. Suppose now $d(Y) = k$, $Y = X_{[J]}$ with $|J| = k$, and $J = (j_1,...,j_k)$. Set $J' = (j_2,...,j_k)$ and define

$$\exp_T(sY)(x) = \begin{cases}\exp_T(s^{\frac{k-1}{k}}X_{[J']})^{-1}\exp_T(s^{\frac{1}{k}}X_{j_1})^{-1} & \\ \qquad\cdot\exp_T(s^{\frac{k-1}{k}}X_{[J']})\exp_T(s^{\frac{1}{k}}X_{j_1})(x) & \text{if } s \ge 0,\\ \exp_T(|s|Y)^{-1}(x) & \text{if } s < 0.\end{cases} \tag{1.6.67}$$

Some useful features of the maps $\exp_T$ are described in the following two lemmas. In Lemma 1.6.9, which is a generalization of [**151**, Lemma 2.21], we shall use the

Campbell-Hausdorff formula

$$\exp(u)\exp(v) = \exp\Big(u + v - \frac{1}{2}[u,v] + S(u,v)\Big),$$

where u and v are non commuting indeterminates and S is a formal sum of commutators of u and v of length at least 3.

LEMMA 1.6.9. *For any commutator $X_{[J]}$, $J = (j_1, \dots, j_k)$, of length $k \geq 1$*

$$\exp_T(sX_{[J]}) = \exp\Big(sX_{[J]} + \operatorname{sgn}(s) \sum_{|I|>k} c_{J,I}|s|^{|I|/k} X_{[I]}\Big), \tag{1.6.68}$$

where the $c_{J,I}$ are suitable constants.

The formal equality (1.6.68) means that, if x belongs to a compact set K and $p > k$ is an integer, then

$$\Big|\exp_T(sX_{[J]})(x) - \exp\Big(sX_{[J]} + \operatorname{sgn}(s) \sum_{k<|I|\leq p} c_{J,I} s^{|I|/k} X_{[I]}\Big)(x)\Big| \leq C s^{(p+1)/k}.$$

Proof. We proceed by induction. Consider first a commutator of length 1, i.e. a vector field X_j, $j = 1, \dots, m$. Applying the Campbell-Hausdorff formula to (1.6.66) we get for $s > 0$

$$\begin{aligned}
\exp_T(sX_j) &= \exp(-sT)\exp(s(X_j + T)) \\
&= \exp\Big(-sT + s(X_j + T) + \frac{1}{2}s^2[T, X_j + T] + \cdots\Big) \\
&= \exp\Big(sX_j + \sum_{|I|>1} c_{(j),I} s^{|I|} X_{[I]}\Big).
\end{aligned}$$

For $s < 0$ note that

$$\exp_T(sX_j) = \exp_T(|s|X_j)^{-1} = \exp\Big(-|s|X_j - \sum_{|I|>1} c_{(j),I}|s|^{|I|} X_{[I]}\Big).$$

We prove now the inductive step. Recall first that an application of the Campbell-Hausdorff formula asserts that, if u and v are non commuting indeterminates, then

$$\exp(v)^{-1}\exp(u)^{-1}\exp(v)\exp(u) = \exp([u,v] + R),$$

where $R = R(u,v)$ denotes a formal series containing commutators (of u and v) of length at least 3. Let $k > 1$, $J = (j_1, ..., j_k)$, $J' = (j_2, ..., j_k)$ and $s \geq 0$. Let also

$$u = s^{1/k} X_{j_1} + \sum_{|I|>1} c_{(j_1),I} s^{|I|/k} X_{[I]} \qquad \text{and}$$
$$v = s^{(k-1)/k} X_{[J']} + \sum_{|I|>k-1} C_{J',I} s^{|I|/k} X_{[I]}.$$

Note that $[u,v] = sX_{[J]} + \widetilde{R}$, where $\widetilde{R}$ is a series containing commutators of order at least $k+1$ of the original vector fields. Thus, by the inductive hypothesis

$$\begin{aligned}
\exp_T(sX_{[J]}) &= \exp_T(s^{\frac{k-1}{k}} X_{[J']})^{-1} \exp_T(s^{\frac{1}{k}} X_{j_1})^{-1} \\
&\qquad \cdot \exp_T(s^{\frac{k-1}{k}} X_{[J']}) \exp_T(s^{\frac{1}{k}} X_{j_1}) \\
&= \exp(v)^{-1} \exp(u)^{-1} \exp(v) \exp(u) \\
&= \exp([u,v] + R)) \\
&= \exp(sX_{[J]} + \widetilde{R} + R)) \\
&= \exp\Big(sX_{[J]} + \sum_{|I|>k} c_{J,I} s^{|I|/k} X_{[I]} \Big),
\end{aligned}$$

for suitable constants $c_{J,I}$. We used the fact that the series R is actually a series of commutators of length at least $k+1$ of the original fields. If $s<0$, formula (1.6.68) follows analogously. $\square$

From now on fix a bounded open set $\Omega_0 \subset \mathbb{R}^n$ and let $\{Y_1, ..., Y_q\}$ be a fixed enumeration of the commutators of length $\leq k$, where k is large enough to ensure that $\mathrm{span}\{X_{[I]}(x,t) : |I| \leq k\}$ has dimension n at each point $(x,t) \in \Omega_0$. Assume also that $Y_q = T$.

Introduce the family of multi-indices $\mathcal{I} = \{I = (i_1, \dots, i_{n-1}) : 1 \leq i_j \leq q-1\}$. Given a multi-index $I \in \mathcal{I}$, set $d(I) = d(Y_{i_1}) + \cdots + d(Y_{i_{n-1}})$ and for $\widetilde{h} = (h, h_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ "small enough" define

$$\begin{aligned}
\Phi_{I,x}(h) &= \exp_T(h_{n-1} Y_{i_{n-1}}) \cdots \exp_T(h_1 Y_{i_1})(x,0), \\
\widetilde{\Phi}_{I,x}(\widetilde{h}) &= \exp(h_n T) \exp_T(h_{n-1} Y_{i_{n-1}}) \cdots \exp_T(h_1 Y_{i_1})(x,0) \qquad (1.6.69)\\
&= (\Phi_{I,x}(h), h_n).
\end{aligned}$$

The form of the fields (1.6.65) guarantees that $\Phi_{I,x}(h) \in \{(x,t) \in \mathbb{R}^n : t = 0\}$ for $h \in \mathbb{R}^{n-1}$. Let also

$$\|h\|_I = \max_{l=1,\dots,n-1} |h_l|^{1/d(Y_{i_l})} \quad \text{and}$$
$$\lambda_I(x) = \det(Y_{i_1}(x,0), \dots, Y_{i_{n-1}}(x,0)),$$

where the vectors Y_{i_l} are thought of as vectors in $\mathbb{R}^{n-1}$.

If $I \in \mathcal{I}$ define $\widetilde{I} = (I, q)$ and set $d(\widetilde{I}) = d(I) + 1$. If $\widetilde{h} = (h, h_n)$ and $(x,t) \in \Omega_0$ define

$$\|\widetilde{h}\|_{\widetilde{I}} = \max\{\|h\|_I, |h_n|\} \quad \text{and}$$
$$\widetilde{\lambda}_{\widetilde{I}}(x,t) = \det(Y_{i_1}(x,t), \dots, Y_{i_{n-1}}(x,t), Y_n(x,t))$$

where $Y_n = T$ and the vectors are thought of as vectors in $\mathbb{R}^n$.

Let d be the C-C metric induced by the vector fields (1.6.65) on $\mathbb{R}^n$ and consider the balls $B((x,0),r) = \{(y,t) \in \mathbb{R}^n : d((x,0),(y,t)) < r\}$ and $\overline{B}(x,r) = \{y \in \mathbb{R}^{n-1} : d((x,0),(y,0)) < r\}$. We now state and prove the structure theorem for the restricted balls $\overline{B}$.

THEOREM 1.6.10. *Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded open set. There exist $r_0 > 0$ and $0 < a < b < 1$ such that for any $(x, 0) \in \Omega_0$, $I \in \mathcal{I}$ and $0 < r < r_0$ such that the inequality*

$$|\lambda_I(x)| r^{d(I)} \geq \frac{1}{2} \max_{J \in \mathcal{I}} |\lambda_J(x)| r^{d(J)} \tag{1.6.70}$$

is satisfied, we have

(i) $\frac{1}{4}|\lambda_I(x)| \leq |J_h \Phi_{I,x}(h)| = |J_{\widetilde{h}} \widetilde{\Phi}_{I,x}(\widetilde{h})| \leq 4|\lambda_I(x)|$ *for every* $\|\widetilde{h}\|_{\widetilde{I}} < br$, *where* $J_h \Phi_{I,x}(h) = \det \frac{\partial}{\partial h} \Phi_{I,x}(h)$.

(ii) $B((x, 0), ar) \subset \widetilde{\Phi}_{I,x}(\{\|\widetilde{h}\|_{\widetilde{I}} < br\}) \subset B((x, 0), r)$.

(iii) $\overline{B}(x, ar) \subset \Phi_{I,x}(\{\|h\|_I < br\}) \subset \overline{B}(x, r)$.

(iv) *The map* $\widetilde{\Phi}_{I,x}$ *is one to one on* $\{\|\widetilde{h}\|_{\widetilde{I}} < br\}$.

REMARK 1.6.11. Inclusions (iii) for the restricted balls are immediate consequence of (ii) and of the structure (1.6.69) of the map $\widetilde{\Phi}$. Indeed, starting from (ii) we get

$$\overline{B}(x, ar) \subset \widetilde{\Phi}_{I,x}(\{\|\widetilde{h}\|_{\widetilde{I}} < br\}) \cap \{t = 0\} = \Phi_{I,x}(\{\|h\|_I < br\}).$$

The opposite inclusion is analogous.

Proof of Theorem 1.6.10. Since $\lambda_I(x) = \widetilde{\lambda}_{\widetilde{I}}(x, 0)$, if (1.6.70) is verified for some $(n-1)-$tuple $I \in \mathcal{I}$ then the $n-$tuple $\widetilde{I} = (I, q)$ satisfies

$$|\widetilde{\lambda}_{\widetilde{I}}(x, 0)| r^{d(\widetilde{I})} \geq \frac{1}{2} \max_{J \in \mathcal{I}} |\widetilde{\lambda}_{\widetilde{J}}(x, 0)| r^{d(\widetilde{J})}. \tag{1.6.71}$$

In [**151**, Theorem 7] it is proved that if $Y_{j_1}, \ldots, Y_{j_n}$ are commutators of degrees $d_1, \ldots, d_n$ which satisfy (1.6.71), then the map $\widetilde{\Phi}^*_{I,x}$ defined by $\widetilde{\Phi}^*_{I,x}(\widetilde{h}) = \exp(h_1 Y_{j_1} + \cdots + h_n Y_{j_n})(x, 0)$ satisfies (i), (ii) and (iv). Moreover in [**150**, Lemmas 3.2-3.6] the following is proved. Assume that the exponential of any commutator Y_j can be approximated by a map $E(sY_j)$ in the sense that

$$E(sY_j) = \exp\Big(sY_j + \operatorname{sgn}(s) \sum_{|I| > d(Y_j)} k_{(j),I} |s|^{|I|/d(Y_j)} X_{[I]}\Big),$$

where the $k_{(j),I}$ are constants and assume also that for a n-tuple of commutators $Y_{j_1}, \ldots, Y_{j_n}$ (1.6.71) holds at a point $(x, 0)$ and for a radius r. Then the map

$$\widetilde{\Phi}_{I,x}(\widetilde{h}) = E(h_n Y_{j_n}) \cdots E(h_1 Y_{j_1})(x, 0)$$

satisfies (i), (ii) and (iv). In view of Lemma 1.6.9 this assertion can be applied to the map $E = \exp_T$ and the Theorem is proved. We also note that the estimate

$$\mu(B((x, 0), r)) \simeq \sum_{I \in \mathcal{I}} |\lambda_I(x)| r^{d(I)} \tag{1.6.72}$$

holds. $\square$

The following factorization theorem will be needed in chapter 3. Define for $\lambda > 0$ and for any vector field X_j

$$\begin{aligned} S_1(\lambda, X_j) &= \exp(\lambda(X_j - T)) \exp(\lambda T), \\ S_2(\lambda, X_j) &= \exp(-\lambda T) \exp(\lambda(X_j + T)). \end{aligned} \tag{1.6.73}$$

THEOREM 1.6.12. *Let $Y = X_{[J]}$ with $J = (j_1, ..., j_k)$. The map $\exp_T(sY)$, $s \in \mathbb{R}$, can be factorized as the composition of a finite number of factors of the form $S_1(h|s|^{\frac{1}{k}}, \tau X_j)$ and $S_2(h|s|^{\frac{1}{k}}, \tau X_j)$, where $\tau \in \{-1, 1\}$, $j = 1, ..., m$ and $1 \le h \le k$. Moreover, the number of factors depends only on k.*

Proof. Since $S_2(h|s|^{\frac{1}{k}}, \tau X_j) = S_1(h|s|^{\frac{1}{k}}, -\tau X_j)^{-1}$, if we prove the claim for $s > 0$ it will also follows for $s < 0$. Without loss of generality we can suppose $s = 1$. First notice that

$$\begin{aligned} S_1(h, \tau X)\exp(T) &= \exp(h(\tau X - T))\exp(hT)\exp(T) \\ &= \exp(T)\exp(T)^{-1}\exp(-\tau X + T)S_1(h+1, \tau X) \\ &= \exp(T)S_2(1, -\tau X)S_1(h+1, \tau X) \end{aligned} \tag{1.6.74}$$

and

$$\begin{aligned} S_2(h, \tau X)\exp(T) &= \exp(hT)^{-1}\exp(h(\tau X + T))\exp(T) \\ &= \exp(T)S_2(h+1, \tau X)\exp(-\tau X - T)\exp(T) \\ &= \exp(T)S_2(h+1, \tau X)S_1(1, -\tau X). \end{aligned} \tag{1.6.75}$$

The proof is by induction on $k = d(Y)$. If $k = 1$ the claim follows directly from definition (1.6.66) with $h = 1$. Let $k = d(Y) > 1$ and let $Y = X_{[J]}$ with $J = (j_1, ..., j_k)$. If $j_1 \neq m$ the claim follows directly from (1.6.67) and the inductive hypothesis on $X_{[J']}$, $J' = (j_2, ..., j_k)$. Suppose $j_1 = m$ and by the inductive hypothesis write

$$\exp_T(X_{[J']}) = \prod_{i=1}^{p} S_{\sigma_i}(h_i, \tau_i X_{j_i})$$

with $\sigma_i \in \{1, 2\}$, $\tau_i \in \{-1, 1\}$, $p \in \mathbb{N}$ less than a constant depending on k, and $1 \le h_i \le k - 1$. Write

$$\begin{aligned} \exp_T(X_{[J]}) &= \exp_T(X_{[J']})^{-1}\exp(T)^{-1}\exp_T(X_{[J']})\exp(T) \\ &= \exp_T(X_{[J']})^{-1}\exp(T)^{-1}\prod_{i=1}^{p} S_{\sigma_i}(h_i, \tau_i X_{j_i})\exp(T). \end{aligned}$$

By (1.6.74) and (1.6.75) $\exp(T)$ can be shifted p times from right to left cancelling $\exp(T)^{-1}$ and the claim follows. □

7. Carnot Groups

In this section we introduce Carnot groups, one of the main classes of C-C spaces. Carnot groups are nilpotent Lie groups which admit a one parameter group of dilations.

7.1. Lie groups. A Lie group is a differentiable manifold G endowed with a group structure which is differentiable in the sense that the product $(x, y) \mapsto x \cdot y$ and the inversion $x \mapsto x^{-1}$ are smooth maps. We shall denote by 0 the identity of the group.

If $g \in G$ let $\tau_g : G \to G$ be the left translation $\tau_g(x) = g \cdot x$. The Lie algebra $\mathfrak{g}$ of G is the set of the vector fields $X \in \Gamma(TG)$, the sections of the tangent bundle, which are left invariant, i.e. such that

$$(Xf)(\tau_g(x)) = X(f \circ \tau_g)(x) \tag{1.7.76}$$

for all $x, g \in G$ and for all $f \in C^\infty(G)$. This set is a vector space, and, for the commutator of left invariant vector fields is a left invariant vector field, it becomes a Lie algebra. This algebra is canonically isomorphic to the tangent space to G at the origin via the identification of X and $X(0)$.

Let $X \in \mathfrak{g}$ and consider the one parameter subgroup $\gamma_X : \mathbb{R} \to G$ which is solution to the equation $\dot\gamma_X(t) = X(\gamma_X(t))$ with initial datum $\gamma_X(0) = 0$. The integral curve γ_X is defined for all $t \in \mathbb{R}$ since left invariant vector fields are complete. The exponential map $\exp : \mathfrak{g} \to G$ is defined by $\exp(X) = \gamma_X(1)$. Define analogously $\exp(X)(g)$ taking g as initial datum instead of the origin. The map exp is a diffeomorphism from a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of 0 in G. If $d\tau_g : T_0G \to T_gG$ denotes the differential of τ_g at the origin, condition (1.7.76) means that $X(g) = d\tau_g X(0)$. It follows that $\exp(X)(g) = \tau_g(\exp(X)) = g \cdot \exp(X)$. In particular

$$\exp(Y) \cdot \exp(X) = \exp(X)(\exp(Y)) \tag{1.7.77}$$

for all $X, Y \in \mathfrak{g}$.

The algebraic structure of $\mathfrak{g}$ determines that of G, and precisely

$$\exp(X) \cdot \exp(Y) = \exp(P(X, Y)) \tag{1.7.78}$$

for any $X, Y \in \mathfrak{g}$, where $P(X, Y)$ is given by the Campbell-Hausdorff formula (1.5.47). The Lie algebra $\mathfrak{g}$ endowed with product $P(X, Y)$ can be checked to be a Lie group. The map $P : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{g}$ is analytic in a neighborhood $\mathfrak{a}$ of the origin $0 \in \mathfrak{g}$ (see [**170**, section 2.15]). Formula (1.7.78) is particularly useful when the Lie algebra is nilpotent, becoming (1.5.47) a finite sum.

By induction define $\mathfrak{g}_1 = \mathfrak{g}$ and $\mathfrak{g}_i = [\mathfrak{g}, \mathfrak{g}_{i-1}]$ for $i > 1$, where $[\mathfrak{g}, \mathfrak{g}_i]$ is the set of the products $[X, Y]$ with $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}_i$. The Lie group G is *nilpotent of step* $k \in \mathbb{N}$ if $\mathfrak{g}_k \neq \{0\}$ and $\mathfrak{g}_{k+1} = \{0\}$. If G is a simply connected nilpotent Lie group and $\mathfrak{g}$ is its Lie algebra the exponential map $\exp : \mathfrak{g} \to G$ is a global diffeomorphism ([**170**, Theorem 3.6.2]). In the sequel G will always be assumed to be connected and simply connected.

7.2. Stratified algebras and groups. A nilpotent Lie group G is *stratified* if its Lie algebra $\mathfrak{g}$ admits a stratification, i.e. there exist linear subspaces $V_1, ..., V_k$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_k, \quad V_i = [V_1, V_{i-1}] \text{ for } i = 2, ..., k \text{ and } V_{k+1} = \{0\}. \tag{1.7.79}$$

V_1 is the first slice of $\mathfrak{g}$ and it generates the whole algebra by iterated brackets. Stratified groups are also called *Carnot* groups.

Fix $\lambda > 0$ and define $\widetilde{\delta}_\lambda : V_1 \to V_1$ by $\widetilde{\delta}_\lambda(X) = \lambda X$. This map can be extended to $\mathfrak{g}$ by $\widetilde{\delta}_\lambda(X) = \lambda^i X$ if $X \in V_i$ and by linearity. The family $(\widetilde{\delta}_\lambda)_{\lambda>0}$ is a group of automorphisms of $\mathfrak{g}$

$$\widetilde{\delta}_\lambda([X, Y]) = [\widetilde{\delta}_\lambda(X), \widetilde{\delta}_\lambda(Y)], \quad \text{and} \quad \widetilde{\delta}_{\lambda\mu}(X) = \widetilde{\delta}_\lambda(\widetilde{\delta}_\mu(X))$$

for all $\lambda, \mu > 0$. In particular for all $X, Y \in \mathfrak{g}$

$$\widetilde{\delta}_\lambda(P(X, Y)) = P(\widetilde{\delta}_\lambda(X), \widetilde{\delta}_\lambda(Y)), \tag{1.7.80}$$

where $P(X, Y)$ is defined in (1.5.47).

The automorphisms $\widetilde{\delta}_\lambda$ induce a group of automorphisms of G via the exponential map. Define $\delta_\lambda : G \to G$ by $\delta_\lambda(x) = \exp(\widetilde{\delta}_\lambda(\exp^{-1}(x)))$. It can be checked that:

(i) $\delta_{\lambda\mu}(x) = \delta_\lambda(\delta_\mu(x))$ for all $\lambda, \mu > 0$ and $x \in G$;
(ii) $\delta_\lambda(x \cdot y) = \delta_\lambda(x) \cdot \delta_\lambda(y)$ for $\lambda > 0$ and $x, y \in G$.

We show for example (ii). Suppose that $x = \exp(X)$ and $y = \exp(Y)$. Then by (1.7.80)

$$\begin{aligned}\delta_\lambda(x \cdot y) &= \exp(\widetilde{\delta}_\lambda(\exp^{-1}(\exp(X) \cdot \exp(Y)))) = \exp(\widetilde{\delta}_\lambda(P(X,Y)))\\ &= \exp(P(\widetilde{\delta}_\lambda(X), \widetilde{\delta}_\lambda(Y))) = \exp(\widetilde{\delta}_\lambda(X)) \cdot \exp(\widetilde{\delta}_\lambda(Y))\\ &= \delta_\lambda(x) \cdot \delta_\lambda(y).\end{aligned}$$

7.3. Exponential coordinates. The underlying manifold of a Carnot group can always be chosen to be $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Fix a vector basis $X_1, ..., X_n$ of a real Lie algebra $\mathfrak{g}$ of a $n-$dimensional Carnot group. If $X, Y \in \mathfrak{g}$ then $X = \sum_{i=1}^n x_i X_i$ and $Y = \sum_{i=1}^n y_i X_i$ for some $x, y \in \mathbb{R}^n$. The coordinates $(x_1, ..., x_n)$ are the *exponential coordinates* of $\exp(X) \in G$. A group law on $\mathbb{R}^n$, which will still be denoted by $\cdot$, can be introduced in the following way

$$x \cdot y = z \quad \text{if and only if} \quad \exp(X) \cdot \exp(Y) = \exp\Big(\sum_{i=1}^n z_i X_i\Big), \tag{1.7.81}$$

which is equivalent to require $P(X,Y) = \sum_{i=1}^n z_i X_i$. With such a product $\mathbb{R}^n$ is a Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$. Since connected and simply connected Lie groups are isomorphic if and only if the corresponding Lie algebras are isomorphic ([**170**, Theorem 2.7.5]), it follows that $(\mathbb{R}^n, \cdot)$ and G are isomorphic. Using the Campbell-Hausdorff formula (1.5.47) the group law can be computed explicitely.

Example 1.7.1. Suppose we have a four dimensional stratified algebra with basis $\{X_1, X_2, X_3, X_4\}$ and generators X_1, X_2. Assume that $[X_1, X_2] = X_3$, $[X_1, X_3] = [X_2, X_3] = X_4$ and all other commutators vanish. We have a stratified algebra of step 3. Write

$$X = \sum_{i=1}^4 x_i X_i \quad \text{and} \quad Y = \sum_{i=1}^4 y_i X_i$$

with $x, y \in \mathbb{R}^4$. By (1.5.48) the formal expression of $P(X,Y)$ reduces to

$$P(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]],$$

where

$$\begin{aligned}[X,Y] &= [x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4]\\ &= (x_1y_2 - x_2y_1)[X_1,X_2] + (x_1y_3 - x_3y_1)[X_1,X_3] + (x_2y_3 - x_3y_2)[X_2,X_3]\\ &= (x_1y_2 - x_2y_1)X_3 + \{(x_1y_3 - x_3y_1) + (x_2y_3 - x_3y_2)\}X_4\\ [X,[X,Y]] &= [x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4, (x_1y_2 - x_2y_1)X_3 + \{\cdots\}X_4]\\ &= (x_1 + x_2)(x_1y_2 - x_2y_1)X_4\\ [Y,[Y,X]] &= [y_1X_1 + y_2X_2 + y_3X_3 + y_4X_4, -(x_1y_2 - x_2y_1)X_3 - \{\cdots\}X_4]\\ &= -(y_1 + y_2)(x_1y_2 - x_2y_1)X_4.\end{aligned}$$

Therefore, the group law in $\mathbb{R}^4$ is

$$\begin{aligned} x\cdot y = \Big(& x_1+y_1, x_2+y_2, x_3+y_3+\frac{1}{2}(x_1y_2-x_2y_1), x_4+y_4 \\ &+\frac{1}{2}\big\{(x_1y_3-x_3y_1)+(x_2y_3-y_2x_3)\big\} \\ &+\frac{1}{12}\big\{(y_1+y_2)(x_2y_1-x_1y_2)+(x_1+x_2)(x_1y_2-x_2y_1)\big\}\Big). \end{aligned}$$

In the general case the group law in $\mathbb{R}^n$ will be written as

$$x\cdot y = P(x,y) = x+y+Q(x,y),$$

where $P=(P_1,...,P_n)$ and $Q=(Q_1,...,Q_n)$ are polynomial functions. If $\mathfrak{g}=V_1\oplus ...\oplus V_k$ is a stratification, set $m_j=\dim(V_j)$, $j=1,...,k$. If i is an index such that $m_1+...+m_{d_i-1}<i\le m_1+...+m_{d_i}$ for some $1\le d_i\le k$ the coordinate x_i will be said to have degree d_i. Group dilations $\delta_\lambda:\mathbb{R}^n\to\mathbb{R}^n$ can be written as

$$\delta_\lambda(x) = (\lambda^{d_1}x_1, \lambda^{d_2}x_2, ..., \lambda^{d_n}x_n). \tag{1.7.82}$$

If $1\le i\le m_1$ then $d_i=1$.

The following Lemma lists some properties of the group product that are of special interest. Thesis (iv) will be useful in Lemma 2.1.4.

LEMMA 1.7.2.
(i) *For all $x\in\mathbb{R}^n$ the inverse element x^{-1} is $-x$.*
(ii) *For all $x,y\in\mathbb{R}^n$ and for all $\lambda>0$ $P(\delta_\lambda(x),\delta_\lambda(y))=\delta_\lambda P(x,y)$.*
(iii) *$P(x,0)=P(0,x)=x$ for all $x\in\mathbb{R}^n$.*
(iv) *$Q_1=...=Q_{m_1}=0$ and $Q_i(x,y)$ for $i>m_1$ is a sum of terms each of which contains a factor $(x_jy_l-x_ly_j)$ for some $1\le j,l<i$.*
(v) *If the coordinate x_i has degree $d_i\ge 2$ then $Q_i(x,y)$ depends only on the coordinate of x and y which have degree strictly less then d_i.*

Proof. Property (i) is a consequence of the fact that $P(X,Y)=0$ if and only if $X=-Y$. Property (ii) simply states that dilations are group automomorphisms.

We prove (iv). Fix the basis $(X_1,...,X_n)$ of $\mathfrak{g}=V_1\oplus...\oplus V_r$, $r\ge 2$ being the step of the group, which gives the exponential coordinates in $\mathbb{R}^n$. $(X_1,...,X_{m_1})$ is a basis of V_1. Let $x,y\in\mathbb{R}^n$ and consider

$$X=\sum_{i=1}^n x_iX_i \quad \text{and} \quad Y=\sum_{i=1}^n y_iX_i.$$

By definition, $x\cdot y=z$ if

$$P(X,Y)=\sum_{i=1}^n z_iX_i,$$

where $P(X,Y)$ is given by the Campbell-Hausdorff formula

$$P(X,Y)=\sum_{k=1}^r\frac{(-1)^{k+1}}{k}\sum_{\alpha_j+\beta_j\ge 1}c_{\alpha\beta}D_{\alpha\beta}(X,Y),$$

where the multi-indeces α and β in the inner sum have length k, $c_{\alpha\beta}$ and $D_{\alpha\beta}(X,Y)$ are as in (1.5.45) and (1.5.44). $D_{\alpha\beta}(X,Y)$ is a commutator of the vector fields X and

Y and has a length that depends on α and β. The sum of the $D_{\alpha\beta}(X,Y)$ with length 1 gives $X+Y$.

By induction on the length $h \geq 2$ of $D_{\alpha\beta}(X,Y)$ we prove that for all multi-indeces α and β

$$D_{\alpha\beta}(X,Y) = \sum_{i=m_1+1}^{n} p^i_{\alpha\beta}(x,y)X_i$$

where each $p^i_{\alpha\beta}(x,y)$ is a polynomial which can be decomposed in a sum of terms each of which contains a factor $(x_jy_l - x_ly_j)$ for some $1 \leq j,l \leq n$. This proves statement (iv) because

$$Q_i(x,y) = \sum_{k=1}^{r} \frac{(-1)^{k+1}}{k} \sum_{\alpha_j+\beta_j \geq 1} c_{\alpha\beta}p^i_{\alpha\beta}(x,y).$$

Notice that there can actually appear only terms $(x_jy_l - x_ly_j)$ with $1 \leq j,l < i$ because $Q_i(x,y)$ is homogeneous of degree d_i with respect to dilations (1.7.82). (This remark also proves (v)).

We prove the inductive base. If $h = 2$ then $D_{\alpha\beta}(X,Y)$ can be assumed to be the form

$$[X,Y] = \sum_{j,l=1}^{n} x_jy_l[X_j,X_l] = \sum_{1\leq j<l\leq n} (x_jy_l - x_ly_j)[X_j,X_l],$$

with $[X_j,X_l] \in V_2 \oplus ... \oplus V_r$, and the inductive base is proved.

If $D_{\alpha\beta}(X,Y)$ is a commutator of length h then we can assume $D_{\alpha\beta}(X,Y) = [X, D_{\bar\alpha\bar\beta}(X,Y)]$ for some multi-indeces $\bar\alpha$ and $\bar\beta$ such that $D_{\bar\alpha\bar\beta}(X,Y)$ is a commutator of length $h-1$. By the inductive hypothesis

$$[X, D_{\bar\alpha\bar\beta}(X,Y)] = \sum_{j=1}^{n}\sum_{i=m_1+1}^{n} x_jp^i_{\bar\alpha\bar\beta}(x,y)[X_j,X_i],$$

with $[X_j,X_i] \in V_2 \oplus ... \oplus V_r$. The inductive step is proved because $x_jp^i_{\bar\alpha\bar\beta}(x,y)$ has the required property.

□

7.4. Left invariant vector fields. The Lie algebra $\mathfrak{g}$ of a nilpotent Lie group structure on $\mathbb{R}^n$ can be thought of as an algebra of left invariant differential operators in $\mathbb{R}^n$ with respect to the group law.

Let $X_1,...,X_n$ be a vector basis of $\mathfrak{g}$, write

$$X_j(x) = \sum_{i=1}^{n} a_{ij}(x)\partial_i, \quad j = 1,...,n,$$

and assume $X_j(0) = \partial_j$. The coefficients $a_{ij} \in C^\infty(\mathbb{R}^n)$ and the product $x \cdot y = P(x,y)$ are linked in the following way. Let $\gamma : (-\delta,\delta) \to \mathbb{R}^n$ be a C^1 curve such that $\gamma(0) = 0$ and $\dot\gamma(0) = \partial_j$. Since X_j is left invariant, if $f \in C^1(\mathbb{R}^n)$ then

$$\begin{aligned} X_jf(x) &= X_j(f\circ\tau_x)(0) = \lim_{t\to 0}\frac{f(P(x,\gamma(t))) - f(P(x,0))}{t} \\ &= \frac{\partial f}{\partial x}(x)\frac{\partial P}{\partial y}(x,0)\dot\gamma(0) = \frac{\partial f}{\partial x}(x)\frac{\partial P}{\partial y_j}(x,0). \end{aligned}$$

The vector fields have polynomial coefficients $a_{ij}(x)$, and precisely

$$X_j(x) = \sum_{i=1}^{n} \frac{\partial P_i}{\partial y_j}(x,0)\partial_i. \tag{1.7.83}$$

As a consequence the following homogeneity property holds

$$a_{ij}(\delta_\lambda(x)) = \lambda^{d_i - d_j} a_{ij}(x), \tag{1.7.84}$$

where d_i and d_j are the degrees of x_i and x_j, respectively.

7.5. Carnot groups as C-C spaces. Let $(\mathbb{R}^n, \cdot)$ be a Carnot structure on $\mathbb{R}^n$ with stratified algebra $\mathfrak{g} = V_1 \oplus ... \oplus V_k$, $k \geq 2$. Let $m = m_1 = \dim(V_1)$ and fix a basis $X = (X_1, ..., X_m)$ of V_1. Chow-Hörmander condition (1.5.41) is verified and X induces a C-C metric d on $\mathbb{R}^n$. By Theorem 1.5.1 for any compact set $K \subset \mathbb{R}^n$ there exists $C > 0$ such that $d(x,y) \leq C|x-y|^{1/k}$ for all $x, y \in K$. Such estimate can be improved in the following way.

Proposition 1.7.3. *For all $x, y, h \in \mathbb{R}^n$ and $\lambda > 0$*
(i) *$d(\tau_h(x), \tau_h(y)) = d(x,y)$;*
(ii) *$d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x,y)$.*

Proof. Statement (i) follows from the fact that $\gamma : [0,T] \to \mathbb{R}^n$ is a subunit curve joining x to y if and only if $\tau_h(\gamma)$ ia a subunit curve joining $\tau_h(x)$ to $\tau_h(y)$.

We prove (ii). Let $\gamma : [0,T] \to \mathbb{R}^n$ be a subunit curve joining x to y

$$\dot\gamma(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)) = \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} h_j(t) a_{ij}(\gamma(t)) \Big) \partial_i.$$

Define $\gamma_\lambda : [0, \lambda T] \to \mathbb{R}^n$ by $\gamma_\lambda(t) = \delta_\lambda(\gamma(t/\lambda))$. Then, by (1.7.84) with $d_j = 1$ if $j = 1, ..., m$

$$\begin{aligned}\dot\gamma_\lambda(t) &= \sum_{i=1}^{n} \lambda^{d_i - 1} \Big(\sum_{j=1}^{m} h_j(t/\lambda) a_{ij}(\gamma(t/\lambda)) \Big) \partial_i \\ &= \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} h_j(t/\lambda) a_{ij}(\gamma_\lambda(t)) \Big) \partial_i = \sum_{j=1}^{m} h_j(t/\lambda) X_j(\gamma_\lambda(t)).\end{aligned}$$

As $\gamma_\lambda(0) = \delta_\lambda(x)$, $\gamma_\lambda(\lambda T) = \delta_\lambda(y)$ and γ_λ is subunit it follows that $d(\delta_\lambda(x), \delta_\lambda(y)) \leq \lambda T$. Being γ arbitrary $d(\delta_\lambda(x), \delta_\lambda(y)) \leq \lambda d(x,y)$ and the converse inequality can be obtained in the same way. $\square$

A C-C ball centered at $x \in \mathbb{R}^n$ with radius $r \geq 0$ will be denoted be $B(x,r)$. Recall that $d_i \geq 1$ is the degree of the coordinate x_i. If $x \in \mathbb{R}^n$ introduce the homogeneous norm

$$\|x\| := \sum_{i=1}^{n} |x_i|^{1/d_i}, \tag{1.7.85}$$

and define the Box

$$\mathrm{Box}(x,r) = \{x \cdot z \in \mathbb{R}^n : \|z\| \leq r\}. \tag{1.7.86}$$

Proposition 1.7.4.

(i) *For all* $x, h \in \mathbb{R}^n$ *and* $r, \lambda > 0$ *we have* $\tau_h B(x,r) = B(\tau_h(x), r)$ *and* $\delta_\lambda B(x,r) = B(\delta_\lambda(x), \lambda r)$.

(ii) *Moreover, there exist* $0 < c_1 < c_2$ *such that* $\mathrm{Box}(x, c_1 r) \subset B(x,r) \subset \mathrm{Box}(x, c_2 r)$ *for all* $x \in \mathbb{R}^n$ *and* $r \geq 0$.

Proof. Statement (i) is a corollary of Proposition 1.7.3. We prove (ii). By compactness there exist $0 < q_1 < q_2$ such that $q_1 \leq \|x\| \leq q_2$ for all $x \in \mathbb{R}^n$ such that $d(x, 0) = 1$. Since $\|\delta_\lambda(x)\| = \lambda\|x\|$ we immediately find $q_1 d(x,0) \leq \|x\| \leq q_2 d(x,0)$ for all $x \in \mathbb{R}^n$ and consequently

$$q_1 d(x,y) \leq \|y^{-1} \cdot x\| \leq q_2 d(x,y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

Since $y \in \mathrm{Box}(x,r)$ if and only if $\|y^{-1} \cdot x\| \leq r$ the claim follows. □

Corollary 1.7.5. *The metric space* $(\mathbb{R}^n, d)$ *is complete and locally compact. Geodesics exist globally.*

Definition 1.7.6. Let $\mathfrak{g} = V_1 \oplus ... \oplus V_k$ be a stratified Lie algebra. Its *homogeneous dimension* is

$$Q = \sum_{i=1}^{k} i \dim(V_i). \tag{1.7.87}$$

If $n \in \mathbb{N}$ is the dimension of $\mathfrak{g}$ as vector space then $Q \geq n$. If $(\mathbb{R}^n, \cdot)$ is the Carnot group associated with the Lie algebra $\mathfrak{g}$ we shall say that its homogeneous dimension is Q.

Proposition 1.7.7. *If* $E \subset \mathbb{R}^n$ *is a Lebesgue measurable set* $|x \cdot E| = |E \cdot x| = |E|$ *and* $|\delta_\lambda E| = \lambda^Q |E|$ *for all* $x \in \mathbb{R}^n$ *and* $\lambda \geq 0$. *Moreover,* $|B(x,r)| = r^Q |B(0,1)|$ *for all* $x \in \mathbb{R}^n$ *and* $r \geq 0$.

Proof. Let $d\tau_x$ and $d\delta_\lambda$ be the differentials of τ_x and δ_λ. The first two statements are a straightforward consequence of $\det(d\tau_x) = 1$ and $\det(\delta_\lambda) = \lambda^Q$. Moreover

$$|B(x,r)| = |\delta_r \tau_x B(0,r)| = r^Q |\tau_x B(0,r)| = r^Q |B(0,1)|.$$

□

8. Heisenberg Group

8.1. Introduction. The Heisenberg group is the most simple non commutative Carnot group and is a privileged object of study in Analysis and Geometry. Consider a $(2n+1)-$dimendional real Lie algebra $\mathfrak{g}$ with basis $\{X_1, ..., X_n, Y_1, ..., Y_n, T\}$ and assume that the non vanishing commutation relations are only

$$[X_i, Y_i] = -4T \quad i = 1, ..., n. \tag{1.8.88}$$

The algebra is stratified $\mathfrak{g} = V_1 \oplus V_2$ with $V_1 = \mathrm{span}\{X_1, ..., X_n, Y_1, ..., Y_n\}$ and $V_2 = \mathrm{span}\{T\}$.

Using exponential coordinates the Lie algebra $\mathfrak{g}$ induces a Lie group structure on $\mathbb{R}^{2n+1}$ as in (1.7.81). Identify $\mathbb{R}^{2n+1} \equiv \mathbb{C}^n \times \mathbb{R}$ and $(x, y, t) \equiv (z, t)$ with $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}^n$. As in Example 1.7.1 using the relations (1.8.88) in the Campbell-Hausdorff formula (1.5.47) (which now is particularly simple being the algebra of step 2) it can be found the group law

$$\begin{aligned}(z,t)\cdot(\zeta,\tau) &= (z+\zeta, t+\tau+2\mathrm{Im}(z\bar{\zeta}))\\ &= \big(z+\zeta, t+\tau+2(\langle y,\xi\rangle - \langle x,\eta\rangle)\big),\end{aligned} \tag{1.8.89}$$

where $z = x+iy$ and $\zeta = \xi + i\eta$. Notice that, denoting by

$$\mathcal{I} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

the unit symplectic matrix, we can also write

$$\langle y,\xi\rangle - \langle x,\eta\rangle = \sum_{j=1}^{n}(y_j\xi_j - x_j\eta_j) = \langle z, \mathcal{I}\zeta\rangle.$$

The center of the group is $Z = \{(z,t) \in \mathbb{R}^{2n+1} : z = 0\}$. Homogeneous dilations $\delta_\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$, $\lambda > 0$, are

$$\delta_\lambda(z,t) = (\lambda z, \lambda^2 t), \tag{1.8.90}$$

and the homogeneous dimension is $Q = 2n+2$.

The Heisenberg Lie algebra can be realized as an algebra of left invariant differential operators on $\mathbb{R}^{2n+1}$. Using formula (1.7.83) it can be found $T = \partial_t$ and

$$X_j = \partial_{x_j} + 2y_j\partial_t, \quad Y_j = \partial_{y_j} - 2x_j\partial_t, \quad j = 1,...,n. \tag{1.8.91}$$

These vector fields clearly satisfy the Chow-Hörmander condition, and a left invariant C-C metric is induced on $\mathbb{R}^{2n+1}$. A ball $B(0,r)$ centered at the origin and with radius $r \geq 0$ behaves like the box

$$\mathrm{Box}(0,r) = \{(z,t) \in \mathbb{R}^{2n+1} : |z| \leq r \text{ and } |t| \leq r^2\}.$$

The Heisenberg group is denoted by $\mathbb{H}^n$. The Heisenberg gradient will be written as

$$\nabla_{\mathbb{H}} = (X_1,...,X_n,Y_1,...,Y_n).$$

8.2. Geodesics in the Heisenberg group. The Heisenberg group endowed with its left invariant C-C metric is a locally compact metric space and geodesics exist globally (Corollary 1.7.5). We shall compute them explicitly.

LEMMA 1.8.1. *Geodesics in $\mathbb{H}^n$ are normal.*

Proof. Let $\mathcal{A} = \mathrm{col}\,[X_1,...,X_n,Y_1,...,Y_n]$ be the matrix of the vector fields (1.8.91) as in (1.1.1), and let $\gamma : [0,1] \to \mathbb{H}^n$ be a geodesic. Then

$$\dot\gamma(s) = \mathcal{A}(\gamma(s))h(s) \quad \text{ for a.e. } s \in [0,1], \tag{1.8.92}$$

with $h = (h_1,h_2)$ and $h_1,h_2 \in \mathrm{L}^2(0,1)^n$. Write $\gamma(s) = (x(s),y(s),t(s))$.

We have to check that the case $\lambda = 0$ in Theorem 1.4.7 may not occur. Since

$$\mathcal{A}h = \sum_{j=1}^{n} h_{1j}X_j + h_{2j}Y_j = (h_1,h_2,2\langle z,\mathcal{I}h\rangle),$$

equations (ii) in Theorem 1.4.7 are

$$\begin{cases} \dot{\xi} = 2\beta h_2 \\ \dot{\eta} = -2\beta h_1 \\ \dot{\tau} = 0, \end{cases} \tag{1.8.93}$$

and therefore $\tau(s) = \beta \in \mathbb{R}$. Suppose by contradiction that the optimal pair $(\gamma(s), h(s))$ corresponds to $\lambda = 0$. From (1.4.39)

$$\begin{cases} \xi + 2\beta y = 0 \\ \eta - 2\beta x = 0. \end{cases} \tag{1.8.94}$$

If $\beta = 0$ then $\xi(s) \equiv \eta(s) \equiv 0$ and this is not possible because of (i) in Theorem 1.4.7. Then $\beta \neq 0$. Differentiating equations (1.8.94), replacing the result in (1.8.93) and simplifying $2\beta \neq 0$ we get

$$\begin{cases} \dot{x}(s) = -h_1(s) \\ \dot{y}(s) = -h_2(s). \end{cases}$$

But from (1.8.92)

$$\begin{cases} \dot{x}(s) = h_1(s) \\ \dot{y}(s) = h_2(s) \end{cases}$$

and thus $h = 0$ almost everywhere. This is not possible unless γ is a constant curve. $\square$

By Lemma 1.8.1 and Proposition 1.4.9 geodesics in $\mathbb{H}^n$ can be found solving the system of Hamilton equations (1.4.37) with Hamiltonian

$$H((z,t),(\zeta,\tau)) = \sum_{j=1}^{n} (\xi_j + 2y_j\tau)^2 + (\eta_j - 2x_j\tau)^2 = |\zeta|^2 + 4\tau^2|z|^2 + 4\tau\langle z, \mathcal{I}\zeta\rangle. \tag{1.8.95}$$

Translations of geoedsics are still geoesics since the metric is left invariant. Therefore, it is enough to study geoedsics starting from the origin. Equations (1.4.37) give

$$\begin{cases} \dot{x} = \xi + 2\tau y & x(0) = 0 \\ \dot{y} = \eta - 2\tau x & y(0) = 0 \\ \dot{t} = 4\tau|z|^2 + 2\langle \mathcal{I}z, \zeta\rangle & t(0) = 0 \\ \dot{\xi} = 2\tau\eta - 4\tau^2 x & \xi(0) = B \\ \dot{\eta} = -2\tau\xi - 4\tau^2 y & \eta(0) = A \\ \dot{\tau} = 0 & \tau(0) = \varphi/4, \end{cases}$$

where $A, B \in \mathbb{R}^n$ and $\varphi \in \mathbb{R}$. The choice ensuring the arclength parametrization turns out to be $|A|^2 + |B|^2 = 1$ (see (1.8.97) below). The solutions are (we are not interested in the dual curve)

$$\begin{cases} x(s) = \dfrac{A(1-\cos\varphi s) + B\sin\varphi s}{\varphi} \\ y(s) = \dfrac{-B(1-\cos\varphi s) + A\sin\varphi s}{\varphi} \\ t(s) = 2\dfrac{\varphi s - \sin\varphi s}{\varphi^2}. \end{cases} \tag{1.8.96}$$

In the limit case $\varphi = 0$ one gets $(x(s), y(s), t(s)) = (Bs, As, 0)$. Finally, notice that

$$\dot\gamma = \sum_{j=1}^{n} (A_j \sin\varphi s + B_j \cos\varphi s) X_j(\gamma) + (A_j \cos\varphi s - B_j \sin\varphi s) Y_j(\gamma). \tag{1.8.97}$$

If $(z,t) \notin Z$, that is $z \neq 0$, then (z,t) and $(0,0)$ can be connected only by one geodesic. On the other side $(0,0)$ and $(0,t)$, $t \neq 0$, can by connected by a continuous family of geodesics even if t is small. Differently from the Riemannian case geodesics in C-C spaces are not locally unique.

8.3. Heisenberg ball. From (1.8.96) a parametrization of the surface of the unitary metric ball centered at the origin can be easily obtained. For the sake of simplicity take $n = 1$ and let $S = \{(x,y,t) \in \mathbb{H}^1 : d((x,y,t),0) = 1\}$.

If in (1.8.96) we choose $s = 1$, $A = \cos\vartheta$ and $B = \sin\vartheta$ we obtain the parametric equations for S

$$\begin{cases} x(\vartheta,\varphi) = \dfrac{\cos\vartheta(1-\cos\varphi) + \sin\vartheta\sin\varphi}{\varphi} \\ y(\vartheta,\varphi) = \dfrac{-\sin\vartheta(1-\cos\varphi) + \cos\vartheta\sin\varphi}{\varphi} \\ t(\vartheta,\varphi) = 2\dfrac{(\varphi - \sin\varphi)}{\varphi^2}, \end{cases} \tag{1.8.98}$$

with $0 \le \vartheta \le 2\pi$ and $-2\pi \le \varphi \le 2\pi$. The surface S is of class C^∞ where $z \neq 0$.

REMARK 1.8.2. The singular antipodal points of the surface S, which have coordinates $(0,0,\pm 1/\pi)$, are Lipschitz points. Indeed, solving $y(\vartheta,\varphi) = 0$ we find

$$\vartheta(\varphi) = \arctan\Big(\frac{\sin\varphi}{1-\cos\varphi}\Big) \quad \text{and} \quad x(\vartheta(\varphi),\varphi) = \frac{\sqrt{2-2\cos\varphi}}{\varphi}.$$

The path $\gamma : [0,2\pi] \to \mathbb{H}^1$ defined by

$$\gamma(\varphi) = \Big(\frac{\sqrt{2-2\cos\varphi}}{\varphi}, 0, 2\frac{\varphi - \sin\varphi}{\varphi^2}\Big)$$

lies in $\partial B(0,1) \cap \{(x,y,t) \in \mathbb{R}^3 : y = 0\}$ and joins the point $(1,0,0)$ to the "north pole" $(0,0,\frac{1}{\pi})$. Its derivative at $\varphi = 2\pi$ is

$$\dot\gamma(2\pi) = \Big(-\frac{1}{2\pi}, 0, -\frac{1}{\pi^2}\Big).$$

This shows that we can put on the "north pole" outside the unitary ball a cone with angular opening $2\arctan(\pi/2)$. □

9. Grushin space

9.1. Grushin metrics. In this section we analyze the C-C metric induced by a family of vector fields not of Hörmander type, metric that has been introduced in [**76**] and [**77**]. We consider $\mathbb{R}^n$ with $n \ge 2$ and fix $1 \le m \le n-1$. We shall write $(x,y) \in \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^n$, $k = n - m$. Consider

$$X_1 = \partial_{x_1}, ..., X_m = \partial_{x_m}, \quad Y_1 = |x|^\alpha \partial_{y_1}, ..., Y_k = |x|^\alpha \partial_{y_k}, \tag{1.9.99}$$

where $\alpha > 0$. If α is a positive even integer the Hörmander condition (1.5.41) can by checked but the C-C metric is defined for any $\alpha > 0$ and is finite because every couple of points in $\mathbb{R}^n$ can be connected by polygonals piecewise integral curves of the vector fields. We shall call the induced C-C metric d on $\mathbb{R}^n$ the Grushin metric.

The metric is invariant with respect to translations in the y variable, precisely for any $x, \xi \in \mathbb{R}^m$ and $y, \eta, h \in \mathbb{R}^k$

$$d((x, y+h), (\xi, \eta+h)) = d((x, y), (\xi, \eta)). \tag{1.9.100}$$

Introduce the one parameter group of dilations $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y)$ for $\lambda > 0$. The distance from the origin is $1-$homogeneus with respect to such dilations. Precisely, if $(x, y) \in \mathbb{R}^n$ and $\lambda > 0$ then $d(0, \delta_\lambda(x, y)) = \lambda d(0, (x, y))$. Indeed, if $\gamma : [0, T] \to \mathbb{R}^n$ is a subunit curve such that $\gamma(0) = 0$ and $\gamma(T) = (x, y)$ the curve $\gamma_\lambda : [0, \lambda T] \to \mathbb{R}^n$ defined by $\gamma_\lambda(t) = \delta_\lambda \gamma(t/\lambda)$ is subunit and joins 0 to $\delta_\lambda(x, y)$.

Grushin metric can be estimated more explicitly on the whole space by means of polygonal integral curves. The following proposition is a special case of [**76**] (see also [**72**]).

Proposition 1.9.1. *Assume $k = 1$ and let $\lambda > 0$. There exists $c \geq 1$ such that for all $P = (x, y)$ and $Q = (\xi, \eta) \in \mathbb{R}^n$ with $|x| \geq |\xi|$*

$$d(P, Q) \leq |x - \xi| + \frac{|y - \eta|}{|x|^\alpha} \leq c\, d(P, Q) \quad \text{if} \quad |x|^{\alpha+1} \geq \lambda |y - \eta|,$$

$$\frac{1}{3} d(P, Q) \leq |x - \xi| + |y - \eta|^{\frac{1}{\alpha+1}} \leq c\, d(P, Q) \quad \text{if} \quad |x|^{\alpha+1} < \lambda |y - \eta|.$$

Proof. Without loss of generality we consider the case $m = 1$ (i.e. $n = 2$). Thanks to (1.9.100) we can suppose $\eta = 0$. Assume moreover $x \geq 0$ and $y \geq 0$.

Since $d((\xi, 0), (x, 0)) = |x - \xi|$ and $|x| \geq |\xi|$, we have to estimate the distance between $(x, 0)$ and (x, y). If $h \geq 0$ set $T(h) := 2h + \frac{y}{(x+h)^\alpha}$ and consider the subunit curve $\gamma_h : [0, T(h)] \to \mathbb{R}^2$

$$\gamma_h(t) = \begin{cases} (x + t, 0) & \text{if } t \in [0, h) \\ (x + h, (x + h)^\alpha t) & \text{if } t \in [h, T(h) - h] \\ (x + h - t, y) & \text{if } t \in (T(h) - h, T(h)]. \end{cases}$$

Notice that $\gamma_h(0) = (x, 0)$ and $\gamma_h(T(h)) = (x, y)$. If $x^{\alpha+1} \geq \lambda y$ choose $h = 0$. By definition (1.1.4)

$$d((x, 0), (x, y)) \leq T(0) = \frac{y}{x^\alpha}.$$

If $x^{\alpha+1} < \lambda y$ choose $h = y^{\frac{1}{\alpha+1}} - x$ (which amounts to solve $T'(h) = 0$ up to a constant before y). Then

$$d((x, 0), (x, y)) \leq T(h) = 2\big(y^{\frac{1}{\alpha+1}} - x\big) + \frac{y}{y^{\frac{\alpha}{\alpha+1}}} \leq 3 y^{\frac{1}{\alpha+1}}.$$

Consider now any subunit curve $\gamma = (\gamma_1, \gamma_2) : [0, T] \to \mathbb{R}^2$ joining $(x, 0)$ to (x, y)

$$\begin{cases} \gamma_1(t) = x + \displaystyle\int_0^t h_1(\tau)\, d\tau \\ \gamma_2(t) = \displaystyle\int_0^t |\gamma_1(\tau)|^\alpha h_2(\tau)\, d\tau, \end{cases}$$

with $\gamma_2(T) = y$, and $h_1(t)^2 + h_2(t)^2 \le 1$ for a.e. $t \in [0,T]$. We assume $h_1(t) = h_2(t) = 1$, weakening in this way the subunit condition. Moreover, by symmetry we can reduce to the case $\gamma(T/2) = y/2$. Thus

$$\frac{y}{2} = \int_0^{T/2} (x+\tau)^\alpha \, d\tau = \frac{1}{\alpha+1}\big((x+T/2)^{\alpha+1} - x^{\alpha+1}\big),$$

and writing $T = xS$

$$\frac{(\alpha+1)y}{2x^{\alpha+1}} = \big((1+S/2)^{\alpha+1} - 1\big). \tag{1.9.101}$$

If $x^{\alpha+1} \ge \lambda y$ the left hand side of (1.9.101) is bounded from above by $(\alpha+1)/(2\lambda)$ and the solution S of the equation has to satisfy $(1+S/2)^{\alpha+1} \le (1+qS)$ for some $q > 0$ depending on α and λ. Thus $T = xS \ge \frac{(\alpha+1)y}{2qx^\alpha}$. As γ is arbitrary it follows that

$$d((x,0),(x,y)) \ge \frac{(\alpha+1)y}{2qx^\alpha}.$$

We consider the case $x^{\alpha+1} < \lambda y$. From (1.9.101)

$$\frac{S}{2} = \Big(\frac{(\alpha+1)y}{2x^{\alpha+1}} + 1\Big)^{\frac{1}{\alpha+1}} - 1 = \frac{y^{\frac{1}{\alpha+1}}}{x}\Big[\Big(\frac{(\alpha+1)}{2} + \frac{x^{\alpha+1}}{y}\Big)^{\frac{1}{\alpha+1}} - \frac{x}{y^{\frac{1}{\alpha+1}}}\Big] \ge \bar q \frac{y^{\frac{1}{\alpha+1}}}{x},$$

for some $\bar q > 0$ depending on α and λ. Whence $T = xS \ge 2\bar q y^{\frac{1}{\alpha+1}}$. As γ is arbitrary it follows that

$$d((x,0),(x,y)) \ge 2\bar q y^{\frac{1}{\alpha+1}}.$$

$\square$

REMARK 1.9.2. If $k > 1$ Proposition 1.9.1 still holds, but 2^k cases should be distinguished according to that $|x|^{\alpha+1} \ge \lambda|y_i - \eta_i|$ or $|x|^{\alpha+1} < \lambda|y_i - \eta_i|$ for $i = 1, ..., k$.

Grushin balls can be represented by suitable boxes. If $(x,y) \in \mathbb{R}^n$ and $r \ge 0$ define

$$\mathrm{Box}((x,y),r) = \{(\xi,\eta) \in \mathbb{R}^n : |x-\xi| \le r \text{ and } |y-\eta| \le r(|x|+r)^\alpha\}. \tag{1.9.102}$$

PROPOSITION 1.9.3. *There exist constants* $0 < c_1 < c_2$ *such that for all* $(x,y) \in \mathbb{R}^n$ *and* $r \ge 0$

$$\mathrm{Box}((x,y),c_1 r) \subset B((x,y),r) \subset \mathrm{Box}((x,y),c_2 r). \tag{1.9.103}$$

Proof. It follows from Proposition 1.9.1. $\square$

COROLLARY 1.9.4. *The metric space* $(\mathbb{R}^n, d)$ *is locally compact and complete. Geodesics exist globally.*

Proof. By Proposition 1.9.1 the metric d is continuous in the Euclidean topology and from Proposition 1.4.3 it follows that $(\mathbb{R}^n, d)$ is locally compact. By Proposition 1.9.3 closed C-C balls are bounded and thus compact (the topology of $(\mathbb{R}^n, d)$ is the Euclidean topology) and from Theorem 1.4.4 geodesics exist globally. $\square$

9.2. Geodesics and balls. Our next task is to compute geodesics solving Hamilton equations (1.4.37). Geodesics in Grushin spaces have been first studied in [**70**]. We have to check that singular geodesics may not occur. We are interested in geodesics starting from $(0,0) \in \mathbb{R}^n$. The Hamiltonian is

$$H((x,y),(\xi,\eta)) = |\xi|^2 + |x|^{2\alpha}|\eta|^2,$$

where $x, \xi \in \mathbb{R}^m$ and $y, \eta \in \mathbb{R}^k$. By Proposition 1.4.9 singular geodesics should verify $H((x(t),y(t)),(\xi(t),\eta(t))) \equiv 0$, being $\gamma(t) = (x(t),y(t))$ the geodesic and $(\xi(t),\eta(t))$ the dual curve given by Theorem 1.4.7. Then $|\xi(t)| \equiv 0$ and since $|\eta(t)| \neq 0$ (by condition (i) in Theorem 1.4.7), it follows that $|x(t)| \equiv 0$. This implies that if $\gamma(0) = (0,0)$ then $\gamma(t) = (0,0)$ for all $t \geq 0$. This is not possible and geodesic must be normal.

We find the system of equations

$$\begin{cases} \dot{x} = \xi, & \dot{\xi} = -\alpha|\eta|^2|x|^{2\alpha-2}x, \\ \dot{y} = |x|^{2\alpha}\eta, & \dot{\eta} = 0. \end{cases}$$

Fix the initial data $x(0) = y(0) = 0$, $\xi(0) = \xi$ and $\eta(0) = \eta$. We look for a solution $x(t) = \varphi(t)\xi$ for some real function $\varphi \geq 0$ such that $\varphi(0) = 0$ and $\dot{\varphi}(0) = 1$. One finds the equation $\ddot{\varphi} + \alpha|\eta|^2\varphi^{2\alpha-1} = 0$ which can be explicitly solved, for example, if $\alpha = 1$. In this case the solution is

$$\varphi(t) = \frac{\sin(|\eta|t)}{|\eta|}, \quad 0 \leq t \leq \frac{\pi}{|\eta|},$$

and geodesics are $\gamma(t) = (x(t),y(t))$ where

$$x(t) = \frac{\sin(|\eta|t)}{|\eta|}\xi, \quad y(t) = \frac{|\xi|^2}{|\eta|^2}\Big(\frac{2|\eta|t - \sin(2|\eta|t)}{4}\Big)\frac{\eta}{|\eta|}, \quad 0 \leq t \leq \frac{\pi}{|\eta|}. \tag{1.9.104}$$

Notice that

$$\dot{\gamma}(t) = \cos(|\eta|t)\sum_{j=1}^{m}\xi_j X_j(\gamma(t)) + \frac{|\xi|\sin(|\eta|t)}{|\eta|}\sum_{i=1}^{k}\eta_i Y_i(\gamma(t)),$$

and therefore γ is parametrized by arclength if $|\xi| = 1$.

Take $m = 2$ and $k = 1$. Write $\xi = (\cos(\varphi),\sin(\varphi))$ and $\eta = \vartheta$. The boundary of the Grushin ball $B((0,0),r) \subset \mathbb{R}^3$, $r > 0$, has the parametrization

$$\begin{cases} x_1(\varphi,\vartheta) = \dfrac{\sin(\vartheta r)}{\vartheta}\cos(\varphi) \\ x_2(\varphi,\vartheta) = \dfrac{\sin(\vartheta r)}{\vartheta}\sin(\varphi) \\ y_1(\varphi,\vartheta) = \dfrac{2\vartheta r - \sin(2\vartheta r)}{4\vartheta^2}, \end{cases} \tag{1.9.105}$$

where $0 \leq \varphi < 2\pi$ and $-\pi \leq r\vartheta \leq \pi$. The parametrization is smooth except that in the "north pole" $(0,0,r^2/(2\pi))$ which is a Lipschitz point.

Take $m = 1$ and $k = 2$. Write $\eta = (\varrho\cos(\vartheta), \varrho\sin(\vartheta))$. The boundary of the Grushin ball $B((0,0),r) \subset \mathbb{R}^3$, $r > 0$, has the parametrization

$$\begin{cases} x_1(\varrho,\vartheta) = \pm\dfrac{\sin(\varrho r)}{\varrho} \\ y_1(\varrho,\vartheta) = \dfrac{2\varrho r - \sin(2\varrho r)}{4\varrho^2}\cos(\vartheta), \\ y_2(\varrho,\vartheta) = \dfrac{2\varrho r - \sin(2\varrho r)}{4\varrho^2}\sin(\vartheta), \end{cases} \tag{1.9.106}$$

where $0 \le \vartheta < 2\pi$ and $0 \le \varrho \le \pi/r$ The parametrization is smooth except that in circular section in the plane $\{x_1 = 0\}$.

10. References and comments

The metric d was introduced in [**77**] and [**64**] to study second order elliptic degenerate differential equations but the construction is usually attributed to Carathéodory [**45**]. The Hölder estimate (1.5.53) of the metric d is deeply linked with the theory of subelliptic operators. The second order differential operator naturally associated with the (selfadjoint) vector fields $X_1, ..., X_m$ is the sum of squares Laplacian

$$\mathcal{L} = -\sum_{j=1}^{m} X_j^2.$$

Fefferman and Phong proved in [**64**] that the Hölder estimate (1.5.53) of exponent $1/k$ for the distance d induced by the vector fields is in fact equivalent to the subelliptic estimate

$$\|u\|_{H^\varepsilon} \le C_K(\langle\mathcal{L}u, u\rangle + \|u\|_2)$$

for all $u \in C_0^1(K)$, $K \subset \mathbb{R}^n$ compact set, where $0 < \varepsilon < 1/k$. Actually, the subelliptic estimate holds for the sharp exponent $\varepsilon = 1/k$ [**117**].

Other different but equivalent definitions of d can be found in [**151**]. C-C spaces have also been extensively studied from the geometric point of view as sub-Riemannian spaces (see for instance [**94**] and the book [**21**] where an extensive bibliography can be found). Proposition 1.1.4 is well known (see for example [**100**]), while in the proof of Theorem 1.1.6 we essentially followed [**111**].

A study of rectifiable curves in Euclidean spaces can be found in [**62**] and for general metric spaces in [**6**]. In this latter book Theorem 1.3.1 is proved. In the proof of Proposition 1.3.3 we essentially followed [**100**]. A general theorem of existence of length minimizing curves in compact metric spaces is proved in [**6**]. Geodesics in C-C spaces have been explicitly computed in [**88**], [**31**], [**70**] and general references to the subject are [**165**], [**142**], [**143**], [**98**], [**1**]. In the monograph [**124**] singular geodesics in the case of rank 2 distributions (sub-Riemannian spaces with metric induced by 2 vector fields) are extensively studied. The most general condition known to rule out singular geodesics is the "strong bracket generating hypothesis" introduced in [**165**] but it applies only to a subclass of vector fields of step 2.

Carnot groups are well known in Harmonic Analysis and in the study of hypoelliptic differential operators as nilpotent or homogeneous groups ([**67**], [**156**] and [**164**]). Many topics in Analysis in groups are dealt with in [**172**]. A beautiful introduction to the Heisenberg group are chapters XII and XIII of [**164**], where particular attention is paid to the links with complex analysis and partial differential equations. Geodesics in the Heisenberg group were first computed in [**31**]. The shape of the Heisenberg ball was studied in [**144**] in order to show that it is not isoperimetric.

CHAPTER 2

Differentiability of Lipschitz maps and eikonal equation for distance functions

In this chapter we study different notions of differentiability of Lipschitz maps and the eikonal equation for C-C metrics. Differentiability of Lipschitz maps in metric spaces is a topic that seems to arouse an increasing interest (see, for instance, [**48**]). As far as C-C spaces in concerned a classical theorem due to P. Pansu [**153**] states that Lipschitz maps between Carnot groups have a differential which is a homogeneous homomorphism. In section 1 we follow the original proof except that in the one dimensional reduction step which has been shortened (see Lemma 2.1.4). Pansu's proof works when the map is defined in an open set. However, the theorem still holds for Lipschitz maps defined on a measurable set (see [**175**] and [**129**]). A weaker but more general result is the differentiability in sense of distribution of real valued Lipschitz functions in quite arbitrary C-C spaces which has been proved in [**81**] and then in [**90**] (see Theorem 2.2.1). Our contribution is a strong differentiability theorem for Lipschitz maps in C-C spaces assuming some structure on the vector fields (see Theorem 2.3.3).

The eikonal equation for the distance from a point in C-C spaces was known to hold in the sense of viscosity solution [**38**]. We improve this result showing that the equation holds almost everywhere [**148**], allowing d to be the distance from a closed set (Theorem 2.6.1). In the special case of the Heisenberg group we prove that the solution is classical because the distance function is regular outside the center of the group [**144**]. Within the study of the distance from a non characteristic surface in the Heisenberg group we also prove a kind of Gauss Lemma stating that the Heisenberg gradient of the distance from a regular surface is the intrinsic normal to the surface (see Lemma 2.5.6 and Theorem 2.5.8).

1. Differentiability of Lipschitz functions between Carnot groups

Let $\mathbb{G} = (\mathbb{R}^n, \cdot, \delta_\lambda, d)$ and $\bar{\mathbb{G}} = (\mathbb{R}^{\bar{n}}, \bar{\cdot}, \bar{\delta}_\lambda, \bar{d})$ be two Carnot groups. In the sequel the group law signs $\cdot$ and $\bar{\cdot}$ will be sometimes omitted.

A map $\varphi : \mathbb{G} \to \bar{\mathbb{G}}$ is a *homogeneous homomorphism* if φ is a group homomorphism and $\varphi(\delta_\lambda(x)) = \bar{\delta}_\lambda(\varphi(x))$ for all $x \in \mathbb{G}$ and $\lambda > 0$. A map $f : \mathbb{G} \to \bar{\mathbb{G}}$ is Lipschitz if there exists a constant $L > 0$ such that $\bar{d}(f(x), f(y)) \le L d(x, y)$ for all $x, y \in \mathbb{G}$.

If $f : \mathbb{G} \to \bar{\mathbb{G}}$, $x, \xi \in \mathbb{G}$ and $t > 0$ define

$$R(x, \xi; t) = \bar{\delta}_{1/t}(f(x)^{-1} f(x\delta_t(\xi))).$$

Definition 2.1.1. A map $f : \mathbb{G} \to \bar{\mathbb{G}}$ is *Pansu-differentiable* (or *differentiable*) at $x \in \mathbb{G}$ if for all $\xi \in \mathbb{G}$ there exists

$$Df(x; \xi) := \lim_{t \downarrow 0} R(x, \xi; t),$$

and the convergence is uniform with respect to ξ. The map $Df(x;\cdot):\mathbb{G}\to\bar{\mathbb{G}}$ is the differential of f at x.

REMARK 2.1.2. If $Df(x;\xi)$ exists then there also exists $Df(x;\delta_\lambda(\xi))=\bar\delta_\lambda Df(x;\xi)$ for all $\lambda>0$. Indeed

$$\bar\delta_{1/t}(f(x)^{-1}f(x\delta_t(\delta_\lambda(\xi))))=\bar\delta_\lambda\bar\delta_{1/\lambda t}(f(x)^{-1}f(x\delta_{\lambda t}(\xi))),$$

and thus

$$\begin{aligned}Df(x;\delta_\lambda(\xi))&=\lim_{t\to0^+}\bar\delta_{1/t}(f(x)^{-1}f(x\delta_t(\delta_\lambda(\xi))))\\&=\bar\delta_\lambda\lim_{t\to0^+}\bar\delta_{1/t}(f(x)^{-1}f(x\delta_t(\xi)))=\bar\delta_\lambda Df(x;\xi).\end{aligned}$$

□

Lipschitz maps between Carnot groups are differentiable almost everywhere and their differential is a homogeneous homomorphism. Here we shall follow Pansu's original proof of this theorem except that in the one dimensional reduction step (Lemma 2.1.4 below). In $\mathbb{G}=\mathbb{R}^n$ we fix the Lebesgue measure and denote by $|E|$ the measure of a measurable set $E\subset\mathbb{G}$.

PROPOSITION 2.1.3. *Let $f:\mathbb{G}\to\bar{\mathbb{G}}$ be a Lipschitz map. If for some $\xi_1,\xi_2\in\mathbb{G}$ the derivatives $Df(x;\xi_1)$ and $Df(x;\xi_2)$ exist for a.e. $x\in\mathbb{G}$, then there also exists $Df(x;\xi_1\xi_2)=Df(x;\xi_1)Df(x;\xi_2)$ for a.e. $x\in\mathbb{G}$.*

Proof. By Remark 2.1.2 we can assume that $d(\xi_1,0)=d(\xi_2,0)=1$. Let $\Omega\subset\mathbb{G}$ be an open set with finite Lebesgue measure and let $\eta>0$. By Lusin and Egorov Theorems there exists a compact set $K\subset\Omega$ such that

(i) $|\Omega\setminus E|\le\eta$;
(ii) $Df(x;\xi_1)$ and $Df(x;\xi_2)$ exist and are continuous at $x\in K$;
(iii) $R(x,\xi_2;t)\to Df(x;\xi_2)$ as $t\downarrow0$ uniformly for $x\in K$.

If we prove the claim for all $x\in K$ we are done. Since δ_λ and $\bar\delta_\lambda$ are group automorphisms, "adding and substracting" $f(x\delta_t(\xi_1))$ we find

$$\begin{aligned}R(x,\xi_1\xi_2;t)&=\bar\delta_{1/t}(f(x)^{-1}f(x\delta_t(\xi_1\xi_2)))=\bar\delta_{1/t}(f(x)^{-1}f(x\delta_t(\xi_1)\delta_t(\xi_2)))\\&=\bar\delta_{1/t}(f(x)^{-1}f(x\delta_t(\xi_1))f(x\delta_t(\xi_1))^{-1}f(x\delta_t(\xi_1)\delta_t(\xi_2)))\\&=\bar\delta_{1/t}(f(x)^{-1}f(x\delta_t(\xi_1)))\,\bar\delta_{1/t}(f(x\delta_t(\xi_1))^{-1}f(x\delta_t(\xi_1)\delta_t(\xi_2)))\\&=R(x,\xi_1;t)\,R(x\delta_t(\xi_1),\xi_2;t).\end{aligned}\tag{2.1.1}$$

and $R(x,\xi_1;t)\to Df(x;\xi_1)$ as $t\to0^+$.

If $\varepsilon>0$ by (iii) there exists $t_\varepsilon>0$ such that

$$\bar d(R(y,\xi_2;t),Df(y;\xi_2))\le\varepsilon$$

for all $y\in K$ as soon as $t\le t_\varepsilon$.

If it were $x\delta_t(\xi_1)\in K$ then $\bar d(Df(x\delta_t(\xi_1);\xi_2),Df(x;\xi_2))\le\varepsilon$ if $t\le t_\varepsilon$ (possibly shrinking t_ε), and

$$\begin{aligned}\bar d(R(x\delta_t(\xi_1),\xi_2;t),Df(x;\xi_2))\le&\,\bar d(R(x\delta_t(\xi_1),\xi_2;t),Df(x\delta_t(\xi_1);\xi_2))\\&+\bar d(Df(x\delta_t(\xi_1);\xi_2),Df(x;\xi_2))\le2\varepsilon,\end{aligned}\tag{2.1.2}$$

which would prove that

$$\lim_{t\to 0} R(x,\xi_1\xi_2;t) = Df(x;\xi_1)Df(x;\xi_2).$$

In general $x\delta_t(\xi_1) \notin K$. Let $B(x,r)$ a C-C ball centered at x with radius r. By the differentiation Theorem in doubling metric spaces 1.6.5

$$\lim_{r\downarrow 0} \frac{|B(x,r)\cap K|}{|B(x,r)|} = 1 \quad \text{and} \quad \lim_{r\downarrow 0} \frac{|B(x,r)\setminus K|}{|B(x,r)|} = 0 \quad \text{for a.e. } x\in K.$$

Let $\lambda(t) = \operatorname{dist}(x\delta_t(\xi_1), K) = d(x\delta_t(\xi_1), \bar{x}(t))$ for some $\bar{x}(t)\in K$ and define $\bar{\xi}_1(t) = \delta_{1/t}(x^{-1}\bar{x}(t))$ so that $\bar{x}(t) = x\delta_t(\bar{\xi}_1(t))$. By Proposition 1.7.3

$$d(x\delta_t(\xi_1), x) = d(\delta_t(\xi_1), 0) = td(\xi_1, 0) = t,$$

and consequently $B(x\delta_t(\xi_1), \lambda(t)) \subset B(x, t+\lambda(t))\setminus K$. Let $Q\geq n$ be the homogeneous dimension of $\mathbb{G}$. By Proposition 1.7.7

$$\Big(\frac{\lambda(t)}{t+\lambda(t)}\Big)^Q = \frac{|B(x,t+\lambda(t))\setminus K|}{|B(x,t+\lambda(t))|} \leq \frac{|B(x,t+\lambda(t))\setminus K|}{|B(x,t+\lambda(t))|}.$$

As the right hand side tends to zero we deduce that

$$\lim_{t\downarrow 0} \frac{\lambda(t)}{t} = 0. \tag{2.1.3}$$

Notice that

$$\lambda(t) = d(x\delta_t(\xi_1), \bar{x}(t)) = d(x\delta_t(\xi_1), x\delta_t(\bar{\xi}_1(t))) = d(\delta_t(\xi_1), \delta_t(\bar{\xi}_1(t))) = td(\xi_1, \bar{\xi}_1(t))$$

and from (2.1.3) it follows that

$$\lim_{t\downarrow 0} d(\xi_1, \bar{\xi}_1(t)) = 0. \tag{2.1.4}$$

We already noticed in (2.1.1) that

$$R(x,\xi_1\xi_2;t) = R(x,\xi_1;t)R(x\delta_t(\xi_1),\xi_2;t).$$

Our aim is to show that $R(x\delta_t(\xi_1),\xi_2;t)$ converges to $Df(x;\xi_2)$. The point $x\delta_t(\xi_1)$ has to be projected on K in order to apply the argument in (2.1.2). Write

$$\begin{aligned} R(x\delta_t(\xi_1),\xi_2;t) = &\bar{\delta}_{1/t}(f(x\delta_t(\xi_1))^{-1} f(x\delta_t(\bar{\xi}_1(t)))) \\ &\bar{\delta}_{1/t}(f(x\delta_t(\bar{\xi}_1(t)))^{-1} f(x\delta_t(\bar{\xi}_1(t))\delta_t(\xi_2))) \\ &\bar{\delta}_{1/t}(f(x\delta_t(\bar{\xi}_1(t))\delta_t(\xi_2))^{-1} f(x\delta_t(\xi_1)\delta_t(\xi_2))) = R_1(t)\cdot R_2(t)\cdot R_3(t). \end{aligned}$$

If $L>0$ is the Lipschitz constant of f we immediately find (again by Proposition 1.7.3)

$$\begin{aligned} \bar{d}(R_1(t),0) &= \bar{d}(\bar{\delta}_{1/t} f(x\delta_t(\bar{\xi}_1(t))), \bar{\delta}_{1/t} f(x\delta_t(\xi_1))) \\ &\leq \frac{L}{t} d(x\delta_t(\bar{\xi}_1(t)), x\delta_t(\xi_1)) \leq L d(\bar{\xi}_1(t), \xi_1) \end{aligned}$$

and analogously

$$\begin{aligned} \bar{d}(R_3(t),0) &= \bar{d}(\bar{\delta}_{1/t} f(x\delta_t(\xi_1)\delta_t(\xi_2)), \bar{\delta}_{1/t} f(x\delta_t(\bar{\xi}_1)\delta_t(\xi_2))) \\ &\leq \frac{L}{t} d(x\delta_t(\xi_1\xi_2), x\delta_t(\bar{\xi}_1(t)\xi_2)) \\ &\leq L d(\xi_1\xi_2, \bar{\xi}_1(t)\xi_2). \end{aligned}$$

By (2.1.4) this shows that both $R_1(t)$ and $R_3(t)$ converge to zero.

Consider now $R_2(t)$. Since $x\delta_t(\bar{\xi}_1(t)) \in K$ the argument in (2.1.2) does apply and thus $\lim_{t\downarrow 0} R_2(t) = Df(x;\xi_2)$. $\square$

The next step is to compute the derivative of Lipschitz curves in a Carnot group according to Definition 2.1.1. The following Lemma should be compared with Theorem 1.3.5.

Let $\mathbb{G} = (\mathbb{R}^n, \cdot, \delta_\lambda, d)$ be a Carnot group and assume that $X = (X_1, ..., X_m)$ is a system of generators of the Lie algebra of the group such that $X_j(0) = \mathrm{e}_j$. We shall denote by $\mathcal{A}$ the matrix of the coefficients of the vector fields.

Lemma 2.1.4. *Let $\gamma : [0,1] \to \mathbb{G}$ be a Lipschitz curve. Then γ is $X-$admissible and if $h \in \mathrm{L}^\infty(0,1)^m$ is its vector of canonical coordinates then*

$$\lim_{t\downarrow 0} \delta_{1/t}(\gamma(s)^{-1} \cdot \gamma(s+t)) = (h_1(s), ..., h_m(s), 0, ..., 0)$$

for a.e. $s \in [0,1]$.

Proof. By abuse of notation we identify h and $(h_1, ..., h_m, 0, ..., 0)$. By Proposition 1.3.3 γ is $X-$admissible and $\dot\gamma(s) = \mathcal{A}(\gamma(s))h(s)$ for a.e. $s \in [0,1]$. Define

$$E = \{s \in [0,1] : \dot\gamma(s) = \mathcal{A}(\gamma(s))h(s) \text{ exists and } s \text{ is a Lebesgue point of } h\}.$$

Let $s \in E$ and assume without loss of generality that $s = 0$. Since the statement is translation invariant we may also assume $\gamma(0) = 0$. We have to prove that

$$\lim_{t\downarrow 0} \delta_{1/t}(\gamma(t)) = (h_1(0), ..., h_m(0), 0, ..., 0).$$

Recall that we write $x \cdot y = P(x,y) = x + y + Q(x,y)$. By formula (1.7.83) for a.e. $t \in [0,1]$

$$\dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)) = \sum_{j=1}^m h_j(s)\mathrm{e}_j + \sum_{j=1}^m h_j(t) \frac{\partial Q(\gamma(t), y)}{\partial y_j}\Big|_{y=0}.$$

We begin with $i = 1, ..., m$. Since $Q_i = 0$ we immediately find

$$\lim_{t\downarrow 0} \frac{\gamma_i(t)}{t} = \lim_{t\downarrow 0} \fint_0^t h_i(s)\,ds = h_i(0).$$

Assume now that the $i-$th coordinate has degree $k \geq 2$ and that the claim has been proved for the degrees $1, 2, ..., k-1$. If we denote by $\bar{Q}_i(x,y)$ the sum of the monomials in $Q_i(x,y)$ in which y appears linearly we can write

$$\sum_{j=1}^m h_j(t) \frac{\partial Q_i(\gamma(t), y)}{\partial y_j}\Big|_{y=0} = \bar{Q}_i(\gamma(t), h(t)).$$

Notice that $Q_i(\gamma, h)$ depends only on the coordinates of γ and h with degree strictly less than k. Moreover, since $\bar{Q}_i$ is homogeneous of degree k each monomial in $\bar{Q}_i(\gamma(t), h(t))$ contains the components $\gamma_1(t), ..., \gamma_{i-1}(t)$ homogeneously of degree $k-1$. Thus $s^{1-k}\bar{Q}_i(\gamma(s), h(s)) = \bar{Q}_i(\delta_{1/s}(\gamma(s)), h(s))$ and

$$\Big|\frac{\gamma_i(t)}{t^k}\Big| \leq \frac{1}{t^k}\int_0^t |\bar{Q}_i(\gamma(s), h(s))|\,ds \leq \fint_0^t |\bar{Q}_i(\delta_{1/s}(\gamma(s)), h(s))|\,ds.$$

By the inductive hypothesis $(\delta_{1/t}(\gamma(t)))_j \to h_j(0)$ as $t \downarrow 0$ for all $j-$coordinates with degree less or equal than $k-1$ and therefore

$$\limsup_{t\downarrow 0} \left|\frac{\gamma_i(t)}{t^k}\right| \leq |\bar{Q}_i(h(0), h(0))|.$$

But $\bar{Q}_i(h(0), h(0)) = 0$ by Lemma 1.7.2 (iv) and the statement is proved.

□

REMARK 2.1.5. Let $V = \{\lambda \mathrm{e}_j; \lambda \in \mathbb{R} \text{ and } j = 1, ..., m\}$. Since the Lie algebra of the group is nilpotent and stratified then by (1.7.81) it follows that there exists $\bar{r} \in \mathbb{N}$ and $C > 0$ such that for every $\xi \in \mathbb{G}$ there exist $\xi_1, ..., \xi_{\bar{r}} \in V$ such that $\xi = \xi_1 \cdot ... \cdot \xi_{\bar{r}}$ and $|\xi_i| \leq C|\xi|$ (see [**67**, Lemma 1.40]).

THEOREM 2.1.6. *Let $f : \mathbb{G} \to \bar{\mathbb{G}}$ be a Lipschitz map. Then $Df(x; \cdot)$ exists for a.e. $x \in \mathbb{G}$ and is a homogeneous homomorphism.*

Proof. Fix $j = 1, ..., m$ and write $\hat{x}_j = (x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n)$. The curve $\gamma_{\hat{x}_j} : \mathbb{R} \to \bar{\mathbb{G}}$ defined by $\gamma_{\hat{x}_j}(t) = f(\exp(tX_j)(\hat{x}_j)$ is Lipschitz, and by Lemma 2.1.4 it is differentiable (according to Definition 2.1.1) at a.e. $t \in \mathbb{R}$. Let $E_j = \{x \in \mathbb{R}^n : \gamma_{\hat{x}_j}$ is differentiable at $x_j\}$ and define $E = \bigcup_{j=1}^m E_j$. By Fubini Theorem $|\mathbb{G} \setminus E| = 0$.

Let $x \in E$ and since the statement is translation invariant assume without loss of generality $x = 0$. Let $K = \partial B(0,1) = \{v \in \mathbb{G} : d(v, 0) = 1\}$. If $v \in K$ we have to prove that there exists

$$Df(0; v) = \lim_{t\downarrow 0} R(0, v; t) = \lim_{t\downarrow 0} \bar{\delta}_{1/t}(f(0)^{-1} \cdot f(\delta_t(v)))$$

and that the convergence is uniform for $v \in K$. Since $\bar{\mathbb{G}}$ with its C-C metric $\bar{d}$ is a complete metric space (Corollary 1.7.5) it is enough to show that for all $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\sup_{v\in K} \bar{d}(R(0, v; t), R(0, v; s)) \leq (1 + 2L)\varepsilon$$

as soon as $0 < s, t \leq t_\varepsilon$. Here L is the Lipschitz constant of f.

Since K is compact we can find $v_1, ..., v_k \in K$ such that $K \subset \bigcup_{i=1}^k B(v_i, \varepsilon)$. Write $v = v_i$ for some i. By Remark 2.1.5 we can write $v = \xi_1 \cdot ... \cdot \xi_{\bar{r}}$ where each ξ_i is of the form $\lambda \mathrm{e}_j$ for some $\lambda \in \mathbb{R}$ and $j = 1, ..., m$. Without loss of generality we can also assume $\lambda = 1$. Now, if $\gamma(t) = f(\exp(tX_j)(0))$

$$Df(0; \xi_i) = \lim_{t\downarrow 0} \delta_{1/t}(f(0)^{-1} f(\delta_t(\xi_i)) = \lim_{t\downarrow 0}(\gamma(0)^{-1} \cdot \gamma(t))$$

exists for all i because $0 \in E$. By Proposition 2.1.3 $Df(0; v)$ exists too and

$$Df(0; v) = Df(0; \xi) \cdot ... \cdot Df(0; \xi_{\bar{r}}).$$

Fix $t_\varepsilon > 0$ such that

$$\sup_{i=1,...,k} \bar{d}(R(0, v_i; t), R(0, v_i; s)) \leq \varepsilon$$

for all $0 < s, t \leq t_\varepsilon$. If $v \in K$ there exists v_i such that $d(v, v_i) \leq \varepsilon$ and

$$\begin{aligned} \bar{d}(R(0, v; t), R(0, v; s)) &\leq \bar{d}(R(0, v; t), R(0, v_i; t)) + \bar{d}(R(0, v_i; t), R(0, v_i; s)) \\ &\quad + \bar{d}(R(0, v_i; s), R(0, v; s)) \\ &\leq (1 + 2L)\varepsilon. \end{aligned}$$

Indeed

$$\begin{aligned}\bar d(R(0,v;t),R(0,v_i;t)) &= \frac{1}{t}\bar d(f(0)^{-1}f(\delta_t(v)),f(0)^{-1}f(\delta_t(v_i)))\\ &\le \frac{L}{t}d(\delta_t(v),\delta_t(v_i))\le L\varepsilon,\end{aligned}$$

and $\bar d(R(0,v;s),R(0,v_i;s))\le L\varepsilon$ by the same estimate. $\square$

Proposition 2.1.7. *Let $f:\mathbb{G}\to\mathbb{R}$ be a Lipschitz map. Then*

$$Df(x;\xi)=\sum_{j=1}^m \xi_j X_j f(x), \tag{2.1.5}$$

for a.e. $x\in\mathbb{G}$ and for all $\xi\in\mathbb{G}$.

Proof. Consider first the case $f\in C^1_0(\mathbb{R}^n)$. Definition 2.1.1 reads

$$Df(x;\xi)=\lim_{t\downarrow 0}\frac{f(x\cdot\delta_t(\xi))-f(x)}{t}=\frac{d}{dt}f(x\cdot\delta_t(\xi))\Big|_{t=0}=\frac{\partial f(x)}{\partial x}\frac{d}{dt}P(x\cdot\delta_t(\xi))\Big|_{t=0},$$

and writing $\bar\xi=(\xi_1,...,\xi_m,0,...,0)$

$$\frac{d}{dt}P(x,\delta_t(\xi))\Big|_{t=0}=\frac{\partial P(x,0)}{\partial y}\frac{d}{dt}\delta_t(\xi)\Big|_{t=0}=\frac{\partial P(x,0)}{\partial y}\bar\xi.$$

On the other hand, by formula (1.7.83) if $j=1,...,m$

$$X_j f(x)=\frac{\partial f(x)}{\partial x}\frac{\partial P(x,0)}{\partial y},$$

and (2.1.5) is proved.

If $f:\mathbb{G}\to\mathbb{R}$ is Lipschitz take $\varphi\in C^1_0(\mathbb{R}^n)$ and $\xi\in\mathbb{R}^n$. By the dominated convergence theorem

$$\begin{aligned}\int_{\mathbb{R}^n}Df(x)(\xi)\varphi(x)\,dx &= \int_{\mathbb{R}^n}\lim_{t\downarrow 0}\frac{f(x\cdot\delta_t(\xi))-f(x)}{t}\varphi(x)\,dx\\ &=\lim_{t\downarrow 0}\int_{\mathbb{R}^n}\frac{f(x\cdot\delta_t(\xi))-f(x)}{t}\varphi(x)\,dx.\end{aligned}$$

Indeed, if $L\ge 0$ is the Lipschitz constant of f then $|f(x\cdot\delta_t(\xi))-f(x)|\le Ld(x\cdot\delta_t(\xi),x)=Ld(\delta_t(\xi),0)=tLd(\xi,0)$.

The Lebesgue measure is left and right invariant so we can perform a change of variable to find

$$\int_{\mathbb{R}^n}\frac{f(x\cdot\delta_t(\xi))-f(x)}{t}\varphi(x)\,dx=\int_{\mathbb{R}^n}f(x)\frac{\varphi(x\cdot(\delta_t(\xi))^{-1})-\varphi(x)}{t}\,dx.$$

Since $(\delta_t(\xi))^{-1}=\delta_t(\xi^{-1})=\delta_t(-\xi)$ the above discussion shows that

$$\lim_{t\downarrow 0}\frac{\varphi(x\cdot(\delta_t(\xi))^{-1})-\varphi(x)}{t}=-\sum_{j=1}^m\xi_jX_j\varphi(x),$$

and integrating by parts we get

$$\int_{\mathbb{R}^n}Df(x;\xi)\varphi(x)\,dx=-\int_{\mathbb{R}^n}f(x)\sum_{j=1}^m\xi_jX_j\varphi(x)\,dx=\int_{\mathbb{R}^n}\varphi(x)\sum_{j=1}^m\xi_jX_jf(x)\,dx,$$

as every X_j is self-adjoint. $\square$

2. Weak derivatives of Lipschitz functions in C-C spaces

Lipschitz functions in general C-C spaces always have weak derivatives along the vector fields that are essentially bounded functions. When the function is the distance function this result was first proved in [**81**], which we shall here follow along with [**75**], and then in [**90**] for a generic Lipschitz function.

THEOREM 2.2.1. *Let $(\mathbb{R}^n, d)$ be a C-C space associated with a family of locally Lipschitz vector fields $X = (X_1, ..., X_m)$. Assume that the metric d is continuous with respect to the Euclidean topology. If $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that for some $L \geq 0$*

$$|f(x) - f(y)| \leq Ld(x, y) \quad \text{for all } x, y \in \mathbb{R}^n, \tag{2.2.6}$$

then the derivatives $X_j f$, $j = 1, ..., m$ exist in distributional sense, are measurable functions and $|Xf(x)| \leq L$ for a.e. $x \in \mathbb{R}^n$.

In the proof of Theorem 2.2.1 a lemma is needed. Let $Y(x) = \sum_{i=1}^n a_i(x)\partial_i$ be a non vanishing locally Lipschitz vector field and consider the Cauchy Problem

$$\begin{cases} \dot{\gamma}_x(t) = Y(\gamma_x(t)) \\ \gamma_x(0) = x \end{cases}$$

If $K \subset \mathbb{R}^n$ is a compact set there exists $T > 0$ such that the solution $\gamma_x(t)$ is defined for all $|t| \leq T$ and $x \in K$. Define $\Phi : K \times [-T, T] \to \mathbb{R}^n$ by $\Phi(x, t) = \gamma_x(t)$. If t is fixed the map $\Phi(\cdot, t)$ is a local diffeomorphism and $\Phi(K \times [-T, T]) \subset K_1$ for some compact set K_1.

LEMMA 2.2.2.

(1) *For any $|t| \leq T$, the map $\Phi(\cdot, t)$ is Lipschitz on K;*
(2) *there exists $C > 0$ such that $|\det J_x\Phi(x, t)| \leq 1 + C|t|$ for a.e. $x \in K$ and for all $|t| \leq T$.*

Proof. Let $M > 0$ be a Lipschitz constant for Y relatively to K_1. Then (we consider $t \geq 0$)

$$\begin{aligned} |\Phi(x, t) - \Phi(y, t)| &= |x + \int_0^t Y(\Phi(x, s))\, ds - y - \int_0^t Y(\Phi(y, s))\, ds| \\ &\leq |x - y| + \int_0^t |Y(\Phi(x, s)) - Y(\Phi(y, s))|\, ds \\ &\leq |x - y| + M \int_0^t |\Phi(x, s) - \Phi(y, s)|\, ds. \end{aligned}$$

By Gronwall Lemma $|\Phi(x, t) - \Phi(y, t)| \leq \bar{M}|x - y|$ with $\bar{M} = \mathrm{e}^{MT}$. By Rademacher Theorem $\Phi(\cdot, t)$ is differentiable for a.e. x. Let $x \in K$ be such a point. From

$$\begin{aligned} |\Phi_i(x + s\mathrm{e}_j, t) - \Phi_i(x, t)| &\leq |s|\delta_{ij} + M \int_0^t |\Phi(x + s\mathrm{e}_j, \tau) - \Phi(x, \tau)|\, d\tau \\ &\leq |s|\delta_{ij} + M_1|s|t \end{aligned}$$

($M_1 := M\bar{M}$) it follows that

$$\Big|\frac{\partial \Phi_i(x,t)}{\partial x_j}\Big| \leq \delta_{ij} + M_1 t$$

and finally for some $C > 0$ we have $\det J_x\Phi(x,t)| \leq 1 + Ct$ for a.e. $x \in K$. □

Proof of Theorem 2.2.1. Let $Y \in \{X_1, ..., X_m\}$. By (2.2.6) the function f is continuous in the Euclidean topology and Yf is a well defined distribution. If Yf is a continuous and linear operator on $\mathrm{L}^1(\mathbb{R}^n)$ it follows that $Yf \in \mathrm{L}^\infty(\mathbb{R}^n)$.

Fix $\bar{x} \in \mathrm{int}(K)$ and let $U = U(\bar{x}, \varepsilon) := \{x \in \mathbb{R}^n : |x - \bar{x}| \leq \varepsilon\} \subset K$ for some $\varepsilon > 0$. The claim is that there exists $C > 0$ such that

$$|\langle Yf, \varphi\rangle| \leq C\|\varphi\|_1 \quad \text{for all } \varphi \in C_0^\infty(U).$$

Such estimate, if proved, will hold by density for all $\varphi \in \mathrm{L}^1(U)$. Integrating by parts

$$\begin{aligned}\langle Yf, \varphi\rangle &= -\int_U f(x) \sum_{i=1}^n \partial_i(a_i(x)\varphi(x))\, dx \\ &= -\int_U f(x) Y\varphi(x)\, dx - \int_U f(x)\varphi(x)\mathrm{div}(Y)\, dx.\end{aligned}$$

The divergence of Y is (essentially) locally bounded and therefore for some $C > 0$ not depending on φ

$$\Big|\int_U f(x)\varphi(x)\mathrm{div}(Y)\, dx\Big| \leq C \sup_{x\in U} |f(x)| \|\varphi\|_1. \tag{2.2.7}$$

In order to estimate the first integral consider

$$\int_U f(x)Y\varphi(x)\, dx = \lim_{t\to 0} \frac{1}{t}\int_U f(x)(\varphi(\Phi(x,t)) - \varphi(x))\, dx.$$

Write $\Phi_t(x) = \Phi(x,t)$, let $\Psi_t(y) = \Phi_t^{-1}(y)$ and perform the change of variable $x = \Psi_t(y)$ to get

$$\int_U f(x)\varphi(\Phi(x,t))\, dx = \int_{\Phi_t(U)} f(\Psi_t(y))\varphi(y)|\det J\Psi_t(y)|\, dy.$$

Being $\varphi \in C_0^\infty(U)$ we may assume $\mathrm{spt}(\varphi) \subset \Phi_t(U) \cap U$ it $|t|$ if small enough, and the integration domain $\Phi_t(U)$ can be replaced with U. By Lemma 2.2.2 the estimate $|\det J\Psi_t(y)| \leq 1 + C|t|$ holds and thus

$$\begin{aligned}\Big|\int_U f(x)Y\varphi(x)\, dx\Big| &= \Big|\lim_{t\to 0}\int_U \Big(f(\Psi_t(x))|\det J\Psi_t(x)| - f(x)\Big)\varphi(x)\, dx\Big| \\ &\leq \limsup_{t\to 0} \frac{1}{|t|}\int_U \Big(|f(\Psi_t(x)) - f(x)| + C|t||f(\Psi_t(x))|\Big)|\varphi(x)|\, dx.\end{aligned}$$

The path $t \to \Psi_t(x)$ is an integral curve of $-Y$. Thus by (2.2.6) $|f(\Psi_t(x)) - f(x)| \leq Ld(\Psi_t(x), x) \leq L|t|$. This yields

$$\frac{1}{|t|}\int_U |f(\Psi_t(x)) - f(x)||\varphi(x)|\, dx \leq L\|\varphi\|_1.$$

Now, for $t > 0$ there exists $\lambda(t) > 0$ such that $\Psi_t(x) \in U(\bar{x}, \varepsilon + \lambda(t))$ for all $x \in U$ and $\lambda(t) \to 0$ as $t \downarrow 0$. Therefore we can write

$$\int_U |f(\Psi_t(x))||\varphi(x)|\, dx \le \|\varphi\|_1 \sup_{x\in U(\bar{x},\varepsilon+\lambda(t))} |f(x)|,$$

and finally

$$\Big| \int_U f(x) Y\varphi(x)\, dx \Big| \le (L + \sup_{x\in U} |f(x)|)\|\varphi\|_1. \tag{2.2.8}$$

By (2.2.7) and (2.2.8)

$$|\langle Yf, \varphi\rangle| \le \Big(L + C \sup_{x\in U(\bar{x},\varepsilon)} |f(x)|\Big)\|\varphi\|_1.$$

and this shows that Yf is a continuous linear functional on $\mathrm{L}^1(U(\bar{x},\varepsilon))$, ad as a consequence $Yf \in \mathrm{L}^\infty(U(\bar{x},\varepsilon))$ and

$$\|Yf, \mathrm{L}^\infty(U(\bar{x},\varepsilon))\| \le L + C \sup_{x\in U(\bar{x},\varepsilon)} |f(x)|. \tag{2.2.9}$$

Let now $x \in \mathbb{R}^n$ be a point such that $f(x) = 0$ (this is not restrictive), $|Xf(x)| > 0$ and $|Xf|$ is approximatively continuous at x. Applying the above argument to the vector field

$$Y(x) = \sum_{j=1}^m \frac{X_j f(x)}{|Xf(x)|} X_j,$$

whose integral curves are $X-$subunit, we find from (2.2.9)

$$|Xf(x)| = |Yf(x)| = \lim_{\varepsilon\downarrow 0} \|Yf, \mathrm{L}^\infty(U(\bar{x},\varepsilon))\| \le L.$$

□

3. Differentiability of Lipschitz functions in C-C spaces

The weak derivatives of a real valued Lipschitz function in a C-C space define a “differential” that exists almost everywhere. But the space needs some more properties.

Let $(\mathbb{R}^n, d)$ the the Carnot-Carathéodory space associated with the vector fields $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$, $m \le n$. The vector fields will be assumed to be of the form

$$X_j(x) = \partial_j + \sum_{i=m+1}^n a_{ij}(x)\partial_i, \quad j + 1, ..., m. \tag{2.3.10}$$

Secondly, $(\mathbb{R}^n, d)$ endowed with Lebesgue measure will be assumed to be a locally homogeneous metric space. Precisely, we assume that for any compact set $K \subset \mathbb{R}^n$ there exist $\delta > 1$ and $r_0 > 0$ such that

$$|B(x, 2r)| \le \delta |B(x, r)| \quad \text{for all } x \in K \text{ and } 0 \le r \le r_0. \tag{2.3.11}$$

Finally, we assume the following Morrey type inequality which will be discussed in chapter 4. For a.e. $x \in \mathbb{R}^n$ there exist $C > 0$, $p \ge 1$ and $r_0 > 0$ such that for all $0 < r < r_0$ and $f \in \mathrm{Lip}(\mathbb{R}^n, d)$

$$|f(x) - f(y)| \le Cr\Big(\fint_{B(x,r)} |Xf(z)|^p\, dz\Big)^{1/p} \quad \text{for all } y \in B(x, r). \tag{2.3.12}$$

Example 2.3.1. Assumptions (2.3.10), (2.3.11) and (2.3.12) hold in any Carnot group. But there are many other C-C spaces satisfying them. Consider, for instance, $\mathbb{R}^3$ endowed with the C-C metric induced by the vector fields $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} + x_1^2 \partial_{x_3}$. Since $[X_1, X_2] = 2x_1\partial_{x_3}$ and $[X_1, [X_1, X_2]] = 2\partial_{x_3}$ the Chow-Hörmander condition is satisfied. If $x_1 \neq 0$ the homogeneous dimension of $(\mathbb{R}^3, d, |\cdot|)$ in a neighborhood of x is 4 and the Morrey inequality (2.3.12) holds. This will be explained in chapter 4.

We introduce a suitable definition of differential. If $x = (x_1, ..., x_n) \in \mathbb{R}^n$ we shall write $\bar{x} = (x_1, ..., x_m) \in \mathbb{R}^m$.

Definition 2.3.2. Let $(\mathbb{R}^n, d)$ be a C-C space associated with a family of locally Lipschitz vector fields $X = (X_1, ..., X_m)$ of the form (2.3.10) A function $f : \mathbb{R}^n \to \mathbb{R}$ is $X-$differentiable at $x \in \mathbb{R}^n$ if there exists a linear transformation $T : \mathbb{R}^m \to \mathbb{R}$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - T(\bar{y} - \bar{x})}{d(x, y)} = 0.$$

The $X-$differential of f at x is $d_X f(x) := T$.

Theorem 2.3.3. *Let $(\mathbb{R}^n, d)$ be a C-C space associated with a family of locally Lipschitz vector fields $X = (X_1, ..., X_m)$. Assume (2.3.10), (2.3.11) and (2.3.12). A Lipschitz function $f \in \mathrm{Lip}(\mathbb{R}^n, d)$ is $X-$differentiable for a.e. $x \in \mathbb{R}^n$ and $d_X f(x) = (X_1 f(x), ..., X_m f(x))$.*

Proof. The proof follows an idea of Calderón [**37**]. By Theorem 2.2.1 the derivatives $X_j f(x)$, $j = 1, ..., m$, exist for a.e. $x \in \mathbb{R}^n$. By (2.3.11) Lebesgue differentiation Theorem 1.6.5 applies and

$$\lim_{r \downarrow 0} \fint_{B(x,r)} |Xf(z) - Xf(x)|^p \, dz = 0 \tag{2.3.13}$$

for a.e. $x \in \mathbb{R}^n$ and for all $p \geq 1$.

Fix $x \in \mathbb{R}^n$ such that $|Xf(x)| < \infty$, (2.3.13) holds and (2.3.12) holds for some $p \geq 1$ which from now on is fixed. Define

$$g(y) = f(y) - \langle Xf(x), \bar{x} - \bar{y} \rangle,$$

and notice that by (2.3.10)

$$Xg(y) = Xf(y) - Xf(x).$$

By (2.3.12) we obtain

$$|g(y) - g(x)| \leq Cr \Big(\fint_{B(x,r)} |Xg(z)|^p \, dz \Big)^{1/p} \quad \text{for all } y \in B(x, r).$$

Choosing $r = 2d(x,y)$ we get

$$\begin{aligned}\frac{|f(y)-f(x)-\langle Xf(x),\bar{x}-\bar{y}\rangle|}{d(x,y)} &= \frac{|g(y)-g(x)|}{d(x,y)} \\ &\le 2C\Big(\fint_{B(x,2d(x,y))} |Xg(z)|^p\,dz\Big)^{1/p} \\ &\le 2C\Big(\fint_{B(x,2d(x,y))} |Xf(z)-Xf(x)|^p\,dz\Big)^{1/p}.\end{aligned}$$

The last term tends to zero as $d(x,y)\to 0$ owing to (2.3.13). □

4. Eikonal equation for the Heisenberg distance

The Heisenberg $\mathbb{H}^n$ group has been introduced in chapter 1. We recall that $\mathbb{H}^n$ is identified with $\mathbb{C}^n\times\mathbb{R}\equiv\mathbb{R}^{2n+1}$. The Heisenberg gradient is $\nabla_{\mathbb{H}} = (X_1,...,X_n,Y_1,...,Y_n)$ where

$$X_j = \partial_{x_j} + 2y_j\partial_t, \quad Y_j = \partial_{y_j} - 2x_j\partial_t, \quad j=1,...,n. \tag{2.4.14}$$

In this section the function $d:\mathbb{H}^n\to[0,+\infty)$ is the Heisenberg C-C distance from the origin. Recall that $Z=\{(z,t)\in\mathbb{H}^n : z=0\}$ is the center of the group. We begin with the following Theorem proved in [**144**].

THEOREM 2.4.1. *The function d is of class C^∞ in $\mathbb{H}^n\setminus Z$ and*

$$|\nabla_{\mathbb{H}} d(z,t)| = 1 \tag{2.4.15}$$

for all $(z,t)\in\mathbb{H}^n$ such that $z\neq 0$.

Proof. For the sake of simplicity we consider the case $n=1$. Set $\Omega=\{(\vartheta,\varphi,\varrho)\in\mathbb{R}^3 : \vartheta\in\mathbb{R},\ -2\pi\le\varphi\varrho\le 2\pi,\ \varrho\ge 0\}$ and define $\Phi:\Omega\to\mathbb{H}^1$ by

$$\Phi(\vartheta,\varphi,\varrho) = (x(\vartheta,\varphi,\varrho), y(\vartheta,\varphi,\varrho), t(\vartheta,\varphi,\varrho)), \tag{2.4.16}$$

where

$$\begin{cases} x(\vartheta,\varphi,\varrho) = \dfrac{\cos\vartheta(1-\cos\varphi\varrho)+\sin\vartheta\sin\varphi\varrho}{\varphi} \\ y(\vartheta,\varphi,\varrho) = \dfrac{-\sin\vartheta(1-\cos\varphi\varrho)+\cos\vartheta\sin\varphi\varrho}{\varphi} \\ t(\vartheta,\varphi,\varrho) = 2\dfrac{(\varphi\varrho-\sin\varphi\varrho)}{\varphi^2}. \end{cases} \tag{2.4.17}$$

We chose $A=\cos\vartheta$, $B=\sin\vartheta$ and $s=\varrho$ in (1.8.96). The range of Φ is $\mathbb{H}^1$. In fact, if $\varrho>0$ is fixed, then equations (2.4.17) with $\vartheta\in[0,2\pi)$ and $-2\pi/\varrho\le\varphi\le 2\pi/\varrho$ parametrize the boundary of the ball $B(0,\varrho)$.

One can compute the determinant of the Jacobian

$$\det J\Phi(\vartheta,\varphi,\varrho) = 4\frac{\varphi\varrho\sin\varphi\varrho - 2(1-\cos\varphi\varrho)}{\varphi^4}.$$

It is easily seen that the equation $s\sin s + 2\cos s = 2$ has the solutions $s=0,\pm 2\pi$ for $|s|\le 2\pi$. This means that

$$\det J\Phi(\vartheta,\varphi,\varrho) = 0$$

if and only if $\varphi\varrho=\pm 2\pi$ or $\varrho=0$ (the case $\varphi=0$ must be excluded). The set of the points $\Phi(\vartheta,\varphi,\varrho)$ with $\varphi\varrho=\pm 2\pi$ is exactly Z.

By the inverse function Theorem the function Φ is a local diffeomorphism in the open set $\{(\vartheta, \varphi, \varrho) \in \mathbb{R}^3 : \varrho > 0 \text{ and } |\varphi\varrho| < 2\pi\}$. Moreover, by the definition of d we have $\Psi(\vartheta, \varphi, \varrho) := d(\Phi(\vartheta, \varphi, \varrho)) \equiv \varrho$. The function Ψ just defined is of class C^∞ and since $d = \Psi \circ \Phi^{-1}$ then d is of class C^∞ in $\mathbb{H}^1 \setminus Z$.

The above discussion is still true in $\mathbb{H}^n$ for $n > 1$. The Heisenberg group satisfies the hypotheses of Theorem 2.2.1, and since d is clearly $1-$Lipschitz

$$|\nabla_{\mathbb{H}} d(z,t)| \leq 1 \tag{2.4.18}$$

for a.e. $(z,t) \in \mathbb{H}^n$. But $|\nabla_{\mathbb{H}} d|$ is continuous on $\mathbb{H}^n \setminus Z$ and (2.4.18) holds for $(z,t) \in \mathbb{H}^n \setminus Z$.

Fix $z \neq 0$ and let $\gamma : [0,T] \to \mathbb{H}^n$ be the geodesic joining 0 to (z,t), which exists and is unique as shown in chapter 1. In particular γ is of class C^∞ and

$$\dot\gamma(s) = \sum_{j=1}^n h_{1j} X_j(\gamma(s)) + h_{2j} Y_j(\gamma(s)), \quad \sum_{j=1}^n h_{1j}(s)^2 + h_{2j}(s)^2 = 1 \quad \text{for all } s \in [0,T].$$

Notice that $\gamma(s) \notin Z$ for all $s > 0$. Differentiating the identity $s = d(\gamma(s))$ we find

$$\begin{aligned} 1 &= \frac{d}{ds} d(\gamma(s)) = \langle Dd(\gamma(s)), \dot\gamma(s) \rangle \\ &= \sum_{j=1}^n h_{1j}(s) X_j d(\gamma(s)) + h_{2j}(s) Y_j d(\gamma(s)) \leq |\nabla_{\mathbb{H}} d(\gamma(s))| \end{aligned}$$

for all $s \in (0,T]$. Choosing $s = T$ we get $|\nabla_{\mathbb{H}} d(z,t)| \geq 1$, which along with (2.4.18) gives $|\nabla_{\mathbb{H}} d(z,t)| = 1$. $\square$

5. Distance from a surface in the Heisenberg group

In this section we study the eikonal equation for the distance from a surface in the Heisenberg group. Let d be the left invariant C-C metric in $\mathbb{H}^n$ and for a closed set $K \subset \mathbb{H}^n$ define the function $d_K : \mathbb{H}^n \to [0, +\infty)$

$$d_K(z,t) = \inf_{(\zeta,\tau) \in K} d((z,t),(\zeta,\tau)).$$

Since d_K is the lower envelope of a family of $1-$Lipschitz functions bounded from below, then d_K is $1-$Lipschitz. By Theorem 2.2.1 $|\nabla_{\mathbb{H}} d_K(z,t)| \leq 1$ for a.e. $(z,t) \in \mathbb{H}^n$. In section 6 we shall prove - in a more general framework - that $|\nabla_{\mathbb{H}} d_K(z,t)| = 1$ for a.e. $(z,t) \in \mathbb{H}^n \setminus K$. In this section we consider the special case when K is a surface which has a “uniform tangent ball” property.

We introduce some more notation. The horizontal space of $\mathbb{H}^n$ is the $2n-$dimensional vector bundle spanned by the vector fields $X_1, ..., X_n, Y_1, ..., Y_n$. Precisely

$$\mathrm{H}(z,t) := \operatorname{span}\{X_1(z,t), ..., X_n(z,t), Y_1(z,t), ..., Y_n(z,t)\} \subset \mathbb{R}^{2n+1}.$$

Define the map $\pi_{(z,t)} : \mathrm{H}(z,t) \to \mathbb{R}^{2n}$

$$\pi_{(z,t)}\Big(\sum_{j=1}^n a_i X_i(z,t) + b_i Y_i(z,t) \Big) = (a_1, ..., a_n, b_1, ..., b_n).$$

Let $\mathcal{A}(z,t)$ be the matrix of the coefficients of the Heisenberg vector fields (2.4.14) as in (1.1.1) and define $\varrho_{(z,t)} : \mathbb{R}^{2n} \to \mathbb{R}^{2n+1}$ by

$$\varrho_{(z,t)}(v) = \mathcal{A}(z,t)v.$$

Notice that if $\varrho^T_{(z,t)} : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ is the transposed map of $\varrho_{(z,t)}$ then

$$\nabla_{\mathbb{H}} f(z,t) = \varrho^T_{(z,t)}(\nabla f(z,t)),$$

for any $f \in C^1(\mathbb{R}^{2n+1})$.

Definition 2.5.1. Let $S \subset \mathbb{R}^{2n+1}$ be a surface of class C^1 locally given by the equation $f = 0$. A point $(z,t) \in S$ is said to be *non characteristic* if $|\nabla_{\mathbb{H}} f(z,t)| \neq 0$. The surface is non characteristic if all its points are non characteristic.

The Euclidean normal to the surface S at a point $(z,t) \in S$ is

$$\nu(z,t) = \frac{\nabla f(z,t)}{|\nabla f(z,t)|}.$$

The point (z,t) is non characteristic if the projection of $\nu(z,t)$ onto the horizontal space $\mathrm{H}(z,t)$ does not vanish.

As a first step we show that a geodesic starting from the center of a ball hits the surface of the ball in the direction given by the projection of the Euclidean normal to the surface onto the horizontal space.

Lemma 2.5.2. *Let $B = B((z_0,t_0),\varrho)$ be the ball of $\mathbb{H}^n$ centered at (z_0,t_0) with radius $\varrho > 0$. Let $(z,t) \in \partial B \setminus (z_0,t_0)\cdot Z$, and let $\gamma : [0,\varrho] \to \mathbb{H}^n$ be the geodesic joining (z_0,t_0) to (z,t). Then*

$$\pi_{(z,t)}(\dot\gamma(\varrho)) = \nabla_{\mathbb{H}} d((z,t),(z_0,t_0)). \tag{2.5.19}$$

Proof. For the sake of simplicity we consider the case $n = 1$ and write $X = X_1$ and $Y = Y_1$. We begin with $(z_0,t_0) = 0$. Let $(z,t) \in \partial B(0,\varrho) \setminus Z$, and fix $\vartheta \in [0,2\pi)$ and $\varphi \in (-2\pi\varrho, +2\pi\varrho)$ such that $\Phi(\vartheta,\varphi,\varrho) = (z,t)$. The map Φ was defined in (2.4.16).

The geodesic γ joining 0 to (z,t) has velocity (recall (1.8.97))

$$\dot\gamma(\tau) = (\cos\vartheta\sin\varphi\tau + \sin\vartheta\cos\varphi\tau)X(\gamma(\tau)) + (\cos\vartheta\cos\varphi\tau - \sin\vartheta\sin\varphi\tau)Y(\gamma(\tau)),$$

for all $0 \le \tau \le \varrho$. Hence,

$$\pi_{(z,t)}(\dot\gamma(\varrho)) = (\cos\vartheta\sin\varphi\varrho + \sin\vartheta\cos\varphi\varrho, \cos\vartheta\cos\varphi\varrho - \sin\vartheta\sin\varphi\varrho). \tag{2.5.20}$$

Write $d(z,t) = d((z,t),0)$. The derivatives $Xd(z,t)$ and $Yd(z,t)$ can be computed by means of the map Φ. Indeed

$$Xd(z,t) = (Xd)(\Phi(\vartheta,\varphi,\varrho)) = (d\Phi^{-1}X)d\circ\Phi(\vartheta,\varphi,\varrho).$$

Here $d\Phi^{-1}$ stands for the differential of Φ^{-1}, map that exists because $(z,t) \notin Z$, and

$$d\Phi^{-1}X = c_1(\vartheta,\varphi,\varrho)\frac{\partial}{\partial\vartheta} + c_2(\vartheta,\varphi,\varrho)\frac{\partial}{\partial\varphi} + c_3(\vartheta,\varphi,\varrho)\frac{\partial}{\partial\varrho}.$$

As $d(\Phi(\vartheta,\varphi,\varrho)) \equiv \varrho$ we find $Xd(\Phi(\vartheta,\varphi,\varrho)) = c_3(\vartheta,\varphi,\varrho)$ and an explicit computation of $c_3(\vartheta,\varphi,\varrho)$ gives

$$Xd(\Phi(\vartheta,\varphi,\varrho)) = \cos\vartheta\sin\varphi\varrho + \sin\vartheta\cos\varphi\varrho.$$

Analogously

$$Yd(\Phi(\vartheta,\varphi,\varrho)) = \cos\vartheta\cos\varphi\varrho - \sin\vartheta\sin\varphi\varrho.$$

By (2.5.20) this proves the thesis (2.5.19) if $(z_0,t_0)=0$.

We study the case $(z_0,t_0)\neq 0$. If $(z,t)\in\partial B((z_0,t_0),\varrho)\setminus(z_0,t_0)\cdot Z$, let $(\zeta,\tau)\in\partial B(0,\varrho)\setminus Z$ be such that $(z_0,t_0)\cdot(\zeta,\tau)=(z,t)$ and consider the geodesic $\widetilde{\gamma}:[0,\varrho]\to\mathbb{H}^1$ joining 0 to (ζ,τ). The curve $\gamma=(z_0,t_0)\cdot\widetilde{\gamma}$ is the geodesic joining (z_0,t_0) to (z,t).

If $\dot{\widetilde{\gamma}}=aX(\widetilde{\gamma})+bY(\widetilde{\gamma})$, by the left invariance of the vector fields X and Y we find

$$\pi_{(z,t)}(\dot{\gamma}(\varrho)) = \pi_{(z,t)}(a(\varrho)X(\gamma(\varrho))+b(\varrho)Y(\gamma(\varrho))) = (a(\varrho),b(\varrho)) = \pi_{(\zeta,\tau)}(\dot{\widetilde{\gamma}}(\varrho)).$$

By the first part of the proof

$$\pi_{(z,t)}(\dot{\gamma}(\varrho)) = \pi_{(\zeta,\tau)}(\dot{\widetilde{\gamma}}(\varrho)) = \nabla_{\mathbb{H}}d(\zeta,\tau),$$

and again by the left invariance of the vector fields

$$\begin{aligned}\nabla_{\mathbb{H}}d(\zeta,\tau) &= (\nabla_{\mathbb{H}}d)((z_0,t_0)^{-1}\cdot(z,t))\\ &= \nabla_{\mathbb{H}}(d((z_0,t_0)^{-1}\cdot(z,t))) = \nabla_{\mathbb{H}}d((z,t),(z_0,t_0)).\end{aligned}$$

□

Since our analysis is local we can assume that $S=\partial E$ where E is an open set in $\mathbb{R}^{2n+1}$. In this way we can define the *signed distance* from S

$$d_S(z,t) = \begin{cases} \inf\limits_{(\zeta,\tau)\in S} d((z,t),(\zeta,\tau)) & \text{if } (z,t)\in E\\ -\inf\limits_{(\zeta,\tau)\in S} d((z,t),(\zeta,\tau)) & \text{if } (z,t)\in\mathbb{R}^{2n+1}\setminus E.\end{cases}$$

DEFINITION 2.5.3. Let $E\subset\mathbb{R}^{2n+1}$ be an open set. A set $K\subset\partial E$ is said to have the *uniform interior ball property* relatively to E if there exists $\varrho_0>0$ such that for all $(z,t)\in K$ there exists $(\zeta,\tau)\in E$ such that $\overline{B((\zeta,\tau),\varrho_0)}\cap\partial E=\{(z,t)\}$. K is said to have the *uniform ball property* if it has the *uniform interior ball property* relatively both to E and to $\mathbb{R}^{2n+1}\setminus\overline{E}$

EXAMPLE 2.5.4. In $\mathbb{H}^1=\mathbb{C}\times\mathbb{R}$ consider $E=\{(z,t)\in\mathbb{H}^1: t>0\}$ and $S=\partial E=\{(z,t)\in\mathbb{H}^1: t=0\}$. We briefly show that $K=\{(z,0)\in S: |z|\geq\varepsilon\}$ has the uniform ball property for any $\varepsilon>0$.

By the parametric equations (1.8.98) for the Heisenberg ball and from Remark (1.8.2) it can be easily computed the total Euclidean size in the vertical direction of $B(0,r)$, which is $2r^2/\pi$. The left translation of $B(0,r)$ by the vector $(0,r^2/\pi)\in\mathbb{C}\times\mathbb{R}$ is indeed an Euclidean translation. Thus

$$\overline{B((0,r^2/\pi),r)}\cap S = \{(z,0)\in S: |z|=2r/\pi\}$$

because $2r/\pi$ is the radial coordinate at which the maximal height in the surface $\partial B(0,r)$ is achieved. Choosing $\varrho_0<\varepsilon\pi/2$ the uniform ball property can be checked for $K\subset S$.

REMARK 2.5.5. If K is a subset of non characteristic points in a surface of class C^2 in $\mathbb{R}^3\equiv\mathbb{H}^1$ then it should have the uniform ball property. At present I am not able to prove (or disprove) this statement.

LEMMA 2.5.6 (Gauss Lemma in $\mathbb{H}^n$). *Let $S = \partial E \subset \mathbb{R}^{2n+1}$ be a surface of class C^1 given by the equation $f = 0$ ($f > 0$ in E) and let $K \subset S$ be a compact set with the uniform ball property. There exists $\varrho_0 > 0$ such that for all $(z,t) \in K$ there exists a geodesic $\gamma : [-\varrho_0, \varrho_0] \to \mathbb{H}^n$ such that $\gamma(0) = (z,t)$, $\varrho = d_S(\gamma(\varrho))$ for all $|\varrho| \le \varrho_0$ and*

$$\pi_{(z,t)}(\dot\gamma(0)) = \frac{\nabla_{\mathbb{H}} f(z,t)}{|\nabla_{\mathbb{H}} f(z,t)|}. \tag{2.5.21}$$

Proof. Without loss of generality we shall prove the claims for $\varrho \ge 0$. There exists $\varrho_0 > 0$ such that for all $(z,t) \in K$ there exists $(\zeta,\tau) \in E$ such that $\overline{B((\zeta,\tau),\varrho_0)} \cap S = \{(z,t)\}$. Let (z,t) and (ζ,τ) be fixed and write $B := B((\zeta,\tau),\varrho_0)$. Since S is of class C^1 by Remark 1.8.2 $(z,t) \notin (\zeta,\tau)\cdot Z$ and thus (z,t) is a regular point of ∂B. It follows that S and ∂B have the same tangent space at (z,t)

$$T_{(z,t)}S = T_{(z,t)}\partial B,$$

and as a consequence they also have the same Euclidean normal at (z,t) with opposite sign. Let $\nu(z,t)$ be the inward unit normal to ∂B at (z,t).

Let $\gamma : [0,\varrho_0] \to \mathbb{R}^{2n+1}$ be the (unique) geodesic such that $\gamma(0) = (z,t)$ and $\gamma(\varrho_0) = (\zeta,\tau)$. As $\overline{B((\zeta,\tau),\varrho_0)} \cap S = \{(z,t)\}$ then $d_S(\gamma(\varrho)) = \varrho$ for all $0 \le \varrho \le \varrho_0$. Moreover, by Lemma 2.5.2

$$\pi_{(z,t)}(\dot\gamma(0)) = -\nabla_{\mathbb{H}} d((z,t),(\zeta,\tau)), \tag{2.5.22}$$

where in the derivatives $\nabla_{\mathbb{H}} d((z,t),(\zeta,\tau))$ the point (ζ,τ) has to be thought as fixed. On the other hand

$$\frac{\nabla d((z,t),(\zeta,\tau))}{|\nabla d((z,t),(\zeta,\tau))|} = -\nu(z,t) = -\frac{\nabla f(z,t)}{|\nabla f(z,t)|},$$

and hence

$$-\nabla_{\mathbb{H}} d((z,t),(\zeta,\tau)) = \frac{|\nabla d((z,t),(\zeta,\tau))||\nabla_{\mathbb{H}} f(z,t)|}{|\nabla f(z,t)|}\frac{\nabla_{\mathbb{H}} f(z,t)}{|\nabla_{\mathbb{H}} f(z,t)|}. \tag{2.5.23}$$

By Theorem 2.4.1 $|\nabla_{\mathbb{H}} d((z,t),(\zeta,\tau))| = 1$ and (2.5.23) implies

$$\frac{|\nabla d((z,t),(\zeta,\tau))||\nabla_{\mathbb{H}} f(z,t)|}{|\nabla f(z,t)|} = 1,$$

so that (2.5.22) and (2.5.23) give (2.5.21). □

REMARK 2.5.7. If $S = \partial E \subset \mathbb{R}^{2n+1}$ is a surface of class C^1 and $K = \{(z,t)\} \subset S$ has the interior ball property then (z,t) is a non characteristic point of S. Indeed, $T_{(z,t)}S = T_{(z,t)}\partial B$ for some ball B and (z,t) is a regular point of ∂B which is non characteristic for ∂B.

THEOREM 2.5.8. *Let $S = \partial E \subset \mathbb{R}^{2n+1}$ be a surface of class C^1 and let $(\bar z,\bar t) \in S$ be a point having a neighborhood in S with the uniform ball property. Then the signed distance d_S is of class C^1 in a neighborhood of $(\bar z,\bar t)$ and moreover $|\nabla_{\mathbb{H}} d_S| = 1$ in this neighborhood.*

Proof. Let $\mathcal{U} \subset S$ be a neighborhood of $(\bar z,\bar t)$ with the uniform ball property. Let γ,λ be variables in $\mathbb{R}^{2n+1}$ and recall the definition of the Heisenberg Hamiltonian $H(\gamma,\lambda)$ in (1.8.95).

Let $(z,t)\in\mathcal{U}$ and notice that $|\nabla_{\mathbb{H}} f(z,t)|\neq 0$ by Remark 2.5.7. Therefore we can consider the Cauchy problem

$$\begin{cases} \dot\gamma = \dfrac{1}{2}\dfrac{\partial H(\gamma,\lambda)}{\partial\lambda} \\ \dot\lambda = -\dfrac{1}{2}\dfrac{\partial H(\gamma,\lambda)}{\partial\gamma} \\ \gamma(0)=(z,t) \\ \lambda(0)=\frac{\nabla f(z,t)}{|\nabla_{\mathbb{H}} f(z,t)|}. \end{cases} \tag{2.5.24}$$

There exists $\varrho_0>0$ such that the solution $(\gamma_{(z,t)}(s),\lambda_{(z,t)}(s))$ of (2.5.24) is defined for $|s|\le\varrho_0$ for all $(z,t)\in\mathcal{U}$. Let (γ,λ) be such a solution. Since $\dot\gamma(s)\in \mathrm{H}(\gamma(s))$ then

$$\dot\gamma(s)=\varrho_{\gamma(s)}(\pi_{\gamma(s)}(\dot\gamma(s))). \tag{2.5.25}$$

Moreover, from (1.4.38) it follows that $\dot\gamma(s)=\varrho_{\gamma(s)}(\varrho^T_{\gamma(s)}(\lambda(s)))$ and thus the following identity holds

$$\pi_{\gamma(s)}(\dot\gamma(s))=\varrho^T_{\gamma(s)}(\lambda(s)).$$

Writing $(\gamma,\lambda)=(\gamma_{(z,t)},\lambda_{(z,t)})$ and taking $s=0$ we finally find

$$\pi_{(z,t)}(\dot\gamma_{(z,t)}(0))=\varrho^T_{(z,t)}(\lambda_{(z,t)}(0))=\frac{\nabla_{\mathbb{H}} f(z,t)}{|\nabla_{\mathbb{H}} f(z,t)|}.$$

Define $\Psi:\mathcal{U}\times[-\varrho_0,\varrho_0]\to\mathbb{R}^{2n+1}$ letting $\Psi((z,t),s)=\gamma_{(z,t)}(s)$. The function Ψ is of class C^1. If we prove that Ψ is a local diffeomorphism it follows that d_S is of class C^1. In order to check this define $\Theta:\mathcal{U}\times[-\varrho_0,\varrho_0]\to[0,+\infty)$ by $\Theta((z,t),s)=d_S(\Psi((z,t),s))$. By Lemma 2.5.6 (take ϱ_0 smaller if necessary) $\Theta((z,t),s)\equiv s$ and so Θ is smooth. Consequently, if Ψ is invertible, $d_S=\Theta\circ\Psi^{-1}$ is of class C^1.

We show that the differential $d\Psi((z,t),0):T_{(z,t)}S\oplus\mathbb{R}\to\mathbb{R}^{2n+1}$ is an isomorphism. It is easy to see that if $v\in T_{(z,t)}S$ then $d\Psi((z,t),0)v=v$. We show that $d\Psi((z,t),0)(\frac{\partial}{\partial s})$ is transversal to $T_{(z,t)}S$.

Let $\varphi\in C^1(\mathbb{R}^{2n+1})$ be a test function and compute

$$\begin{aligned} d\Psi((z,t),0)\Big(\frac{\partial}{\partial s}\Big)\varphi &= \frac{d}{ds}\varphi\circ\Psi((z,t),s)\Big|_{s=0} = d\varphi(\Psi((z,t),0))\frac{\partial\Psi}{\partial s}((z,t),0) \\ &= d\varphi(z,t)\dot\gamma_{(z,t)}(0) = d\varphi(z,t)\varrho_{(z,t)}\Big(\frac{\nabla_{\mathbb{H}} f(z,t)}{|\nabla_{\mathbb{H}} f(z,t)|}\Big). \end{aligned}$$

This shows that

$$d\Psi((z,t),0)\Big(\frac{\partial}{\partial s}\Big)=\varrho_{(z,t)}\Big(\frac{\nabla_{\mathbb{H}} f(z,t)}{|\nabla_{\mathbb{H}} f(z,t)|}\Big).$$

Assume by contradiction that

$$d\Psi((z,t),0)\Big(\frac{\partial}{\partial s}\Big)\in T_{(z,t)}S.$$

Since $\nabla f(z,t)$ is orthogonal to $T_{(z,t)}S$, it follows that

$$0=\langle\varrho_{(z,t)}(\nabla_{\mathbb{H}} f(z,t)),\nabla f(z,t)\rangle=\langle\nabla_{\mathbb{H}} f(z,t),\varrho^T_{(z,t)}\nabla f(z,t)\rangle=|\nabla_{\mathbb{H}} f(z,t)|^2,$$

and thus $\nabla_{\mathbb{H}} f(z,t)=0$. But this is not possible because (z,t) is non characteristic by Remark 2.5.7.

Finally, we show that $|\nabla_{\mathbb{H}} d_S| = 1$ in a neighborhood of $(\bar{z}, \bar{t})$. Take (z,t) in a neighborhood of $(\bar{z}, \bar{t})$ in $\mathbb{H}^n$ where d_S is of class C^1. Let $T = d_S(z,t) > 0$ and let $\gamma : [0,T] \to \mathbb{H}^n$ be a geodesic such that $\gamma(T) = (z,t)$ and $d_S(\gamma(s)) = s$ for all $0 \le s \le T$. This identity can be differentiated at $s = T$ to find

$$\begin{aligned} 1 &= \langle \nabla d_S(z,t), \dot{\gamma}(T) \rangle = \langle \nabla d_S(z,t), \varrho_{(z,t)}(\pi_{(z,t)}(\dot{\gamma}(T))) \rangle \\ &= \langle \varrho^T_{(z,t)} \nabla d_S(z,t), \pi_{(z,t)}(\dot{\gamma}(T)) \rangle = \langle \nabla_{\mathbb{H}} d_S(z,t), \pi_{(z,t)}(\dot{\gamma}(T)) \rangle. \end{aligned}$$

We used (2.5.25). As a consequence $1 \le |\nabla_{\mathbb{H}} d_S(z,t)| |\pi_{(z,t)}(\dot{\gamma}(T))| \le |\nabla_{\mathbb{H}} d_S(z,t)|$. Theorem 2.2.1 gives the opposite inequality a.e. and by continuity $|\nabla_{\mathbb{H}} d_S| = 1$ in a neighborhood of $(\bar{z}, \bar{t})$. □

6. Eikonal equation for distance functions in C-C spaces

Let $(\mathbb{R}^n, d)$ be a C-C space induced by the vector fields $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$. We shall assume the following hypotheses:

(H1) The metric d is continuous with respect to the Euclidean topology of $\mathbb{R}^n$.
(H2) The metric space $(\mathbb{R}^n, d)$ is complete.

If (H1) holds then by Theorem 1.4.2 hypothesis (H2) is equivalent to require the boundedness of C-C balls with respect to the Euclidean metric.

Let $K \subset \mathbb{R}^n$ and define the distance function from K

$$d_K(x) := \inf_{y \in K} d(x,y).$$

If $d_K(x) = \lim_{k\to\infty} d(x, y_k)$, we can assume that $y_k \in K \cap B(x,r)$ for some $r > d_K(x)$ and for all $k \in \mathbb{N}$. If K is closed then $K \cap \overline{B(x,r)}$ is compact (by (H1) and (H2)), and - possibly extracting a subsequence - we can assume that $y_k \to y \in K$. Hence $d(x,y) = d_K(x)$ and $d_K(x) = \min_{y\in K} d(x,y)$.

The function $d_K : (\mathbb{R}^n, d) \to \mathbb{R}$ is $1-$Lipschitz and Theorem 2.2.1 implies that the derivatives $X_j d_K$, $j = 1, ..., m$, exist almost everywhere. Moreover

$$|X d_K(x)| \le 1 \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{2.6.26}$$

In order to reach equality in (2.6.26) for a.e. $x \in \mathbb{R}^n \setminus K$ we need the global existence of geodesics and a chain rule to differentiate the distance function along geodesics. Such tools are at hand in the following cases:

(C1) the vector fields $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ satisfy the conditions (2.3.10), (2.3.11) and (2.3.12);
(C2) the vector fields $X_1, ..., X_m$ are of Grushin type as in (1.9.99);
(C3) the vector fields $X_1, ..., X_m$ are smooth and $\mathrm{span}\{X_1(x), ..., X_m(x)\} = \mathbb{R}^n$ at every $x \in \mathbb{R}^n$.

All Carnot groups and many other C-C spaces induced by Hörmander vector fields are in Case (C1). Case (C3) is essentially the Riemannian one. In all these cases we are able to prove the following Theorem ([**148**]).

Theorem 2.6.1. *Let $(\mathbb{R}^n, d)$ be the C-C space induced by a family of vector fields $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ that satisfy (C1), (C2) or (C3). Assume (H1) and (H2). Let $K \subset \mathbb{R}^n$ be a closed set and let d_K be the distance from K. Then*

$$|X d_K(x)| = 1 \tag{2.6.27}$$

for a.e. $x \in \mathbb{R}^n \setminus K$.

Proof. We shall deal in detail with Case (C1). By (H2) and Theorem 1.4.4 (i) geodesics exist globally. We shall write $f = d_K$ and $\bar{x} = (x_1, ..., x_m) \in \mathbb{R}^m$ if $x \in \mathbb{R}^n$. The function $f : (\mathbb{R}^n, d) \to [0, +\infty)$ is $1-$Lipschitz. By Theorem 2.3.3

$$\lim_{y \to x} \frac{f(y) - f(x) - \langle Xf(x), \bar{y} - \bar{x} \rangle}{d(x, y)} = 0 \tag{2.6.28}$$

for a.e. $x \in \mathbb{R}^n$, and moreover by Theorem 2.2.1

$$|Xf(x)| \leq 1 \tag{2.6.29}$$

for a.e. $x \in \mathbb{R}^n$.

Let $x \in \mathbb{R}^n \setminus K$ be a point such that (2.6.28) and (2.6.29) hold. By (H1) and (H2) there exists $x_0 \in K$ such that $d(x, x_0) = d_K(x) := T > 0$. Take a geodesic $\gamma \in \mathrm{Lip}([0, T]; \mathbb{R}^n)$ such that $\gamma(0) = x$ and $\gamma(T) = x_0$. Notice that

$$d_K(\gamma(t)) = d(\gamma(t), x_0) = T - t$$

for all $t \in [0, T]$. Since

$$\dot{\gamma}(t) = \sum_{j=1}^{m} h_j(t) X_j(\gamma(t)), \qquad \text{for a.e. } t \in [0, T]$$

with $h = (h_1, ..., h_m)$ measurable coefficients such that $|h(t)| \leq 1$ for a.e. $t \in [0, T]$, from the special form of the vector fields (2.3.10) it follows that

$$\bar{\gamma}(t) = \bar{\gamma}(0) + \int_0^t h(s)\, ds \quad \text{for all } t \in [0, T],$$

where $\bar{\gamma} = (\gamma_1, ..., \gamma_m)$, and thus

$$|\bar{\gamma}(t) - \bar{\gamma}(0)| \leq t \quad \text{for all } t \in [0, T] \tag{2.6.30}$$

As $t = d(\gamma(t), \gamma(0))$, from (2.6.28) it follows

$$f(\gamma(t)) - f(\gamma(0)) = \langle Xf(x), \bar{\gamma}(t) - \bar{\gamma}(0) \rangle + o(t). \tag{2.6.31}$$

On the other hand

$$f(\gamma(t)) - f(\gamma(0)) = d_K(\gamma(t)) - d_K(\gamma(0)) = (T - t) - T = -t, \tag{2.6.32}$$

so that (2.6.32), (2.6.31) and (2.6.30) all together give

$$1 = |\langle Xf(x), (\bar{\gamma}(t) - \bar{\gamma}(0))/t \rangle + o(1)| \leq |Xf(x)| + o(1),$$

for all $t \in (0, T]$, and letting $t \downarrow 0$ we find $|Xf(x)| \geq 1$. This inequality and the converse one (2.6.29) prove that $|Xf(x)| = 1$ for a.e. $x \in \mathbb{R}^n \setminus K$. The proof is ended in Case (C1).

In Case (C2) the vector fields are of the form (2.3.10) outside a vector subspace and Theorem 2.3.3 still applies on the complement of this subspace.

CHAPTER 3

Regular domains and trace on boundaries in C-C spaces

The first part of this chapter deals with regular domains in C-C spaces (general references on the subject are [**174**], [**44**], [**89**], [**92**], [**42**], [**43**], [**65**], [**66**], [**147**]), while the second part is devoted to trace theorems for sub-elliptic gradients (related references are [**60**], [**71**], [**25**], [**13**], [**146**], [**56**]).

Section 1 is a survey of results concerning regular domains in metric spaces. We first introduce John domains (see Definition 3.1.1, [**112**] and also [**100**] for general references) and uniform domains (see Definition 3.1.10, [**113**], [**136**], [**135**], [**169**], [**174**], [**42**]). In homogeneous spaces with geodesics the class of John domains equals that of Boman domains (see Theorems 3.1.8 and 3.1.9, [**32**] and [**89**]) introduced in [**27**]. This is of particular relevance because conditions involving chain of balls (as in the definition of Boman domains) are a key technical tool in proving several theorems in Functional Analysis such as global Sobolev-Poincaré inequalities, compactness theorems of Rellich-Kondrachov type, optimal potential estimates, relative isoperimetric inequalities (see chapter 4, [**109**], [**126**], [**74**], [**78**], [**89**], [**128**], [**100**]). The definition of uniform domain can be rephrased in the language of chain of balls, too. Indeed, it implies the Harnack chain condition (see Definition 3.1.17 and Proposition 3.1.18) which in the Euclidean space is relevant in the study of the non-tangential boundary behavior of harmonic functions (see [**110**]). Similar results have also been established in C-C spaces in [**42**] (see also [**65**] and [**66**]).

In section 2 we show that a smooth domain $\Omega \subset \mathbb{R}^n$ with non characteristic boundary with respect to a system of Hörmander vector fields $X_1, ..., X_m$ is a uniform domain in the metric space $(\mathbb{R}^n, d)$, being d the C-C metric induced by $X_1, ..., X_m$ (see Theorem 3.2.1). The results proved in section 3 are concerned with C-C spaces of Grushin type: we introduce a class of admissible domains possibly with characteristic boundary (see Definition 3.3.1) which are uniform (Theorem 3.3.3). In section 4 we prove that a connected, bounded open set $\Omega \subset \mathbb{R}^n$ with boundary of class $C^{1,1}$ is a uniform domain in the metric space $(\mathbb{R}^n, d)$, being d the metric associated with a Carnot group structure of step 2 (Theorem 3.4.2). Finally, in section 5 we give a sufficient condition for a connected, bounded open set of class C^2 in a Carnot group of step 3 to be a John domain (see Definition 3.5.2 and Theorem 3.5.5).

The second part of the chapter deals with trace theorems. In section 6 we prove a trace theorem on non characteristic boundaries for sub-elliptic gradient associated with a system of Hörmander vector fields (see Theorem 3.6.4). In section 7, within the framework of the Grushin plane we prove a trace theorem for domains that have "flat" boundary at characteristic points (see Theorem 3.7.5) and we show that this result is sharp (see Proposition 3.7.6).

1. Regular domains in metric spaces

We consider a metric space (M, d). If $\gamma : [0, L] \to M$ and $0 \le a \le b \le L$ we shall denote by $\gamma_{[a,b]}$ the restricted function $\gamma_{[a,b]} : [0, b-a] \to M$ defined by $\gamma_{[a,b]}(t) = \gamma(t+a)$. A *domain* $\Omega \subset M$ is a connected open set. The metric space (M, d) will be said *with geodesics* if every couple of point $x, y \in M$ can be connected by a continuous rectifiable curve with length $d(x, y)$.

DEFINITION 3.1.1. Let (M, d) be a metric space. A bounded open set $\Omega \subset M$ is a *John domain* if there exist $x_0 \in \Omega$ and $C > 0$ such that for every $x \in \Omega$ there exists a continuous rectifiable curve parametrized by arclength $\gamma : [0, T] \to \Omega$, $T \ge 0$, such that $\gamma(0) = x$, $\gamma(T) = x_0$ and

$$\operatorname{dist}(\gamma(t); \partial\Omega) \ge Ct. \tag{3.1.1}$$

DEFINITION 3.1.2. Let (M, d) be a metric space. A bounded open set $\Omega \subset M$ is a *weak John domain* if there exist $x_0 \in \Omega$ and $0 < C \le 1$ such that for every $x \in \Omega$ there exists a continuous curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$\operatorname{dist}(\gamma(t); \partial\Omega) \ge C d(\gamma(t), x). \tag{3.1.2}$$

REMARK 3.1.3. If (M, d) is a metric space with geodesics every ball $B(x_0, r)$, $x_0 \in M$ and $r > 0$, is a John domain with constant $C = 1$ in (3.1.1).

DEFINITION 3.1.4. Let (M, d) be a metric space. A set $E \subset M$ satisfies the *interior* (*exterior*) *corkscrew condition* if there exist $r_0 > 0$ and $k \ge 1$ such that for all $0 < r \le r_0$ and $x \in \partial E$ there exists $y \in E$ ($y \in M \setminus E$) such that

$$\frac{r}{k} \le \operatorname{dist}(y; \partial E) \quad \text{and} \quad d(x, y) \le r.$$

A set E satisfies the *corkscrew condition* if it satisfies both the interior and the exterior corkscrew condition. The constant k will be called the *corkscrew constant* of E .

Clearly, if Ω is a John domain then it satisfies the interior corkscrew condition.

PROPOSITION 3.1.5. *Let (M, d, μ) be a doubling metric space with arcwise connected balls. If $E \subset M$ satisfies the interior corkscrew condition then there exist $r_0 > 0$ and $C > 0$ such that for all $x \in \partial E$ and $0 \le r \le r_0$*

$$\mu(E \cap B(x, r)) \ge C\mu(B(x, r)).$$

Proof. Fix $x \in \partial E$ and $0 < r \le r_0$. There exists $y \in E$ such that $d(x, y) \le r/4$ and $\operatorname{dist}(y; \partial E) \ge r/(4k)$, $k \ge 1$ being given by Definition 3.1.4, as well as $r_0 > 0$. Since balls are arcwise connected $B(y, r/4k) \subset E$ and therefore $B(y, r/(4k)) \subset E \cap B(x, r)$. Moreover, $B(x, r) \subset B(y, 2r)$. By Proposition 1.6.3

$$\mu(E \cap B(x, r)) \ge \mu(B(y, r/(4k))) \ge C\mu(B(y, 2r)) \ge C\mu(B(x, r)),$$

where $C > 0$ is a constant that does not depend on x. $\square$

THEOREM 3.1.6. *Let (M, d) be a doubling metric space with geodesics. Then $\Omega \subset M$ is a weak John domain if and only if it is a John domain.*

The proof of Theorem 3.1.6 can be found in [**100**, Proposition 9.6] and for the Euclidean case in [**136**, Lemma 2.7].

Definition 3.1.7. An open set $\Omega \subset M$ is a *Boman domain* if there exists a covering $\mathcal{F}$ of Ω with balls and there exist $N \geq 1$, $\lambda > 1$ and $\nu \geq 1$ such that

(i) $\lambda B \subset \Omega$ for all $B \in \mathcal{F}$;
(ii) $\sum_{B\in\mathcal{F}} \chi_{\lambda B}(x) \leq N$ for all $x \in \Omega$;
(iii) there exists $B_0 \in \mathcal{F}$ such that for any $B \in \mathcal{F}$ there exist $B_1, ..., B_k$ such that $B_k = B$, $\mu(B_i \cap B_{i+1}) \geq 1/N \max\{\mu(B_i), \mu(B_{i+1})\}$ and $B \subset \nu B_i$ for all $i = 0, 1, ..., k$.

Under additional hypotheses on the metric space the definition of John domain is equivalent to that of Boman domain (see [**32**] and [**89**, section 6]). In the proof of the following two theorems we shall essentially follow [**89**].

Theorem 3.1.8. *Let (M, d, μ) be a doubling metric space. If $\Omega \subset M$, $\Omega \neq M$, is a weak John domain then it is a Boman domain.*

Proof. We shall denote by $\delta > 0$ the doubling constant. By Whitney Covering Theorem there exists a family $\mathcal{B}$ of disjoint balls such that for some $\alpha > 1$ and $\beta \in (0, 1)$ with $\alpha\beta < 1$

(i) $4\alpha B \subset \Omega$ for all $B \in \mathcal{B}$;
(ii) $\Omega = \bigcup_{B\in\mathcal{B}} \alpha B$;
(iii) $r = \beta \mathrm{dist}(x; \partial\Omega)$ for all $B = B(x, r) \in \mathcal{B}$.

Since M is doubling we can also assume that there exists $N \geq 1$ such that

(iv) $\sum_{B\in\mathcal{B}} \chi_{4\alpha B}(x) \leq N$ for all $x \in \Omega$.

We show that $\mathcal{F} = \{2\alpha B : B \in \mathcal{B}\}$ is a covering of Ω that satisfies the conditions in Definition 3.1.7 with $\lambda = 2$. Let $x_0 \in \Omega$ be the John center given in Definition 3.1.1 and let $B_0 \in \mathcal{F}$ be a ball containing x_0. If $B = B(x, r) \in \mathcal{F}$ we have to find a chain of balls joining B to B_0. By hypothesis there exists a continuous curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$\mathrm{dist}(\gamma(t); \partial\Omega) \geq C d(\gamma(t), x) \tag{3.1.3}$$

for all $t \in [0, 1]$ and for some $C > 0$ depending only on Ω. By compactness and by (ii) there exist $B_1, ..., B_k = B \in \mathcal{F}$ such that $\gamma([0, 1]) \subset \bigcup_{i=0}^{k} 1/2B_i$ and $1/2B_i \cap 1/2B_{i+1} \neq \emptyset$ for all $i = 0, 1, ..., k-1$. Let $r_i := r(B_i)$ be the radius of B_i. By (iii)

$$\frac{r_{i+1}}{2\alpha\beta} = \mathrm{dist}(x_{i+1}; \partial\Omega) \geq \mathrm{dist}(x_i; \partial\Omega) - d(x_i, x_{i+1}) \geq \frac{r_i}{2\alpha\beta} - \frac{r_i}{2} - \frac{r_{i+1}}{2},$$

and thus

$$r_{i+1}\Big(\frac{1}{2\alpha\beta} + \frac{1}{2}\Big) \geq r_i\Big(\frac{1}{2\alpha\beta} - \frac{1}{2}\Big).$$

The argument is symmetric. Letting $\Lambda = (1 + \alpha\beta)/(1 - \alpha\beta)$ with $0 < \alpha\beta < 1$ we find for any $i = 0, 1, ..., k-1$

$$\Lambda^{-1} r_{i+1} \leq r_i \leq \Lambda r_{i+1}. \tag{3.1.4}$$

The constant Λ depends only on the covering. Since $B_i \subset B(x_{i+1}, d(x_i, x_{i+1}) + r_i)$ and by (3.1.4) $d(x_i, x_{i+1}) + r_i \leq r_i/2 + r_{i+1}/2 + r_i \leq r_{i+1}(1/2 + 3\Lambda/2) := \Lambda_1 r_{i+1}$ we get $B_i \subset \Lambda_1 B_{i+1}$ and analogously $B_{i+1} \subset \Lambda_1 B_i$. As a consequence, there exists $m \geq 1$ depending only on the covering and on the doubling constant such that

$$m^{-1}\mu(B_{i+1}) \leq \mu(B_i) \leq m\mu(B_{i+1}). \tag{3.1.5}$$

We claim that

$$\mu(B_i \cap B_{i+1}) \geq \frac{1}{\delta^2 m} \max\{\mu(B_i), \mu(B_{i+1})\}. \tag{3.1.6}$$

Assume that $r_i \leq r_{i+1}$ and let $y \in 1/2B_i \cap 1/2B_{i+1}$. Then $B(y, r_i/2) \subset B_i \cap B_{i+1}$ and $B_i \subset B(y, 2r_i)$. Thus by the doubling property of μ

$$\mu(B_i \cap B_{i+1}) \geq \mu(B(y, r_i/2)) \geq \frac{1}{\delta^2}\mu(B(y, 2r_i)) \geq \frac{1}{\delta^2}\mu(B_i).$$

By (3.1.5) the claim (3.1.6) is proved.

The open set Ω will be proved to be a Boman domain if we show that there exists a constant $\nu \geq 1$ such that

$$B = B(x, r) \subset \nu B_i \quad \text{for all } i = 0, 1, ..., k.$$

We claim that there exists $\bar{\nu} \geq 1$ depending only on the covering and on the John constant C in (3.1.3) such that

$$r \leq \bar{\nu} r_i \quad \text{for all } i = 0, 1, ..., k. \tag{3.1.7}$$

Fix i and let $t \in [0, 1]$ be such that $\gamma(t) \in B_i$. Then by (iii)

$$\frac{r}{2\alpha\beta} = \operatorname{dist}(x; \partial\Omega) \leq d(x, x_i) + \operatorname{dist}(x_i; \partial\Omega)$$

and by the weak John condition (3.1.3)

$$\begin{aligned} d(x, x_i) &\leq \operatorname{dist}(x; B_i) + r_i \leq d(x, \gamma(t)) + r_i \leq \frac{1}{C}\operatorname{dist}(\gamma(t); \partial\Omega) + r_i \\ &\leq \frac{1}{C}(r_i + \operatorname{dist}(x_i; \partial\Omega)) + r_i. \end{aligned}$$

All together we find

$$\frac{r}{2\alpha\beta} \leq (1 + 1/C)\operatorname{dist}(x_i; \partial\Omega) + (1 + 1/C)r_i \leq (1 + 1/C)(1 + 1/(2\alpha\beta))r_i$$

and (3.1.7) holds with $\bar{\nu} = (1 + 1/C)(1 + 2\alpha\beta)$.

Finally, if $z \in B(x, r)$ then by (3.1.7), (3.1.3) and by (ii)

$$\begin{aligned} d(z, x_i) &\leq d(z, x) + d(x, \gamma(t)) + d(\gamma(t), x_i) \leq r + 1/C\operatorname{dist}(\gamma(t); \partial\Omega) + r_i \\ &\leq (1 + \bar{\nu})r_i + 1/C(r_i + \operatorname{dist}(x_i; \partial\Omega)) \leq [(1 + \bar{\nu}) + 1/C(1 + 1/(2\alpha\beta))]r_i. \end{aligned}$$

This shows that $B(x, r) \subset \nu B_i$ with $\nu = [(1 + \bar{\nu}) + 1/C(1 + 1/(2\alpha\beta))]$. □

THEOREM 3.1.9. *Let (M, d, μ) be a doubling metric space with geodesics. If $\Omega \subset M$ is a Boman domain then it is a John domain.*

Proof. By Proposition 3.1.6 it will be enough to show that Ω is a weak John domain. There exists a covering $\mathcal{F}$ of Ω with balls such that for some $\lambda > 1$ and $\nu \geq 1$

(i) $\lambda B \subset \Omega$ for all $B \in \mathcal{F}$;

(ii) there exists $B_0 \in \mathcal{F}$ such that for all $B \in \mathcal{F}$ there exists a chain of balls $B_1, ..., B_k = B \in \mathcal{F}$ such that $B_i \cap B_{i+1} \neq \emptyset$ and $B \subset \nu B_i$ for all $i = 0, 1, ..., k-1$.

Conditions (i) and (ii) are the only properties of Boman domains used in the proof. Let $B_0 = B(x_0, r_0)$. We have to show that there exists $C > 0$ such that any $x \in \Omega$ can be joined to x_0 by a curve satisfying (3.1.2). Let $B \in \mathcal{F}$ be a ball containing x and let $B_1, ..., B_k = B \in \mathcal{F}$ be a chain of balls satisfying (i) and (ii). If $B_i = B(x_i, r_i)$ we claim that

$$r_i \leq \frac{1}{\lambda - 1} \mathrm{dist}(B_i; \partial\Omega) \quad \text{for all } i = 0, 1, ..., k. \tag{3.1.8}$$

Fix $\varepsilon > 0$. There exist $y \in B_i$ and $z \in \partial\Omega$ such that $d(y, z) \leq \mathrm{dist}(B_i; \partial\Omega) + \varepsilon$ and there exists a geodesic $\bar\gamma : [0, T] \to M$ such that $\bar\gamma(0) = y$, $\bar\gamma(T) = z$ and $\mathrm{length}(\bar\gamma) = d(y, z)$. Since $\lambda B \subset \Omega$ there exists $t \in [0, T]$ such that $d(\bar\gamma(t), x_i) = \lambda r_i$. Now, since $d(\bar\gamma(t), x_i) \geq (\lambda - 1) r_i + d(y, x_i)$ we have

$$(\lambda - 1) r_i \leq d(\bar\gamma(t), x_i) - d(y, x_i) \leq d(\bar\gamma(t), y),$$

and from $d(\bar\gamma(t), y) + d(\bar\gamma(t), z) \leq \mathrm{length}(\bar\gamma) = d(y, z)$ it follows that

$$(\lambda - 1) r_i + \mathrm{dist}(\lambda B_i; \partial\Omega) \leq d(\bar\gamma(t), y) + d(\bar\gamma(t), z) = d(y, z) \leq \mathrm{dist}(B_i; \partial\Omega) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we get (3.1.8).

Let $y_i \in B_i \cap B_{i+1}$, $i = 0, 1, ..., k-1$. Join by geodesics x to x_k, x_k to y_{k-1}, y_{k-1} to x_{k-1},..., x_1 to y_0 and y_0 to x_0 and let $\gamma : [0, T] \to \Omega$ be the curve obtained joining all such geodesics. Let $t \in [0, T]$ and assume that $\gamma(t) \in B_i$. Then, since $B \subset \nu B_i$ and using (3.1.8)

$$\begin{aligned} d(\gamma(t), x) &\leq d(\gamma(t), x_i) + d(x_i, x) \leq (1 + \nu) r_i \\ &\leq \frac{1+\nu}{\lambda - 1} \mathrm{dist}(B_i; \partial\Omega) \leq \frac{1+\nu}{\lambda - 1} \mathrm{dist}(\gamma(t); \partial\Omega), \end{aligned}$$

and (3.1.2) holds with $C = (\lambda - 1)/(1 + \nu)$. □

We introduce now uniform domains.

Definition 3.1.10. Let (M, d) be a metric space. A (bounded) domain $\Omega \subset M$ is a *uniform domain* if there exists $\varepsilon > 0$ such that for all $x, y \in \Omega$ there exists a continuous rectifiable curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$,

$$\mathrm{length}(\gamma) \leq \frac{1}{\varepsilon} d(x, y), \tag{3.1.9}$$

and for all $t \in [0, 1]$

$$\mathrm{dist}(\gamma(t); \partial\Omega) \geq \varepsilon \min\{\mathrm{length}(\gamma_{[0,t]}), \mathrm{length}(\gamma_{[t,1]})\}. \tag{3.1.10}$$

Definition 3.1.11. Let (M, d) be a metric space. A (bounded) domain $\Omega \subset M$ is a *weak uniform domain* if there exists $\varepsilon > 0$ such that for every $x, y \in \Omega$ there exists a continuous curve $\gamma : [0, 1] \to \Omega$ that $\gamma(0) = x$, $\gamma(1) = y$,

$$\mathrm{diam}(\gamma) \leq \frac{1}{\varepsilon} d(x, y), \tag{3.1.11}$$

and for all $t \in [0, 1]$

$$\mathrm{dist}(\gamma(t); \partial\Omega) \geq \varepsilon \min\{\mathrm{diam}(\gamma_{[0,t]}), \mathrm{diam}(\gamma_{[t,1]})\}. \tag{3.1.12}$$

Uniform and weak uniform domains correspond respectively to the domains defined by “length cigars” and “diameter cigars” in [**169**]. If conditions (3.1.9) and (3.1.10) hold for all $x, y \in \Omega$ such that $d(x, y) \leq \delta$ for some $\delta > 0$ then Ω is also called $(\varepsilon, \delta)-$*domain*. If Ω is a bounded $(\varepsilon, \delta)-$domain then it is uniform. This is a consequence of the following localization lemma (see [**169**, Theorem 4.1]).

Lemma 3.1.12. *Let (M, d) be a metric space, $\Omega \subset M$ be a bounded open set and $0 < r < \operatorname{diam}(\Omega)$. If for any $z \in \partial\Omega$ and for all $x, y \in \Omega \cap B(z, r)$ there exists a continuous and rectifiable curve $\gamma : [0, 1] \to \Omega$ joining x to y and such that hold* (3.1.9) *and* (3.1.10) *for some $\varepsilon > 0$ not depending on z, then Ω is a uniform domain.*

Recall that a metric space (M, d) endowed with a Borel measure μ is Ahlfors-regular if there exist $Q > 0$ and $\alpha > 0$ such that for all $x \in M$ and $r \geq 0$

$$\alpha^{-1} r^Q \leq \mu(B(x, r)) \leq \alpha r^Q. \tag{3.1.13}$$

The following Theorem uses a “packing argument” introduced in [**136**]. It is stated and proved for Ahlfors regular metric spaces but it holds for doubling spaces.

Theorem 3.1.13. *Let (M, d, μ) be an Ahlfors regular metric space with geodesics. If $\Omega \subset M$ is a weak uniform domain then it is a uniform domain.*

Proof. There exists $\varepsilon > 0$ such that for all $x, y \in \Omega$ there exists a continuous curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$ and (3.1.11) and (3.1.12) hold.

Let $\bar{t} \in [0, 1]$ be such that $\operatorname{diam}(\gamma_{[0,\bar{t}]}) = \operatorname{diam}(\gamma_{[\bar{t},1]}) =: \delta$. We shall construct a rectifiable curve $\kappa : [0, 1] \to \Omega$ joining x to $\gamma(\bar{t})$ such that

$$\operatorname{length}(\kappa) \leq \frac{1}{\varepsilon_1} d(x, y), \tag{3.1.14}$$

and for $t \in [0, 1]$

$$\operatorname{dist}(\kappa(t); \partial\Omega) \geq \varepsilon_1 \operatorname{length}(\kappa_{[0,t]}), \tag{3.1.15}$$

for some $\varepsilon_1 > 0$ depending on ε, β and Q.

Let $T = (1 + \bar{t})\delta$. The function $\varphi : [0, \bar{t}] \to [0, T]$ defined by $\varphi(t) = (1 + t)\operatorname{diam}(\gamma_{[0,t]})$ is continuous and increasing. Define the reparameterization $\bar{\kappa} : [0, T] \to M$ by $\bar{\kappa}(s) = \gamma(\varphi^{-1}(s))$. Since $d(\gamma(t), x) \leq \operatorname{diam}(\gamma_{[0,t]}) \leq \varphi(t)$ we have

$$d(\bar{\kappa}(s), x) \leq s \quad \text{for all } s \in [0, T]. \tag{3.1.16}$$

Moreover, by (3.1.12)

$$\begin{aligned}\operatorname{dist}(\bar{\kappa}(s); \partial\Omega) = \operatorname{dist}(\gamma(\varphi^{-1}(s); \partial\Omega)) &\geq \varepsilon \min\{\operatorname{diam}(\gamma_{[0,\varphi^{-1}(s)]}), \operatorname{diam}(\gamma_{[\varphi^{-1}(s),1]})\} \\ &\geq \varepsilon \operatorname{diam}(\gamma_{[0,\varphi^{-1}(s)]}) = \frac{\varepsilon s}{1 + \varphi^{-1}(s)},\end{aligned}$$

and thus

$$\operatorname{dist}(\bar{\kappa}(s); \partial\Omega) \geq \frac{\varepsilon s}{2}. \tag{3.1.17}$$

Now define $t_0 = T$, $t_1 = \inf\{t \in [0, t_0] : d(\bar{\kappa}(t), \bar{\kappa}(t_0)) \leq \varepsilon t_0/4\}$ and by induction for any $i \in \mathbb{N}$ let $t_{i+1} = \inf\{t \in [0, t_i] : d(\bar{\kappa}(t), \bar{\kappa}(t_i)) \leq \varepsilon t_i/4\}$. Let $x_i = \bar{\kappa}(t_i)$ and notice that by (3.1.17) $B(x_i, \varepsilon t_i/2) \subset \Omega$ for all $i \in \mathbb{N}$.

We use the “packing argument” of [**136**]. We claim that there exists $\bar{k} \in \mathbb{N}$ such that

$$t_{i+k} \leq \frac{1}{2} t_i \quad \text{for all } k \geq \bar{k} \text{ and for all } i \in \mathbb{N}. \tag{3.1.18}$$

Assume that

$$t_j > \frac{1}{2}t_i \quad \text{for } j = i+1, ..., k. \tag{3.1.19}$$

We have to find an upper bound for k independent from i. If $i \le j < h \le i+k$ then $x_h \notin B(x_j, \varepsilon t_j/4)$ and the balls $B_j := B(x_j, \varepsilon t_j/8)$, $j = i+1, ..., i+k$, are disjoint. Moreover, if $\lambda = (1+\varepsilon/8)$ then $B_j \subset B(x, \lambda t_i)$. Indeed, if $z \in B_j$ then by (3.1.16) and using $t_j \le t_i$

$$d(z,x) \le d(z,x_j) + d(x_j,x) \le \varepsilon t_j/8 + t_j \le (1+\varepsilon/8)t_i.$$

Then

$$\alpha(\lambda t_i)^Q \ge \mu(B(x,\lambda t_i)) \ge \mu\Big(\bigcup_{j=i+1}^{k} B_j\Big) = \sum_{j=i+1}^{k} \mu(B_j) \ge \frac{1}{\alpha}\sum_{j=i+1}^{k}\Big(\frac{\varepsilon t_j}{4}\Big)^Q \ge \frac{1}{\alpha}k\Big(\frac{\varepsilon t_i}{8}\Big)^Q,$$

and thus

$$k \le \alpha^2\Big(\frac{8\lambda}{\varepsilon}\Big)^Q$$

This proves the claim (3.1.18).

Let $\kappa_1 : [0,L) \to \Omega$ be the rectifiable curve obtained joining by geodesics parametrized by arc length x_0 to x_1, x_1 to x_2,..., x_i to x_{i+1} and so on. If $L < +\infty$ then κ_1 can be completed letting $\kappa_1(L) = x$. For any $i \in \mathbb{N}$ fixed let $\tau_i \in [0,L)$ be such that $\kappa_1(\tau_i) = x_i$ and let L_i be the length of κ_1 restricted to $[\tau_i, L)$. Then

$$L_i = \sum_{j=i}^{+\infty} d(x_j,x_{j+1}) = \frac{\varepsilon}{4}\sum_{j=i}^{+\infty} t_j = \frac{\varepsilon}{4}\sum_{k=0}^{\bar{k}-1}\sum_{j=0}^{+\infty} t_{i+k+j\bar{k}} \le \frac{\varepsilon}{4}\sum_{k=0}^{\bar{k}-1}\sum_{j=0}^{+\infty}\frac{1}{2^j}t_{i+k} \le \frac{\varepsilon\bar{k}t_i}{2}. \tag{3.1.20}$$

Now, since $t_0 = T = (1+\bar{t})\delta \le 2\mathrm{diam}(\gamma_{[0,\bar{t}]}) \le 2\mathrm{diam}(\gamma)$ and $\mathrm{diam}(\gamma) \le 1/\varepsilon d(x,y)$ by (3.1.11), when $i = 0$ we find

$$\mathrm{length}(\kappa_1) \le \frac{\varepsilon\bar{k}t_0}{2} \le \bar{k}d(x,y). \tag{3.1.21}$$

Let $\kappa : [0,L] \to \Omega$ be the continuous rectifiable curve parametrized by arc length defined by $\kappa(t) = \kappa_1(L-t)$. By (3.1.21) κ satisfies (3.1.14). Moreover, if $\kappa(t) \in B_j$ then by (3.1.20)

$$\mathrm{length}(\kappa_{[0,t]}) \le L_j + \frac{\varepsilon t_j}{4} \le \varepsilon\Big(\frac{1}{4} + \frac{\bar{k}}{2}\Big)t_j$$

and by (3.1.17)

$$\mathrm{dist}(\kappa(t);\partial\Omega) \ge \mathrm{dist}(x_j;\partial\Omega) - d(x_j,\kappa(t)) \ge \frac{\varepsilon t_j}{2} - \frac{\varepsilon t_j}{4} = \frac{\varepsilon t_j}{4},$$

so that

$$\mathrm{dist}(\kappa(t);\partial\Omega) \ge \varepsilon_1 \mathrm{length}(\kappa_{[0,t]})$$

with

$$\varepsilon_1 = \min\Big\{\frac{1}{\bar{k}}, \frac{\varepsilon}{4+2\bar{k}}\Big\}.$$

$\square$

REMARK 3.1.14. The following Proposition will be used in section 4 in Theorem 3.4.2. The curves constructed in this Theorem satisfy

$$d(\gamma(0),\gamma(t)) \simeq \mathrm{diam}(\gamma_{[0,t]}). \tag{3.1.22}$$

PROPOSITION 3.1.15. *Let (M,d) be a doubling metric space with geodesics and let $\Omega \subset M$ be an open set. Assume that there exists $\lambda > 0$ and $r > 0$ such that for any $z \in \partial\Omega$ and for all $x, y \in B(z,r) \cap \Omega$ there exist two John curves $\gamma_x, \gamma_y : [0,1] \to \Omega$ starting respectively from x and y, with John constant λ, such that $\gamma_x(1) = \gamma_y(1)$ and*

$$\max\{\mathrm{diam}(\gamma_x), \mathrm{diam}(\gamma_y)\} \leq \frac{1}{\lambda} d(x,y).$$

Assume also that (3.1.22) *holds. Then Ω is a uniform domain.*

Proof. Let γ be the curve sum of γ_x and γ_y. First of all

$$\mathrm{diam}(\gamma) \leq \mathrm{diam}(\gamma_x) + \mathrm{diam}(\gamma_y) \leq \frac{2}{\lambda} d(x,y).$$

Consider now a point $\gamma(t)$ and assume that $\gamma(t) = \gamma_x(t)$. Then

$$\begin{aligned}\mathrm{dist}(\gamma(t);\partial\Omega) &= \mathrm{dist}(\gamma_x(t);\partial\Omega) \geq \lambda d(\gamma_x(t),x) \\ &\simeq \lambda \mathrm{diam}\big((\gamma_x)_{[0,t]}\big) \geq \lambda \min\big\{\mathrm{diam}\big((\gamma_x)_{[0,t]}\big), \mathrm{diam}\big((\gamma_x)_{[t,1]}\big)\big\}.\end{aligned}$$

If $\gamma(t)$ is in γ_y the estimate is the same. The claim follows from Lemma 3.1.12 and Theorem 3.1.13. □

DEFINITION 3.1.16 (Harnack chain). Let (M,d) a metric space, let $\Omega \subset M$ be a domain and let $\alpha \geq 1$. A relatively compact set $K \subset \Omega$ is $\alpha-$*non tangential* in Ω if

$$\frac{1}{\alpha}\mathrm{dist}(K;\partial\Omega) \leq \mathrm{diam}(K) \leq \alpha \mathrm{dist}(K;\partial\Omega).$$

A sequence of balls $B_0, B_1, ..., B_k \subset \Omega$ is a $\alpha-$Harnack chain of Ω if

(i) $B_i \cap B_{i-1} \neq \emptyset$ for all $i = 1, ..., k$;
(ii) every ball B_i is $\alpha-$non tangential.

DEFINITION 3.1.17. Let (M,d) be a metric space. A bounded domain $\Omega \subset M$ is a *Harnack domain* if there exists $\alpha \geq 1$ such that for all $\eta > 0$ and for all $x, y \in \Omega$ such that $\mathrm{dist}(x;\partial\Omega) \geq \eta$, $\mathrm{dist}(y;\partial\Omega) \geq \eta$ and $d(x,y) \leq C\eta$ for some $C > 0$ there exists a $\alpha-$Harnack chain $B_0, B_1, ..., B_k \subset \Omega$ such that $x \in B_0$, $y \in B_k$ and k depends on C but not on η.

PROPOSITION 3.1.18. *Let (M,d) be a metric space and assume that there exists $0 < \delta \leq 2$ such that $\mathrm{diam}(B(x,r)) \geq \delta r$ for all $x \in X$ and $r \geq 0$. If $\Omega \subset M$ is a uniform domain then it is a Harnack domain.*

Proof. Let Ω be a uniform domain with constant $\varepsilon > 0$. Let $\eta > 0$ and take $x, y \in \Omega$ such that $\mathrm{dist}(x;\partial\Omega) \geq \eta$, $\mathrm{dist}(y;\partial\Omega) \geq \eta$ and $d(x,y) \leq C\eta$ for some constant $C > 0$. Let $\gamma : [0,1] \to \Omega$ be a continuous rectifiable curve such that $\gamma(0) = x$, $\gamma(1) = y$, $\mathrm{length}(\gamma) \leq d(x,y)/\varepsilon$ and (3.1.10) holds. Let $\bar{t} \in (0,1)$ be such that $\mathrm{length}(\gamma_{[0,\bar{t}]}) = \mathrm{length}(\gamma_{[\bar{t},1]})$. We shall construct a Harnack chain of balls joining x to $\gamma(\bar{t})$.

Define $B_0 = B(x, \mathrm{dist}(x;\partial\Omega)/4)$. Clearly,

$$\mathrm{diam}(B_0) \le \frac{1}{2}\mathrm{dist}(x;\partial\Omega) \le \mathrm{dist}(B_0;\partial\Omega),$$

and moreover

$$\mathrm{diam}(B_0) \ge \frac{\delta}{4}\mathrm{dist}(x;\partial\Omega) \ge \frac{\delta}{4}\mathrm{dist}(B_0;\partial\Omega).$$

With $\alpha_0 = \max\{1, 4/\delta\} = 4/\delta$ the ball B_0 is α_0-non tangential in Ω.

Define

$$t_1 = \sup\{t \in [0,\bar t] : \gamma(t) \in B_0\}, \quad x_1 = \gamma(t_1),$$
$$r_1 = \frac{1}{2}\mathrm{dist}(x_1;\partial\Omega), \quad B_1 = B(x_1, r_1).$$

By (3.1.10) $\mathrm{dist}(x_1;\partial\Omega) \ge \varepsilon \mathrm{length}(\gamma_{[0,t_1]})$ and

$$\mathrm{length}(\gamma_{[0,t_1]}) \ge d(x, x_1) \ge \frac{1}{4}\mathrm{dist}(x;\partial\Omega) \ge \frac{\eta}{4}.$$

Then

$$r_1 = \frac{1}{2}\mathrm{dist}(x;\partial\Omega) \ge \frac{\varepsilon}{2}\mathrm{length}(\gamma_{[0,t_1]}) \ge \frac{\varepsilon\eta}{8}. \tag{3.1.23}$$

Since $\mathrm{diam}(B_1) \le 2r_1 = \mathrm{dist}(x_1;\partial\Omega)$

$$2\mathrm{dist}(B_1;\partial\Omega) \ge \mathrm{dist}(x_1;\partial\Omega) \ge \mathrm{diam}(B_1),$$

and since by hypothesis $\delta r_1 \le \mathrm{diam}(B_1)$

$$\mathrm{dist}(B_1;\partial\Omega) \le \mathrm{dist}(x_1;\partial\Omega) = 2r_1 \le \frac{2}{\delta}\mathrm{diam}(B_1).$$

With $\alpha := \max\{\alpha_0, 2, \delta/2\} = \alpha_0$ both B_0 and B_1 are $\alpha-$non tangential.

By induction assume that x_{k-1} and B_{k-1} have been already defined. Now, if $d(x_{k-1}, \gamma(\bar t)) < \varepsilon\eta/8$ we stop. Otherwise we define

$$t_k = \sup\{t \in [0,\bar t] : \gamma(t) \in B_{k-1}\}, \quad x_k = \gamma(t_k),$$
$$r_k = \frac{1}{2}\mathrm{dist}(x_k;\partial\Omega), \quad B_k = B(x_k, r_k).$$

Exactly as above we have

$$2\mathrm{dist}(B_k;\partial\Omega) \ge \mathrm{dist}(x_k;\partial\Omega) \ge \mathrm{diam}(B_k),$$

and moreover

$$\mathrm{dist}(B_k;\partial\Omega) \le \mathrm{dist}(x_k;\partial\Omega) = 2r_k \le \frac{2}{\delta}\mathrm{diam}(B_k).$$

This shows that B_k is $\alpha-$non tangential.

By (3.1.10), arguing as in (3.1.23)

$$\mathrm{dist}(x_k;\partial\Omega) \ge \varepsilon\mathrm{length}(\gamma_{[0,t_k]}) \ge \varepsilon\mathrm{length}(\gamma_{[0,t_1]}) \ge \varepsilon d(x, x_1) \ge \frac{\varepsilon\eta}{4},$$

and thus

$$r_k \ge \frac{\varepsilon\eta}{8}. \tag{3.1.24}$$

Assume that $d(x,y) \le C\eta$ for some $C > 0$. We claim that that there exists $k \in \mathbb{N}$ depending on C but not on η such that $d(x_k, \gamma(\bar t)) < \varepsilon\eta/8$ so that $B_0, B_1, ..., B_k$ cover $\gamma([0,\bar t])$.

First of all by (3.1.9)

$$\text{length}(\gamma_{[0,\bar{t}]}) = \frac{1}{2}\text{length}(\gamma) \le \frac{1}{2\varepsilon} d(x,y) \le \frac{C\eta}{2\varepsilon}.$$

Moreover, for any i in the inductive definition using (3.1.24) we obtain

$$\text{length}(\gamma_{[t_i,t_{i-1}]}) \ge d(x_i, x_{i-1}) \ge r_i \ge \frac{\varepsilon\eta}{8}$$

where $t_0 = 0$, so that

$$\text{length}(\gamma_{[t_k,0]}) = \sum_{i=1}^{k} \text{length}(\gamma_{[t_i,t_{i-1}]}) \ge \frac{k\varepsilon\eta}{8}.$$

The condition on k which proves the claim is

$$\frac{k\varepsilon\eta}{8} \ge \frac{C\eta}{2\varepsilon} \quad \Leftrightarrow \quad k \ge \frac{4C}{\varepsilon^2}.$$

Thus k can be chosen independently from η.

$\square$

2. Non characteristic boundary

In this section we begin the study of regular domains in C-C spaces. We shall prove that a bounded smooth domain without characteristic points is uniform with respect to the C-C metric induced by a system of Hörmander vector fields $X = (X_1, ..., X_m)$.

If $\Omega \subset \mathbb{R}^n$ is an open set with regular boundary, $x \in \partial\Omega$ and $\Phi = 0$ is a local equation for $\partial\Omega$ in a neighborhood of x, then the point x is non characteristic if there exists $j = 1, ..., m$ such that $X_j\Phi(x) \ne 0$. If every $x \in \partial\Omega$ is non characteristic then Ω is said to be non characteristic.

THEOREM 3.2.1. *Let $(\mathbb{R}^n, d)$ be the C-C space associated with a family $X = (X_1, ..., X_m)$ of Hörmander vector fields and let $\Omega \subset \mathbb{R}^n$ be a (Euclidean) bounded domain with boundary of class C^∞. If Ω is non characteristic then it is a uniform domain.*

Before proving Theorem 3.2.1 we shall establish some Lemmata. First we recall that a non characteristic surface can be made flat by a diffeomorphism and that a resulting transversal vector field can be orthogonalized and the other ones can be made lie on the surface.

LEMMA 3.2.2. *Let $\mathcal{U} \subset \mathbb{R}^n$ be a neighborhood of $0 \in \mathbb{R}^n$ and let $Y \in C^\infty(\mathcal{U}; \mathbb{R}^n)$ be a vector field such that $\langle Y(0), \mathrm{e}_n \rangle \ne 0$. Let $x_n = g(x_1, \ldots, x_{n-1}) = g(x')$ be a function of class C^∞ such that $g(0) = 0$ and $\nabla g(0) = 0$. Possibly shrinking $\mathcal{U}$, there exists a diffeomorphism $\Phi \in C^\infty(\mathcal{U}; \mathbb{R}^n)$ such that $d\Phi(x)Y(x) = \mathrm{e}_n$ for all $x \in \mathcal{U}$ and $\Phi(x', g(x')) = (x', 0)$ for all $(x', g(x')) \in \mathcal{U}$.*

The proof of Lemma 3.2.2 can be essentially found in [**85**] where even less regularity is required.

REMARK 3.2.3. Let $\widetilde{X}_1, ..., \widetilde{X}_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ satisfy the Hörmander condition and induce the C-C metric $\widetilde{d}$. Write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and assume the vector fields are of the form

$$\widetilde{X}_j = b_j(x)\partial_n + \sum_{i=1}^{n-1} a_{ij}(x)\partial_i, \; j = 1, ..., m-1, \quad \widetilde{X}_m = \partial_n.$$

The new family of vector fields

$$X_j = \sum_{i=1}^{n-1} a_{ij}(x)\partial_i, \; j = 1, ..., m-1, \quad X_m = \partial_n. \tag{3.2.25}$$

still satisfies the Hörmander condition. Moreover, if d is the corresponding C-C metric and $K \subset \mathbb{R}^n$ is a compact set, there exist c_1 and c_2 such that on K

$$c_1\widetilde{d} \le d \le c_2\widetilde{d} \quad \text{and} \quad c_1|\widetilde{X}u| \le |Xu| \le c_2|\widetilde{X}u| \tag{3.2.26}$$

for all $u \in C^1$. A proof of the equivalence between d and $\widetilde{d}$ can be found in [**85**]. Actually, it can be proved that each one of the two equivalences in (3.2.26) implies the other one (see [**100**, Theorem 11.11]).

The notations I, $\mathcal{I}$, $\|\xi\|_I$ with $\xi \in \mathbb{R}^{n-1}$, $\Phi_{I,x'}$, Y_i, $d(Y_i)$, S_1 and S_2 have been introduced in chapter 1, section 6, subsection 6.3.

LEMMA 3.2.4. *Let $(\mathbb{R}^n, d)$ be the C-C spaces induced by the Hörmander vector fields $X_1, ..., X_m \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ of the form (1.6.65). Let $K \subset \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n = 0\}$ be a bounded set. There exists a constant $\alpha \ge 1$ such that for all $x', y' \in K$ there exists a rectifiable curve parametrized by arclength curve $\gamma : [0, t_0] \to \mathbb{R}^n$ such that:*

(i) *$\gamma(0) = (x', 0)$, $\gamma(t_0) = (y', 0)$ and $t_0 \le \alpha d((x', 0), (y', 0))$;*
(ii) *if $\gamma = (\gamma_1, ..., \gamma_n)$ then $\gamma_n(t) \ge 0$ for all $t \in [0, t_0]$;*

Proof. Let $K \subset \Omega_0$ for some bounded open set $\Omega_0 \subset \mathbb{R}^n$. Let $\bar{k}$ be the minimum length of commutators that assures the Hörmander condition on Ω_0.

We are in order to apply Theorem 1.6.10. Fix the constants $0 < a < b$ and $r_0 > 0$ as in Theorem 1.6.10. Possibly using a covering argument assume that $d((y', 0), (x', 0)) \le ar_0$. There exists a multi-index $I = (i_1, ..., i_{n-1}) \in \mathcal{I}$ satisfying (1.6.70) and there exists $\xi \in \mathbb{R}^{n-1}$ such that $y' = \Phi_{I,x'}(\xi)$. Suppose for the sake of simplicity that $\xi_k \ge 0$ for all $k = 1, ..., n-1$.

By Theorem 1.6.12 we can write

$$\Phi_{I,x'}(\xi) = \prod_{k=1}^{n-1}\prod_{l=1}^{N_k} S_{\sigma_{lk}}(d_{lk}\xi_k^{1/d(Y_{i_k})}, \tau_{lk}X_{j_{lk}})(x'),$$

with $\sigma_{lk} \in \{1, 2\}$, $\tau_{lk} \in \{-1, 1\}$, $d_{lk} \le \bar{k}$, $j_{lk} \in \{1, ..., m-1\}$ and N_k less than a constant not depending on x' and y'.

We show how to define the curve γ relatively to the factor $S_{\sigma_{lk}}(d_{lk}\xi_k^{1/d(Y_{i_k})}, \tau_{lk}X_{j_{lk}})$. If, for instance, $\sigma_{lk} = 1$ then consider

$$\exp(d_{lk}\xi_k^{1/d(Y_{i_k})}(\tau_{lk}X_{j_{lk}} - X_m)) \exp(d_{lk}\xi_k^{1/d(Y_{i_k})}X_m)(\bar{x}')$$

for some $\bar{x}' \in \mathbb{R}^{n-1}$. Set $\bar{t} = d_{lk}\xi_k^{1/d(Y_{i_k})}$ and define $\gamma : [0, 2\bar{t}] \to \mathbb{R}^n$ by

$$\gamma(t) = \begin{cases} \exp(tX_m)(\bar{x}') & \text{if } 0 \le t \le \bar{t} \\ \exp((t-\bar{t})(\tau_{lk}X_{j_{lk}} - X_m))(\exp(\bar{t}X_m)(\bar{x})) & \text{if } \bar{t} \le t \le 2\bar{t} \end{cases}$$

The curve γ obtained joining curves of the type just defined is rectifiable and is parametrized over an interval whose total length is bounded by $C\|\xi\|_I$ with C constant not depending on $x', y' \in K$. By (1.6.73) claim (ii) is verified. Moreover, by Theorem 1.6.10

$$\|\xi\|_I \le \frac{b}{a} d((x', 0), (y', 0)),$$

and claim (i) is verified with $\alpha = Cb/a$. □

Lemma 3.2.5. *Let $(\mathbb{R}^n, d)$ be C-C spaces induced by the Hörmander vector fields $X_1, ..., X_m \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Let $\Omega \subset \mathbb{R}^n$ be an open set with C^∞ boubdary. If $x_0 \in \partial\Omega$ is a non characteristic point then there exists a neighborhood $\mathcal{U}$ of x_0 such that for all $x, y \in \Omega \cap \mathcal{U}$ there exists a continuous rectifiable curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$ and (3.1.9), (3.1.10) hold.*

Proof. By Lemma 3.2.2 and Remark 3.2.3 we can assume without loss of generality that $x_0 = 0$, $\Omega = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$ and $X_1, ..., X_m$ are of the form (1.6.65).

Let $\mathcal{U}$ be a bounded neighborhood of the origin, let $x, y \in \Omega \cap \mathcal{U}$ and assume that $x_n \le y_n$. Define $\delta := d(x, y)$ and notice that $\delta \ge y_n - x_n$. If $\bar{x} := (x', x_n + \delta)$ and $\bar{y} := (y', x_n + \delta)$ then $d(\bar{x}, \bar{y}) \le d(\bar{x}, x) + d(x, y) + d(y, \bar{y}) \le 3\delta$. By Lemma 3.2.4 there exists a rectifiable curve parametrized by arclength $\bar{\gamma} : [0, \bar{T}] \to \Omega$ such that $\bar{\gamma}(0) = (\bar{x})$, $\bar{\gamma}(\bar{T}) = (\bar{y})$, $\bar{\gamma}_n(t) \ge x_n + \delta$ for all $t \in [0, \bar{T}]$ and $\bar{T} \le \alpha d(\bar{x}, \bar{y}) \le 3\alpha\delta$. We can also assume $\bar{T} \ge \delta$. Let $T := \delta + \bar{T} + (y_n - x_n)$ and define $\gamma : [0, T] \to \Omega$ by

$$\gamma(t) = \begin{cases} (x', x_n + t) & \text{if } 0 \le t \le \delta \\ \bar{\gamma}(t - \delta) & \text{if } \delta \le t \le \delta + \bar{T} \\ (y', x_n + \delta - t) & \text{if } \delta + \bar{T} \le t \le \delta + \bar{T} + (y_n - x_n). \end{cases}$$

Since $T \le 2\delta + \bar{T} \le (2 + 3\alpha)\delta = (2 + 3\alpha)d(x, y)$ and γ is parametrized by arclength then $\text{length}(\gamma) \le T =\le (2 + 3\alpha)d(x, y)$ and condition (3.1.9) holds (possibly up to a reparametrization of γ on $[0, 1]$).

We have to check condition (3.1.10). As $\text{length}(\bar{\gamma}) \ge \delta$ then

$$\min\{\text{length}(\gamma_{[0,t]}), \text{length}(\gamma_{[t,T]})\} = \begin{cases} \text{length}(\gamma_{[0,t]}) & \text{if } t \in [0, \delta] \\ \text{length}(\gamma_{[t,T]}) & \text{if } t \in [\delta + \bar{T}, T]. \end{cases}$$

It will be enough to prove that for all $t \in [0, \delta + \bar{T}]$

$$\text{dist}(\gamma(t); \partial\Omega) \ge \varepsilon \text{length}(\gamma_{[0,t]}) \tag{3.2.27}$$

for some uniform constant $\varepsilon > 0$. If $t \in [0, \delta]$

$$\text{dist}(\gamma(t); \partial\Omega) = x_n + t \ge t = \text{length}(\gamma_{[0,t]}).$$

If $t \in [\delta, \delta + \bar{T}]$ then

$$\text{dist}(\gamma(t); \partial\Omega) = x_n + \delta \ge \delta,$$

whereas

$$\text{length}(\gamma_{[0,t]}) \le \delta + \bar{T} \le (1 + 3\alpha)\delta.$$

Therefore we get (3.2.27) with $\varepsilon = 1/(1 + 3\alpha)$. □

Proof of Theorem 3.2.1. By Lemma 3.2.5 and Theorem 3.1.12 the thesis immediately follows.

3. John and uniform domains in Grushin space

Let $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$ and consider the vector fields

$$X_1 = \partial_{x_1}, \dots, X_{n-1} = \partial_{x_{n-1}}, X_n = |x|^\alpha \partial_y, \tag{3.3.28}$$

where $\alpha > 0$. Let $(\mathbb{R}^n, d)$ the C-C space associated with such vector fields (see chapter 1 section 9).

DEFINITION 3.3.1. Let $\Omega \subset \mathbb{R}^n$ be a connected open set with Lipschitz boundary such that $\partial\Omega$ is of class C^1 in a neighborhood of every point $(0, y) \in \partial\Omega$.

A point $(0, y) \in \partial\Omega$ will be said *flat* if there exist a neighborhood $\mathcal{V}$ of $(0, y)$ and a neighborhood $\mathcal{U}$ of the origin in $\mathbb{R}^{n-1}$ such that $\partial\Omega \cap \mathcal{V} = \{(x, \varphi(x)) : x \in \mathcal{U}\}$ for some $\varphi \in C^1(\mathcal{U}; \mathbb{R})$ with $\nabla\varphi(0) = 0$. A flat point $(0, y) \in \partial\Omega$ will be said $\alpha-$*admissible* if there exists a constant $C > 0$ such that

$$|\nabla\varphi(x)| \leq C|x|^\alpha \quad \text{for all } x \in \mathcal{U}. \tag{3.3.29}$$

Finally, Ω will be said $\alpha-$*admissible* if flat points in $\partial\Omega$ are $\alpha-$admissible or if Ω has no flat points.

REMARK 3.3.2. Let $\Omega = \{(x, y) \in \mathbb{R}^n : y > \varphi(x)\}$ where $\varphi \in C^1(\mathbb{R}^{n-1})$ is a function such that $\varphi(0) = 0$, $|\nabla\varphi(x)| \leq c|x|^\alpha$ for all $x \in \mathbb{R}^{n-1}$ and for some $c \geq 0$. The surface $\partial\Omega \subset (\mathbb{R}^n, d)$ is bilipschitz equivalent to $(\mathbb{R}^{n-1}, |\cdot|)$. Indeed consider $\Phi : (\mathbb{R}^{n-1}, |\cdot|) \to (\mathbb{R}^n, d)$ defined by $\Phi(x) = (x, \varphi(x))$. If $x, \bar{x} \in \mathbb{R}^{n-1}$ are such that $|\bar{x}| \leq |x|$ then

$$|\varphi(x) - \varphi(\bar{x})| \leq c|x|^\alpha |x - \bar{x}| \leq 2c|x|^{\alpha+1},$$

and by Proposition 1.9.1

$$\begin{aligned} |x - \bar{x}| \leq d(\Phi(x), \Phi(\bar{x})) &= d((x, \varphi(x)), (\bar{x}, \varphi(\bar{x}))) \\ &\leq |x - \bar{x}| + \frac{|\varphi(x) - \varphi(\bar{x})|}{|x|^\alpha} \leq (1 + c)|x - \bar{x}|. \end{aligned}$$

THEOREM 3.3.3. *If $\Omega \subset \mathbb{R}^n$ is a bounded $\alpha-$admissible domain then it is a uniform domain in $(\mathbb{R}^n, d)$.*

Proof. By Theorem 3.1.12 the uniformity is a local property of the boundary. If $(x, y) \in \partial\Omega$ and $x \neq 0$ then $\partial\Omega$ is Lipschitz in a neighborhood of (x, y) and the uniform property in this neighborhood follows as for Euclidean Lipschitz domains in $\mathbb{R}^n$. We have to check that there exist connecting curves that satisfy conditions (3.1.9) and (3.1.10) in a neighborhood of a point $(0, y) \in \partial\Omega$. We may assume $y = 0$. Let $\mathcal{U} \subset \mathbb{R}^{n-1}$ be a neighborhood of 0 and let $\varphi \in C^1(\mathcal{U})$ be a function such that $\{(x, \varphi(x)) : x \in \mathcal{U}\} = \partial\Omega \cap \mathcal{V}$, being $\mathcal{V} \subset \mathbb{R}^n$ a neighborhood of 0, $\varphi(0) = 0$ and

$$|\nabla\varphi(x)| \leq k|x|^\alpha \quad \text{for all } x \in \mathcal{U}.$$

We can assume $\mathcal{U} = \{x \in \mathbb{R}^{n-1} : |x| < r_0\}$ for some $r_0 > 0$ and $\Omega \cap \mathcal{V} = \{(x, y) \in \mathbb{R}^n : x \in \mathcal{U}, \, y > \varphi(x)\}$.

Let $(x, y), (\xi, \eta) \in \Omega$ with $x, \xi \in \mathcal{U}$. Assume that $|\xi| \leq |x|$ and $\eta \leq y$. We shall distinguish two cases:

(A) $|x|^{\alpha+1} \geq |y-\eta|$;
(B) $|x|^{\alpha+1} < |y-\eta|$.

First, we discuss how to connect the points (x,η) and (x,y), $y \geq \eta$, by a rectifiable curve whose length is comparable with the distance between the points. In case (A) take the curve

$$\gamma(t) = \exp(tX_n)(x,\eta) = (x, \eta + |x|^\alpha t), \quad 0 \leq t \leq \frac{|y-\eta|}{|x|^\alpha}, \tag{3.3.30}$$

whose length is (by Theorem 1.9.1)

$$\text{length}(\gamma) = \frac{|y-\eta|}{|x|^\alpha} \leq c\, d((x,y),(x,\eta)). \tag{3.3.31}$$

In Case (B) the curve is constructed in the following way. We introduce a parameter $\beta > 0$ that will be fixed later. Let

$$T = \frac{2}{\beta}\left[|x|^{\alpha+1} + \frac{\beta(\alpha+1)}{2}|y-\eta|\right]^{1/(\alpha+1)} - \frac{2|x|}{\beta}, \tag{3.3.32}$$

and define $\gamma : [0,T] \to \mathbb{R}^n$ by ($\mathrm{v} := x/|x|$)

$$\begin{aligned}\gamma(t) &= \exp(t(X_n + \beta \sum_{i=1}^n X_i))(x,\eta) \\ &= \left(x + \beta t\mathrm{v}, \eta + \frac{1}{\beta(\alpha+1)}(|x+\beta t\mathrm{v}|^{\alpha+1} - |x|^{\alpha+1})\right),\end{aligned} \tag{3.3.33}$$

if $0 \leq t \leq T/2$ and

$$\gamma(t) = \exp(t(X_n - \beta \sum_{i=1}^n X_i))(\gamma(T/2))$$

if $T/2 < t \leq T$. It can be checked that $\gamma(T) = (x,y)$. The length of γ is estimated by

$$\text{length}(\gamma) \leq kT \leq \bar{k}|y-\eta|^{1/(\alpha+1)} \leq c\bar{k}d((x,y),(x,\eta)), \tag{3.3.34}$$

where $\bar{k}$ is a constant that depends only on α and β.

Let now $(x,y),(\xi,\eta) \in \Omega$ be such that $|\xi| \leq |x| \leq r_0/2$ and $y \geq \eta$, and write $d := d((x,y),(\xi,\eta))$. Let $\lambda > 0$ be a constant that will be fixed later and fix $\delta > 0$ such that

$$d((x,y),(x,y+\delta)) = \lambda d. \tag{3.3.35}$$

The points (x,y) and (ξ,η) will be connected by a rectifiable curve γ piecewise defined in the following way

(1) a path $\gamma^{(1)}$ which joins (x,y) to $(x,y+\delta)$;
(2) a path $\gamma^{(2)}$ which joins $(x,y+\delta)$ to $(\xi,y+\delta)$;
(3) a path $\gamma^{(3)}$ which joins $(\xi,y+\delta)$ to (ξ,η).

We begin with (1). In Case (A), that is $|x|^{\alpha+1} \geq \delta$, take $\gamma^{(1)}(t) = (x, y+|x|^\alpha t)$ with $0 \leq t < \delta/|x|^\alpha$. By (3.3.31) and (3.3.35) $\text{length}(\gamma^{(1)}) \leq c\, d((x,y),(x,y+\delta)) = c\lambda d$.

We claim that there exists a constant $k_1 > 0$ such that the inequality

$$d(\gamma(t),\partial\Omega) \geq \varepsilon d(\gamma(t),(x,y)), \quad 0 \leq t \leq \delta/|x|^\alpha \tag{3.3.36}$$

holds as soon as $\varepsilon \leq k_1$.

Since $d(\gamma(t),(x,y)) \leq t$, we have to show that $d(\gamma(t),\partial\Omega) \geq \varepsilon t$. This is true if and only if $B(\gamma(t),\varepsilon t)\cap\partial\Omega = \emptyset$, and by (1.9.103) a sufficient condition is $\mathrm{Box}(\gamma(t),c_2\varepsilon t)\cap\partial\Omega = \emptyset$, and this amounts to

$$\varphi(x+\mathrm{v}) < y + |x|^\alpha t - c_2\varepsilon t(|x| + c_2\varepsilon t)^\alpha$$

for all $t \in [0,\delta/|x|^\alpha]$ and $|\mathrm{v}_i| \leq c_2\varepsilon t$, $i = 1,\dots,n-1$, and since $y > \varphi(x)$ we find the stronger condition

$$L := t|x|^\alpha \geq c_2\varepsilon t(|x| + c_2\varepsilon t)^\alpha + \varphi(x+\mathrm{v}) - \varphi(x) := R. \tag{3.3.37}$$

By case (A) $t \leq \delta/|x|^\alpha \leq |x|$ and using (3.3.29) (we can assume $x+\mathrm{v} \in \mathcal{U}$ and $\varepsilon \leq 1$)

$$\begin{aligned}|\varphi(x+\mathrm{v}) - \varphi(x)| &\leq k|\mathrm{v}||x+\mathrm{v}|^\alpha \leq k|\mathrm{v}|(|x| + |\mathrm{v}|)^\alpha \leq kc_2\varepsilon t(|x| + c_2\varepsilon t)^\alpha \\ &\leq kc_2\varepsilon t(1+c_2)^\alpha|x|^\alpha.\end{aligned}$$

Analogously $(|x| + c_2\varepsilon t)^\alpha \leq (1+c_2)^\alpha|x|^\alpha$. Thus

$$L = t|x|^\alpha \geq \varepsilon t|x|^\alpha c_2(1+c_2)^\alpha(1+k) \geq R$$

as soon as $\varepsilon \leq k_1 := [c_2(1+c_2)^\alpha(1+k)]^{-1}$.

In case (B) we choose $\gamma^{(1)} : [0,T] \to \mathbb{R}^n$ of the form (3.3.33) for a suitable $\beta > 0$ with T as in (3.3.32). We claim that there exists a constant $k_2 > 0$ such that if $\varepsilon \leq \beta \leq k_2$ then $d(\gamma^{(1)}(t),\partial\Omega) \geq \varepsilon t$ for all $0 \leq t \leq T$. It suffices to show that $\mathrm{Box}(\gamma^{(1)}(t),c_2\varepsilon t)\cap\partial\Omega = \emptyset$ for $0 \leq t \leq T/2$, that is

$$\varphi(x+\beta t\mathrm{v}+\mathrm{w}) < y + \frac{1}{\beta(\alpha+1)}(|x+\beta t\mathrm{v}|^{\alpha+1} - |x|^{\alpha+1}) - c_2\varepsilon t(|x+\beta t\mathrm{v}| + c_2\varepsilon t)^\alpha,$$

for all $0 \leq t \leq T/2$ and $|\mathrm{w}_i| \leq c_2\varepsilon t$ ($\mathrm{v} := x/|x|$). Since $y > \varphi(x)$ we find the stronger condition

$$\begin{aligned}L :&= \varphi(x+\beta t\mathrm{v}+\mathrm{w}) - \varphi(x) + c_2\varepsilon t(|x+\beta t\mathrm{v}| + c_2\varepsilon t)^\alpha \\ &\leq \frac{1}{\beta(\alpha+1)}\big(|x+\beta t\mathrm{v}|^{\alpha+1} - |x|^{\alpha+1}\big) := R.\end{aligned} \tag{3.3.38}$$

Now,

$$\begin{aligned}|\varphi(x+\beta t\mathrm{v}+\mathrm{w}) - \varphi(x)| &\leq k|\beta t\mathrm{v}+\mathrm{w}||x(1+(\beta t + c_2\varepsilon t)/|x|)|^\alpha \\ &\leq kt(\beta + c_2\varepsilon)(|x| + t(\beta + c_2\varepsilon))^\alpha,\end{aligned}$$

and taking $\varepsilon \leq \beta$

$$\begin{aligned}L &\leq k\beta t(1+c_2)(|x| + \beta t(1+c_2))^\alpha + c_2\beta t(|x| + \beta t(1+c_2))^\alpha \\ &\leq \beta t(k(1+c_2) + c_2)(|x| + \beta t(1+c_2))^\alpha \\ &\leq \beta t\bar{k}_1(|x| + \beta t)^\alpha,\end{aligned}$$

with $\bar{k}_1$ depending on c_1, c_2 and α.

On the other hand, by the mean value theorem

$$\begin{aligned}R &\geq \frac{1}{\beta(\alpha+1)}\big(|x+\beta t\mathrm{v}|^{\alpha+1} - |x+(\beta/2)t\mathrm{v}|)^{\alpha+1}\big) \\ &\geq \frac{1}{\beta}|(\beta/2)t\mathrm{v}||x+(\beta/2)t\mathrm{v}|^\alpha \\ &\geq t\bar{k}_2(|x| + \beta t)^\alpha.\end{aligned}$$

Thus (3.3.38) holds as soon as $\beta \le k_2 := \bar{k}_2/\bar{k}_1$.

The definition of $\gamma^{(3)}$ is similar to that of $\gamma^{(1)}$. The estimate of the distance from the boundary is identical, while the estimate of the length follows from (3.3.35). Indeed

$$\begin{aligned} d((\xi,\eta),(\xi,y+\delta)) &\le d((\xi,\eta),(x,y)) + d((x,y),(x,y+\delta)) + d((x,y+\delta),(\xi,y+\delta)) \\ &= (2+\lambda)d, \end{aligned}$$

and (recall (3.3.34))

$$\text{length}(\gamma^{(3)}) \le \bar{k}c\, d((\xi,\eta),(\xi,y+\delta)) \le \bar{k}c(2+\lambda)d.$$

The curve $\gamma^{(2)}$ is the horizontal line

$$\gamma^{(2)}(t) := \left(x + \frac{\xi - x}{|\xi - x|}t, y+\delta\right), \quad 0 \le t \le |\xi - x|.$$

Clearly, $\text{length}(\gamma^{(2)}) = |\xi - x| \le d$.

The total length of γ can be now easily estimated

$$\text{length}(\gamma) = \text{length}(\gamma^{(1)}) + \text{length}(\gamma^{(2)}) + \text{length}(\gamma^{(3)}) \le Ld$$

where L is a constant depending only on k and α. This is (3.1.9).

By the analysis of $\gamma^{(1)}$ and by (3.3.35)

$$d((x,y+\delta),\partial\Omega) \ge \min\{k_1,k_2\} d((x,y+\delta),(x,y)) = \min\{k_1,k_2\}\lambda d = d,$$

if we choose $\lambda = \min\{k_1,k_2\}^{-1}$. On the other hand

$$\begin{aligned} d(\gamma^{(2)}(t),\partial\Omega) &\ge d((x,y+\delta),\partial\Omega) \ge d((x,y),(\xi,\eta)) \\ &\ge L^{-1}\text{length}(\gamma) \ge L^{-1}\min\{d(\gamma^{(2)}(t),(x,y)), d(\gamma^{(2)}(t),(\xi,\eta))\}. \end{aligned}$$

This proves (3.1.10) relatively to $\gamma^{(2)}$ and the proof is ended. □

We shall now show that the condition of $\alpha-$admissibility is sharp in the sense that a domain of class C^1 that is not $\alpha-$admissible is not a John domain. We consider in $\mathbb{R}^2$ the vector fields $X_1 = \partial_x$ and $X_2 = |x|^\alpha \partial_y$.

Remark 3.3.4. It can be easily seen that in the metric space we are dealing with the Definition of John domain 3.1.1 could have been equivalently given requiring that any x in the closure of Ω can be connected with x_0 in such a way that (3.1.1) holds.

Proposition 3.3.5. *Let $\Omega \subset \mathbb{R}^2$ be a C^1 domain, assume that $0 \in \partial\Omega$ is a characteristic point and that in a neighborhood of 0 we have $\Omega = \{y > \varphi(x)\}$ where $\varphi \in C^1(-\delta,\delta)$ is a function such that $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\varphi(x) > c|x|^\beta$ for all $x \in (-\delta,\delta)$, for some $c > 0$ and for some $\beta < \alpha + 1$. Then Ω is not a John domain in $(\mathbb{R}^2, d)$.*

Proof. In view of Remark 3.3.4 it will be enough to prove that for any $\varepsilon > 0$, for any $t_0 > 0$ and for any rectifiable continuous curve parametrized by arclength $\gamma : [0,t_0] \to \mathbb{R}^2$ such that $\gamma(0) = 0$ there exists $t \in [0,t_0]$ such that $\text{dist}(\gamma(t);\partial\Omega) < \varepsilon t$. Because of Lemma 1.9.3 this is implied by

$$\text{Box}(\gamma(t), c_1\varepsilon t) \cap \partial\Omega \ne \emptyset, \tag{3.3.39}$$

with $c_1 > 0$. Consider a curve γ solution of

$$\begin{cases} \dot\gamma = h_1 X_1(\gamma) + h_2 X_2(\gamma) \\ \gamma(0) = 0 \end{cases}$$

with the condition $h_1^2 + h_2^2 = 1$ a.e. , that is

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) = \Big(\int_0^t h_1(s)\, ds, \int_0^t h_2(s) \Big| \int_0^s h_1(\tau) d\tau \Big|^\alpha ds \Big).$$

Relation (3.3.39) is implied by

$$\varphi(\gamma_1(t) + c_1\varepsilon t) > \gamma_2(t)$$

and thus by

$$L := c\Big(\Big| \int_0^t h_1(s)\, ds \Big| + c_1\varepsilon t \Big)^\beta > \int_0^t h_2(s) \Big| \int_0^s h_1(\tau) d\tau \Big|^\alpha ds := R. \tag{3.3.40}$$

But $L \geq c(c_1\varepsilon t)^\beta$ and since $|h_1|, |h_2| \leq 1$

$$R \leq \int_0^t s^\alpha\, ds = \frac{t^{\alpha+1}}{\alpha+1}.$$

Inequality (3.3.40) holds if $c(\alpha+1)(c_1\varepsilon t)^\beta > t^{\alpha+1}$, which is true for all $t > 0$ small enough since $\beta < \alpha + 1$. □

Example 3.3.6. Carnot-Carathéodory balls need not be uniform domains. Consider in $\mathbb{R}^2$ the Grushin vector fields $X_1 = \partial_x$ and $X_2 = x\partial_y$ and let $(\mathbb{R}^2, d)$ be the induced C-C space. Let $B = B(0,1)$ be the C-C ball centered at the origin with radius 1. The ball B is $x-$ and $y-$symmetric, and using the geodesics equations (1.9.104) it can be shown that

$$\partial B \cap \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\} = \Big\{ (x(\vartheta), y(\vartheta)) = \Big(\frac{\sin\vartheta}{\vartheta}, \frac{2\vartheta - \sin 2\vartheta}{4\vartheta^2} \Big) : 0 \leq \vartheta \leq \pi \Big\}.$$

Since

$$(x'(\pi), y'(\pi)) = \Big(-\frac{1}{\pi}, -\frac{1}{\pi^2} \Big)$$

we can put above and outside B a cone with axis in the direction $(0,1)$, vertex at the "north pole" $N = (0, 1/(2\pi))$ and angular opening $2\arctan\pi$. If $0 < \arctan\beta < \pi/2 - \arctan\pi$ then $(x, 2/\pi + \beta x) \in B$ for all $0 < x \leq x_0$ for some $x_0 > 0$ depending on β.

Consider the points $P = (x, 2/\pi + \beta x)$ and $Q = (-x, 2/\pi + \beta x)$ with $0 < x \leq x_0$. Then $d(P,Q) = 2x$. If $\gamma : [0,T] \to B$ is any rectifiable curve such that $\gamma(0) = P$ and $\gamma(T) = Q$

$$\text{length}(\gamma) \geq d(P,N) + d(N,Q).$$

By Proposition 1.9.1 (with $m = 1$, $k = 1$, $\alpha = 1$ and $\lambda = \beta$), as $x^2 < \beta x$ if $x < \beta$, we have

$$d(P,N) = d(N,Q) \simeq x + (\beta x)^{1/2}.$$

Thus $d(P,N) = d(N,Q) \geq Cx^{1/2}$ for some $C > 0$ and for all small x. Therefore we find for all $0 < x \leq x_0$

$$\text{length}(\gamma) \geq 2Cx^{1/2} = \frac{C}{x^{1/2}} d(P,Q).$$

This shows that condition (3.1.9) can not hold.

4. Uniform domains in groups of step 2

In this section we study uniform domains in homogeneous groups of step 2. We shall work in $\mathbb{R}^n$ endowed a left invariant metric induced by a system of vector fields $X = (X_1, ..., X_m)$ which generates a stratified Lie algebra of step 2. In $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^q$ we denote $x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^q$ and by abuse of notation we shall write $x' = (x', 0)$ and $x'' = (0, x'')$. We say that x' are the variables of the first slice and that x'' are the variables of the second slice.

The vector fields can be assumed to be of the form

$$X_j = \partial_j + \sum_{k=m+1}^{n} q_{jk}\partial_k, \quad j = 1, ..., m,$$

where $q_{jk} = q_{jk}(x')$ are homogeneous polynomials of degree 1 in the variables x'. Introduce the group law

$$\begin{aligned} x \cdot y &= x + y + Q(x, y), \\ &= \big(x_1 + y_1, \dots, x_m + y_m, x_{m+1} + y_{m+1} + Q_{m+1}(x, y), \dots, Q_n(x, y)\big), \end{aligned} \tag{3.4.41}$$

where $Q = (Q_1, ..., Q_n)$ with $Q_1 = ... = Q_m = 0$, and $Q_j = Q_j(x', y')$, $j = m+1, ..., n$, are homogeneous polynomials of degree 2 that can be assumed to satisfy

$$|Q_j(x', y')| \le C|x'||y'|. \tag{3.4.42}$$

We may assume that the vector fields $X_1, \dots, X_m$ are left invariant with respect to the introduced law.

We denote by d the Carnot-Carathéodory distance induced on $\mathbb{R}^n$ by $X_1, ..., X_m$ and by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^n$ with radius $r \ge 0$. We also introduce in $\mathbb{R}^n$ the following continuous homogeneous norm

$$\|x\| = |x'| + |x''|^{1/2}. \tag{3.4.43}$$

By a standard argument it can be proved that

$$d(x, y) \simeq \|y^{-1} \cdot x\|. \tag{3.4.44}$$

Letting $\mathrm{Box}(x, r) = \{x \cdot y \in \mathbb{R}^n : \|y\| \le r\}$ by (3.4.44) there exists $c > 1$ such that for all $x \in \mathbb{R}^n$ and $r \ge 0$

$$\mathrm{Box}(x, c^{-1}r) \subset B(x, r) \subset \mathrm{Box}(x, cr).$$

Definition 3.4.1. Let $S \subset \mathbb{R}^n$ be a hypersurface of class C^1 given in a neighborhood $\mathcal{U}$ of $x_0 \in S$ by the local equation $\Phi = 0$ where $\Phi \in C^1(\mathcal{U})$. The point x_0 is *characteristic* if $X_1\Phi(x_0) = ... = X_m\Phi(x_0) = 0$.

We denote by e_j the $j-$th coordinate versor and if $x = \sum_{i=1}^n x_i e_i \in \mathbb{R}^n$ and $j \in \{1, \dots, n\}$ we write

$$\hat{x}_j = \sum_{1 \le i \le n, i \ne j} x_i e_i.$$

Theorem 3.4.2. *Any connected, bounded open set $\Omega \subset \mathbb{R}^n$ with boundary of class $C^{1,1}$ is a nta domain in the metric space $(\mathbb{R}^n, d)$.*

Proof. The proof will be split in several numbered small steps.

1. We claim that for all $x_0 \in \partial\Omega$ there exists a neighborhood $\mathcal{U}$ of x_0 such that for all $x, y \in \mathcal{U} \cap \Omega$ there exist continuous curves γ_x and $\gamma_y : [0,1] \to \Omega$ satisfying hypotheses of Proposition 3.1.15.

2. Let $\mathcal{U}$ be a neighborhood of x_0 and let $\Phi \in C^{1,1}(\mathcal{U})$ be a function such that $\partial\Omega \cap \mathcal{U} = \{x \in \mathcal{U} : \Phi(x) = 0\}$. We shall distinguish two cases:

(C1) $|X_1\Phi(x_0)| = ... = |X_m\Phi(x_0)| = 0$ (x_0 is a characteristic point of $\partial\Omega$);

(C2) $|X_1\Phi(x_0)| + ... + |X_m\Phi(x_0)| > 0$ (x_0 is a non characteristic point of $\partial\Omega$).

We notice that if $x \in \partial\Omega \cap \mathcal{U}$ then, possibly shrinking $\mathcal{U}$, the translated surface $x^{-1} \cdot (\partial\Omega \cap \mathcal{U})$ can be expressed in parametric form by an equation of the type $y_j = \varphi(\hat{y}_j)$ for $\hat{y}_j$ belonging to a neighborhood of the origin in $\mathbb{R}^{n-1}$ and φ of class $C^{1,1}$. If we are in Case 1 we have to choose $j \in \{m+1, ..., n\}$, while if we are in Case 2 we can choose $j \in \{1, ..., m\}$.

3. **Case 1.** We consider an open set $\{y \in \mathbb{R}^n : y_j > \varphi(\hat{y}_j)\}$ where $j > m$ and $\varphi \in C^{1,1}(\mathbb{R}^{n-1})$ is a function such that $\varphi(0) = 0$. Define

$$\nu_i = -\partial_i\varphi(0), \quad \text{for } i = 1, ..., m, \qquad \text{and} \quad \nu = (\nu_1, ..., \nu_m, 0, ..., 0).$$

Write also

$$\varphi(\hat{y}_j) = -\sum_{i=1}^{m} \nu_i y_i + \psi(\hat{y}_j)$$

where ψ can be written by the Taylor formula in the following form

$$\psi(\hat{y}_j) = \varphi(\hat{y}_j) - \sum_{i=1}^{m} \partial_i\varphi(0)y_j = \sum_{i>m, i\neq j} \partial_i\varphi(0)y_i + O(|\hat{y}_j|^2),$$

and satisfies the growth estimate

$$|\psi(\hat{y}_j)| \lesssim \|\hat{y}_j\|^2. \tag{3.4.45}$$

Here we used the homogeneous norm introduced in (3.4.43) and the fact that y belongs to a bounded set.

Our construction will take place in two main steps. In the first step we define "canonical" John curves starting from points near the boundary. In the second step we join points near the boundary by curves satisfying the hypotheses of Lemma 3.1.15.

4. *First step.* Define

$$N_1 = \frac{\nu_1}{|\nu|}, ..., N_m = \frac{\nu_1}{|\nu|}, \quad \text{and} \quad N = (N_1, ..., N_m, 0, ..., 0),$$

and if $\nu = 0$ simple set $N = 0$. For $\sigma > 0$ let $t_1 = \sigma|\nu|$. Fix $x = x_j e_j$ with $x_j \geq 0$ and define the continuous curve $\gamma : [0,1] \to \mathbb{R}^n$

$$\gamma(t) = \begin{cases} x \cdot tN = tN + x_j e_j, & \text{if } 0 \leq t \leq t_1, \\ x \cdot (t_1 N) + (t - t_1)e_j = t_1 N + (t - t_1 + x_j)e_j, & \text{if } t_1 \leq t \leq 1. \end{cases} \tag{3.4.46}$$

5. We claim that there exist $\sigma, \lambda \in (0,1)$ such that for all $t \in [0,1]$

$$\text{dist}(\gamma(t); \partial\Omega) \geq \lambda d(\gamma(t), x). \tag{3.4.47}$$

If $0 \leq t \leq t_1$ then $d(\gamma(t), x) \simeq \|x^{-1} \cdot \gamma(t)\| = \|tN\| = t$, and (3.4.47) is equivalent to

$$\text{Box}(\gamma(t), \lambda t) \cap \{y_j = \varphi(\hat{y}_j)\} = \varnothing, \tag{3.4.48}$$

which is implied by

$$\langle \nu, y\rangle + y_j \geq |\psi(\hat{y}_j)|, \quad \text{for all } y \in \text{Box}(\gamma(t), \lambda t). \tag{3.4.49}$$

Points in $\text{Box}(\gamma(t), \lambda t)$ are of the form

$$\gamma(t)\cdot h = (tN + x_j e_j)\cdot h = tN + x_j e_j + h + Q\,(tN, h) \tag{3.4.50}$$

with $\|h\| \leq \lambda t$ (Q does not depend on the variables on the second slice) and thus we have to check that

$$\langle \nu, tN + h\rangle + x_j + h_j + Q_j(tN, h) \geq \left|\psi\big((\widehat{\gamma(t)\cdot h})_j\big)\right|$$

which is guaranteed by

$$t|\nu| + \langle \nu, h\rangle + x_j \geq |h_j| + \left|Q_j(tN, h)\right| + \left|\psi\big((\widehat{\gamma(t)\cdot h})_j\big)\right|.$$

Now, since $|\langle \nu, h\rangle| \leq \lambda|\nu| t$ then $t|\nu| + \langle \nu, h\rangle \gtrsim t|\nu|$ as soon as $\lambda < 1/2$. Moreover $|h_j| \leq t^2$ and by (3.4.45)

$$\begin{aligned}\left|\psi\big((\widehat{\gamma(t)\cdot h})_j\big)\right| &\lesssim \left\|(\widehat{\gamma(t)\cdot h})_j\right\|^2 = \|tN + \hat{h}_j + \hat{Q}_j(tN, h)\|^2\\ &\lesssim t^2 + \|h\|^2 + \|Q(tN, h)\|^2 \lesssim t^2.\end{aligned}$$

Moreover $|Q_j(tN, h)| \lesssim \lambda t^2 \lesssim t^2$. Thus (3.4.48) is implied by

$$\varepsilon_0(t|\nu| + x_j) \geq t^2, \tag{3.4.51}$$

where ε_0 is a small but absolute constant. Since $x_j \geq 0$, (3.4.51) holds provided that $t \leq \sigma|\nu|$ and $\sigma \leq \varepsilon_0$. Our claim is proved if $0 \leq t \leq t_1$.

6. We study the case $t \geq t_1$. Notice that in this case

$$d(\gamma(t), x) \simeq \|x^{-1}\cdot\gamma(t)\| \simeq t_1 + (t - t_1)^{1/2} =: \delta(t). \tag{3.4.52}$$

Let $a = (t - t_1)^{1/2}$ so that $\delta(t) = t_1 + a$. We shall sometimes write δ instead of $\delta(t)$. We claim that there exists $0 < \lambda < 1$ such that the John property $\text{Box}(\gamma(t), \lambda\delta(t)) \cap \{y_j = \varphi(\hat{y}_j)\} = \varnothing$ holds for all $t \geq t_1$.

Points in $\text{Box}(\gamma(t), \lambda\delta)$ are of the form

$$\begin{aligned}\gamma(t)\cdot h &= (t_1 N + (t - t_1 + x_j)e_j)\cdot h\\ &= t_1 N + (t - t_1 + x_j)e_j + h + Q(t_1 N, h),\end{aligned} \tag{3.4.53}$$

with $\|h\| \leq \lambda\delta$. Thus, the John property is ensured by

$$\langle \nu, t_1 N + h\rangle + (t - t_1) + x_j + Q_j(t_1 N, h) \geq \left|\psi\left(t_1 N + \hat{h}_j + \hat{Q}_j(t_1 N, h)\right)\right|,$$

which (write $t - t_1 = a^2$) is a consequence of the following stronger inequality

$$t_1|\nu| + a^2 + x_j \geq |\nu|\|h\| + |Q_j(t_1 N, h)| + |\psi(z)|, \tag{3.4.54}$$

where z denotes the argument of ψ in the previous equation.

Now, $|\nu|\|h\| \lesssim \lambda|\nu| t_1 + \lambda|\nu| a$ and $\lambda|\nu| t_1$ can be absorbed in the left hand side, as soon as $\lambda \leq \frac{1}{2}$. We also note that

$$|Q_j(t_1 N, h)| \leq t_1\|h\| \leq t_1\lambda\delta \leq t_1^2 + \lambda\delta^2$$

Moreover

$$\|z\| \lesssim t_1 + \|h\| + \|\hat{Q}_j(t_1 N, h)\| \lesssim t_1 + \lambda\delta + (t_1\lambda\delta)^{1/2} \simeq t_1 + \lambda\delta,$$

and by (3.4.45)

$$|\psi(z)| \lesssim \|z\|^2 \lesssim t_1^2 + \lambda\delta^2 + \lambda t_1 \delta \lesssim t_1^2 + \lambda a^2 + \lambda t_1 a \\ \simeq t_1^2 + \lambda\delta^2 \simeq t_1^2 + \lambda a^2.$$

Since the term λa^2 can be absorbed in the left hand side and $x_j \geq 0$, then (3.4.54) will follow if we prove that for all $a \geq 0$

$$\varepsilon_0 \left(t_1|\nu| + a^2\right) \geq t_1^2 + \lambda|\nu|a$$

where $\varepsilon_0 > 0$ is a small but absolute constant. Replacing $t_1 = \sigma|\nu|$ we get

$$\varepsilon_0 \left(\sigma|\nu|^2 + a^2\right) \geq \sigma^2|\nu|^2 + \lambda|\nu|a. \tag{3.4.55}$$

now, since $\sigma^2|\nu|^2 + \lambda|\nu|a \leq (\sigma^2 + \frac{\lambda}{2})|\nu|^2 + \frac{\lambda}{2}a^2$ (3.4.55) holds for all $a \geq 0$ provided $\sigma^2 + 2\lambda^2 < \varepsilon_0\sigma$ and $2\lambda^2 \leq \varepsilon_0$.

7. *Second step.* We prove that, given x and y in the open set $\{z_j > \varphi(\hat{z}_j)\}$ there exists a continuous curve connecting them and satisfying (3.1.9) and (3.1.10). Without loss of generality we can assume that $x = x_j e_j$ with $x_j \geq 0$ and $y = y_j e_j + \hat{y}_j$ with $y_j > \varphi(\hat{y}_j)$. In the first step the "canonical" John curve starting from x has been defined in (3.4.46). The parameters ν, N and $t_1 = \sigma|\nu|$ are defined as in the first step and are relative to x. The constant σ does not depend on x.

8. Our next task is to write the curve starting from y. First we notice that, letting $\Phi(\xi) = \xi_j - \varphi_(\hat{\xi}_j)$, we have for $i = 1, ..., m$

$$X_i\Phi(\xi) = -\partial_i\varphi(\hat{\xi}_j) + \sum_{k>m} q_{ik}(\xi)\partial_k\Phi(\xi),$$

and hence

$$\nu_i = -\partial_i\varphi(0) = X_i\Phi(0). \tag{3.4.56}$$

Let now $w = \hat{y}_j + \varphi(\hat{y}_j)e_j$. We look for the parameters $\nu_i, i = 1, \ldots, m$ of the curve starting from $w^{-1} \cdot y = (y_j - \varphi(\hat{y}_j))e_j$ relatively to the translated boundary $w^{-1} \cdot \{z_j = \varphi(\hat{z}_j)\}$. Denote these parameters by $\bar{\nu}_1, ..., \bar{\nu}_m$. Then we find by left invariance

$$\begin{aligned}\bar{\nu}_i &= (X_i\Phi)(\hat{y}_j + \varphi(\hat{y}_j)e_j) = -\partial_i\varphi(\hat{y}_j) + \sum_{k>m} q_{ik}(y)\frac{\partial}{\partial\xi_k}\left(\xi_j - \varphi(\hat{\xi}_j)\right)\big|_{\hat{\xi}_j=\hat{y}_j, \xi_j=\varphi(\hat{y}_j)} \\ &= -\partial_i\varphi(\hat{y}_j) + q_{ij}(y') - \sum_{k>m, k\neq j} q_{ik}(y)\partial_k\varphi(\hat{y}_j).\end{aligned} \tag{3.4.57}$$

Define

$$\bar{N} = \left(\frac{\bar{\nu}_1}{|\bar{\nu}|}, ..., \frac{\bar{\nu}_m}{|\bar{\nu}|}, 0, ..., 0\right), \quad \text{and} \quad \bar{t}_1 = \sigma|\bar{\nu}|.$$

The "canonical" John curve γ_y starting from y can be defined (by left translation of (3.4.46)) in the following way . If $0 \leq t \leq \bar{t}_1$ let

$$\begin{aligned}\gamma_y(t) &= \left(\hat{y}_j + \varphi(\hat{y}_j)e_j\right) \cdot \left(t\bar{N} + (y_j - \varphi(\hat{y}_j))e_j\right) \\ &= \hat{y}_j + t\bar{N} + y_j e_j + Q(y, t\bar{N}),\end{aligned} \tag{3.4.58}$$

and if $t \geq \bar{t}_1$ let

$$\begin{aligned}\gamma_y(t) &= \big(\hat{y}_j + \varphi(\hat{y}_j)e_j\big) \cdot \big(\bar{t}_1\bar{N} + (t - \bar{t}_1 + y_j - \varphi(\hat{y}_j))e_j\big) \\ &= \hat{y}_j + \bar{t}_1\bar{N} + (t - \bar{t}_1 + y_j)e_j + Q(y, \bar{t}_1\bar{N}).\end{aligned} \tag{3.4.59}$$

9. Denote by γ_x and γ_y the curves starting from x and y. The curves γ_x and γ_y can not be expected to meet as Proposition 3.1.15 requires. Thus we enlarge the curve γ_x by constructing a curvilinear cone around it. Define

$$\delta(t) = \begin{cases} t & \text{if } 0 \leq t \leq t_1, \\ t_1 + (t - t_1)^{1/2} & \text{if } t \geq t_1, \end{cases}$$

and recall that $\delta(t) \simeq d(\gamma_x(t), x)$. For $\lambda > 0$ let $\mathcal{U}(\lambda) = \{h \in \mathbb{R}^n : \|h\| \leq \lambda\}$, and if $h = (h', h'') \in \mathcal{U}(\lambda)$ define $h_t = (\delta(t)h', \delta(t)^2h'')$. As $h \in \mathcal{U}(\lambda)$ the family of curves

$$\gamma_x^h(t) = \gamma_x(t) \cdot h_t = \begin{cases} tN + x_je_j + h_t + Q(tN, h_t) & \text{if } 0 \leq t \leq t_1, \\ t_1N + (t - t_1 + x_j)e_j + h_t + Q(t_1N, h_t) & \text{if } t \geq t_1 \end{cases}$$

forms a curvilinear cone with core γ_x. By the triangle inequality, if λ is small enough, then for any $h \in \mathcal{U}(\lambda)$, the curve $t \mapsto \gamma_x^h(t)$ is a John curve starting from x. ¿From now on we assume that λ has been fixed small enough in order to ensure this property.

10. Two cases must be distinguished:

(A) $d(x, y) \leq \eta|\nu|$;
(B) $d(x, y) > \eta|\nu|$.

The parameter $0 < \eta < 1$ will be fixed later. Note that if 0 is a characteristic point, then Case A is empty.

11. *Study of Case A.* We claim that there exist $\eta > 0$ and $M > 1$ such that for all x and y there exists $h \in \mathcal{U}(\lambda)$ such that $\gamma_y(Md(x, y)) = \gamma_x^h(Md(x, y))$. A correct choice of $0 < \eta < 1$ and $M > 1$ will show that the two curves meet in their first tract (see condition (3.4.68)).

Without loss of generality we can assume $|\nu| \leq |\bar{\nu}|$ (otherwise the roles of x and y should be interchanged). If $t \leq t_1 = \sigma|\nu|$ then $t \leq \bar{t}_1 = \sigma|\bar{\nu}|$ and $\gamma_y(t) = \gamma_x^h(t)$ reads

$$\hat{y}_j + t\bar{N} + y_je_j + Q(y, t\bar{N}) = tN + x_je_j + h_t + Q(tN, h_t). \tag{3.4.60}$$

We have to show that for any $\lambda > 0$ the solution $h = (h', h'')$ of this equation belongs to $\mathcal{U}(\lambda)$ if $t = Md(x, y)$ and M is great enough.

As $t \leq t_1$ then $\delta(t) = t$ and $h_t = (th', t^2h'')$. Projecting (3.4.60) along the first m components we get the equation $y' + t\bar{N} = tN + h_t'$ that is

$$th' = y' + t(\bar{N} - N). \tag{3.4.61}$$

Replacing $t = Md(x, y)$ we find that the solution h' satisfies

$$|h'| \leq \frac{|y'|}{Md(x, y)} + |N - \bar{N}|. \tag{3.4.62}$$

First of all notice that $d(x, y) \simeq \|(-x_je_j) \cdot (y_je_j + \hat{y}_j)\| \geq |y'|$, which gives $|y'| \leq d(x, y)$. Moreover, using the inequality

$$\Big|\frac{v}{|v|} - \frac{w}{|w|}\Big| \leq 2\frac{|v - w|}{|v|} \quad \text{if } v, w \in \mathbb{R}^n \setminus \{0\},$$

and the explicit form (3.4.56) and (3.4.57) of ν and $\bar{\nu}$ we get

$$\begin{aligned} |N-\bar{N}| &\le 2\frac{|\nu-\bar{\nu}|}{|\nu|} \\ &\le \frac{2}{|\nu|}\sum_{i=1}^{m}\Big|\partial_i\varphi(0)-\partial_i\varphi(\hat{y}_j)+q_{ij}(y)-\sum_{k>m,\,k\neq j}q_{ik}(y)\partial_k\varphi(\hat{y}_j)\Big| \\ &\lesssim \frac{1}{|\nu|}\big(|\hat{y}_j|+|y'|\big)\lesssim\frac{d(x,y)}{|\nu|}. \end{aligned} \tag{3.4.63}$$

The last string of estimates follows from the boundedness and the Lipschitz continuity of $\partial_i\varphi$ and from

$$\begin{aligned} d(x,y) &\simeq \|(-x_je_j)\cdot(\hat{y}_j+y_je_j)\| \\ &= \|\hat{y}_j+(y_j-x_j)e_j+Q(-x_je_j,\hat{y}_j+y_je_j)\| \\ &= \|\hat{y}_j+(y_j-x_j)e_j\|\ge\|\hat{y}_j\|\gtrsim|\hat{y}_j|, \end{aligned} \tag{3.4.64}$$

because y lies in a bounded set.

Putting (3.4.63) into (3.4.62) and using Case A we get

$$|h'|\lesssim\frac{1}{M}+\frac{d(x,y)}{|\nu|}\le\frac{1}{M}+\eta. \tag{3.4.65}$$

This shows that $|h'|\le\lambda$ as soon as M is great enough and η is small enough.

We project now (3.4.60) along the components of the second slice obtaining

$$\hat{y}''_j+y_je_j+tQ(y,\bar{N})=x_je_j+h''_t+tQ(N,h'_t).$$

Here $h''_t=t^2h''$ and $h'_t=th'$ where h' is the vector determined in (3.4.61) and satisfies the estimate (3.4.65). The last equation has a unique solution h'' which satisfies

$$|h''|\le\frac{|\hat{y}''_j|+|y_j-x_j|}{t^2}+\frac{1}{t}|Q(y,\bar{N})|+|Q(N,h')|.$$

Here we have to replace $t=Md(x,y)$ but first we notice that by (3.4.64),

$$d(x,y)=\|\hat{y}_j+(y_j-x_j)e_j\|\ge|\hat{y}''_j|^{1/2}+|y_j-x_j|^{1/2}. \tag{3.4.66}$$

Moreover $|Q(y,\bar{N})|\lesssim|y'|\lesssim d(x,y)$ and by (3.4.65)

$$|Q(N,h')|\lesssim|h'|\lesssim\frac{1}{M}+\eta,$$

Putting all these estimates together we find

$$|h''|\lesssim\frac{1}{M^2}+\frac{1}{M}+\eta. \tag{3.4.67}$$

Thus $|h''|\le\lambda$ as soon as M is great enough and η is small enough.

Our claim will be proved if we show that the choice of M and η is compatible with the condition $Md(x,y)\le t_1=\sigma|\nu|$. As we are in Case A then $d(x,y)\le\eta|\nu|$ and we find the stronger condition

$$M\eta\le\sigma \tag{3.4.68}$$

which can be satisfied.

12. In view of Proposition 3.1.15 we have to estimate the diameter of the curves γ_x^h and γ_y. First of all by (3.4.46) we have $\operatorname{diam}(\gamma_x) = Md(x,y)$. Moreover, if $0 \le s,t \le Md(x,y)$ and $\|h\| \le 1$

$$\begin{aligned} d(\gamma_x^h(s),\gamma_x^h(t)) &\le d(\gamma_x^h(s),\gamma_x(s)) + d(\gamma_x(s),\gamma_x(t)) + d(\gamma_x(t),\gamma_x^h(t)) \\ &\lesssim \|h_s\| + \operatorname{diam}(\gamma_x) + \|h_t\| \le 3Md(x,y), \end{aligned}$$

and thus $\operatorname{diam}(\gamma_x^h) \lesssim d(x,y)$.

13. *Study of Case B.* In this case the points x and y satisfy $d(x,y) \ge \eta|\nu|$ where $\eta > 0$ is from now on a fixed constant. Recall that $t_1 = \sigma|\nu|$ and $\bar t_1 = \sigma|\bar\nu|$, and for $R > 0$ let

$$t_x = t_1 + R^2 d(x,y)^2 \quad \text{and} \quad t_y = \bar t_1 + R^2 d(x,y)^2.$$

As above let $\mathcal{U}(\lambda) = \{h \in \mathbb{R}^n : \|h\| \le \lambda\}$ and write $h_t = (\delta(t)h', \delta(t)^2 h'')$ where now $\delta(t) = t_1 + (t-t_1)^{1/2} \simeq d(\gamma(t),x)$ for $t \ge t_1$.

14. We claim that there exists $R > 0$ such that for all x,y there exists $h \in \mathcal{U}(\lambda)$ (λ is the parameter fixed at the end of 9.) such that $\gamma_y(t_y) = \gamma_x^h(t_x)$ (the times t_x and t_y depend on R).

This equation gives

$$\hat y_j + \bar t_1 \bar N + (t_y - \bar t_1 + y_j)e_j + Q(y, \bar t_1 \bar N) = t_1 N + (t_x - t_1 + x_j)e_j + h_{t_x} + Q(t_1 N, h_{t_x}).$$

Replacing $\bar t_1 = \sigma|\bar\nu|$, $t_1 = \sigma|\nu|$, $t_y - \bar t_1 = R^2 d(x,y)^2$ and $t_x - t_1 = R^2 d(x,y)^2$ we find

$$\begin{aligned} &\hat y_j + \sigma\bar\nu + (R^2 d(x,y)^2 + y_j)e_j + \sigma|\bar\nu|Q(y,\bar N) \\ &\quad = \sigma\nu + (R^2 d(x,y)^2 + x_j)e_j + h_{t_x} + \sigma|\nu|Q(N, h_{t_x}). \end{aligned} \tag{3.4.69}$$

Projecting this equation along the coordinates of the first slice we get

$$y' + \sigma\bar\nu = \sigma\nu + h'_{t_x}, \tag{3.4.70}$$

and the solution h'_{t_x} satisfies

$$|h'_{t_x}| \le |y'| + \sigma|\nu| + \sigma|\bar\nu|.$$

We use $|y'| \le d(x,y)$ and $\sigma|\nu| \le \sigma/\eta d(x,y)$ (this is Case B). By (3.4.57)

$$\begin{aligned} |\bar\nu_i| &\le |\partial_i\varphi(\hat y_j)| + |q_{ij}(y)| + \sum_{k>m, k\ne j} |q_{ik}(y)\partial_k\varphi(\hat y_j)| \\ &\lesssim |\partial_i\varphi(0)| + |\partial_i\varphi(0) - \partial_i\varphi(\hat y_j)| + |y'| \\ &\lesssim |\nu| + d(x,y) \lesssim \frac{d(x,y)}{\eta}, \end{aligned} \tag{3.4.71}$$

(φ has Lipschitz continuous and bounded derivatives) and ultimately we obtain for some great but absolute constant C_0

$$|h'_{t_x}| \le C_0 \frac{d(x,y)}{\eta} = C_0 d(x,y) \tag{3.4.72}$$

(the parameter η has been fixed in 11. and can be considered from now on an absolute constant).

Projecting (3.4.69) along the coordinates of the second slice we have

$$\hat y''_j + y_j e_j + \sigma|\bar\nu|Q(y,\bar N) = x_j e_j + h''_{t_x} + \sigma|\nu|Q(N, h_{t_x}).$$

Thus

$$h''_{t_x} = \hat{y}''_j + (y_j - x_j)e_j + \sigma|\bar{\nu}|Q(y, \bar{N}) - \sigma|\nu|Q(N, h'_{t_x}),$$

where h'_{t_x} satisfies (3.4.72). Notice that by (3.4.66) $|\hat{y}''_j| + |y_j - x_j| \lesssim d(x,y)^2$ and moreover, taking into account (3.4.72) and Case B

$$\sigma|\nu||Q(N, h'_{t_x})| \lesssim |\nu||h'_{t_x}| \lesssim d(x,y)^2.$$

By (3.4.71)

$$\sigma|\bar{\nu}||Q(y', \bar{N})| \lesssim |\bar{\nu}||y'| \lesssim d(x,y)^2,$$

and hence $|h''_{t_x}| \lesssim d(x,y)^2$. Finally

$$\|h\| = \frac{\|h_{t_x}\|}{\delta(t_x)} = \frac{|h'_{t_x}| + |h''_{t_x}|^{1/2}}{\delta(t_x)} \lesssim \frac{d(x,y)}{\delta(t_x)} = \frac{d(x,y)}{t_1 + (t_x - t_1)^{1/2}} \le \frac{1}{R},$$

and $\|h\| \le \lambda$ as soon as $R \ge C_0/\lambda$ where C_0 is a great but absolute constant. Our claim is proved and the proof of the Theorem in the characteristic case is ended.

15. The estimates for $\mathrm{diam}(\gamma_x^h)$ and $\mathrm{diam}(\gamma_y)$ can be obtained as in 12.

16. **Case 2.** We now study the non characteristic case. Assume without loss of generality that $\Omega = \{y \in \mathbb{R}^n : y_j > \varphi(\hat{y}_j)\}$ where $j \in \{1, ..., m\}$ and $\varphi \in C^{1,1}(\mathbb{R}^{n-1})$ is a function such that $\varphi(0) = 0$. Let $\nu_i = -\partial_i\varphi(0)$ if $i = 1, ..., m$ with $i \neq j$, and $\nu_j = 1$. Finally write $\nu = (\nu_1, ..., \nu_m, 0, ..., 0)$.

17. *First step.* We construct John curves starting from near the boundary. The function ψ defined by

$$\psi(\hat{y}_j) = \varphi(\hat{y}_j) + \sum_{i=1,\dots,m, i\neq j} \nu_i y_i \tag{3.4.73}$$

satisfies

$$\begin{aligned} |\psi(\hat{y}_j)| &= \Big|\varphi(\hat{y}_j) - \sum_{i=1,\dots,m,i\neq j} \partial_i\varphi(0) y_i\Big| \\ &= \Big|\sum_{i>m} \partial_i\varphi(0)y_i + O(|\hat{y}_j|^2)\Big| \lesssim \|\hat{y}_j\|^2, \end{aligned} \tag{3.4.74}$$

because y belongs to a bounded set.

Fix a point $x \in \Omega$ of the form $x = x_j e_j$ with $x_j > 0$. For $t \ge 0$ define the curve starting from x

$$\gamma_x(t) = x \cdot t\nu = x \cdot \Big(te_j + t\sum_{i=1,\dots m, i\neq j} \nu_i e_i\Big). \tag{3.4.75}$$

Note first that $d(\gamma(t), x) \simeq \|t\nu\| = t|\nu| \simeq t$.

18. We claim that there exist $t_0 > 0$ and $0 < \lambda < 1$ such that for all $0 \le t \le t_0$

$$\mathrm{dist}(\gamma(t); \partial\Omega) \ge \lambda t. \tag{3.4.76}$$

The John condition (3.4.76) is equivalent to $\mathrm{Box}(\gamma(t), \lambda t) \cap \partial\Omega = \varnothing$.

Points in $\mathrm{Box}(\gamma(t), \lambda t)$ are of the form

$$\begin{aligned} x \cdot t\nu \cdot h &= x_j e_j \cdot (t\nu + h + tQ(\nu, h')) \\ &= x_j e_j + t\nu + h + tQ(\nu, h') + Q(x_j e_j, t\nu + h') \equiv z, \end{aligned}$$

where $h \in \mathbb{R}^n$ and $\|h\| \le \lambda t$. We have to check that (z is defined in the last equation)

$$x_j + t + h_j > \varphi(\hat{z}_j) = -\sum_{i\le m, i\ne j} \nu_i(t\nu_i + h_i) + \psi(\hat{z}_j),$$

by (3.4.73). Since $|h_k| < \lambda t$, $k = 1, \dots, m$, if $\lambda > 0$ is small enough, the last inequality is ensured by

$$x_j + (1-\lambda)t + t\sum_{i\le m, i\ne j}(\nu_i^2 - \lambda|\nu_i|) \ge |\psi(\hat{z}_j)|,$$

which is implied by

$$\varepsilon_0(x_j + t) \ge |\psi(\hat{z}_j)|. \tag{3.4.77}$$

The right hand side of (3.4.77) can be estimated by (3.4.74)

$$\begin{aligned} |\psi(\hat{z}_j)| &\lesssim \|\hat{z}_j\|^2 = \big\|t\hat{\nu}_j + \hat{h}_j + tQ(\nu, h') + Q(x_je_j, t\nu + h')\big\|^2 \\ &\lesssim t^2 + \lambda t^2 + \|tQ(\nu, h')\|^2 + \|Q(x_je_j, t\nu + h')\|^2 \\ &\lesssim t^2 + (t|\nu|\lambda t) + (x_j|t\nu + h'|) \lesssim t^2 + x_jt, \end{aligned}$$

where we used $|\nu| \lesssim 1$. Then (3.4.77) is ensured by

$$\varepsilon_0(x_j + t) \ge t^2 + x_jt,$$

where $\varepsilon_0 > 0$ is a small but absolute constant. This inequality is trivially satisfied as soon as $t \le \varepsilon_0$.

19. *Second step.* We prove the uniform condition. Given two points $x, y \in \Omega$ we have to connect them by curves γ_x and γ_y satisfying the hypotheses of Proposition 3.1.15. Assume that $x = x_je_j$ with $x_j > 0$ and write $y = \hat{y}_j + y_je_j$ with $y_j > \varphi(\hat{y}_j)$.

We first notice that if $d(x, y) < \mathrm{dist}(x; \partial\Omega)$ then x and y can be connected simply by a geodesic. Therefore, without loss of generality we can assume that

$$d(x, y) \ge \mathrm{dist}(x; \partial\Omega). \tag{3.4.78}$$

20. We claim that there exists a constant $C_0 > 0$ such that

$$x_j \le C_0 d(x, y) \tag{3.4.79}$$

for all $x = x_je_j, y \in \Omega$ satisfying (3.4.78) and lying in a bounded set (say the unit Euclidean ball with center at the origin). Indeed, if $\xi = \hat{\xi}_j + \varphi(\hat{\xi}_j)e_j \in \partial\Omega$ then

$$\begin{aligned} d(x, \xi) &\simeq \big\|(-x_je_j)\cdot\big(\hat{\xi}_j + \varphi(\hat{\xi}_j)e_j\big)\big\| \\ &\simeq |\varphi(\hat{\xi}_j) - x_j| + |\hat{\xi}_j'| + \big|\xi'' + Q\big(-x_je_j, \hat{\xi}_j + \varphi(\hat{\xi}_j)e_j\big)\big|^{1/2} \\ &= |\varphi(\hat{\xi}_j) - x_j| + |\hat{\xi}_j'| + \big|\xi'' + Q\big(-x_je_j, \hat{\xi}_j'\big)\big|^{1/2}. \end{aligned}$$

We used here the bilinearity of Q and the property $0 = (-e_j)\cdot e_j = -Q(e_j, e_j)$. In order to prove (3.4.79) it will be enough to show that

$$x_j \le C_0\Big(|\varphi(\hat{\xi}_j) - x_j| + |\hat{\xi}_j'| + \big|\xi'' + Q\big(-x_je_j, \hat{\xi}_j'\big)\big|^{1/2}\Big). \tag{3.4.80}$$

By the Lipschitz continuity of φ we find

$$\begin{aligned}
x_j &\le |x_j - \varphi(\hat\xi_j)| + |\varphi(\hat\xi_j)| \lesssim |x_j - \varphi(\hat\xi_j)| + |\hat\xi_j| \\
&= |x_j - \varphi(\hat\xi_j)| + |\hat\xi_j'| + |\xi''| \\
&\lesssim |x_j - \varphi(\hat\xi_j)| + |\hat\xi_j'| + |\xi'' + Q(-x_je_j, \hat\xi_j')| + |Q(-x_je_j, \hat\xi_j')| \\
&\lesssim |x_j - \varphi(\hat\xi_j)| + |\hat\xi_j'| + |\xi'' + Q(-x_je_j, \hat\xi_j')|^{1/2} + x_j|\hat\xi_j'| \\
&\lesssim |x_j - \varphi(\hat\xi_j)| + |\hat\xi_j'| + |\xi'' + Q(-x_je_j, \hat\xi_j')|^{1/2}.
\end{aligned}$$

We used here the fact that all the involved vectors lie in a bounded set. Our claim (3.4.79) is proved.

21. Our next step is to compute the "canonical" John curve starting from a generic point $y \in \Omega$. The point y and the boundary of Ω will be translated by a suitable vector $\eta \in \mathbb{R}^n$ in such a way that $\eta \cdot y$ lies in the half axis $\{\alpha e_j : \alpha > 0\}$. Using the equations of the translated surface the correct vector of parameters $\bar\nu$ can be computed and the curve starting from y will be defined as $\gamma_y(t) = \eta^{-1} \cdot (\eta \cdot y) \cdot (t\bar\nu) = y \cdot (t\bar\nu)$ for $t \ge 0$.

22. We claim that there exist $\varrho > 0$ and $C_0 > 1$ such that for all $y \in \Omega \cap \{|y| \le \varrho\}$ there exists $\eta \in \mathbb{R}^n$ such that:

(i) $\eta \cdot \partial\Omega$ contains the origin;
(ii) $\eta \cdot y$ belongs to $\{\lambda e_j : \lambda > 0\}$;
(iii) $|\eta| \le C_0|\hat y_j|$.

We look for $\eta = (\eta', \eta'')$. If η'' is given, we can define η' by the equation

$$\eta' = -\hat y_j' - \varphi(\hat y_j' - \eta'')e_j \tag{3.4.81}$$

and (i) is satisfied. Indeed the point $z := \hat y_j' - \eta'' + \varphi(\hat y_j' - \eta'')e_j \in \partial\Omega$ and

$$\eta \cdot z = \eta' + \eta'' + \hat y_j' - \eta'' + \varphi(\hat y_j' - \eta'')e_j + Q(\eta', \hat y_j' + \varphi(\hat y_j' - \eta'')e_j) = 0,$$

by (3.4.81).

We shall soon prove that the implicit equation

$$\eta'' + y'' + Q(-\hat y_j' - \varphi(\hat y_j' - \eta'')e_j, \hat y_j' + y_je_j) = 0, \tag{3.4.82}$$

has a solution η''. This ensures that the vector $\eta = (\eta', \eta'')$ satisfies (ii). Indeed

$$\begin{aligned}
\eta \cdot y &= \big(- \hat y_j' + \eta'' - \varphi(\hat y_j' - \eta'')e_j\big) \cdot \big(\hat y_j' + y'' + y_je_j\big) \\
&= y'' + \eta'' + \big(y_j - \varphi(\hat y_j' - \eta'')\big)e_j + Q\big(- \hat y_j' - \varphi(\hat y_j' - \eta'')e_j, \hat y_j' + y_je_j\big),
\end{aligned}$$

which belongs to the $j-$th axis if and only if (3.4.82) holds.

We prove the existence of the solution η''. First notice that by the bilinearity of Q

$$\begin{aligned}
Q\big(- \hat y_j' - \varphi(\hat y_j' - \eta'')e_j, y'\big) &= Q\big(- \hat y_j' - y_je_j + \big(y_j - \varphi(\hat y_j' - \eta'')\big)e_j, y'\big) \\
&= Q\big(\big(y_j - \varphi(\hat y_j' - \eta'')\big)e_j, y'\big) \\
&= \big(y_j - \varphi(\hat y_j' - \eta'')\big)Q(e_j, y').
\end{aligned}$$

The map $y' \mapsto Q(e_j, y')$ is linear and does not depend on y_j). Thus equation (3.4.82) is equivalent to the equation

$$\eta'' + y'' + \big(y_j - \varphi(\hat y_j' - \eta'')\big)Q(e_j, \hat y_j') = 0. \tag{3.4.83}$$

We show that there exists $\varrho > 0$ such that if $y \in \Omega$ and $|y| \le \varrho$ then (3.4.83) has a solution η'' satisfying

$$|\eta''| \le 2|\hat{y}_j|. \tag{3.4.84}$$

We use a fixed point argument. Letting $F(\eta'') = -y'' - \big(y_j - \varphi(\hat{y}'_j - \eta'')\big)Q(e_j, \hat{y}'_j)$ equation (3.4.83) becomes $F(\eta'') = \eta''$. Let $D = \{\eta \in \mathbb{R}^n : |\eta| \le C_0|\hat{y}_j|\}$. If we show that for some $C_0 > 0$ $F(D) \subseteq D$, then the continuous map F has a fixed point by Brouwer theorem. Indeed

$$\begin{aligned} |F(\eta'')| &\le |y''| + |Q(e_j, \hat{y}'_j)|\big|\big(y_j - \varphi(\hat{y}'_j - \eta'')\big)\big| \\ &\le |\hat{y}_j| + C|\hat{y}'_j|\big(|y_j| + |\hat{y}'_j| + |\eta''|\big) \\ &\le |\hat{y}_j|(1 + 4C|y|) \le 2|\hat{y}_j|, \end{aligned}$$

as soon as $|y| \le \varrho = 1/(4C)$ (here the constant C depends only on the surface). Moreover by (3.4.81) and by (3.4.84)

$$|\eta'| = |\hat{y}'_j| + |\varphi(\hat{y}'_j - \eta'')| \lesssim |\hat{y}'_j| + |\eta''| \lesssim |\hat{y}_j|.$$

This proves claim (iii).

23. We compute $\bar{\nu}$ by a left translation argument. Let $\Phi(y) = y_j - \varphi(\hat{y}_j)$. The parameters ν at the point $y = 0$ are given by $\nu_i = X_i\Phi(0)$, $i = 1, ..., m$. Then for any point $\xi = \hat{\xi}_j + \varphi(\hat{\xi}_j)e_j$ belonging to the surface $\{\Phi = 0\}$ the parameters $\nu_i = \nu_i(\xi)$ are given by

$$\begin{aligned} \nu_i(\xi) &= (X_i\Phi)(\xi) \\ &= \begin{cases} \Big(\partial_j + \sum_{k>m} q_{jk}(\xi')\partial_k\Big)\Phi(\xi) = 1 - \sum_{k>m} q_{jk}(\xi')\partial_k\varphi(\hat{\xi}_j) & \text{if } i = j, \\ \Big(\partial_i + \sum_{k>m} q_{ik}(\xi')\partial_k\Big)\Phi(\xi) = -\partial_i\varphi(\hat{\xi}) - \sum_{k>m} q_{ik}(\xi')\partial_k\varphi(\hat{\xi}_j) & \text{if } i \le m,\ i \ne j. \end{cases} \end{aligned}$$

Let $\eta \in \mathbb{R}^n$ be a vector relative to y as in the claims (i), (ii) and (iii) in 22. The correct value of the parameters is given by the evaluation of the previous equation at the point $-\eta$ (this is because the point $-\eta$ is taken to the origin by the left translation τ_η). Define $\bar{\nu}_i = \nu_i(-\eta)$. Set $\bar{\nu} = (\bar{\nu}_1, ..., \bar{\nu}_m, 0, ..., 0)$. We claim that

$$|\nu - \bar{\nu}| \lesssim |\hat{y}_j| \tag{3.4.85}$$

If $i \ne j$, by the Lipschitz continuity of φ and by claim (iii)

$$\begin{aligned} |\bar{\nu}_i - \nu_i| &= \Big| -\partial_i\varphi(-\hat{\eta}_j) - \sum_{k=m+1}^{n} q_{ik}(-\eta')\partial_k\varphi(-\hat{\eta}_j) + \partial_i\varphi(0)\Big| \\ &\lesssim |\partial_i\varphi(0) - \partial_i\varphi(-\hat{\eta}_j)| + \sum_{k=m+1}^{n} \big|q_{ik}(-\eta')\big|\big|\partial_k\varphi(-\hat{\eta}_j)\big| \\ &\lesssim |\hat{\eta}_j| + |\eta'| \simeq |\eta| \lesssim |\hat{y}_j|. \end{aligned}$$

The estimate of the $j-$th component of $\nu - \bar{\nu}$ is even easier and we skip it.

24. Let γ_x be the curve starting from $x = x_je_j$ defined in (3.4.75) and let γ_y be the curve starting from $y \in \Omega$ defined for $t \ge 0$ by

$$\gamma_y(t) = y \cdot (t\bar{\nu}) = \hat{y}'_j + t\bar{\nu} + y_je_j + y'' + Q(y', t\bar{\nu}),$$

where $\bar\nu$ is the vector of parameters discussed above. We now construct a cone with core γ_x. For $\lambda > 0$ let $\mathcal{U}(\lambda) = \{h \in \mathbb{R}^n : \|h\| \le \lambda\}$ and for $t \ge 0$ define $h_t = (th', t^2 h'')$. Note that $h_t = t\|h\| \simeq d(\gamma_x(t), \gamma_x(0))\|h\|$. Finally let

$$\begin{aligned}\gamma_x^h(t) &= x_j e_j \cdot (t\nu) \cdot h_t = x_j e_j \cdot (t\nu + h_t + Q(t\nu, h_t'))\gamma_x(t) \cdot h_t \\ &= x_j e_j + t\nu + h_t + tQ(\nu, h_t') + Q(x_j e_j, t\nu + h_t').\end{aligned}$$

25. We claim that there exist $M > 0$ and $\varrho > 0$ such that for all $x = x_j e_j \in \Omega$ and for all $y \in \Omega$ such that $|\hat y_j| \le \varrho$ there exists $h \in \mathcal{U}(\lambda)$ such that $\gamma_x^h(Md(x,y)) = \gamma_y(Md(x,y))$. Here λ is a parameter small enough to ensure that for all $h \in \mathcal{U}(\lambda)$ γ_x^h is a John curve with constant λ.

Equality $\gamma_y(t) = \gamma_x^h(t)$ reads

$$\hat y_j' + t\bar\nu + y_j e_j + y'' + tQ(y', \bar\nu) = x_j e_j + t\nu + h_t + tQ(\nu, h_t') + x_j Q(e_j, t\nu + h_t'). \tag{3.4.86}$$

Projecting this equation along the coordinates of the first slice we get

$$\hat y_j' + t\bar\nu + y_j e_j = x_j e_j + t\nu + th' \tag{3.4.87}$$

and the solution h' satisfies $|h'| \le \frac{1}{t}\{|\hat y_j'| + |y_j - x_j| + t|\nu - \bar\nu|\}$. Replacing $t = Md(x,y)$ we find

$$|h'| \le \frac{|\hat y_j'| + |y_j - x_j|}{Md(x,y)} + |\nu - \bar\nu|.$$

By the equivalence

$$d(x,y) \simeq \|(-x)\cdot y\| \simeq |y_j - x_j| + |\hat y_j'| + \left|y'' + Q(-x_j e_j, \hat y_j')\right|^{1/2}, \tag{3.4.88}$$

and by (3.4.85) we obtain for some absolute constant C_0

$$|h'| \le C_0\Big(\frac{1}{M} + \varrho\Big) \tag{3.4.89}$$

as soon as $|\hat y_j| \le \varrho$.

We project now (3.4.86) along the coordinates of the second slice obtaining

$$y'' + tQ(y', \bar\nu) = h_t'' + tQ(\nu, h_t') + x_j Q(e_j, t\nu + h_t'),$$

where $h_t' = th'$ and h' satisfies (3.4.89). We deduce that

$$|h_t''| \le |y''| + t|Q(y', \bar\nu)| + t|Q(\nu, h_t')| + x_j|Q(e_j, t\nu + h_t')|.$$

We estimate separately each term in the right hand side. By (3.4.88) and (3.4.79)

$$|y''| \le |y'' + Q(-x_j e_j, \hat y_j')| + |Q(-x_j e_j, \hat y_j')| \lesssim d(x,y)^2 + x_j|\hat y_j'| \lesssim d(x,y)^2,$$

Moreover $|Q(y', \bar\nu)| \lesssim |y'| \lesssim d(x,y)$ and by (3.4.89)

$$|Q(\nu, h_t')| \lesssim |h_t'| \lesssim t\Big(\frac{1}{M} + \varrho\Big).$$

The vectors ν and $\bar\nu$ are bounded. Finally, again by (3.4.79) $x_j|Q(e_j, t\nu + h_t')| \lesssim td(x,y)$. Then

$$|h_t''| \lesssim d(x,y)^2 + td(x,y) + t^2\Big(\frac{1}{M} + \varrho\Big),$$

and replacing $t = Md(x,y)$ we finally get

$$|h''| \lesssim \frac{1}{M^2} + \frac{1}{M} + \varrho,$$

which shows that $\|h\| \leq \lambda$ if M is great and ϱ is small enough.

$\square$

5. John domains in a group of step 3

In this section we study John domains in groups of step 3. In order to make explicit computations we shall study the simplest Carnot group of step 3 whose Lie algebra has the lowest dimension, which is 4.

Consider in $\mathbb{R}^4$ the vector fields

$$\begin{aligned}
X_1 &= \partial_1 - \frac{1}{2}x_2\partial_3 - \Big\{\frac{1}{12}(x_1x_2 + \alpha x_2^2) + \frac{1}{2}x_3\Big\}\partial_4,\\
X_2 &= \partial_2 + \frac{1}{2}x_1\partial_3 + \Big\{\frac{1}{12}(x_1^2 + \alpha x_1x_2) - \frac{\alpha}{2}x_3\Big\}\partial_4,\\
X_3 &= \partial_3 + \frac{1}{2}(x_1 + \alpha x_2)\partial_4\\
X_4 &= \partial_4,
\end{aligned}$$

where $\alpha \in \mathbb{R}$ is a real parameter. The commutation relations are

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = \alpha X_4,$$

and all other commutators vanish. Thus, for any $\alpha \in \mathbb{R}$ the vector fields X_1, X_2 are generators of a Lie algebra of differential operators in $\mathbb{R}^4$ of step 3. It can be checked that the following group law on $\mathbb{R}^4$ makes X_1, X_2, X_3 and X_4 left invariant.

$$\begin{aligned}
x \cdot y = \Big(&x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1y_2 - x_2y_1),\\
&x_4 + y_4 + \frac{1}{12}\big\{(y_1 + \alpha y_2)(x_2y_1 - x_1y_2) + (x_1 + \alpha x_2)(x_1y_2 - x_2y_1)\big\}\\
&\quad + \frac{1}{2}\big\{(x_1y_3 - x_3y_1) + \alpha(x_2y_3 - y_2x_3)\big\}\Big).
\end{aligned}$$

Notice that $x^{-1} = -x$. Introduce the abbreviations

$$\begin{aligned}
q_1(x_1, x_2, x_3) &= -\Big\{\frac{1}{12}(x_1x_2 + \alpha x_2^2) + \frac{1}{2}x_3\Big\}\\
q_2(x_1, x_2, x_3) &= \Big\{\frac{1}{12}(x_1^2 + \alpha x_1x_2) - \frac{\alpha}{2}x_3\Big\}\\
q_3(x_1, x_2) &= \frac{1}{2}(x_1 + \alpha x_2),
\end{aligned} \tag{3.5.90}$$

and

$$\begin{aligned}
Q_3(x_1, x_2, y_1, y_2) &= \frac{1}{2}(x_1y_2 - x_2y_1)\\
Q_4(x_1, x_2, x_3, y_1, y_2, y_3) &= \frac{1}{12}\big\{(y_1 + \alpha y_2)(x_2y_1 - x_1y_2) + (x_1 + \alpha x_2)(x_1y_2 - x_2y_1)\big\}\\
&\quad + \frac{1}{2}\big\{(x_1y_3 - x_3y_1) + \alpha(x_2y_3 - y_2x_3)\big\},
\end{aligned}$$

in such a way that

$$x \cdot y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + Q_3(x_1, x_2, y_1, y_2),$$
$$x_4 + y_4 + Q_4(x_1, x_2, x_3, y_1, y_2, y_3)).$$

We denote by d the Carnot-Carathéodory distance induced on $\mathbb{R}^4$ by X_1 and X_2 and by $B(x, r)$ the open ball centered at $x \in \mathbb{R}^4$ with radius $r \geq 0$. Define also the following homogeneous norm in $\mathbb{R}^4$

$$\|x\| = |x_1| + |x_2| + |x_3|^{1/2} + |x_4|^{1/3}.$$

By a standard argument it can be proved that

$$d(x, y) \simeq \|y^{-1} \cdot x\|. \tag{3.5.91}$$

Define the Box

$$\text{Box}(x, r) = \{x \cdot y \in \mathbb{R}^4 : \|y\| \leq r\}. \tag{3.5.92}$$

By (3.5.91) it follows that there exists $c > 1$ such that for all $x \in \mathbb{R}^n$ and $r \geq 0$

$$\text{Box}(x, c^{-1}r) \subset B(x, r) \subset \text{Box}(x, cr).$$

Let $S \subset \mathbb{R}^4$ be a 3−dimensional surface of class C^1. If $x_0 \in S$ there exists a neighborhood $\mathcal{U}$ of x_0 in $\mathbb{R}^4$ and there exists $\Phi \in C^1(\mathcal{U}; \mathbb{R})$ such that $S \cap \mathcal{U} = \{x \in \mathcal{U} : \Phi(x) = 0\}$ and $\nabla\Phi \neq 0$ on $S \cap \mathcal{U}$. A point $x \in S \cap \mathcal{U}$ is said to be *characteristic* if and only if $X_1\Phi(x) = X_2\Phi(x) = 0$. From a geometric point of view this means that X_1 and X_2 belong to the tangent spaces to S at x.

Definition 3.5.1. A characteristic point $x \in S \cap \mathcal{U}$ is *of first type* if $X_3\Phi(x) \neq 0$, is of *second type* if $X_3\Phi(x) = 0$.

If $x \in S \cap \mathcal{U}$ is a characteristic point of second type then $X_4\Phi(x) = \partial_4\Phi(x)$ can not be 0. Otherwise it would be $X_1\Phi = \cdots = X_4\Phi = 0$ at x and this is impossible because $\nabla\Phi \neq 0$ and $X_1, \ldots, X_4$ are independent at each point.

We are interested in expressing S in parametric form in a neighborhood of $x_0 \in S$ after a translation that takes x_0 to the origin. Notice that x_0 is a characteristic point (of first, second type) of S if and only if 0 is a characteristic point (of first, second type) of the translated surface $x_0^{-1} \cdot S$. Indeed, $\Phi(x_0 \cdot x) = 0$ is a local equation for $x_0^{-1} \cdot S$ at 0 and since X_1, X_2, X_3, X_4 are left invariant

$$X_j(\Phi(x_0 \cdot x))\Big|_{x=0} = X_j\Phi(x_0), \quad j = 1, ..., 4.$$

If x_0 is a characteristic point of second type there is only one possible parametrization of S in a neighborhood of x_0: the variable x_4 must be given in terms of the variables x_1, x_2, x_3. Such a choice of parametrization will be also possible in a neighborhood of x_0. Let $x \in S$ be a point near x_0 and let now $\mathcal{V} \subset \mathbb{R}^4$ be a neighborhood of 0. There exist $D \subset \mathbb{R}^3$ open neighborhood of $0 \in \mathbb{R}^3$ and $\varphi \in C^1(D; \mathbb{R})$ such that

$$(x^{-1} \cdot S) \cap \mathcal{V} = \{(y_1, y_2, y_3, \varphi(y_1, y_2, y_3)) \in \mathbb{R}^4 : (y_1, y_2, y_3) \in D\}.$$

Notice that $\varphi(0) = 0$. We say that φ is a local parametrization of S at x of second type.

Definition 3.5.2. A connected, bounded open set $\Omega \subset \mathbb{R}^4$ is *admissible* if

(i) Ω is of class C^2;

(ii) there exists $k \geq 0$ such that for any $x_0 \in \partial\Omega$ characteristic point of second type there exists $\mathcal{U} \subset \mathbb{R}^4$ open neighborhood of x_0 such that if $\Phi = 0$ is a local equation for $\partial\Omega \cap \mathcal{U}$ then for all $x \in \partial\Omega \cap \mathcal{U}$

$$\begin{aligned} &|X_1^2\Phi(x)| + |X_2^2\Phi(x)| + |(X_1X_2 + X_2X_1)\Phi(x)| \\ &\qquad \leq k(|X_1\Phi(x)|^{1/2} + |X_2\Phi(x)|^{1/2} + |X_3\Phi(x)|). \end{aligned} \tag{3.5.93}$$

REMARK 3.5.3. The needed flatness of $\partial\Omega$ at characteristic points of first type is guaranteed by the assumption that $\partial\Omega$ is of class C^2. Inequality (3.5.93) becomes trivial as soon as we are away from the characteristic set of second type.

REMARK 3.5.4. The meaning of (3.5.93) near a characteristic point of the second type can be clarified representing the surface in parametric form as follows. Let $\Phi = 0$ be a local equation for $\partial\Omega$ in a neighborhood of a characteristic point of second type $x_0 \in \partial\Omega$. Take a point in $\partial\Omega$ belonging to this neighborhood and assume without loss of generality it is the origin. Assume that (3.5.93) holds. Since $\partial_4\Phi(0) = X_4\Phi(0) \neq 0$, by the implicit function Theorem there exist a neighborhood D of the origin in $\mathbb{R}^3$ and a function $\varphi \in C^2(D)$ such that $\varphi(0) = 0$ and (we write $x = (x_1, x_2, x_3)$)

$$\Phi(x, \varphi(x)) = 0 \quad \text{for all } x \in D. \tag{3.5.94}$$

If we apply the vector fields X_1, X_2 and X_3 to identity (3.5.94) and evaluate the expressions thus obtained at $x = 0$ we obtain

$$\begin{cases} \partial_1\Phi(0) + \partial_4\Phi(0)\partial_1\varphi(0) = 0, \\ \partial_2\Phi(0) + \partial_4\Phi(0)\partial_2\varphi(0) = 0, \\ \partial_3\Phi(0) + \partial_4\Phi(0)\partial_3\varphi(0) = 0. \end{cases}$$

Since $\partial_4\Phi(0) \neq 0$ we get

$$|\partial_1\varphi(0)| = \frac{|\partial_1\Phi(0)|}{|\partial_4\Phi(0)|}, \quad |\partial_2\varphi(0)| = \frac{|\partial_2\Phi(0)|}{|\partial_4\Phi(0)|}, \quad |\partial_3\varphi(0)| = \frac{|\partial_3\Phi(0)|}{|\partial_4\Phi(0)|}.$$

Applying the second order operators X_1^2, X_2^2 and $X_1X_2 + X_2X_1$ to the identity (3.5.94) and evaluating the expressions thus obtained at $x = 0$ we get

$$\begin{cases} \partial_1^2\Phi + [2\partial_{14}\Phi + \partial_4^2\Phi\,\partial_1\varphi]\partial_1\varphi + \partial_4\Phi\,\partial_1^2\varphi = 0, \\ \partial_2^2\Phi + [2\partial_{24}\Phi + \partial_4^2\Phi\,\partial_2\varphi]\partial_2\varphi + \partial_4\Phi\,\partial_2^2\varphi = 0, \\ 2\partial_{12}\Phi + [2\partial_{24}\Phi + \partial_4^2\Phi\,\partial_2\varphi]\partial_1\varphi + [2\partial_{14}\Phi + \partial_4^2\Phi\,\partial_1\varphi]\partial_2\varphi + 2\partial_4\Phi\,\partial_{12}\varphi = 0. \end{cases}$$

All square brackets are bounded functions and $\partial_4\Phi$ is away from 0. Thus

$$\begin{cases} |\partial_1^2\varphi(0)| \lesssim |\partial_1^2\Phi(0)| + |\partial_1\varphi(0)|, \\ |\partial_2^2\varphi(0)| \lesssim |\partial_2^2\Phi(0)| + |\partial_2\varphi(0)|, \\ |\partial_{12}\varphi(0)| \lesssim |\partial_{12}\Phi(0)| + |\partial_1\varphi(0)| + |\partial_2\varphi(0)|. \end{cases}$$

The signs "$\lesssim$" mean that the estimates are uniform in a neighborhood of the characteristic points x_0 we are considering.

Now let $x \in \mathbb{R}^4$ and consider $x \to \Phi(x)$. Applying X_1, X_2, X_3, X_1^2, X_2^2 and $X_1X_2 + X_2X_1$ to Φ and evaluating at $x = 0$ we see that

$$\begin{cases} X_1\Phi(0) = \partial_1\Phi(0),\ X_2\Phi(0) = \partial_2\Phi(0),\ X_3\Phi(0) = \partial_3\Phi(0), \\ X_1^2\Phi(0) = \partial_1^2\Phi(0),\ X_2^2\Phi(0) = \partial_2^2\Phi(0),\ (X_1X_2 + X_2X_1)\Phi(0) = 2\partial_{12}\Phi(0). \end{cases}$$

Using all these estimates, from (3.5.93) we deduce that the function φ which parametrizes $\partial\Omega$ satisfies

$$|\partial_1^2\varphi(0)| + |\partial_2^2\varphi(0)| + |\partial_{12}^2\varphi(0)| \leq \bar{k}(|\partial_1\varphi(0)|^{1/2} + |\partial_2\varphi(0)|^{1/2} + |\partial_3\varphi(0)|), \tag{3.5.95}$$

where now $\bar{k}$ is a new constant that depends on k and that is uniform in a neighborhood of the characteristic point of second type we are considering.

By Taylor formula with $\varphi(0) = 0$

$$\begin{aligned}\varphi(y) =& \partial_1\varphi(0)y_1 + \partial_2\varphi(0)y_2 + \partial_3\varphi(0)y_3 \\ &+ \frac{1}{2}\partial_1^2\varphi(0)y_1^2 + \frac{1}{2}\partial_2^2\varphi(0)y_2^2 + \partial_{12}^2\varphi(0)y_1y_2 + O(\|y\|^3),\end{aligned}$$

where $\|y\| = |y_1| + |y_2| + |y_3|^{1/2}$, and (3.5.93) implies (possibly with a new uniform constant $\bar{k}$)

$$\begin{aligned}&|\varphi(y) - \partial_1\varphi(0)y_1 - \partial_2\varphi(0)y_2 - \partial_3\varphi(0)y_3| \leq \\ &\qquad \leq \bar{k}\big(\|y\|^3 + (|\partial_1\varphi(0)|^{1/2} + |\partial_2\varphi(0)|^{1/2} + |\partial_3\varphi(0)|)(y_1^2 + y_2^2)\big).\end{aligned} \tag{3.5.96}$$

If $0 \in \partial\Omega$ is a characteristic point of second type, i.e. $\partial_1\varphi(0) = \partial_2\varphi(0) = \partial_3\varphi(0) = 0$, then (3.5.96) gives the growth condition $|\varphi(y)| \leq \bar{k}\|y\|^3$. If this is not the case, then a quadratic term $(y_1^2 + y_2^2)$ is admitted, but its coefficient must disappear in a way controlled by $|\partial_1\varphi|^{1/2} + |\partial_2\varphi|^{1/2} + |\partial_3\varphi|$. The constant $\bar{k}$ should be uniform.

THEOREM 3.5.5. *If $\Omega \subset \mathbb{R}^4$ is an admissible domain then it is a John domain in $(\mathbb{R}^4, d)$.*

Proof. We shall construct "canonical" John curves starting from points near the boundary $\partial\Omega$. The proof will be split in several numbered small steps.

1. Fix a point $x_0 \in \partial\Omega$, let $\mathcal{U} \subset \mathbb{R}^4$ be a neighborhood of x_0 and let $\Phi \in C^2(\mathcal{U}; \mathbb{R})$ be a local equation for $\partial\Omega \cap \mathcal{U}$. We shall distinguish three cases:

(C1) $X_1\Phi(x_0) = X_2\Phi(x_0) = 0$, and $X_3\Phi(x_0) \neq 0$ (x_0 is a characteristic point of first type);

(C2) $X_1\Phi(x_0) = X_2\Phi(x_0) = X_3\Phi(x_0) = 0$ (x_0 is a characteristic point of second type);

(C3) $|X_1\Phi(x_0)| + |X_2\Phi(x_0)| > 0$ (x is a non characteristic point of $\partial\Omega$);

2. **Case 1.** After a translation $0 \in \partial\Omega$ can be assumed to be near x_0. Thus, in a neighborhood of 0 the surface $\partial\Omega$ admits a parametrization of first type, i.e. there exists a function $\varphi = \varphi(y_1, y_2, y_4)$ of class C^2 such that $\varphi(0) = 0$ and we have $\partial\Omega = \{y_3 = \varphi(y_1, y_2, y_4)\}$. Define

$$\nu_1 = -\partial_1\varphi(0),\ \nu_2 = -\partial_2\varphi(0),\ \nu = (\nu_1, \nu_2),\ N_1 = \frac{\nu_1}{|\nu|},\ N_2 = \frac{\nu_2}{|\nu|},$$

and if $\nu = 0$ simply set $N_1 = N_2 = 0$. Moreover let $\psi(y) = \varphi(y) + \nu_1y_1 + \nu_2y_2$. Then, using a Taylor expansion for $\varphi(y)$ we have

$$|\psi(y)| = |\varphi(y) + \nu_1y_1 + \nu_2y_2| \lesssim y_1^2 + y_2^2 + |y_4|. \tag{3.5.97}$$

Consider now a point $x = (0, 0, x_3, 0) \in \Omega$ with $0 < x_3 \leq 1$. We shall define a continuous path $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x$ and $\mathrm{dist}(\gamma(t); \partial\Omega) \geq \lambda d(\gamma(t), x)$

for all $t \in [0,1]$ and for some $\lambda > 0$ depending only on Ω. The path will be made by two pieces.

3. *First piece.* Let $\sigma > 0$ and define $t_1 = \sigma|\nu|$. For $t \in [0, t_1]$ we define

$$\gamma(t) = (0, 0, x_3, 0) \cdot (tN_1, tN_2, 0, 0).$$

Notice that $d(\gamma(t), x) \simeq t$.

4. We claim that there exist $0 < \sigma, \lambda < 1$ absolute constants such that for all $t \leq t_1$

$$\mathrm{Box}(\gamma(t), \lambda t) \subset \Omega. \tag{3.5.98}$$

Condition (3.5.98) is equivalent to the John property for γ in this first piece. The first piece is trivial if $\nu = 0$. Points in $\mathrm{Box}(\gamma(t), \lambda t)$ are of the form

$$\begin{aligned}\gamma(t) \cdot h &= (0, 0, x_3, 0) \cdot (tN_1, tN_2, 0, 0) \cdot (h_1, h_2, h_3, h_4)\\ &= (0, 0, x_3, 0) \cdot (tN_1 + h_1, tN_2 + h_2, h_3 + Q_3(tN_1, tN_2, h_1, h_2),\\ &\qquad\qquad , h_4 + Q_4(tN_1, tN_2, 0, h_1, h_2, h_3))\\ &= \Big(tN_1 + h_1, tN_2 + h_2, x_3 + h_3 + Q_3(tN_1, tN_2, h_1, h_2),\\ &\qquad , h_4 + Q_4(tN_1, tN_2, 0, h_1, h_2, h_3)\\ &\qquad + Q_4\big(0, 0, x_3, tN_1 + h_1, tN_2 + h_2, h_3 + Q_3(tN_1, tN_2, h_1, h_2)\big)\Big)\end{aligned}$$

with $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \leq \lambda t$.

Now, $\gamma(t) \cdot h \in \Omega$ provided that (recall that $\varphi(z) = -\nu_1 z_1 - \nu_2 z_2 + \psi(z)$)

$$\begin{aligned}x_3 + h_3 + Q_3(tN_1, tN_2, h_1, h_2) &\geq -\nu_1(tN_1 + h_1) - \nu_2(tN_2 + h_2) +\\ &+ \psi\Big(tN_1 + h_1, tN_2 + h_2, h_4 + Q_4(tN_1, tN_2, 0, h_1, h_2, h_3)\\ &\quad + Q_4\big(0, 0, x_3, tN_1 + h_1, tN_2 + h_2, h_3 + Q_3(tN_1, tN_2, h_1, h_2)\big)\Big).\end{aligned} \tag{3.5.99}$$

Since $\nu_1 N_1 + \nu_2 N_2 = |\nu|$ last inequality is guaranteed by

$$x_3 + |\nu| t \geq |h_1||\nu_1| + |h_2||\nu_2| + |h_3| + |Q_3(tN_1, tN_2, h_1, h_2)| + |\psi(z)|,$$

where $z = (z_1, z_2, z_4)$ denotes the argument of ψ in (3.5.99). Note that $|h_1||\nu_1| + |h_2||\nu_2| \leq \lambda|\nu|t$ and this term can be absorbed in the left hand side if λ is small. Moreover $|h_3| \leq \lambda t^2$ and $|Q_3(tN_1, tN_2, h_1, h_2)| \lesssim \lambda t^2$. Then, in order to prove inclusion (3.5.98) it will be enough to show that

$$\varepsilon_0(x_3 + |\nu| t) \geq \lambda t^2 + |\psi(z)|$$

for some $\varepsilon_0 > 0$ small but absolute. We estimate z_1, z_2 and z_4. Clearly, $|z_1| = |tN_1 + h_1| \lesssim t$ and $|z_2| = |tN_2 + h_2| \lesssim t$. Moreover,

$$\begin{aligned}|z_4| &= \big|h_4 + Q_4(tN_1, tN_2, 0, h_1, h_2, h_3)\\ &\qquad + Q_4\big(0, 0, x_3, tN_1 + h_1, tN_2 + h_2, h_3 + Q_3(tN_1, tN_2, h_1, h_2)\big)\big|\\ &\lesssim \lambda t^3 + x_3 t\end{aligned}$$

because $Q_4(0, 0, x_3, \xi_1, \xi_2, \xi_3) = 1/2\{(-x_3\xi_1) + \alpha(-\xi_2 x_3)\}$.

Thus by (3.5.97)

$$|\psi(z)| \lesssim z_1^2 + z_2^2 + |z_4| \lesssim t^2 + \lambda t^3 + x_3 t \simeq t^2 + x_3 t,$$

because $\lambda t^3 \lesssim t^2$ (we assume $t \leq 1$).

We finally have to prove the inequality

$$\varepsilon_0(x_3 + |\nu| t) \geq t^2 + x_3 t, \tag{3.5.100}$$

which holds if $t \leq \sigma|\nu|$ with $\sigma > 0$ small depending only on Ω (we used $x_3 t \leq x_3 \sigma \nu \lesssim \sigma$).

5. *Second piece.* From now up to the end of Case 1, $t_1 = \sigma|\nu|$ will be fixed. For $t \geq t_1$ define

$$\gamma(t) = (0, 0, x_3, 0) \cdot (t_1 N_1, t_1 N_2, t - t_1, 0),$$

and note that $d(\gamma(t), x) \simeq t_1 + (t - t_1)^{1/2}$. Write $b = (t - t_1)^{1/2}$ and $\delta(t) = t_1 + b$.

6. We claim that there exists a positive $\lambda < 1$ such that for all $t_1 \leq t \leq 1$

$$\mathrm{Box}(\gamma(t), \lambda\delta(t)) \subset \Omega. \tag{3.5.101}$$

Condition (3.5.101) is equivalent to the John property for γ in its second piece.

Points in $\mathrm{Box}(\gamma(t), \lambda\delta(t))$ have the form

$$\begin{aligned}\gamma(t) \cdot h &= (0, 0, x_3, 0) \cdot (t_1 N_1, t_1 N_2, b^2, 0) \cdot (h_1, h_1, h_3, h_4) \\ &= \big(t_1 N_1, t_1 N_2, x_3 + b^2, Q_4(0, 0, x_3, t_1 N_1, t_1 N_2, b^2)\big) \cdot (h_1, h_2, h_3, h_4) \\ &= \Big(t_1 N_1 + h_1, t_1 N_2 + h_2, x_3 + b^2 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2), \\ &\qquad , Q_4(0, 0, x_3, t_1 N_1, t_1 N_2, b^2) + h_4 + Q_4(t_1 N_1, t_1 N_2, x_3 + b^2, h_1, h_2, h_3)\Big),\end{aligned}$$

with $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \leq \lambda\delta(t)$. Now, $\gamma(t) \cdot h \in \Omega$ provided that

$$\begin{aligned}x_3 + &b^2 + h_3 + Q_3(t_1 N_1, t_1 N_2, h_1, h_2) \geq -\nu_1(t_1 N_1 + h_1) - \nu_2(t_1 N_2 + h_2) \\ &+ \psi\Big(t_1 N_1 + h_1, t_1 N_2 + h_2, Q_4(0, 0, x_3, t_1 N_1, t_1 N_2, b^2) + h_4 \\ &\qquad + Q_4(t_1 N_1, t_1 N_2, x_3 + b^2, h_1, h_2, h_3)\Big),\end{aligned} \tag{3.5.102}$$

which is implied by

$$t_1|\nu| + x_3 + b^2 \geq |\nu_1||h_1| + |\nu_2||h_2| + |h_3| + |Q_3(t_1 N_1, t_1 N_2, h_1, h_2)| + |\psi(z)|, \tag{3.5.103}$$

where $z = (z_1, z_2, z_4)$ is the argument of ψ in (3.5.102). In order to prove (3.5.103) note that $|\nu_1||h_1| + |\nu_2||h_2| \lesssim \lambda|\nu|\delta(t) \simeq \lambda|\nu| t_1 + \lambda|\nu| b$. The term $\lambda|\nu| t_1$ can be put in the left hand side. Moreover $|h_3| \leq \lambda\delta^2(t) \lesssim \lambda t_1^2 + \lambda b^2$ and $|Q_3(t_1 N_1, t_1 N_2, h_1, h_2)| \leq \lambda t_1 \delta(t) \lesssim \lambda t_1^2 + \lambda b^2$. The term λb^2 can also be absorbed in the left hand side.

Claim (3.5.101) will be proved if we show that for some uniform constant $\varepsilon_0 > 0$ and for $t_1 \leq t \leq 1$

$$\varepsilon_0(t_1|\nu| + x_3 + b^2) \geq \lambda|\nu| b + \lambda t_1^2 + |\psi(z)|. \tag{3.5.104}$$

We estimate z_1, z_2 and z_4. First of all $|z_1| = |t_1 N_1 + h_1| \lesssim t_1 + \lambda\delta(t) \lesssim t_1 + \lambda b$ and the same estimate holds for $|z_2|$. Moreover, writing δ instead of $\delta(t)$

$$\begin{aligned}|z_4| &= \big|Q_4(0, 0, x_3, t_1 N_1, t_1 N_2, b^2) + h_4 + Q_4(t_1 N_1, t_1 N_2, x_3 + b^2, h_1, h_2, h_3)\big| \\ &\lesssim x_3 t_1 + \lambda\delta^3 + \lambda t_1^2 \delta + \lambda t_1 \delta^2 + \lambda(x_3 + b^2)\delta \\ &\simeq x_3 t_1 + \lambda(t_1 + b)^3 + \lambda t_1^2(t_1 + b) + \lambda t_1(t_1 + b)^2 + \lambda(x_3 + b^2)(t_1 + b) \\ &\simeq x_3 t_1 + \lambda t_1^3 + \lambda b^3 + \lambda x_3 t_1 + \lambda x_3 b \simeq x_3 t_1 + \lambda t_1^3 + \lambda b^3 + \lambda x_3 b.\end{aligned}$$

Then by (3.5.97)

$$|\psi(z)| \lesssim |z_1|^2 + |z_2|^2 + |z_4| \leq t_1^2 + \lambda b^2 + x_3 t_1 + \lambda t_1^3 + \lambda b^3 + \lambda x_3 b.$$

Thus (3.5.104) is implied by

$$\begin{aligned} \varepsilon_0(t_1|\nu| + x_3 + b^2) &\geq \lambda|\nu|b + t_1^2 + \lambda b^2 + x_3 t_1 + \lambda t_1^3 + \lambda b^3 + \lambda x_3 b \\ &\simeq \lambda|\nu|b + t_1^2 + \lambda b^2 + x_3 t_1 + \lambda b^3 + \lambda x_3 b. \end{aligned} \tag{3.5.105}$$

Inequality (3.5.105) holds for $b = 0$. This has been proved in (3.5.100) with $t = t_1$.

Taking a smaller constant in the left hand side of (3.5.100) we can assert that (3.5.105) is guaranteed by

$$\varepsilon_0(t_1|\nu| + x_3 + b^2) \geq \lambda|\nu|b + \lambda b^2 + \lambda b^3 + \lambda x_3 b. \tag{3.5.106}$$

We can estimate the right hand side using $|\nu| \leq 1$ and $b \leq 1$ getting

$$\lambda(|\nu|b + b^2 + b^3 + x_3 b) \leq \lambda(|\nu|^2 + b^2 + x_3).$$

Recalling now that $t_1 = \sigma|\nu|$ inequality (3.5.106) is proved.

7. **Case 2.** Let $x_0 \in \partial\Omega$ be a characteristic point of second type. After a translation we can assume that $0 \in \partial\Omega$ is near x_0. Thus, in a neighborhood of 0 the surface $\partial\Omega$ admits a parametrization of second type, i.e. there exists a function $\varphi = \varphi(y_1, y_2, y_3)$ of class C^2 such that $\varphi(0) = 0$ and in a neighborhood of 0 we have $\partial\Omega = \{y_4 = \varphi(y_1, y_2, y_3)\}$. Define

$$\begin{aligned} &\nu_1 = -\partial_1\varphi(0),\ \nu_2 = -\partial_2\varphi(0),\ \nu_3 = -\partial_3\varphi(0),\ \nu = (\nu_1, \nu_2), \\ &N_1 = \frac{\nu_1}{|\nu|},\ N_2 = \frac{\nu_2}{|\nu|},\ N_3 = \frac{\nu_3}{|\nu_3|} = \operatorname{sgn}(\nu_3). \end{aligned}$$

If $\nu = 0$ simply set $N_1 = N_2 = 0$. If $\nu_3 = 0$ set $N_3 = 0$. Moreover let $\psi(y) = \varphi(y) + \nu_1 y_1 + \nu_2 y_2 + \nu_3 y_3$. By (3.5.96) ψ satisfies the following growth condition

$$|\psi(y)| \lesssim \|y\|^3 + (|\nu|^{1/2} + |\nu_3|)(y_1^2 + y_2^2). \tag{3.5.107}$$

We shall now construct the John curve starting from $x = x_4 e_4$. Without loss of generality (the map $z \mapsto z + \mu e_4$, $\mu \in \mathbb{R}$ is a left translation) assume that $x = 0 \in \partial\Omega$. We have to define a continuous path $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = 0$ and $\operatorname{dist}(\gamma(t); \partial\Omega) \geq \lambda d(\gamma(t), 0)$ for all $t \in [0,1]$ and for some $\lambda > 0$ depending only on Ω. We split the path in three pieces.

8. *First piece.* For $\sigma > 0$ let

$$t_1 = \sigma \min\Big\{|\nu|^{1/2}, \frac{|\nu|}{|\nu_3|}\Big\}, \tag{3.5.108}$$

and if $t \in [0, t_1]$ define

$$\gamma(t) = (N_1 t, N_2 t, 0, 0).$$

Note that $d(\gamma(t), 0) = t$.

9. We claim that there exist positive constants $\sigma, \lambda < 1$ such that for all $t \in [0, t_1]$ the following John property holds

$$\operatorname{Box}(\gamma(t), \lambda t) \subset \Omega. \tag{3.5.109}$$

Points in $\mathrm{Box}(\gamma(t), \lambda t)$ are of the form

$$\begin{aligned}\gamma(t) \cdot h &= (N_1 t, N_2 t, 0, 0) \cdot (h_1, h_2, h_3, h_4) \\ &= (N_1 t + h_1, N_2 t + h_2, h_3 + Q_3(N_1 t, N_2 t, h_1, h_2), \\ &\qquad\qquad\qquad , h_4 + Q_4(N_1 t, N_2 t, 0, h_1, h_2, h_3)),\end{aligned}$$

with $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \leq \lambda t$. Now, $\gamma(t) \cdot h \in \Omega$ if

$$\begin{aligned}h_4 + Q_4(N_1 t, N_2 t, 0, h_1, h_2, h_3) > &- \nu_1(N_1 t + h_1) - \nu_2(N_2 t + h_2) \\ &- \nu_3(h_3 + Q_3(N_1 t, N_2 t, h_1, h_2)) \\ &+ \psi\big(N_1 t + h_1, N_2 t + h_2, h_3 + Q_3(N_1 t, N_2 t, h_1, h_2)\big),\end{aligned}$$

which is implied by

$$\begin{aligned}|\nu| t \geq\ & |\nu_1||h_1| + |\nu_2||h_2| + |\nu_3||h_3| + |\nu_3||Q_3(N_1 t, N_2 t, h_1, h_2)| \\ &+ \big|\psi\big(N_1 t + h_1, N_2 t + h_2, h_3 + Q_3(N_1 t, N_2 t, h_1, h_2)\big)\big| \\ &+ |h_4| + |Q_4(N_1 t, N_2 t, 0, h_1, h_2, h_3)|.\end{aligned} \tag{3.5.110}$$

Recall that $|\nu_1||h_1| + |\nu_2||h_2| \leq \lambda |\nu| t$, $|h_3| \leq \lambda t^2$, $|Q_3(N_1 t, N_2 t, h_1, h_2)| \leq \lambda t^2$, $|h_4| \leq \lambda t^3$ and $|Q_4(N_1 t, N_2 t, 0, h_1, h_2, h_3)| \leq \lambda t^3$. If $z = (z_1, z_2, z_3)$ is the argument of ψ in (3.5.110) then we get

$$\|z\| = \|(N_1 t + h_1, N_2 t + h_2, h_3 + Q_3(N_1 t, N_2 t, h_1, h_2))\| \lesssim t + \lambda t \simeq t,$$

and by (3.5.107)

$$|\psi(z)| \lesssim \|z\|^3 + (|\nu|^{1/2} + |\nu_3|)(z_1^2 + z_2^2) \leq t^3 + (|\nu|^{1/2} + |\nu_3|) t^2$$

We finally get the following inequality which is stronger than (3.5.110)

$$\varepsilon_0 |\nu| t \geq \lambda |\nu| t + \lambda |\nu_3| t^2 + t^3 + (|\nu|^{1/2} + |\nu_3|) t^2,$$

where $\varepsilon_0 < 1$ is an absolute constant. Dividing by t we have to show that for some $\varepsilon_0 > 0$

$$\varepsilon_0 |\nu| \geq t^2 + (|\nu|^{1/2} + |\nu_3|) t. \tag{3.5.111}$$

($\lambda |\nu|$ has been absorbed in the left hand side). It will be enough to determine all t that solve the following two inequalities

$$t^2 < \varepsilon_0 |\nu| \quad \text{and} \quad t(|\nu|^{1/2} + |\nu_3|) < \varepsilon_0 |\nu|.$$

The first one gives $t \leq \varepsilon_0 |\nu|^{1/2}$ and the second one is consequently solved by $t|\nu_3| \leq \varepsilon_0 |\nu|$. Claim (3.5.109) is proved if t_1 is as in (3.5.108) for a small absolute constant $\sigma > 0$.

10. *Second piece.* From now on t_1 is fixed as in (3.5.108). For $\eta > 0$ let

$$t_2 = \eta \max\{|\nu|, |\nu_3|^2\}, \tag{3.5.112}$$

and if $t \in [t_1, t_1 + t_2]$ define

$$\gamma(t) = (t_1 N_1, t_1 N_2, (t - t_1) N_3, 0).$$

Notice that

$$\delta(t) := t_1 + (t - t_1)^{1/2} \simeq d(\gamma(t), 0). \tag{3.5.113}$$

If $\nu_3 = 0$, then the third piece is trivial. In the sequel we shall sometimes write δ instead of $\delta(t)$. Moreover, let $b = (t - t_1)^{1/2}$.

11. We claim that there exist positive constants $\eta, \lambda < 1$ such that for all $t \in [t_1, t_1 + t_2]$ the following John property for γ holds

$$\mathrm{Box}(\gamma(t), \lambda\delta(t)) \subset \Omega. \tag{3.5.114}$$

Points in $\mathrm{Box}(\gamma(t), \lambda\delta)$ are of the form

$$\begin{aligned}\gamma(t)\cdot h &= (t_1N_1, t_1N_2, (t-t_1)N_3, 0)\cdot(h_1, h_2, h_3, h_4)\\ &= (t_1N_1 + h_1, t_1N_2 + h_2, (t-t_1)N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2),\\ &\qquad , h_4 + Q_4(t_1N_1, t_1N_2, (t-t_1)N_3, h_1, h_2, h_3),\end{aligned}$$

with $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \le \lambda\delta$. Now, $\gamma(t)\cdot h \in \Omega$ if

$$\begin{aligned}h_4 + Q_4(t_1N_1, t_1N_2, b^2, h_1, h_2, h_3) \ge &-\nu_1(t_1N_1 + h_1) - \nu_2(t_1N_2 + h_2)\\ &- \nu_3(b^2N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2))\\ &+ \psi(t_1N_1 + h_1, t_1N_2 + h_2, b^2N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2)),\end{aligned}$$

which is implied by

$$\begin{aligned}|\nu|t_1 + |\nu_3|b^2 \ge\ &|h_4| + |Q_4(t_1N_1, t_1N_2, t - t_1, h_1, h_2, h_3)|\\ &+ |\nu_1||h_1| + |\nu_2||h_2| + |\nu_3||h_3| + |\nu_3|\big|Q_3(t_1N_1, t_1N_2, h_1, h_2)\big|\\ &+ \big|\psi(t_1N_1 + h_1, t_1N_2 + h_2, b^2N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2))\big|.\end{aligned} \tag{3.5.115}$$

We estimate now the right hand side: $|h_4| \le \lambda\delta^3$, $|Q_4(t_1N_1, t_1N_2, b^2, h_1, h_2, h_3)| \le \lambda\delta^3 + b^2\lambda\delta \simeq \lambda\delta^3$, $|\nu_1||h_1| + |\nu_2||h_2| \le \lambda|\nu|\delta$, $|h_3| \le \lambda\delta^2$ and $|Q_3(t_1N_1, t_1N_2, h_1, h_2)| \le \lambda t_1\delta \le \lambda\delta^2$.

Let $z = (z_1, z_2, z_3)$ be the argument of ψ in (3.5.115). Then $|z_1| = |t_1N_1 + h_1| \lesssim t_1 + \lambda\delta$ and analogously $|z_2| \lesssim t_1 + \lambda\delta$. Moreover, as $b \le \delta$

$$\begin{aligned}\|(z_1, z_2, z_3)\| &= \|(t_1N_1 + h_1, t_1N_2 + h_2, b^2N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2)\|\\ &\simeq t_1 + \lambda\delta + b + \lambda\delta + (t_1\lambda\delta)^{1/2} \simeq t_1 + \lambda\delta + b \simeq \delta.\end{aligned}$$

This furnishes

$$\begin{aligned}|\psi(z)| &\lesssim \delta^3 + (|\nu|^{1/2} + |\nu_3|)(t_1 + \lambda\delta)^2\\ &\lesssim \delta^3 + |\nu|^{1/2}t_1^2 + \lambda|\nu|^{1/2}\delta^2 + |\nu_3|t_1^2 + \lambda|\nu_3|\delta^2,\end{aligned}$$

and (3.5.115) is guaranteed by

$$|\nu|t_1 + |\nu_3|b^2 \ge \lambda\delta^3 + \lambda|\nu|\delta + \lambda|\nu_3|\delta^2 + \delta^3 + |\nu|^{1/2}t_1^2 + \lambda|\nu|^{1/2}\delta^2 + |\nu_3|t_1^2 + \lambda|\nu_3|\delta^2. \tag{3.5.116}$$

Replacing $\delta = t_1 + b$ we get

$$\begin{aligned}\varepsilon_0(|\nu|t_1 + |\nu_3|b^2) \ge\ &(t_1 + b)^3 + \lambda|\nu|(t_1 + b) + \lambda|\nu_3|(t_1 + b)^2 + |\nu|^{1/2}t_1^2\\ &+ \lambda|\nu|^{1/2}(t_1 + b)^2 + |\nu_3|t_1^2,\end{aligned}$$

where ε_0 is a small but absolute constant. Possibly changing ε_0 it will be enough to show that

$$\begin{aligned}\varepsilon_0(|\nu|t_1 + |\nu_3|b^2) \geq t_1^3 + b^3 + \lambda|\nu|t_1 + \lambda|\nu|b + \lambda|\nu_3|t_1^2 + \lambda|\nu_3|b^2 \\ + |\nu|^{1/2}t_1^2 + \lambda|\nu|^{1/2}t_1^2 + \lambda|\nu|^{1/2}b^2 + |\nu_3|t_1^2.\end{aligned}$$

Now, $\lambda|\nu|t_1$ and $\lambda|\nu_3|b^2$ can be absorbed in the left hand side, and $\lambda|\nu_3|t_1^2 + |\nu_3|t_1^2 \simeq |\nu_3|t_1^2$. Then

$$\varepsilon_0(|\nu|t_1 + |\nu_3|b^2) \geq t_1^3 + b^3 + \lambda|\nu|b + |\nu_3|t_1^2 + |\nu|^{1/2}t_1^2 + \lambda|\nu|^{1/2}b^2. \tag{3.5.117}$$

Inequality (3.5.117) holds with $b = 0$ by (3.5.111) with $t = t_1$. It will be enough to show that

$$\varepsilon_0(|\nu|t_1 + |\nu_3|b^2) \geq b^3 + \lambda|\nu|b + \lambda|\nu|^{1/2}b^2. \tag{3.5.118}$$

12. In order to prove (3.5.118) the following two cases must be distinguished:

(2A) $|\nu_3| \leq |\nu|^{1/2}$;
(2B) $|\nu_3| > |\nu|^{1/2}$.

13. Case 2A. In this case $t_1 = \sigma|\nu|^{1/2}$ and (3.5.118) becomes (with a smaller ε_0)

$$\varepsilon_0(|\nu|^{3/2} + |\nu_3|b^2) \geq b^3 + \lambda|\nu|b + \lambda|\nu|^{1/2}b^2.$$

By the trivial estimate $|\nu_3|b^2 \geq 0$ and letting $\lambda = 1$ in the right hand side we get the stronger inequality

$$\varepsilon_0|\nu|^{3/2} \geq b^3 + |\nu|b + |\nu|^{1/2}b^2.$$

Setting $b = |\nu|^{1/2}a$ (this can be done because in Case 2A it should be $\nu \neq 0$) we find $\varepsilon_0 \geq a^3 + a^2 + a$ which holds for all $0 \leq a < a_0$. Then (3.5.118) holds for all $0 \leq b \leq a_0|\nu|^{1/2}$ and consequently our claim (3.5.114) holds for all $t \leq t_1 + a_0^2|\nu|$.

14. Case 2B. Here $t_1 = \sigma|\nu|/|\nu_3|$. The term $\lambda|\nu|^{1/2}b^2$ in the right hand side of (3.5.118) is less than $\varepsilon_0|\nu_3|b^2$ and can be absorbed in the left hand side. Then we get the inequality (with a possibly smaller ε_0)

$$\varepsilon_0\Big(\frac{|\nu|^2}{|\nu_3|} + |\nu_3|b^2\Big) \geq b^3 + \lambda|\nu|b$$

that is

$$\varepsilon_0(|\nu|^2 + |\nu_3|^2b^2) \geq b^3|\nu_3| + \lambda|\nu||\nu_3|b.$$

Now, $\lambda|\nu||\nu_3|b \leq \frac{\lambda}{2}|\nu|^2 + \frac{\lambda}{2}|\nu_3|^2b^2$ and both these terms can be absorbed in the left hand side if λ is suitable. Thus it suffices to solve

$$\varepsilon_0(|\nu|^2 + |\nu_3|^2b^2) \geq b^3|\nu_3|.$$

Setting $|\nu| = 0$ we find $b \leq \varepsilon_0|\nu_3|$ which gives the correct choice $t_2 = \varepsilon_0^2|\nu_3|^2$, as declared in (3.5.112). Claim (3.5.114) is proved in Case 2B too.

15. *Third piece.* From now on t_2 is fixed as in (3.5.112). If $t \geq t_1 + t_2$ define

$$\gamma(t) = (t_1N_1, t_1N_2, t_2N_3, t - (t_1 + t_2)),$$

and notice that

$$\delta(t) := t_1 + t_2^{1/2} + (t - (t_1 + t_2))^{1/3} \simeq d(\gamma(t), 0).$$

As before we shall sometimes write δ instead of $\delta(t)$. Moreover, let $a = (t-(t_1+t_2))^{1/3}$.

16. We claim that there exists $\lambda < 1$ such that the following John property for γ holds for all $t_1 + t_2 \le t \le 1$

$$\operatorname{Box}(\gamma(t), \lambda\delta(t)) \subset \Omega. \tag{3.5.119}$$

Points in $\operatorname{Box}(\gamma(t), \lambda\delta(t))$ are of the form

$$\begin{aligned}
\gamma(t) \cdot h &= (t_1N_1, t_1N_2, t_2N_3, a^3) \cdot (h_1, h_2, h_3, h_4) \\
&= \Big(t_1N_1 + h_1, t_1N_2 + h_2, t_2N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2), \\
&\qquad a^3 + h_4 + Q_4(t_1N_1, t_1N_2, t_2N_3, h_1, h_2, h_3)\Big),
\end{aligned}$$

where $h = (h_1, h_2, h_3, h_4)$ and $\|h\| \le \lambda\delta$. Now, $\gamma(t) \cdot h \in \Omega$ if

$$\begin{aligned}
a^3 + h_4 + Q_4(t_1N_1, t_1N_2, t_2N_3, h_1, h_2, h_3) &\ge -\nu_1(t_1N_1 + h_1) - \nu_2(t_1N_2 + h_2) \\
&\quad - \nu_3\big(t_2N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2)\big) \\
&\quad + \big|\psi\big(t_1N_1 + h_1, t_1N_2 + h_2, t_2N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2)\big)\big|.
\end{aligned}$$

As usual we find the stronger inequality

$$\begin{aligned}
|\nu|t_1 + |\nu_3|t_2 + a^3 \ge |h_4| &+ |Q_4(t_1N_1, t_1N_2, t_2N_3, h_1, h_2, h_3)| \\
&+ |\nu_1||h_1| + |\nu_2||h_2| + |\nu_3||h_3| + |\nu_3|\big|Q_3(t_1N_1, t_1N_2, h_1, h_2))\big| \\
&+ \big|\psi(t_1N_1 + h_1, t_1N_2 + h_2, t_2N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2))\big|.
\end{aligned}$$

In the right hand side we can estimate $|h_4|, |Q_4| \le \lambda\delta^3$, $|\nu_1||h_1| + |\nu_2||h_2| \le \lambda|\nu|\delta$, $|h_3| \le \lambda\delta^2$ and $|Q_3(t_1N_1, t_1N_2, h_1, h_2)| \le \lambda t_1\delta \lesssim \lambda\delta^2$.

Let $z = (z_1, z_2, z_3)$ be the argument of ψ. Then $|z_1| = |t_1N_1 + h_1| \le t_1 + \lambda\delta$ and $|z_2| \le t_1 + \lambda\delta$. Moreover

$$\begin{aligned}
\|z\| &= \|(t_1N_1 + h_1, t_2N_2 + h_2, t_2N_3 + h_3 + Q_3(t_1N_1, t_1N_2, h_1, h_2))\| \\
&\le t_1 + \lambda\delta + t_2^{1/2} + \lambda\delta + (t_1\lambda\delta)^{1/2} \simeq t_1 + \lambda\delta + t_2^{1/2}.
\end{aligned}$$

By (3.5.107)

$$\begin{aligned}
|\psi(z)| &\le \|z\|^3 + (|\nu|^{1/2} + |\nu_3|)(z_1^2 + z_2^2) \\
&\le (t_1 + t_2^{1/2} + \lambda\delta)^3 + (|\nu|^{1/2} + |\nu_3|)(t_1 + \lambda\delta)^2 \\
&\simeq t_1^3 + t_2^{3/2} + \lambda\delta^3 + t_1^2|\nu|^{1/2} + t_1^2|\nu_3| + \lambda|\nu|^{1/2}\delta^2 + \lambda|\nu_3|\delta^2.
\end{aligned}$$

Ultimately we have to show that

$$\begin{aligned}
\varepsilon_0(t_1|\nu| + t_2|\nu_3| + a^3) \ge \lambda\delta^3 &+ \lambda|\nu|\delta + \lambda|\nu_3|\delta^2 + t_1^3 + t_2^{3/2} \\
&+ t_1^2|\nu|^{1/2} + t_1^2|\nu_3| + \lambda|\nu|^{1/2}\delta^2.
\end{aligned}$$

Notice that $|\nu|^{1/2}\delta^2 \le \frac{1}{2}(|\nu|\delta + \delta^3)$ and thus the term $\lambda|\nu|^{1/2}\delta^2$ in the right hand side can be deleted. Now, writing $\delta = t_1 + t_2^{1/2} + a$ we get

$$\begin{aligned}
\varepsilon_0(t_1|\nu| + t_2|\nu_3| + a^3) \ge \lambda t_1^3 &+ \lambda t_2^{3/2} + \lambda a^3 + \lambda|\nu|t_1 + \lambda|\nu|t_2^{1/2} + \lambda|\nu|a + \lambda|\nu_3|t_1^2 \\
&+ \lambda|\nu_3|t_2 + \lambda|\nu_3|a^2 + t_1^3 + t_2^{3/2} + t_1^2|\nu|^{1/2} + t_1^2|\nu_3|, \quad ,
\end{aligned}$$

and letting absorbe $\lambda|\nu|t_1$, $\lambda|\nu_3|t_2$ and λa^3 by the left hand side we find the stronger inequality

$$\begin{aligned}\varepsilon_0(t_1|\nu| + t_2|\nu_3| + a^3) \geq t_1^3 + t_2^{3/2} + \lambda|\nu|t_2^{1/2} + \lambda|\nu|a \\ + \lambda|\nu_3|a^2 + t_1^2|\nu|^{1/2} + t_1^2|\nu_3|.\end{aligned}$$

Such inequality holds if $a = 0$ (let $b^2 = t_2$ in (3.5.117)). Thus it will be enough to prove that for a small but absolute constant ε_0

$$\varepsilon_0(t_1|\nu| + t_2|\nu_3| + a^3) \geq \lambda|\nu|a + \lambda|\nu_3|a^2, \tag{3.5.120}$$

for all $a \geq 0$.

We distinguish Case 2A and Case 2B.

17. Case 2A. In this case $|\nu_3| \leq |\nu|^{1/2}$, $t_1 = \sigma|\nu|^{1/2}$ and $t_2 = \eta|\nu|$. Using $t_2|\nu_3| \geq 0$ in the left hand side of (3.5.120), replacing $t_1 = \sigma|\nu|^{1/2}$ and using also $|\nu_3| \leq |\nu|^{1/2}$ in the right hand side we get the stronger inequality

$$\varepsilon_0(|\nu|^{3/2} + a^3) \geq \lambda(|\nu|a + |\nu|^{1/2}a^2)$$

which holds because 3 and 3/2 are Hölder conjugate exponents.

18. Case 2B. Here $|\nu_3| > |\nu|^{1/2}$, $t_1 = \sigma|\nu|/|\nu_3|$ and $t_2 = \eta|\nu_3|^2$. in the left hand side of (3.5.120) we use $t_1|\nu| \geq 0$ and put $t_2 = \eta|\nu_3|^2$. In the right hand side we estimate $|\nu| \leq |\nu_3^2|$. Thus we find the stronger inequality

$$\varepsilon_0(|\nu_3|^3 + a^3) \geq \lambda|\nu_3|^2a + \lambda|\nu_3|a^2,$$

which holds for all $a \geq 0$.

19. **Case 3.** This is the non characteristic case and can be analyzed as in Theorem 3.4.2.

□

Example 3.5.6. Using Theorem 3.5.5 we construct an example of John domain $\Omega \subset \mathbb{R}^4$ with respect to the metric structure of the group of step 3 considered in section 5. Let $g \in C^2(0,1) \cap C([0,1])$ be a function such that

$$g(t) = \begin{cases} 1 - t^{1/4} & \text{if } 0 \leq t \leq 1/4, \\ (1-t)^{1/4} & \text{if } 3/4 \leq t \leq 1. \end{cases}$$

Such a function can be chosen with the additional property $g'(t) > 0$ for all $t \in (0,1)$. Let

$$N(x_1, x_2, x_3) = (x_1^2 + x_2^2)^6 + x_3^6,$$

and define the open set

$$\Omega = \{x \in \mathbb{R}^4 : |x_4| < g(N(x_1, x_2, x_3))\}. \tag{3.5.121}$$

We notice that if $N(x_1, x_2, x_3) \geq 3/4$ then $\partial\Omega$ has equation $(x_1^2 + x_2^2)^6 + x_3^6 + x_4^4 = 1$. If $N(x_1, x_2, x_3) \leq 1/4$ then $\partial\Omega$ has equation $|x_4| + [(x_1^2 + x_2^2)^6 + x_3^6]^{1/4} = 1$. Therefore the boundary $\partial\Omega$ is globally of class C^1 and is of class C^2 where $N(x_1, x_2, x_3) \neq 0$.

We show that the points $(0,0,0,\pm 1) \in \partial\Omega$ are the only characteristic points of second type of $\partial\Omega$. Indeed, let $\Phi(x_1,x_2,x_3,x_4) = g(N(x)) - x_4$ and compute

$$X_1\Phi(x) = g'(N(x))X_1N(x) - q_1(x)$$
$$X_2\Phi(x) = g'(N(x))X_2N(x) - q_2(x)$$
$$X_3\Phi(x) = g'(N(x))X_3N(x) - q_3(x),$$

where q_1, q_2 and q_3 are defined in (3.5.90). We have $x_1q_1(x) + x_2q_2(x) = -x_3q_3(x)$ and

$$X_1N(x) = 12x_1(x_1^2+x_2^2)^6 - 3x_2x_3$$
$$X_2N(x) = 12x_2(x_1^2+x_2^2)^6 + 3x_1x_3,$$

and thus $x_1X_1N(x) + x_2X_2N(x) = 12(x_1^2+x_2^2)^6$. Then

$$x_1X_1\Phi + x_2X_2\Phi = g'(N(x))12(x_1^2+x_2^2)^6 + x_3q_3 = 0,$$

but

$$x_3X_3\Phi = g'(N(x))6x_3^6 - x_3q_3(x) = 0,$$

and summing up the last two equations we finally get

$$g'(N(x))\big(12(x_1^2+x_2^2)^6 + 6x_3^6\big) = 0,$$

which implies $x_1 = x_2 = x_3 = 0$, as $g'(N(x)) \neq 0$.

In order to apply Theorem 3.5.5 we have to check that, letting $\Phi(x) = N(x)^{1/4} - x_4$, there exists a constant $k > 0$ such that

$$|X_1^2\Phi| + |X_2^2\Phi| + |(X_1X_2 + X_2X_1)\Phi| \le k(|X_1\Phi|^{1/2} + |X_2\Phi|^{1/2} + |X_3\Phi|)$$

for all $x \in \partial\Omega$ such that $0 < N(x) \le 1/4$. We note that away from the origin the function Φ is smooth and moreover it is homogeneous of degree 3 with respect to the dilations $(x_1,x_2,x_3,x_4) \to (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4)$. Then the derivatives $X_1\Phi$ and $X_2\Phi$ are homogeneous of degree 2 and their square roots $|X_1\Phi|^{1/2}$ and $|X_2\Phi|^{1/2}$ are homogeneous of degree 1. Analogously, $X_1^2\Phi$, $X_2^2\Phi$, $X_3\Phi$ and $(X_1X_2 + X_2X_1)\Phi$, being derivatives of degree 2, are homogeneous of degree 1. Then the function $H = H(x_1,x_2,x_3)$ defined by

$$H = \frac{|X_1^2\Phi| + |X_2^2\Phi| + |(X_1X_2 + X_2X_1)\Phi|}{|X_1\Phi|^{1/2} + |X_2\Phi|^{1/2} + |X_3\Phi|}$$

is homogeneous of degree 0. We showed above that $|X_1\Phi(x)|^{1/2} + |X_2\Phi(x)|^{1/2} + |X_3\Phi(x)| > 0$ for all $N(x) > 0$, and thus by the $0-$homogeneity

$$\sup_{0<N(x)\le 1/4} H(x) = \max_{N(x)=1/4} H(x) = k < +\infty.$$

6. Trace theorem for Hörmander vector fields

In this section we prove the trace theorem for Hörmander vector fields. We begin with some Lemmata.

LEMMA 3.6.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set with C^∞ boundary. Let $K \subset \partial\Omega$ be a compact set of non characteristic points with respect to the vector fields $X_1, ..., X_m \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfying the Hörmander condition. If $\mu = \mathcal{H}^{n-1} \llcorner \partial\Omega$, then there exist $r_0 > 0$, $0 < m_1 < m_2$ such that*

$$m_1 \frac{|B(x,r)|}{r} \leq \mu(B(x,r)) \leq m_2 \frac{|B(x,r)|}{r} \tag{3.6.122}$$

for all $x \in K$ and for all $0 < r < r_0$.

Proof. In view of Lemma 3.2.2 and Remark 3.2.3 $X_1, ..., X_m$ can be assumed to be of the form (3.2.25) and $K \subset \partial\Omega \subset \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t = 0\}$. The Lemma follows from (ii) and (iii) in Theorem 1.6.10. □

Next, we recall Hardy inequality.

PROPOSITION 3.6.2. *Let $0 < r \leq +\infty$. If $1 < p < \infty$ and if $f \in \mathrm{L}^p(0,r)$ then*

$$\int_0^r \Big(\frac{1}{t}\int_0^t |f(x)|\,dx\Big)^p dt \leq \Big(\frac{p}{p-1}\Big)^p \int_0^r |f(x)|^p\,dx. \tag{3.6.123}$$

Finally, we need the following formula for integration of "radial functions".

LEMMA 3.6.3. *Let $d : \mathbb{R}^n \to [0,\infty)$ be a Lipschitz function such that $|\{x \in \mathbb{R}^n : d(x) < \lambda\}| = \sigma\lambda^Q$ for some $Q > 0$, $\sigma > 0$, for all $\lambda > 0$, and $|\nabla d(x)| \neq 0$ for a.e. $x \in \mathbb{R}^n$. Then*

$$\int_{\{d(x)<r\}} \varphi(d(x))\,dx = \sigma Q \int_0^r \varphi(\lambda)\lambda^{Q-1}\,d\lambda \tag{3.6.124}$$

for all measurable functions $\varphi \geq 0$, $r > 0$.

Proof. For $\varepsilon > 0$ let $g_\varepsilon(x) = \chi_{\{|\nabla d| > \varepsilon\}}(x)$ and by the coarea formula write

$$\int_{\{d(x)<\lambda\}} g_\varepsilon(x)\varphi(d(x))\,dx = \int_0^\lambda \varphi(r) \int_{\{d(x)=r\}} \frac{g_\varepsilon(x)}{|\nabla d(x)|} d\mathcal{H}^{n-1}(x)\,dr.$$

Since $\mathcal{H}^{n-1}(\{d(x) = r\} \cap \{\nabla d(x) = 0\}) = 0$ for a.e. $r > 0$, by monotone convergence we get

$$\int_{\{d(x)<\lambda\}} \varphi(d(x))\,dx = \int_0^\lambda \varphi(r) \int_{\{d(x)=r\}} \frac{1}{|\nabla d(x)|} d\mathcal{H}^{n-1}(x)\,dr,$$

for all $\lambda > 0$. Choosing $\varphi = 1$ we find

$$\sigma\lambda^Q = |\{x \in \mathbb{R}^n : d(x) < \lambda\}| = \int_0^\lambda \int_{\{d(x)=r\}} \frac{1}{|\nabla d(x)|} d\mathcal{H}^{n-1}(x)\,dr,$$

and taking the derivative we obtain for a.e. $\lambda > 0$

$$\sigma Q\lambda^{Q-1} = \int_{\{d(x)=\lambda\}} \frac{1}{|\nabla d(x)|} d\mathcal{H}^{n-1}(x),$$

which gives the proof. □

We are now ready to prove the main theorem of this section. In the next theorem $X = (X_1, ..., X_m)$ is a system of Hörmander vector fields of the form 3.2.25 and d is the induced C-C metric on $\mathbb{R}^n$. We shall write $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and for the sake of simplicity we contract the notation writing $x = (x, 0)$. Let $\mu = \mathcal{H}^{n-1} \llcorner \{t = 0\}$ be the Lebesgue measure on $\mathbb{R}^{n-1}$.

THEOREM 3.6.4. *Let $1 < p < \infty$, $s = 1 - \frac{1}{p}$ and let $\mathcal{U} \subset \mathbb{R}^{n-1}$ be a bounded open set. If $\lambda > 0$ and $t_0 > 0$ there exist $C > 0$ and $\delta_0 > 0$, such that*

$$\int\limits_{\mathcal{U}\times\mathcal{U}\cap\{d(x,y)<\delta_0\}} \frac{|u(x,0) - u(y,0)|^p \, dxdy}{d(x,y)^{ps}\mu(B(x, d(x,y)))} \leq C \int\limits_{\mathcal{U}_\lambda\times(0,t_0)} |Xu(x,t)|^p \, dxdt \tag{3.6.125}$$

for all $u \in C^1(\mathcal{U}_\lambda \times (0, t_0)) \cap C(\mathcal{U}_\lambda \times [0, t_0))$, where $\mathcal{U}_\lambda = \{y \in \mathbb{R}^{n-1} : \operatorname{dist}(y, \mathcal{U}) < \lambda\}$.

Proof. Let $\mathcal{U} \subset \Omega_0$ for some bounded open set $\Omega_0 \subset \mathbb{R}^n$ and let $k \in \mathbb{N}$ be the minimal length of the commutators which ensures the Hörmander condition on Ω_0. Fix $r_0 > 0$ and $0 < a < b$ by Theorem 1.6.10. Define

$$N(p, \delta_0; \mathcal{U}) = \int_{\mathcal{U}\times\mathcal{U}\cap\{d(x,y)<\delta_0\}} \frac{|u(x,0) - u(y,0)|^p}{d(x,y)^{ps}\mu(B(x, d(x,y)))} \, dxdy.$$

Let $\mathcal{I}$ be the set of the multi-indices I defined in chapter 1, section 6, subsection 6.3 and write

$$\begin{aligned} N(p, \delta_0; \mathcal{U}) &= \int\limits_{\mathcal{U}} dx \int\limits_{\mathcal{U}\cap\{d(x,y)<\delta_0\}} \frac{|u(x,0) - u(y,0)|^p}{d(x,y)^{ps}\mu(B(x, d(x,y)))} \, dy \\ &\leq \sum_{I\in\mathcal{I}} \int\limits_{\mathcal{U}} dx \int\limits_{\mathcal{U}\cap A_I(x)\cap\{d(x,y)<\delta_0\}} \frac{|u(x,0) - u(y,0)|^p \, dy}{d(x,y)^{ps}\mu(B(x, d(x,y)))} \\ &= \sum_{I\in\mathcal{I}} \int\limits_{\mathcal{U}} f_I(x) \, dx, \end{aligned} \tag{3.6.126}$$

where f_I is defined by the last equality and we introduced the annulus

$$\begin{aligned} A_I(x) := \Big\{ y \in \mathbb{R}^{n-1} : {}& |\lambda_I(x)| (2d(x,y)/a)^{d(I)} \\ & \geq \frac{1}{2} \max_{J\in\mathcal{I}} |\lambda_J(x)| (2d(x,y)/a)^{d(J)} \Big\}. \end{aligned}$$

Fix $\delta_0 \leq ar_0/2$. By Theorem 1.6.10 the map $y = \Phi_{I,x}(h)$ is one-to-one on the set $\{h \in \mathbb{R}^{n-1} : \|h\|_I < (2b/a)d(x, \bar{y})\}$ where $\bar{y} \in A_I(x)$ is such that $d(x, \bar{y}) = \min\{\delta_0, \max_{y\in A_I(x)} d(x,y)\}$ (the condition $d(x, \bar{y}) \leq \delta_0$ amounts to $2d(x, \bar{y})/a < r_0$ and ensures that Theorem 1.6.10 can be applied), and moreover $\Phi_{I,x}(\{h \in \mathbb{R}^{n-1} : \|h\|_I < (2b/a)d(x, \bar{y})\}) \supset \overline{B}(x, 2d(x, \bar{y})) \supset A_I(x)$. By the same theorem statement (iii)

$$\begin{aligned} \overline{B}(x, 2d(x,y)) &\subset \Phi_{I,x}\big(\{h \in \mathbb{R}^{n-1} : \|h\|_I < (2b/a)d(x,y)\}\big) \\ &\subset \overline{B}(x, 2d(x,y)/a) \end{aligned}$$

for all $y \in A_I(x)$ and $d(x,y) < \delta_0$, i.e. $2d(x,y)/a < r_0$. Thus

$$\|h\|_I < \frac{2b}{a} d(x, \Phi_{I,x}(h)) \leq \frac{2b}{a} \delta_0 \leq br_0. \tag{3.6.127}$$

Set $H_{I,\delta_0}(x) = \Phi_{I,x}^{-1}(\mathcal{U} \cap A_I(x) \cap \{d(x,y) < \delta_0\})$. Thus, by the first inequality of (3.6.127),

$$f_I(x) \le C \int_{H_{I,\delta_0}(x)} \frac{|u(x,0) - u(\Phi_{I,x}(h),0)|^p \; |J_h\Phi_{I,x}(h)|}{\|h\|_I^{ps}\, \mu(B(x, C\|h\|_I))}\, dh. \tag{3.6.128}$$

Note that (1.6.72) furnishes the estimate $\mu(B(x,C\|h\|_I)) \ge C|\lambda_I(x)|\|h\|^{d(I)}$. Letting $\eta = 2b\delta_0/a$ and recalling that $|J_h\Phi_{I,x}(h)| \simeq |\lambda_I(x)|$ from (3.6.126) and (3.6.128) we get

$$\begin{aligned} N(p;\delta_0;\mathcal{U}) &\le C \sum_{I\in\mathcal{I}} \int_{\mathcal{U}} dx \int_{\{\|h\|_I<\eta\}} \frac{|u(x,0) - u(\Phi_{I,x}(h),0)|^p}{\|h\|_I^{ps+d(I)}}\, dh \\ &= C \sum_{I\in\mathcal{I}} \int\limits_{\{\|h\|_I<\eta\}} \frac{dh}{\|h\|_I^{ps+d(I)}} \int\limits_{\mathcal{U}} |u(x,0) - u(\Phi_{I,x}(h),0)|^p\, dx. \end{aligned} \tag{3.6.129}$$

If $I = (i_1, ..., i_{n-1})$ and $\|h\|_I < \eta$ set $z_0(x) = x$ and define $z_l(x) = \prod_{j=1}^{\kappa} \exp_T(h_jY_{i_j})(x)$ for $\kappa = 1, ..., n-1$, in such a way that $z_{n-1}(x) = \Phi_{I,x}(h)$. Thus, fixed a constant λ', $0 < \lambda' < \lambda$

$$\begin{aligned} &\int_{\mathcal{U}} |u(x,0) - u(\Phi_{I,x}(h),0)|^p\, dx \\ &\quad\le C \sum_{\kappa=1}^{n-1} \int_{\mathcal{U}} |u(z_{\kappa-1}(x),0) - u(z_\kappa(x),0)|^p\, dx \\ &\quad\le C \sum_{\kappa=1}^{n-1} \int_{\mathcal{U}} \Big| u\Big(\prod_{j=1}^{l-1} \exp_T(h_jY_{i_j})(x), 0\Big) \\ &\qquad\qquad - u\Big(\exp_T(h_lY_{i_l}) \prod_{j=1}^{l-1} \exp_T(h_jY_{i_j})(x), 0\Big)\Big|^p\, dx \\ &\quad\le C \sum_{\kappa=1}^{n-1} \int_{\mathcal{U}_{\lambda'}} |u(\xi,0) - u(\exp_T(h_\kappa Y_{i_\kappa})(\xi),0)|^p\, d\xi, \end{aligned} \tag{3.6.130}$$

where in each integral we performed the change of variable $\xi = z_{\kappa-1}(x)$ which has Jacobian greater than a positive constant. Moreover, $\xi \in \mathcal{U}_{\lambda'}$ if δ_0 is small enough. Then, we have to estimate a finite number of integrals of the form

$$\int_{\mathcal{U}_{\lambda'}} |u(x,0) - u(\exp_T(t(h)Y)(x),0)|^p\, dx,$$

with $d(Y) \le k$ and $|t(h)|^{1/d(Y)} \le \|h\|_I$. By Lemma 1.6.12 we can write $\exp_T(tY) = \prod_{i=1}^p S_{\sigma_i}(q_i|t|^{1/d(Y)}, \tau_iX_{j_i})$ with $\sigma_i \in \{1,2\}$, $\tau_i \in \{-1,1\}$, $1 \le q_i \le k$, p less than an absolute constant and S_1, S_2 as in (1.6.73). With triangle inequalities and changes of variable quite similar to the ones in (3.6.130) we are led to the estimate of integrals of one of the two types

$$\begin{aligned}&\int_{\mathcal{U}_{\lambda'}} |u(\exp(q|t(h)|^{1/d(Y)}T)(x),0)-u(x,0)|^p dx \qquad \text{or}\\ &\int_{\mathcal{U}_{\lambda'}} |u(\exp(q|t(h)|^{1/d(Y)}(\tau X_j+T))(x),0)-u(x,0)|^p dx,\end{aligned} \tag{3.6.131}$$

with $j=1,...,m-1$, $1\le q\le k$ and $|t(h)|^{1/d(Y)}\le \|h\|_I$. If we consider, for instance, an integral of the second type with $\tau=1$ the computation in (3.6.129) can be concluded in the following way (recall that $ps+d(I)=p-1+d(I)$):

$$\begin{aligned}
&\int_{\{\|h\|_I<\eta\}} \frac{dh}{\|h\|_I^{ps+d(I)}}\int_{\mathcal{U}_{\lambda'}} |u(\exp(q|t(h)|^{1/d(Y)}(X_j+T)(x,0)))-u(x,0)|^p\,dx\\
&\le C\int_{\{\|h\|_I<\eta\}} \frac{dh}{\|h\|_I^{ps+d(I)}}\int_{\mathcal{U}_{\lambda'}}\Big(\int_0^{k\|h\|_I}|Xu(\exp(t(X_j+T))(x,0)|\,dt\Big)^p dx\\
&\le C\int_{\{\|h\|_I<\eta\}} \frac{dh}{\|h\|_I^{ps+d(I)}}\Big(\int_0^{k\|h\|_I}\Big(\int_{\mathcal{U}_{\lambda'}}|Xu(\exp(t(X_j+T))(x,0)|^p dx\Big)^{\frac1p}dt\Big)^p\\
&= C\int_0^{k\eta}\frac{dr}{r^p}\Big(\int_0^r\Big(\int_{\mathcal{U}_{\lambda'}}|Xu(\exp(t(X_j+T))(x,0)|^p\,dx\Big)^{\frac1p}dt\Big)^p\\
&\le C\int_0^{k\eta}\int_{\mathcal{U}_{\lambda'}}|Xu(\exp(t(X_j+T))(x,0))|^p\,dx\,dt.
\end{aligned}$$

We used the Minkowski inequality, formula (3.6.124) and Hardy inequality (3.6.123).

Finally, write $\exp(t(X_j+T))(x,0)=\Theta(x,t)$ and perform the change of variable $(\xi,\tau)=\Theta(x,t)$. Since $\Theta(x,0)=(x,0)$ then

$$\frac{\partial\Theta(x,t)}{\partial x\partial t}\Big|_{t=0}=\begin{pmatrix} I_{n-1} & X_j(x,0)\\ 0 & 1\end{pmatrix}$$

and thus Θ is a change of variable on the rectangle $\mathcal{U}_{\lambda'}\times(0,\varrho_0)$, where ϱ_0 is suitably small. Choosing δ_0 small we obtain $k\eta\le\varrho_0$ and $\Theta(x,t)\in\mathcal{U}_\lambda\times(0,t_0)$ for all $(x,t)\in\mathcal{U}_{\lambda'}\times(0,k\eta)$. Then

$$\int_{\mathcal{U}_{\lambda'}\times(0,k\eta)}|Xu(\Theta(x,t))|^p\,dxdt\le C\int_{\mathcal{U}_\lambda\times(0,t_0)}|Xu(\xi,\tau)|^p\,d\xi d\tau.$$

Integrals of the first type in (3.6.131) can be treated in the same way and the proof of the Theorem is concluded. □

Corollary 3.6.5. *Let $X_1,...,X_m\in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ satisfy the Hörmander condition and let $\Omega\subset\mathbb{R}^n$ be a bounded open set with $\partial\Omega$ of class C^∞ and non characteristic. Let $1<p<\infty$ and $s=1-\frac1p$. There exist constants $C,\delta_0>0$ such that*

$$\int_{\partial\Omega\times\partial\Omega\cap\{d(x,y)<\delta_0\}}\frac{|u(x)-u(y)|^p\,d\mu(x)d\mu(y)}{d(x,y)^{ps}\mu(B(x,d(x,y)))}\le C\int_\Omega|Xu(x)|^p\,dx$$

for all $u \in C^1(\Omega) \cap C(\overline{\Omega})$*, where* $\mu = \mathcal{H}^{n-1} \llcorner \partial\Omega$.

Proof. The proof follows from Theorem 3.6.4 using a standard covering argument, Lemma 3.2.2 and Remark 3.2.3. $\square$

EXAMPLE 3.6.6 (Trace on subgroups of $\mathbb{H}^n$). Consider the Heisenberg group $\mathbb{H}^n$, $n \geq 1$. The homogeneous norm $\|z,t\| = (|z|^4 + t^2)^{1/4}$ is equivalent to the C-C metric d (see Proposition 1.7.4). The integer $Q = 2n+2$ is the "dimension" of $\mathbb{H}^n$ and $|B((z,t),r)| = cr^Q$ for some $c > 0$ and for all $(z,t) \in \mathbb{H}^n$ and $r \geq 0$.

Consider the half space $\Omega = \{(x,y,t) \in \mathbb{H}^n : x_j > 0\}$ for some $j = 1,...,n$ with boundary $\partial\Omega = \{(x,y,t) \in \mathbb{H}^n : x_j = 0\}$. Actually, the hyperplane $\partial\Omega$ is a subgroup of $\mathbb{H}^n$ and all its points are non characteristic. If $\mu = \mathcal{H}^{2n} \llcorner \partial\Omega$ then $\mu(B((z,t),r)) = mr^{Q-1}$ for some $m > 0$ and for all $(z,t) \in \partial\Omega$. Using the technique developed in this section it can be proved that there exists a constant $C > 0$ such that ($1 < p < \infty$ and $s = 1 - \frac{1}{p}$)

$$\int_{\partial\Omega\times\partial\Omega} \frac{|u(z,t) - u(\zeta,\tau)|^p \, d\mu(z,t) d\mu(\zeta,\tau)}{\|(\zeta,\tau)^{-1}\cdot(z,t)\|^{ps+Q-1}} \leq C \int_\Omega |\nabla_{\mathbb{H}} u(z,t)|^p \, dzdt$$

for all $u \in C^1(\Omega) \cap C(\overline{\Omega})$, where $\nabla_{\mathbb{H}}$ is the Heisenberg gradient.

7. Trace theorem in the Grushin space

7.1. Trace theorem. In this section we focus our attention on the Grushin plane where we prove that the trace estimate holds for domains which are sufficiently "flat" at characteristic points.

Let d be the C-C metric induced on $\mathbb{R}^2$ by the vector fields

$$X_1 = \partial_x \quad \text{and} \quad X_2 = |x|^\alpha \partial_y, \ \ \alpha > 0.$$

If $(x,y) \in \mathbb{R}^2$ and $r \geq 0$ let $B((x,y),r) = \{(\xi,\eta) \in \mathbb{R}^2 : d((x,y),(\xi,\eta)) < r\}$. Moreover, define the "box"

$$\mathrm{Box}((x,y),r) = [x-r, x+r] \times [y - r(|x|+r)^\alpha, y + r(|x|+r)^\alpha].$$

Such boxes are equivalent to C-C balls as shown in chapter 1 section 9. We recall the main results concerning them (see [**77**]).

LEMMA 3.7.1. *There exist constants* $0 < c_1 < c_2$ *such that for all* $(x,y) \in \mathbb{R}^2$ *and* $r \geq 0$

$$\mathrm{Box}((x,y),c_1 r) \subset B((x,y),r) \subset \mathrm{Box}((x,y),c_2 r). \tag{3.7.132}$$

LEMMA 3.7.2. *Let* $\lambda > 0$. *For all* (x,y), $(\xi,\eta) \in \mathbb{R}^2$ *with* $|x| \geq |\xi|$

$$d((x,y),(\xi,\eta)) \simeq |x-\xi| + \frac{|y-\eta|}{|x|^\alpha} \quad \textit{if} \quad |x|^{\alpha+1} \geq \lambda|y-\eta|, \tag{3.7.133}$$

$$d((x,y),(\xi,\eta)) \simeq |x-\xi| + |y-\eta|^{\frac{1}{\alpha+1}} \quad \textit{if} \quad |x|^{\alpha+1} < \lambda|y-\eta|, \tag{3.7.134}$$

where the equivalence constants depend on λ.

DEFINITION 3.7.3. Let $\Omega \subset \mathbb{R}^2$ be an open set with $\partial\Omega$ of class C^1. A point $(0,y_0) \in \partial\Omega$ is said to be $\alpha-$*admissible*, $\alpha > 0$, if one of the following two conditions holds:

(i) (Non characteristic case). There exist $\delta > 0$ and $\psi \in C^1(y_0 - \delta, y_0 + \delta)$ such that $\psi(y_0) = 0$ and

$$\partial\Omega \cap (-\delta, \delta) \times (y_0 - \delta, y_0 + \delta) = \{(\psi(y), y) : |y - y_0|, |\psi(y)| < \delta\}.$$

(ii) (Characteristic case). There exist $\delta > 0$ and $c > 0$ such that

$$\partial\Omega \cap (-\delta, \delta) \times (y_0 - \delta, y_0 + \delta) = \{(x, \varphi(x)) \in \mathbb{R}^2 : |x| < \delta\},$$

where $\varphi \in C^1(-\delta, \delta)$ and $|\varphi'(x)| \le c|x|^\alpha$ for all $x \in (-\delta, \delta)$.

Finally, Ω is said to be $\alpha-$admissible if all the points of $\partial\Omega \cap \{x = 0\}$ are $\alpha-$admissible.

Let $\Omega \subset \mathbb{R}^2$ be an open set of class C^1 and let $\nu(x, y)$ be the unit normal to $\partial\Omega$ at $(x, y) \in \partial\Omega$. Consider the modulus of the "projected" normal

$$\begin{aligned}|X\nu(x,y)| &= \Big(\langle X_1(x,y), \nu(x,y)\rangle^2 + \langle X_2(x,y), \nu(x,y)\rangle^2\Big)^{\frac{1}{2}} \\ &= \Big(\nu_1(x,y)^2 + |x|^{2\alpha}\nu_2(x,y)^2\Big)^{\frac{1}{2}},\end{aligned}$$

and define the measure $\mu = |X\nu|\mathcal{H}^1 \llcorner \partial\Omega$. The measure μ is the one that appears in the left hand side of the trace estimates.

In the sequel we shall use the equivalence

$$\int_I |\xi|^\alpha d\xi \simeq |I| \max_{\xi \in I} |\xi|^\alpha \tag{3.7.135}$$

for any interval $I \subset \mathbb{R}$, where the equivalence constants depend only on $\alpha > 0$.

Lemma 3.7.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with $\partial\Omega$ of class C^1 and suppose it is $\alpha-$*admissible. *Then there exist $0 < m_1 < m_2$ and $r_0 > 0$ such that*

$$m_1 \frac{|B((x,y),r)|}{r} \le \mu(B(x,y),r) \le m_2 \frac{|B((x,y),r)|}{r} \tag{3.7.136}$$

for all $(x, y) \in \partial\Omega$ and for all $0 < r < r_0$,

Proof. Since away from the set $\{x = 0\}$ we are essentially in a Euclidean situation it suffices to prove (3.7.136) for $(x, y) \in \partial\Omega$ belonging to a neighborhood of an $\alpha-$admissible point.

Suppose first that $(0, 0) \in \partial\Omega$ is an $\alpha-$admissible point of type (i) (non characteristic). In a neighborhood of the origin $\partial\Omega$ is the graph of a function $\psi \in C^1(-\delta, \delta)$ in the variable y. If $\delta > 0$ and $r > 0$ are small, then the graph of ψ meets $\partial\mathrm{Box}((\psi(y), y), r)$ on its horizontal edges. This is ensured by $|\psi(y) - \psi(y - r(|\psi(y)| + r)^\alpha)| < r$, which holds true provided y and r are small enough. Now

$$\begin{aligned}\mu(\mathrm{Box}((\psi(y), y), r)) &\simeq \int_{y - r(|\psi(y)|+r)^\alpha}^{y + r(|\psi(y)|+r)^\alpha} d\eta \\ &= 2r(|\psi(y)| + r)^\alpha = \frac{|\mathrm{Box}((\psi(y), y), r)|}{2r},\end{aligned}$$

and (1.9.103) gives the proof of the required estimate.

Suppose now that $(0,0) \in \partial\Omega$ is an $\alpha-$admissible point of type (ii). Let $\varphi \in C^1(-\delta,\delta)$ be the function whose graph represents $\partial\Omega$ and such that $|\varphi'(x)| \le c|x|^\alpha$ for all $|x| < \delta$ and for some $c \ge 0$. Then, if $y = \varphi(x)$ and $|x| \le \delta/2$

$$\nu(x,y) = \frac{(\varphi'(x),-1)}{\sqrt{1+\varphi'(x)^2}}, \qquad \text{and} \qquad |X\nu(x,y)| = \frac{\sqrt{|x|^{2\alpha}+\varphi'(x)^2}}{\sqrt{1+\varphi'(x)^2}} \simeq |x|^\alpha.$$

By Lemma 1.9.3 $\mu(\mathrm{Box}((x,y),c_1r)) \le \mu(B((x,y),r)) \le \mu(\mathrm{Box}((x,y),c_2r))$, and, supposing for instance $0 \le x \le \delta/2$ and $0 < r < \delta/(2c_2)$

$$\begin{aligned}\mu(\mathrm{Box}((x,y),c_2r)) &= \int_{\mathrm{Box}((x,y),c_2r)\cap\partial\Omega} |X\nu| d\mathcal{H}^1 \le C\int_{x-c_2r}^{x+c_2r} |\xi|^\alpha\, d\xi \\ &\le 2Cc_2r(x+c_2r)^\alpha \simeq \frac{|\mathrm{Box}((x,y),c_2r)|}{r}.\end{aligned}$$

The estimate from above in (3.7.136) follows by Lemma 1.9.3. In order to prove the opposite inequality assume without loss of generality that the constant c relative to φ is greater than 1 and that $x \ge 0$. Introduce the new box

$$\begin{aligned}\overline{\mathrm{Box}}((x,y),c_1r) := &\left[x-\frac{c_1}{c}r, x+\frac{c_1}{c}r\right] \\ &\quad\times [y-c_1r(x+c_1r)^\alpha, y+c_1r(x+c_1r)^\alpha] \\ &\subset \mathrm{Box}((x,y),c_1r).\end{aligned}$$

Since $|\varphi(x+\frac{c_1}{c}r)-\varphi(x)| \le c_1r(x+c_1r)^\alpha$, the graph of φ meets $\partial\overline{\mathrm{Box}}((x,y),c_1r)$ on its left and right vertical edges. Thus

$$\begin{aligned}\mu(B((x,y)),r)) \ge \mu(\overline{\mathrm{Box}}((x,y),c_1r) &= \int_{\overline{\mathrm{Box}}((x,y),c_1r)\cap\partial\Omega} |X\nu| d\mathcal{H}^1 \\ &\ge C\int_{x-\frac{c_1}{c}r}^{x+\frac{c_1}{c}r} |\xi|^\alpha\, d\xi \simeq C\frac{c_1}{c}r(x+\frac{c_1}{c}r)^\alpha \\ &\simeq \frac{|\mathrm{Box}((x,y),c_1r)|}{r},\end{aligned}$$

which is the required estimate. We also used (3.7.135). □

THEOREM 3.7.5. *Let $X_1 = \partial_x$ and $X_2 = |x|^\alpha\partial_y$, $\alpha > 0$. Let $1 < p < \infty$ and $s = 1-\frac{1}{p}$. If $\Omega \subset \mathbb{R}^2$ is a bounded open set of class C^1 which is $\alpha-$admissible, then there exist $C > 0$ and $\delta_0 > 0$ such that*

$$\int\limits_{\partial\Omega\times\partial\Omega\cap\{d(z,\zeta)<\delta_0\}} \frac{|u(z)-u(\zeta)|^p \;\, d\mu(z)d\mu(\zeta)}{d(z,\zeta)^{ps}\mu(B(z,d(z,\zeta)))} \le C\int\limits_\Omega |Xu(x,y)|^p\, dxdy$$

for all $u \in C^1(\Omega)\cap C(\overline{\Omega})$.

Proof. Since away from the set $\{x=0\}$ we are essentially in the Euclidean case, it suffices to prove the estimate in a neighborhood of an $\alpha-$admissible point which may assumed to be the origin. Denote by $\mathcal{U}$ the intersection of $\partial\Omega$ with a small fixed neighborhood of $(0,0)$. Recalling that, by Lemma 3.7.4, $d(z,\zeta)^{ps}\mu(B(z,d(z,\zeta))) \simeq d(z,\zeta)^{ps-1}|B(z,d(z,\zeta))|$, we have to prove that

$$\begin{aligned} N(p;\mathcal{U}) &:= \int_{\mathcal{U}\times\mathcal{U}} \frac{|u(z)-u(\zeta)|^p}{d(z,\zeta)^{ps-1}|B(z,d(z,\zeta))|} d\mu(z)d\mu(\zeta) \\ &\le C\int_\Omega |Xu(x,y)|^p\,dxdy. \end{aligned}$$

The $\alpha-$admissible point can be of type (i) or of type (ii).

Type (i). We may assume that $\mathcal{U} = \{(\psi(y),y) : |y| < \delta\}$ for some $\delta > 0$ and $\psi \in C^1(-\delta,\delta)$ with $\psi(0) = 0$, and that Ω lies in the region $\{x > \psi(y)\}$. Write $z = (\psi(y),y)$ and $\zeta = (\psi(\eta),\eta)$, and notice that, by the doubling property of the Lebesgue measure, which follows from Lemma 1.9.3, $|B(z,d(z,\zeta))| \simeq |B(\zeta,d(z,\zeta))|$. Thus the kernel is essentially symmetric and the integration can be performed without loss of generality on the set $\{|\psi(\eta)| < |\psi(y)|\}$

$$\begin{aligned} N(p;\mathcal{U}) &\simeq \int_{\{|y|<\delta,\,|\eta|<\delta,\,|\psi(\eta)|<|\psi(y)|\}} \frac{|u(z)-u(\zeta)|^p}{d(z,\zeta)^{ps-1}|B(z,d(z,\zeta))|} dyd\eta \\ &= \int_A \frac{|u(z)-u(\zeta)|^p dyd\eta}{d(z,\zeta)^{ps-1}|B(z,d(z,\zeta))|} + \int_B \frac{|u(z)-u(\zeta)|^p dyd\eta}{d(z,\zeta)^{ps-1}|B(z,d(z,\zeta))|} \\ &:= I_A + I_B, \end{aligned}$$

where we let

$$\begin{aligned} A &= \{(y,\eta) : |y| < \delta,\ |\eta| < \delta,\ |\psi(\eta)| < |\psi(y)|, |\psi(y)|^{\alpha+1} \ge |y-\eta|\}, \\ B &= \{(y,\eta) : |y| < \delta,\ |\eta| < \delta,\ |\psi(\eta)| < |\psi(y)|, |\psi(y)|^{\alpha+1} < |y-\eta|\}. \end{aligned}$$

We begin with the estimate of I_A. If $(y,\eta) \in A$ then

$$\begin{aligned} d(z,\zeta) &\simeq |\psi(y)-\psi(\eta)| + \frac{|y-\eta|}{|\psi(y)|^\alpha} \\ &= \frac{|y-\eta|}{|\psi(y)|^\alpha}\Big(1 + |\psi(y)|^\alpha \frac{|\psi(y)-\psi(\eta)|}{|y-\eta|}\Big) \simeq \frac{|y-\eta|}{|\psi(y)|^\alpha} \end{aligned}$$

and

$$|B(z,d(z,\zeta))| \simeq d(z,\zeta)^2\big(|\psi(y)| + d(z,\zeta)\big)^\alpha \simeq d(z,\zeta)^2|\psi(y)|^\alpha.$$

Without loss of generality assume $y > \eta$. Let $\eta = y - h$ and write (recall that $1 + ps = p$)

$$\begin{aligned} I_A &\simeq \int\limits_A \frac{|u(\psi(y),y)-u(\psi(\eta),\eta)|^p}{|y-\eta|^p}|\psi(y)|^{p\alpha-\alpha}dyd\eta \\ &\le \int\limits_0^{2\delta} \frac{dh}{|h|^p} \int\limits_{\{|\psi(y)|^{\alpha+1}>|h|,\,|y|<\delta\}} |u(\psi(y),y)-u(\psi(y-h),y-h)|^p|\psi(y)|^{p\alpha-\alpha}dy. \end{aligned}$$

We shall connect the points $(\psi(y),y)$ and $(\psi(y-h),y-h)$ by the curves

$$\begin{aligned} \gamma_1(t) &:= \exp(t(X_1 - bX_2))(\psi(y),y) \\ &= \Big(\psi(y)+t, y - b\int_0^t |\psi(y)+\tau|^\alpha d\tau\Big) := \Psi_1(t,y), \end{aligned}$$

where $b = \min\{1, 1/L\}$, $L := \sup_{|y|<\delta} |\psi'(y)|$, and

$$\begin{aligned}\gamma_2(t) :&= \exp(tX_1)(\psi(y-h), y-h)\\ &= (\psi(y-h)+t, y-h) := \Psi_2(t, y-h).\end{aligned}$$

In order to reach the height $y-h$, the curve γ_1 needs a time t_1 such that

$$\int_0^{t_1} |\psi(y)+\tau|^\alpha d\tau = \frac{|h|}{b}. \tag{3.7.137}$$

By (3.7.135) the left hand side is greater than $Ct_1|\psi(y)|^\alpha$ and then $t_1 \leq C|h|/|\psi(y)|^\alpha$. The time t_2 such that $\gamma_2(t_2) = \gamma_1(t_1)$ can also be estimated by $|h|/|\psi(y)|^\alpha$. Indeed

$$t_2 = |\psi(y)+t_1-\psi(y-h)| \leq L|h| + t_1 \leq C\frac{|h|}{|\psi(y)|^\alpha}.$$

The choice of the parameter b guarantees that $\gamma_1(t) \in \Omega$ for all $|y| < \delta$ and $0 < t \leq t_1$. In fact this happens if and only if

$$\psi\Big(y - b\int_0^t |\psi(y)+\tau|^\alpha d\tau\Big) < \psi(y)+t. \tag{3.7.138}$$

This last inequality is a consequence of the following

$$\Big|\psi\Big(y - b\int_0^t |\psi(y)+\tau|^\alpha d\tau\Big) - \psi(y)\Big| \leq Lb\int_0^t |\psi(y)+\tau|^\alpha d\tau < t.$$

Since $\Psi_1(t_1, y) = \Psi_2(t_2, y-h)$ then $|u(\psi(y), y) - u(\psi(y-h), y-h)|$ is less than

$$\begin{aligned}&|u(\psi(y), y) - u(\Psi_1(t_1, y))| + |u(\psi(y-h), y-h)) - u(\Psi_2(t_2, y-h))|\\ &\qquad \leq C\Big(\int_0^{t_1} |Xu(\Psi_1(t,y))|dt + \int_0^{t_2} |Xu(\Psi_2(t, y-h))|dt\Big),\end{aligned}$$

and we find

$$\begin{aligned}I_A \leq C\Big[&\int_0^{2\delta} \frac{dh}{|h|^p} \int_{(-\delta,\delta)\cap\{|\psi(y)|^{\alpha+1}\geq|h|\}} |\psi(y)|^{p\alpha-\alpha}\Big(\int_0^{t_1} |Xu(\Psi_1(t,y))|dt\Big)^p dy\\ &+ \int_0^{2\delta} \frac{dh}{|h|^p} \int_{(-\delta,\delta)\cap\{|\psi(y)|^{\alpha+1}\geq|h|\}} |\psi(y)|^{p\alpha-\alpha}\Big(\int_0^{t_2} |Xu(\Psi_2(t,y-h))|dt\Big)^p dy\Big]\\ := C[&I_A^{(1)} + I_A^{(2)}].\end{aligned}$$

We shall estimate $I_A^{(1)}$ and $I_A^{(2)}$ by the same technique and we begin with $I_A^{(1)}$. Letting in the inner integral $\tau = |\psi(y)|^\alpha t$, recalling that $t_1 \leq C|h|/\ |\psi(y)|^\alpha$ and using the

Minkowski inequality we find

$$
\begin{aligned}
I_A^{(1)} &\le \int_0^{2\delta} \frac{dh}{|h|^p} \int_{(-\delta,\delta)\cap\{|\psi(y)|^{\alpha+1}\ge|h|\}} \frac{dy}{|\psi(y)|^\alpha} \Big(\int_0^{C|h|} |Xu(\Psi_1(\tau/|\psi(y)|^\alpha, y))| d\tau \Big)^p \\
&\le \int_0^{2\delta} \Big(\frac{dh}{|h|} \int_0^{C|h|} \Big(\int_{(-\delta,\delta)\cap\{|\psi(y)|^{\alpha+1}\ge|h|\}} \frac{|Xu(\Psi_1(\tau/|\psi(y)|^\alpha, y))|^p dy}{|\psi(y)|^\alpha} \Big)^{\frac{1}{p}} d\tau \Big)^p.
\end{aligned}
$$

Since $\{|\psi(y)|^{\alpha+1} \ge |h|\} \subset \{C|\psi(y)|^{\alpha+1} \ge \tau\}$ the last integral is estimated by an integral of the form $\int_0^{2\delta} \Big(\frac{1}{|h|} \int_0^{C|h|} |f(\tau)| d\tau \Big)^p dh$ with f not depending on h. So we can apply the Hardy inequality to get

$$
\begin{aligned}
I_A^{(1)} &\le C \int_0^{2\delta} \int_{(-\delta,\delta)\cap\{C|\psi(y)|^{\alpha+1}\ge\tau\}} \frac{|Xu(\Psi_1(\tau/|\psi(y)|^\alpha, y))|^p}{|\psi(y)|^\alpha} dy\, d\tau \\
&\le C \int_{-\delta}^{\delta} \int_0^{C\delta} |Xu(\Psi_1(t,y))|^p dt\, dy.
\end{aligned}
$$

We let $\tau/|\psi(y)|^\alpha = t$ and we used $\tau/|\psi(y)|^\alpha \le C|\psi(y)| \le C|y| \le C\delta$. The Jacobian matrix of Ψ_1 is

$$
\frac{\partial \Psi_1(y,t)}{\partial y \partial t} = \begin{pmatrix} 1 & \psi'(y) \\ -b|\psi(y)+t|^\alpha & 1 - b(|\psi(y)+t|^\alpha - |\psi(y)|^\alpha)\psi'(y) \end{pmatrix}.
$$

By the same argument used in the proof of (3.7.138) we can see that if $\delta > 0$ is small, then $\Psi_1((0, C\delta)\times(-\delta,\delta)) \subset \Omega$. Moreover $|J\Psi_1(t,y)| = |1 + b\psi'(y)|\psi(y)|^\alpha| \simeq 1$. Then

$$
I_A^{(1)} \le C \int_\Omega |Xu(x,y)|^p dx dy.
$$

We estimate now $I_A^{(2)}$. Note first that if $\delta > 0$ is small and $(y,\eta) \in A$, we have

$$
|\psi(y)| \le 2|\psi(\eta)|. \tag{3.7.139}
$$

Indeed $L|\psi(y)|^{\alpha+1} \ge L|y-\eta| \ge |\psi(y)-\psi(\eta)| \ge |\psi(y)| - |\psi(\eta)|$, and thus $|\psi(\eta)| \ge |\psi(y)| - L|\psi(y)|^{\alpha+1} \ge 1/2|\psi(y)|$ if $\delta > 0$ is small. Taking (3.7.139) into account with $\eta = y - h$, recalling that $t_2 \le C|h|/|\psi(y)|^\alpha \le C|h|/|\psi(y-h)|^\alpha$ and letting $\tau = |\psi(y-h)|^\alpha t$ in the inner integral we find that $I_A^{(2)}$ is smaller than

$$
\begin{aligned}
&\int_0^{2\delta} \frac{dh}{|h|^p} \int_{(-\delta,\delta)\cap\{C|\psi(y-h)|^{\alpha+1}\ge|h|\}} \frac{dy}{|\psi(y-h)|^\alpha} \Big(\int_0^{C|h|} \Big| Xu\Big(\Psi_2\Big(\frac{\tau}{|\psi(y-h)|^\alpha}, y-h\Big)\Big)\Big| d\tau \Big)^p \\
&\le \int_0^{2\delta} \Big(\frac{dh}{|h|} \int_0^{C|h|} \Big(\int_{(-\delta,\delta)\cap\{C|\psi(y-h)|^{\alpha+1}\ge|h|\}} \frac{|Xu(\Psi_2(\frac{\tau}{|\psi(y-h)|^\alpha}, y-h))|^p}{|\psi(y-h)|^\alpha} dy \Big)^{1/p} d\tau \Big)^p \\
&\le \int_0^{2\delta} \Big(\frac{dh}{|h|} \int_0^{C|h|} \Big(\int_{(-3\delta,\delta)\cap\{C|\psi(y)|^{\alpha+1}\ge|h|\}} \frac{|Xu(\Psi_2(\tau/|\psi(y)|^\alpha, y))|^p}{|\psi(y)|^\alpha} dy \Big)^{1/p} d\tau \Big)^p.
\end{aligned}
$$

Since $\{C|\psi(y)|^{\alpha+1} \geq |h|\} \subset \{C|\psi(y)|^{\alpha+1} \geq \tau\}$, we can apply the Hardy inequality to get

$$\begin{aligned} I_A^{(2)} &\leq C \int_0^{2\delta} \int_{(-3\delta,\delta)\cap\{C|\psi(y)|^{\alpha+1}\geq\tau\}} \frac{|Xu(\Psi_2(\tau/|\psi(y)|^\alpha, y))|^p}{|\psi(y)|^\alpha} dy\, d\tau \\ &\leq C \int_{-3\delta}^{\delta} \int_0^{C\delta} |Xu(\Psi_2(t,y))|^p dt\, dy. \end{aligned}$$

Since $|J\Psi_2(t,y)| = 1$ the estimate for $I_A^{(2)}$ follows.

We now turn to the estimate of I_B. Writing again $z = (\psi(y), y)$ and $\zeta = (\psi(\eta), \eta)$, if $(y,\eta) \in B$ then

$$d(z,\zeta) \simeq |\psi(y) - \psi(\eta)| + |y-\eta|^{1/(\alpha+1)} \simeq |y-\eta|^{1/(\alpha+1)}$$

because $\psi \in C^1$ and $|y-\eta| \leq 2\delta$. Moreover starting from the inequality $|\psi(y)| \leq |y-\eta|^{1/(\alpha+1)}$ which defines B, we find

$$\begin{aligned} |B(z, d(z,\zeta))| &\simeq d(z,\zeta)^2(|\psi(y)| + d(z,\zeta))^\alpha \\ &\simeq |y-\eta|^{2/(\alpha+1)}(|\psi(y)| + |y-\eta|^{1/(\alpha+1)})^\alpha \\ &\simeq |y-\eta|^{(\alpha+2)/(\alpha+1)}. \end{aligned}$$

Assume $\eta < y$, let $\eta = y - h$ and write

$$\begin{aligned} I_B &\simeq \int\limits_B \frac{|u(\psi(y),y) - u(\psi(\eta),\eta)|^p}{|y-\eta|^{1+\frac{ps}{\alpha+1}}} dy d\eta \\ &\leq C \int\limits_0^{2\delta} \frac{dh}{|h|^{1+\frac{ps}{\alpha+1}}} \int\limits_{\{|\psi(y)|^{\alpha+1}<|h|,\, |y|<\delta\}} |u(\psi(y),y) - u(\psi(y-h), y-h)|^p dy. \end{aligned}$$

The points $(\psi(y), y)$ and $(\psi(y-h), y-h)$ can be connected by the curves $\gamma_1(t) := \exp(t(X_1 - bX_2))(\psi(y), y) = \Psi_1(t,y)$ and $\gamma_2(t) := \exp(tX_1)(\psi(y-h), y-h) = \Psi_2(t, y-h)$. In order to reach the height $y-h$, the curve γ_1 needs a time t_1 such that (3.7.137) holds. By (3.7.135)

$$\int_0^{t_1} |\psi(y) + \tau|^\alpha d\tau \simeq t_1 \max_{\tau\in[\psi(y),\psi(y)+t_1]} |\tau|^\alpha \geq t_1 \Big(\frac{t_1}{2}\Big)^\alpha.$$

This yields $t_1 \leq C|h|^{1/(\alpha+1)}$. The time t_2 such that $\gamma_2(t_2) = \gamma_1(t_1)$ can also be estimated by $|h|^{1/(\alpha+1)}$. By the triangle inequality we get

$$\begin{aligned} I_B \leq C\Big[&\int_0^{2\delta} \frac{dh}{|h|^{1+\frac{ps}{\alpha+1}}} \int_{-\delta}^{\delta} \Big(\int_0^{t_1} |Xu(\Psi_1(t,y))| dt \Big)^p dy \\ &+ \int_0^{2\delta} \frac{dh}{|h|^{1+\frac{ps}{\alpha+1}}} \int_{-\delta}^{\delta} \Big(\int_0^{t_2} |Xu(\Psi_2(t, y+h))| dt \Big)^p dy \Big] \\ := C[&I_B^{(1)} + I_B^{(2)}]. \end{aligned}$$

Now, by the Minkowski inequality

$$\begin{aligned} I_B^{(1)} &\leq \int_0^{2\delta} \frac{dh}{|h|^{1+\frac{ps}{\alpha+1}}} \Big(\int_0^{C|h|^{1/(\alpha+1)}} \Big(\int_{-\delta}^{\delta} |Xu(\Psi_1(t,y))|^p dy \Big)^{1/p} dt \Big)^p \\ &\leq C \int_0^{(2\delta)^{1/(\alpha+1)}} \frac{dr}{r^p} \Big(\int_0^{Cr} \Big(\int_{-\delta}^{\delta} |Xu(\Psi_1(t,y))|^p dy \Big)^{1/p} dt \Big)^p \\ &\leq \int_{(0,(2\delta)^{1/(\alpha+1)})\times(-\delta,\delta)} |Xu(\Psi_1(t,y))|^p dt dy. \end{aligned}$$

We used $s = 1 - 1/p$, the change of variable $r = h^{1/(\alpha+1)}$ and the Hardy inequality.

The estimate of $I_B^{(2)}$ is analogous to the one of $I_A^{(2)}$. This ends the trace estimates for $\alpha-$admissible points of type (i).

Type (ii). Write $\mathcal{U} = \{(x, \varphi(x)) \in \partial\Omega : |x| < \delta\}$ for some $\varphi \in C^1(-\delta, \delta)$ such that $|\varphi'(x)| \leq c|x|^\alpha$ for some $c \geq 0$ and for all $x \in (-\delta, \delta)$. Write $z = (x, \varphi(x))$, $\zeta = (y, \varphi(y))$, and observe that

$$N(p;\mathcal{U}) \simeq \int_{|x|<\delta,\, |y|<\delta} \frac{|u(z) - u(\zeta)|^p \, |xy|^\alpha}{d(z,\zeta)^{ps-1}|B(z, d(z,\zeta))|} \, dxdy.$$

Since the integrand is symmetric up to equivalence constants, the integration may take place on the set $\{|x| < |y| < \delta\}$. Since $|\varphi'(y)| \leq c|y|^\alpha$ we have $|\varphi(y) - \varphi(x)| \leq c|y - x| \, |y|^\alpha \leq 2c|y|^{\alpha+1}$. Then on the mentioned set the C-C metric behaves as

$$d(z,\zeta) \simeq |y - x| + \frac{|\varphi(y) - \varphi(x)|}{|y|^\alpha} \simeq |y - x|.$$

By Lemma 1.9.3

$$|B(z, d(z,\zeta)| \simeq |y - x|^2 (|x| + |y - x|)^\alpha \simeq |y - x|^2 \, |y|^\alpha,$$

and, since $ps - 1 = p - 2$, we get

$$N(p;\mathcal{U}) \simeq \int_{\{|x|<|y|<\delta\}} \frac{|u(x,\varphi(x)) - u(y,\varphi(y))|^p \, |x|^\alpha}{|y - x|^p} \, dxdy.$$

By symmetry it suffices to consider the integration on $A_1 := \{0 < x < y < \delta\}$ and $A_2 := \{x > 0, \, -\delta < y < -x\}$. Set $h = y - x$ and write

$$\begin{aligned} I_{A_1} &= \int_{\{0<x<y<\delta\}} \frac{|u(x,\varphi(x)) - u(y,\varphi(y))|^p \, |x|^\alpha}{|y - x|^p} \, dxdy \\ &\leq \int_0^\delta \frac{dh}{|h|^p} \int_0^\delta |u(x,\varphi(x)) - u(x + h, \varphi(x + h))|^p \, |x|^\alpha \, dx. \end{aligned}$$

We shall connect the points $(x, \varphi(x))$ and $(x + h, \varphi(x) + h)$ by the paths

$$\begin{aligned} \gamma_1(t) &:= \exp(t(bX_1 + X_2))(x, \varphi(x)) \\ &= \Big(x + bt, \varphi(x) + \int_0^t |x + b\tau|^\alpha d\tau \Big) := \Phi_1(x,t), \end{aligned}$$

for $0 \leq t \leq t_1 := |h|/b$ (here $b \in (0,1)$ is a fixed number such that $2^{\alpha+1}cb < 1$), and

$$\begin{aligned} \gamma_2(t) &: = \exp(t(X_2))(x + h, \varphi(x + h)) \\ &= (x + h, \varphi(x + h) + (x + h)^\alpha t) := \Phi_2(x + h, t). \end{aligned}$$

If $t = t_1$, γ_1 reaches the height $\varphi(x) + \int_0^{|h|/b}(x+b\tau)^\alpha d\tau$. Thus the curve γ_2 needs the time $t_2 = \frac{1}{(x+h)^\alpha}|\varphi(x) - \varphi(x+h) + \int_0^{|h|/b}(x+b\tau)^\alpha d\tau|$ to reach the same height. The hypothesis on φ and (3.7.135) give the estimate $t_2 \le C|h|$.

The choice of b ensures that $\gamma_1(t) \in \Omega$ for all $t \in (0, t_1]$. In fact this amounts to

$$\varphi(x+bt) < \varphi(x) + \int_0^t |x+b\tau|^\alpha d\tau.$$

In view of $|\varphi(x+bt) - \varphi(x)| \le cbt(x+bt)^\alpha$ and $\int_0^t (x+b\tau)^\alpha d\tau \ge \int_0^{t/2}(x+b\tau)^\alpha d\tau \ge t/2(x+bt/2)^\alpha$ the inequality is implied by $cb(x+bt)^\alpha < 1/2(x+bt/2)^\alpha$ which holds true if $2^{\alpha+1}cb < 1$.

By the triangle inequality

$$\begin{aligned} I_{A_1} \le C\Big[&\int_0^\delta \frac{dh}{|h|^p}\int_0^\delta \Big(\int_0^{C|h|} |Xu(\Phi_1(x,t))|\, dt\Big)^p |x|^\alpha dx + \\ &+ \int_0^\delta \frac{dh}{|h|^p}\int_0^\delta \Big(\int_0^{C|h|} |Xu(\Phi_2(x+h,t))|\, dt\Big)^p |x|^\alpha dx\Big] \\ :=\ & C[I_{A_1}^{(1)} + I_{A_1}^{(2)}]. \end{aligned}$$

Now, by Minkowski and Hardy

$$\begin{aligned} I_{A_1}^{(1)} &\le \int_0^\delta \Big(\frac{1}{|h|}\int_0^{C|h|}\Big(\int_0^\delta |Xu(\Phi_1(x,t))|^p |x|^\alpha dx\Big)^{1/p} dt\Big)^p dh \\ &\le C\int_{(0,\delta)\times(0,\delta)} |Xu(\Phi_1(x,t))|^p |x|^\alpha dx dt \le C\int_\Omega |Xu(x,y)|^p\, dxdy. \end{aligned}$$

The last inequality follows from the fact that if $\delta > 0$ is small then Φ_1 is one-to-one, $\Phi_1((0,\delta)\times(0,\delta)) \subset \Omega$ and

$$\frac{\partial \Phi_1(x,t)}{\partial t \partial x} = \begin{pmatrix} 1 & b \\ \varphi'(x) + \frac{1}{b}[(x+bt)^\alpha - x^\alpha] & (x+bt)^\alpha \end{pmatrix}.$$

Thus $|J\Phi_1(x,t)| = |x^\alpha - b\varphi'(x)| \ge |x|^\alpha - b|\varphi'(x)| \ge (1-bc)|x|^\alpha \ge (1 - 2^{-(\alpha+1)})|x|^\alpha$, and the estimate for $I_{A_1}^{(1)}$ follows.

Analogously, recalling that $t_2 \le C|h|$ and $|x| \le |x+h|$

$$\begin{aligned} I_{A_1}^{(2)} &\le \int_0^\delta \Big(\frac{1}{h}\int_0^{C|h|}\Big(\int_0^\delta |Xu(\Phi_2(x+h,t))|^p |x+h|^\alpha dx\Big)^{1/p} dt\Big)^p dh \\ &\le \int_0^\delta \Big(\frac{1}{|h|}\int_0^{C|h|}\Big(\int_0^{2\delta} |Xu(\Phi_2(x,t))|^p |x|^\alpha dx\Big)^{1/p} dt\Big)^p dh \\ &\le C\int_{(0,2\delta)\times(0,\delta)} |Xu(\Phi_2(x,t))|^p |x|^\alpha dx dt. \end{aligned}$$

Since $|J\Phi_2(x,t)| = |x|^\alpha$, the change of variable $(\xi,\tau) = \Phi_2(x,t)$ ends the estimate for $I_{A_1}^{(2)}$.

The integral on the set $A_2 = \{0 < x < \delta, -\delta < y < -x\}$ can be treated in the same way of I_{A_1}, letting $y = x + h$ and using the curves

$$\gamma_1(t) = \exp(t(-bX_1 + X_2))(x, \varphi(x)),$$
$$\gamma_2(t) = \exp(tX_2)(x + h, \varphi(x + h)).$$

□

7.2. Analysis of a counterexample. The hypothesis of $\alpha-$admissibility for the domain Ω in Theorem 3.7.5 is necessary. More precisely, there exist domains of class C^1 that are not $\alpha-$admissible for which the trace estimate (3.7.5) fails.

Let $\alpha > 0$, fix $\beta \in (0, \alpha + 1)$ and consider the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : |x|^\beta < y < 1\}.$$

Except that at the points $(\pm 1, 1)$ the boundary $\partial\Omega$ is of class C^1. These points are not important, problems stem from the boundary point $(0, 0)$ which is not $\alpha-$admissible.

We shall consider the case $p = 2$. As usual write $z = (x, y)$ and $\zeta = (\xi, \eta)$.

PROPOSITION 3.7.6. *Let $\alpha > 0$ and $\beta \in (0, \alpha + 1)$. There exists $\gamma > 0$ such that the function $u(x, y) = y^{-\gamma}$ satisfies*

$$I := \int_\Omega |Xu|^2\, dxdy < +\infty$$

and

$$N := \int_{\partial\Omega\times\partial\Omega} \frac{|u(z) - u(\zeta)|^2}{d(z, \zeta)\mu(B(z, d(z, \zeta)))} d\mu(z)d\mu(\zeta) = +\infty.$$

Proof. We compute first I. Indeed

$$I = \gamma^2 \int_0^1 y^{-2\gamma-2}\Big(\int_{-y^{1/\beta}}^{y^{1/\beta}} |x|^{2\alpha}\, dx\Big) dy = \frac{2\gamma^2}{2\alpha + 1} \int_0^1 y^{-2\gamma-2+(2\alpha+1)/\beta}\, dy,$$

and

$$I < +\infty \quad \Leftrightarrow \quad -2\gamma - 2 + (2\alpha + 1)/\beta > -1 \quad \Leftrightarrow \quad \gamma < \frac{2\alpha + 1 - \beta}{2\beta}. \tag{3.7.140}$$

Now we shall estimate N but first some remarks on $d(z, \zeta)$ and $\mu(B(z, d(z, \zeta)))$ are in order. Let $z = (x, x^\beta) \in \partial\Omega$ with $0 < x < 1$ and let $r > 0$. Assume that

$$r \geq x^{\beta/(\alpha+1)}. \tag{3.7.141}$$

From (3.7.141) it follows that $x^\beta \leq r^{\alpha+1} \leq r(x + r)^\alpha$ and thus $x^\beta - r(x + r)^\alpha \leq 0$. This means that

$$\mathrm{Box}(z, r) \cap \{y \leq 0\} \neq \emptyset, \tag{3.7.142}$$

i.e. the box $\mathrm{Box}(z, r)$ meets the lower half plane.

Analogously, since $\beta < \alpha + 1$ we find $x \leq x^{\beta/(\alpha+1)} \leq r$ and thus $x - r \leq 0$. This means that

$$\mathrm{Box}(z, r) \cap \{x \leq 0\} \neq \emptyset, \tag{3.7.143}$$

i.e. the box $\mathrm{Box}(z, r)$ meets the left half plane.

We now claim that, for r and x sufficiently small the right part $\{(t,t^\beta):0<t<1\}$ of the boundary of Ω meets $\partial\mathrm{Box}(z,r)$ at its upper horizontal edge. This is equivalent to show that $(x+r)^\beta \geq x^\beta + r(x+r)^\alpha$, which holds because

$$(x+r)^\beta - x^\beta \geq Cr(x+r)^{\beta-1} \geq r(x+r)^\alpha$$

for $x, r \leq \sigma_0$, where σ_0 is a suitable constant (we have used $\beta < \alpha+1$). We also note that the x-coordinate of the intersection point $\{(t,t^\beta):0<t<1\}\cap\partial\mathrm{Box}(z,r)$ is $(x^\beta + r(x+r)^\alpha)^{1/\beta}$. Then from (3.7.142) and (3.7.143)

$$\begin{aligned}\mu(\mathrm{Box}(z,r)) &\simeq \mu(\mathrm{Box}(z,r)\cap\{(\xi,\eta):\xi\geq 0\})\\ &\simeq \int_0^{(x^\beta+r(x+r)^\alpha)^{1/\beta}} |\xi|^{\beta-1}\,d\xi \simeq x^\beta + r(x+r)^\alpha.\end{aligned}$$

Since $r \leq x+r \leq 2r$ then $x+r\simeq r$ and $\mu(\mathrm{Box}(z,r)) \simeq x^\beta + r^{\alpha+1}$. But $r^{\alpha+1} \leq x^\beta + r^{\alpha+1} \leq 2r^{\alpha+1}$ and this proves that if (3.7.141) holds then

$$\mu(\mathrm{Box}(z,r)) \simeq r^{\alpha+1}. \tag{3.7.144}$$

We shall now briefly discuss $d(z,\zeta)$ where $z=(x,x^\beta)$ and $\zeta=(\xi,\xi^\beta)$. Assume that $0<x<\xi$ and that

$$\xi^{\alpha+1} \leq \xi^\beta - x^\beta. \tag{3.7.145}$$

From (3.7.134)

$$d(z,\zeta) \simeq (\xi - x) + (\xi^\beta - x^\beta)^{1/(\alpha+1)},$$

and using the equivalence $\xi^\beta - x^\beta \simeq (\xi-x)\xi^{\beta-1}$ we get

$$\begin{aligned}d(z,\zeta) &\simeq (\xi-x)^{1/(\alpha+1)}\big((\xi-x)^{\alpha/(\alpha+1)} + \xi^{(\beta-1)/(\alpha+1)}\big)\\ &\simeq (\xi-x)^{1/(\alpha+1)}\xi^{(\beta-1)/(\alpha+1)}.\end{aligned} \tag{3.7.146}$$

In the last equivalence we used again $\beta<\alpha+1$.

Recalling (3.7.141) and (3.7.145) we define

$$\begin{aligned}D = \{(z,\zeta)\in\partial\Omega\times\partial\Omega : 0<x<\xi<\sigma_0,\ \xi^{\alpha+1}\leq \xi^\beta - x^\beta,\\ \sigma_0 \geq d(z,\zeta)\geq x^{\beta/(\alpha+1)}\}.\end{aligned}$$

Then, by (3.7.144) and Lemma 1.9.3

$$N \geq \int_D \frac{|u(z)-u(\zeta)|^2}{d(z,\zeta)\mu(B(z,d(z,\zeta)))}d\mu d\mu \simeq \int_D \frac{|u(z)-u(\zeta)|^2}{d(z,\zeta)^{\alpha+2}}d\mu d\mu := M.$$

By (3.7.146) there is a positive constant $k>0$ such that

$$d(z,\zeta) \geq \Big(\frac{(\xi-x)\xi^{\beta-1}}{k}\Big)^{1/(\alpha+1)}$$

and thus $\{(z,\zeta)\in\partial\Omega\times\partial\Omega : 0<x<\xi<\sigma_0,\ \xi^{\alpha+1}\leq\xi^\beta-x^\beta,\ (\xi-x)\xi^{\beta-1}\geq kx^\beta\}\subset D$. Then, letting

$$E = \{(x,\xi): 0<x<\xi<\sigma_0,\ \xi^{\alpha+1}\leq \xi^\beta - x^\beta,\ (\xi-x)\xi^{\beta-1}\geq kx^\beta\}.$$

we have

$$
\begin{aligned}
M &\simeq \int_E \frac{|x^{-\beta\gamma} - \xi^{-\beta\gamma}|^2 |x\xi|^{\beta-1}}{\big((\xi - x)^{1/(\alpha+1)}\xi^{(\beta-1)/(\alpha+1)}\big)^{\alpha+2}}\, dx d\xi \\
&\simeq \int_E \frac{(\xi^{\beta\gamma} - x^{\beta\gamma})^2}{x^{2\beta\gamma-\beta+1}\xi^{\varphi(\alpha,\beta,\gamma)}(\xi - x)^{(\alpha+2)/(\alpha+1)}}\, dx d\xi,
\end{aligned}
$$

where $\varphi(\alpha, \beta, \gamma) = 2\beta\gamma - \beta + 1 + (\alpha + 2)(\beta - 1)/(\alpha + 1)$.

In order to separate the integration variables we perform in the last integral the change of variable $x = \xi t$. The integration domain E changes in the following way. The relation $0 < x < \xi < \sigma_0$ gives $0 < t < 1$, the relation $(\xi - x)\xi^{\beta-1} \geq kx^\beta$ gives $(1 - t) \geq kt^\beta$, and finally the relation $\xi^{\alpha+1} \leq \xi^\beta - x^\beta$ gives $t^\beta \leq 1 + \xi^{\alpha-\beta+1}$ which is implied by the first one. This shows that in the new integral we may integrate on the square $\{(t, \xi) : 0 < t, \xi < \delta\}$ where $\delta > 0$ is a small but positive constant. Thus we find

$$
M \geq \int_0^\delta \frac{d\xi}{\xi^{\varphi(\alpha,\beta,\gamma)-\beta+(\alpha+2)/(\alpha+1)}} \int_0^\delta \frac{(1 - t^{\beta\gamma})^2}{t^{2\beta\gamma-\beta+1}(1 - t)^{(\alpha+2)/(\alpha+1)}}\, dt.
$$

If $\psi(\alpha, \beta, \gamma) := \varphi(\alpha, \beta, \gamma) - \beta + (\alpha + 2)/(\alpha + 1) \geq 1$ then $M = +\infty$, which implies $N = +\infty$. Now, $\psi(\alpha, \beta, \gamma) = 2\beta\gamma - 2\beta + \beta(\alpha+2)/(\alpha+1) + 1$, and hence $\psi(\alpha, \beta, \gamma) \geq 1$ if and only if $\gamma \geq \alpha/(2\alpha + 2)$. Finally

$$
\gamma \geq \frac{\alpha}{2(\alpha + 1)} \quad \Rightarrow \quad N = +\infty. \tag{3.7.147}
$$

Notice that if $\beta \in (0, \alpha + 1)$ then

$$
\frac{\alpha}{2(\alpha + 1)} < \frac{2\alpha + 1 - \beta}{2\beta},
$$

and therefore we can choose

$$
\gamma \in \Big[\frac{\alpha}{2(\alpha + 1)}, \frac{2\alpha + 1 - \beta}{2\beta}\Big).
$$

The interval becomes empty when $\beta = \alpha + 1$, i.e. exactly when the domain Ω becomes $\alpha-$admissible. With such a choice $I < +\infty$ by (3.7.140) and $N = +\infty$ by (3.7.147). $\square$

CHAPTER 4

Anisotropic Sobolev spaces and functions with bounded $X-$variation

1. Anisotropic Sobolev spaces

The theory of Sobolev spaces associated with vector fields has been deeply developed in the last years and it is not possible here to give an exhaustive account of the existing literature. We just mention the papers and books [**26**], [**47**], [**67**], [**77**], [**75**], [**81**], [**89**], [**172**]. For the theory of Sobolev spaces in metric spaces we refer to [**91**], [**99**] and [**100**]. More references can be found in the paper [**100**].

1.1. Introduction. Let $X = (X_1, ..., X_m)$ be a family of locally Lipschitz continuous vector fields and let $\Omega \subset \mathbb{R}^n$ be an open set. We introduce an anisotropic Sobolev space associated with X. If $1 \le p \le +\infty$ define

$$\begin{aligned} \mathrm{W}^{1,p}_X(\Omega) = \big\{u \in \mathrm{L}^p(\Omega) : &\text{there exists } X_j u \in \mathrm{L}^p(\Omega) \text{ for } j = 1, ..., m, \\ &\text{in distributional sense}\big\}. \end{aligned} \tag{4.1.1}$$

We shall write $\mathrm{H}^1_X(\Omega) := \mathrm{W}^{1,2}_X(\Omega)$. If $u \in \mathrm{W}^{1,p}_X(\Omega)$ we shall denote by

$$Xu = (X_1 u, ..., X_m u)$$

the weak $X-$gradient of u. The natural norm on $\mathrm{W}^{1,p}_X(\Omega)$ is

$$\|u\|_{\mathrm{W}^{1,p}_X(\Omega)} = \|u\|_{\mathrm{L}^p(\Omega)} + \sum_{j=1}^m \|X_j u\|_{\mathrm{L}^p(\Omega)}. \tag{4.1.2}$$

It is easy to show that the normed space in this way obtained is complete.

PROPOSITION 4.1.1. *Endowed with the norm* (4.1.2) $\mathrm{W}^{1,p}_X(\Omega)$, $1 \le p \le +\infty$, *is a Banach space and* $\mathrm{H}^1_X(\Omega)$ *is a Hilbert space with the natural inner product.*

We can also define a space $\mathrm{H}^{1,p}_X(\Omega)$ as the closure of $\mathrm{W}^{1,p}_X(\Omega) \cap C^\infty(\Omega)$ in the norm (4.1.2). As in the classical theory of Sobolev spaces $\mathrm{W}^{1,p}_X(\Omega)$ and $\mathrm{H}^{1,p}_X(\Omega)$ turn out to be equal. The following theorem of Meyers-Serrin type was proved in [**81**], [**51**] and [**89**] (see also [**100**, Theorem 11.9]) but in the case of vector fields of class C^1 its proof had been already given in [**86**].

THEOREM 4.1.2. *Let* $X = (X_1, ..., X_m)$ *be a system of locally Lipschitz vector fields and let* $\Omega \subset \mathbb{R}^n$ *be an open set. If* $u \in \mathrm{W}^{1,p}_X(\Omega)$, $1 \le p < +\infty$, *then there exists* $(u_h)_{h\in\mathbb{N}} \subset \mathrm{W}^{1,p}_X(\Omega) \cap C^\infty(\Omega)$ *such that*

$$\lim_{h\to\infty} \|u_h - u\|_{\mathrm{W}^{1,p}_X(\Omega)} = 0.$$

If u has compact support in Ω Theorem 4.1.2 can be proved by a Friedrichs regularization technique. The general case follows by a partition of unity.

1.2. Poincaré inequalities. One of the crucial tools in the theory of Sobolev spaces is the Poincaré inequality for balls. In the setting of C-C spaces the Poincaré inequality was first proved in [**77**] for a special class of vector fields and then in [**109**] for Hörmander vector fields. A simple proof in Carnot groups was given in [**171**] and a proof that implies all the known results has been recently given in [**122**]. The Poincaré inequality is also central in the theory of Sobolev spaces in abstract metric spaces [**100**].

Theorem 4.1.3. *Let $X = (X_1, ..., X_m)$ be a system of Hörmander or of Grushin's type* (1.9.99) *vector fields on $\mathbb{R}^n$. If $K \subset \mathbb{R}^n$ is a compact set there exist $C > 0$, $r_0 > 0$ and $\lambda \geq 1$ such that*

$$\int_B |u - u_B|\, dx \leq rC \int_{\lambda B} |Xu|\, dx \tag{4.1.3}$$

for all $u \in C^1_0(\mathbb{R}^n)$ and for all $B = B(x, r)$ with $x \in K$ and $0 \leq r \leq r_0$, where $\lambda B = B(x, \lambda r)$.

Proof. Since it is particularly elegant we give the proof for Carnot groups of [**171**]. Let a structure of Carnot group induced by $X = (X_1, ..., X_m)$ be given on $\mathbb{R}^n$. The group product of $x, y \in \mathbb{R}^n$ will be denoted by $x \cdot y$ and $Q \geq n$ will stand for the homogeneous dimension of the group so that by (1.7.7) $|B(x, r)| = kr^Q$ for all $x \in \mathbb{R}^n$ and $r \geq 0$ with $k = |B(0, 1)|$.

Fix $B = B(x_0, r)$ with $x_0 \in \mathbb{R}^n$ and $r > 0$ and let $u \in C^1_0(\mathbb{R}^n)$. Notice that

$$\int_B |u(x) - u_B|\, dx = \int_B \left| \fint_B (u(x) - u(y))dy \right| dx \leq \fint_B \int_B |u(x) - u(y)|\, dxdy.$$

We perform in the inner integral the change of variable $z = y^{-1} \cdot x$, which has Jacobian identically 1, getting

$$\int_B |u(x) - u_B|\, dx \leq \fint_B \int_{y^{-1} \cdot B} |u(y \cdot z) - u(y)|\, dzdy \leq \fint_B \int_{B(0,2r)} |u(y \cdot z) - u(y)|\, dzdy.$$

Indeed, if $y \in B$ then $y^{-1} \cdot B \subset B(0, 2r)$ because by Proposition 1.7.3 for all $z \in B$ we have $d(y^{-1} \cdot z, 0) = d(z, y) \leq d(z, x_0) + d(x_0, y) \leq 2r$.

Let now $z \in B(0, 2r)$ be fixed, let $\delta = d(0, z)$ and take a geodesic $\gamma : [0, \delta] \to \mathbb{R}^n$ such that $\gamma(0) = 0$ and $\gamma(\delta) = z$ with $\delta \leq 2r$. For some $h \in \mathrm{L}^\infty(0, \delta)^m$

$$\dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)) \quad \text{and} \quad |h(t)| \leq 1 \quad \text{for a.e. } t \in [0, \delta].$$

Then

$$\begin{aligned} u(y \cdot z) - u(y) &= \int_0^\delta \frac{d}{dt} u(y \cdot \gamma(t))\, dt = \int_0^\delta \langle Du(y \cdot \gamma(t)), \frac{d}{dt}(y \cdot \gamma(t)) \rangle\, dt \\ &= \int_0^\delta \langle Du(y \cdot \gamma(t)), \sum_{j=1}^m h_j(t) X_j(y \cdot \gamma(t)) \rangle\, dt \\ &= \int_0^\delta \langle Xu(y \cdot \gamma(t)), h(t) \rangle\, dt. \end{aligned}$$

We used the left invariance of $X_1, ..., X_m$. As $\|h\|_\infty \leq 1$ we obtain

$$\begin{aligned}\int_B |u(x) - u_B|\, dx &\leq \fint_B \int_{B(0,2r)} \int_0^\delta |Xu(y\cdot\gamma(t))|\, dt dz dy \\ &\leq \int_0^\delta \int_{B(0,2r)} \fint_B |Xu(y\cdot\gamma(t))|\, dy dz dt.\end{aligned}$$

The curve γ depends on z. Since $\gamma(t) \in B(0, 2r)$ for all $t \in [0, \delta]$, if $y \in B$ then $y \cdot \gamma(t) \in 3B = B(x_0, 3r)$. Indeed

$$d(y\cdot\gamma(t), x_0) \leq d(y\cdot\gamma(t), y) + d(y, x_0) = d(\gamma(t), 0) + d(y, x_0) \leq 3r.$$

Thus we finally get

$$\begin{aligned}\int_B |u(x) - u_B|\, dx &\leq \frac{1}{|B(0,r)|}\int_0^\delta \int_{B(0,2r)} \int_{3B} |Xu(y)|\, dy dz dt \\ &\leq 2r\frac{|B(0,2r)|}{|B(0,r)|}\int_{3B} |Xu(y)|\, dy = r2^{Q+1}\int_{3B} |Xu(y)|\, dy.\end{aligned}$$

□

REMARK 4.1.4. The constant λ in λB in the rhs of (4.1.3) can be chosen $\lambda = 1$ because Carnot-Carathéodory balls are John domains (see Theorem 4.1.7 below).

REMARK 4.1.5. The Poincaré inequality (4.1.3) holds for more general systems of vector fields than the ones mentioned in the statement of Theorem 4.1.3. A proof of (4.1.3) is known when C-C balls can be represented by "almost exponential maps" (see [**122**]).

Fix a compact set $K \subset \mathbb{R}^n$ and $r_0 > 0$. Let Q be the homogeneous dimension of $(\mathbb{R}^n, d)$ relatively to K defined as in Definitions 1.6.2 and 1.6.4 for balls $B(x, r)$ with $x \in K$ and $0 \leq r \leq r_0$. Such Q is well defined because the C-C spaces we are dealing with are doubling metric spaces, at least locally (see Remark 1.6.8 for the Hörmander case). If $1 \leq p < Q$ define the conjugate exponent

$$p^* = \frac{pQ}{Q-p}. \tag{4.1.4}$$

The Poincaré inequality (4.1.3) can be improved to the following Sobolev-Poincaré inequality for balls (see [**127**], [**78**], [**74**], [**89**], [**100**]).

THEOREM 4.1.6. *Let $X = (X_1, ..., X_m)$ be a system of Hörmander or of Grushin's type* (1.9.99) *vector fields on $\mathbb{R}^n$. If $K \subset \mathbb{R}^n$ is a compact set there exist $C > 0$ and $r_0 > 0$ such that if $1 \leq p < Q$*

$$\Big(\fint_B |u - u_B|^{p^*}\, dx\Big)^{1/p^*} \leq rC\Big(\fint_B |Xu|^p\, dx\Big)^p \tag{4.1.5}$$

for all $u \in C_0^1(\mathbb{R}^n)$ and for all $B = B(x, r)$ with $x \in K$ and $0 \leq r \leq r_0$, where $\lambda B = B(x, \lambda r)$.

Let $(\mathbb{R}^n, d)$ be a C-C space associated with $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and assume that there exists $\delta > 0$ such that for all $x \in \mathbb{R}^n$ and $r \geq 0$

$$|B(x, 2r)| \leq \delta |B(x, r)| \tag{4.1.6}$$

and let $Q \geq n$ be the homogeneous dimension introduced in Definition 1.6.4. Recall the definition of John domain 3.1.1. The following Sobolev-Poincaré inequality has been proved in the setting of C-C spaces in [**78**] and [**89**] but the result holds for Sobolev spaces defined in abstract metric spaces that support a Poincaré inequality for balls and have the doubling property (see [**100**]).

THEOREM 4.1.7. *Let $(\mathbb{R}^n, d, |\cdot|)$ be C-C space associated with $X = (X_1, ..., X_m)$, with the doubling property* (4.1.6) *and with homogeneous dimension Q. Assume that the Poincaré inequality* (4.1.3) *holds. If $1 \leq p < Q$ and $\Omega \subset \mathbb{R}^n$ is a John domain (with small diameter) then there exists $C > 0$ such that*

$$\left(\fint_\Omega |u - u_\Omega|^{p^*}\, dx\right)^{1/p^*} \leq C \mathrm{diam}(\Omega) \left(\fint_\Omega |Xu|^p\, dx\right)^{1/p} \tag{4.1.7}$$

for all $u \in C^1_0(\mathbb{R}^n)$, where u_Ω is the mean of u over Ω.

From the study of John and uniform domains in C-C spaces contained in chapter 3 the following corollary immediately follows.

COROLLARY 4.1.8. *Consider the following cases:*

(i) *$(\mathbb{R}^n, d)$ is a Carnot group of step 2 with homogeneous dimension $Q \geq n$, $X = (X_1, ..., X_m)$ is a system of generators of the Lie algebra of the group and $\Omega \subset \mathbb{R}^n$ is a connected, bounded open set of class $C^{1,1}$.*

(ii) *$(\mathbb{R}^4, d)$ is the Carnot group of step 3 introduced in chapter 3 section 5 with homogeneous dimension $Q = 7$, $X = (X_1, X_2)$ are the generators of the Lie algebra of the group and $\Omega \subset \mathbb{R}^4$ is a connected, bounded open set which is admissible according to Definition 3.5.2.*

(iii) *$(\mathbb{R}^n, d)$ is the Grushin space induced by $X_1 = \partial_{x_1}, ..., X_{n-1} = \partial_{x_{n-1}}, X_n = |x|^\alpha \partial_y$ where $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\alpha > 0$, $\Omega \subset \mathbb{R}^n$ is a connected, bounded open set with Lipschitz boundary which is $\alpha-$admissible according to Definition 3.3.1 and $Q = n + \alpha$.*

(iv) *$(\mathbb{R}^n, d)$ is the C-C spaces induced by a system $X = (X_1, ..., X_m)$ of Hörmander vector fields, $\Omega \subset \mathbb{R}^n$ is a connected, bounded open set of class C^∞ with small diameter and without characteristic points on its boundary and Q is the homogeneous dimension relative to Ω.*

In cases (i), (ii), (iii) and (iv) the Sobolev-Poincaré inequality (4.1.7) *holds for $1 \leq p < Q$.*

EXAMPLE 4.1.9. Consider the Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$. We write $(x, y, t) = (z, t) \in \mathbb{R}^{2n} \times \mathbb{R} = \mathbb{H}^n$. Recall that $Q = 2n + 2$ is the homogeneous dimension of the group. We give a counterexample to the Sobolev-Poincaré inequality (4.1.3) and at the same time we formulate a problem related to the regularity of the integration domain Ω.

Let $\Omega = \{(z, t) \in \mathbb{H}^n : |z|^\beta < t < 1\}$ where $\beta \geq 1$ is a real parameter. The domain $\partial\Omega$ is not smooth when $|z| = t = 1$ but this is not important because we are interested in the characteristic point $(0, 0)$. If $\beta \geq 2$ then Ω is of class C^2 in a neighborhood of $0 \in \partial\Omega$ and by Corollary 4.1.8 it supports the Sobolev-Poincaré inequality (4.1.7) (for functions with support in a neighborhood of the origin).

We consider the case $1 \leq \beta < 2$. Let $u(z,t) = t^{-\gamma}$. In view of a possible counterexample to (4.1.7) we look for an exponent $\gamma > 0$ such that

$$\int_\Omega |\nabla_{\mathbb{H}} u|^p \, dzdt < +\infty \quad \text{but} \quad \int_\Omega |u|^q \, dzdt = +\infty, \tag{4.1.8}$$

where $1 \leq p < Q$, for some $q \geq 1$ which should be less than $p^* = pQ/(Q-p)$. We have

$$\begin{aligned} \int_\Omega |\nabla_{\mathbb{H}} u|^p \, dzdt &\simeq \int_0^1 t^{-p(\gamma+1)} \int_{|z|<t^{1/\beta}} |z|^p \, dz \, dt \\ &\simeq \int_0^1 t^{-p(\gamma+1)+(Q-2+p)/\beta} \, dt < +\infty \quad \Leftrightarrow \quad \gamma < \frac{Q-(2-\beta)-p(\beta-1)}{\beta p}. \end{aligned}$$

On the other hand

$$\int_\Omega |u|^q \, dzdt \simeq \int_0^1 t^{-q\gamma+(Q-2)/\beta} \, dt = +\infty \quad \Leftrightarrow \quad \gamma \geq \frac{Q-(2-\beta)}{\beta q}.$$

An exponent γ ensuring (4.1.8) can be found if the following condition holds

$$\frac{Q-(2-\beta)}{\beta q} < \frac{Q-(2-\beta)-p(\beta-1)}{\beta p}, \tag{4.1.9}$$

which also gives

$$q > \frac{p(Q-(2-\beta))}{Q-(2-\beta)-p(\beta-1)}. \tag{4.1.10}$$

If $\beta = 2$ (4.1.10) becomes $q > p^*$ which is exactly what one should expect. If $\beta = 1$ (4.1.10) becomes $q > p$.

If $\beta < 2$ we can find $q < p^*$ such that (4.1.9) holds and an exponent $\gamma > 0$ ensuring (4.1.8) does exist. If (4.1.10) does not hold the counterexample does not work. This analysis suggests the following problem. Let $1 \leq \beta < 2$ and let $\Omega = \{(z,t) \in \mathbb{H}^n : |z|^\beta < t < 1\}$. If

$$q = \frac{p(Q-(2-\beta))}{Q-(2-\beta)-p(\beta-1)}$$

then the following Sobolev-Poincaré inequality holds

$$\Big(\int_\Omega |u-u_\Omega|^q \, dzdt\Big)^{1/q} \leq C\Big(\int_\Omega |\nabla_{\mathbb{H}} u|^p \, dzdt\Big)^p. \tag{4.1.11}$$

In the Euclidean case similar Poincaré inequalities in Hölder domains have been proved in [**162**] and [**101**].

1.3. Potential estimate and Morrey inequality. The Poincaré inequality (4.1.3) implies the following estimate of potential type.

THEOREM 4.1.10. *Let $X = (X_1, ..., X_m)$ be a system of Hörmander or of Grushin's type* (1.9.99) *vector fields on $\mathbb{R}^n$. Let $B \subset \mathbb{R}^n$ be a ball in the C-C metric d with (small) radius $r > 0$. There exists $C > 0$ such that*

$$|u(x) - u_B| \leq Cr \int_B |Xu(y)| \frac{d(x,y)}{|B(x,d(x,y))|} \, dy \tag{4.1.12}$$

for all $x \in B$ and $u \in C_0^1(\mathbb{R}^n)$.

The potential estimate (4.1.12) holds in much more general situations than the ones stated in Theorem 4.1.10 (see [**100**]). In the case of Hörmander vector fields formula (4.1.12) was first proved using the structure theorem for C-C balls of [**151**] and the estimates for the Green function of the corresponding sub-elliptic Laplacians of [**157**] (see [**50**], [**78**], [**41**] and see also [**74**]). But it became soon clear that (4.1.12) can also be directly obtained from the Poincaré inequality (4.1.3) (see [**79**] and [**85**]). Actually, the Poincaré inequality itself can be proved by (4.1.12) using the doubling property of C-C balls and so it is equivalent to the potential estimate. That integration in the right hand side of (4.1.12) may take place on the ball B and not only on a larger ball τB with $\tau > 1$ has been recently shown in [**128**] (at least in the cases considered here).

Let $X = (X_1, ..., X_m)$ be a system of Hörmander or Grushin type vector fields. Fix a compact set $K \subset \mathbb{R}^n$ and $r_0 > 0$. Let Q be the homogeneous dimension of $(\mathbb{R}^n, d)$ relatively to K defined as in Definitions 1.6.2 and 1.6.4 for balls $B(x, r)$ with $x \in K$ and $0 \le r \le r_0$. The following Theorem is proved in [**127**] (see also [**141**] for a Morrey inequality involving non smooth vector fields). We shall give the proof since a weak form of the Morrey inequality has been used in chapter 2.

THEOREM 4.1.11. *Let K, r_0 and Q be fixed as above and let $p > Q$. There exists a constant $C > 0$ such that for all $B = B(x_0, r)$ with $x_0 \in K$ and $0 \le r \le r_0$ and for all $u \in \mathrm{W}^{1,p}_X(B)$*

$$|u(x) - u(y)| \le Cr\Big(\fint_B |Xu(z)|^p\, dz\Big)^{1/p} \tag{4.1.13}$$

for a.e. $x, y \in B$.

Proof. Without loss of generality assume that $u \in C^1_0(\mathbb{R}^n)$. Notice that $|u(x) - u(y)| \le |u(x) - u_B| + |u_B - u(y)|$ and thus it will be enough to estimate $|u(x) - u_B|$ using Theorem 4.1.10. By Hölder inequality

$$\begin{aligned}|u(x) - u_B| &\le C \int_B |Xu(z)| \frac{d(x,z)}{|B(x, d(x,z))|}\, dz \\ &\le C\Big(\int_B |Xu(z)|^p\, dz\Big)^{1/p}\Big(\int_B \frac{d(x,z)^{p'}}{|B(x, d(x,z))|^{p'}}\, dz\Big)^{1/p'}.\end{aligned}$$

Now, the integration domain $B = B(x_0, r)$ in the last integral can be replaced by $B(x, 2r)$ and letting $A_k = \{z \in \mathbb{R}^n : 2^{-k}r \le d(x,z) \le 2^{-k+1}r\}$

$$\begin{aligned}\int_{B(x,2r)} \frac{d(x,z)^{p'}}{|B(x, d(x,z))|^{p'}}\, dz &= \sum_{k=0}^{+\infty} \int_{A_k} \frac{d(x,z)^{p'}}{|B(x, d(x,z))|^{p'}}\, dz \\ &\le \sum_{k=0}^{+\infty} \frac{r^{p'}}{2^{p'(k-1)}} \frac{|B(x, 2^{-k+1}r)|}{|B(x, 2^{-k}r)|^{p'}}.\end{aligned}$$

By the doubling property $|B(x, 2^{-k+1}r)| \le C|B(x, 2^{-k}r)|$ and by Proposition 1.6.3 $|B(x, 2^{-k}r)| \ge C2^{-kQ}|B(x,r)|$ where C is a positive constant not depending on x.

Thus

$$\int_B \frac{d(x,z)^{p'}}{|B(x,d(x,z))|^{p'}}\,dz \le Cr^{p'} \sum_{k=0}^{+\infty} \frac{|B(x,2^{-k}r)|^{1-p'}}{2^{p'(k-1)}}$$
$$\le Cr^{p'}|B(x,r)|^{1-p'} \sum_{k=0}^{+\infty} \frac{1}{2^{k(p'-Q(p'-1))}}.$$

Notice that $p' - Q(p'-1) > 0$ if and only if $p > Q$ and thus the last sum converges. We finally find

$$|u(x) - u_B| \le Cr\Big(\fint_B |Xu(z)|^p\,dz\Big)^{1/p},$$

and the claim is proved. □

1.4. Compactness. We end this section on Sobolev spaces stating a Compactness Theorem which is very useful in applications. Let $X = (X_1, ..., X_m)$ be a system of vector fields of Hörmander or Grushin type and consider the C-C space $(\mathbb{R}^n, d)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $Q \ge n$ be the homogeneous dimension of Ω according to Definitions 1.6.2 and 1.6.4. If $1 \le p < Q$ we denote by $p^* = pQ/(Q-p)$ the Sobolev conjugate exponent.

THEOREM 4.1.12. *Let Ω be a John domain in the metric space $(\mathbb{R}^n, d)$ with small diameter. Then:*

(i) *if $1 \le p < Q$ and $1 \le q < p^*$ the embedding $\mathrm{W}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^q(\Omega)$ is compact;*
(ii) *if $p \ge Q$ and $q \ge 1$ the embedding $\mathrm{W}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^q(\Omega)$ is compact.*

The assumption "with small diameter" can be omitted if $(\mathbb{R}^n, d)$ is a Carnot group or a C-C space of Grushin type. A proof of Theorem 4.1.12 can be found in [**89**] or in [**100**] where an argument is used that works on general metric spaces with the doubling property and with the Poincaré inequality (see also [**115**]). Other related references are [**54**], [**81**], [**125**], [**133**], [**156**].

Using our study of John domains in chapter 3 we immediately get the following Corollary.

COROLLARY 4.1.13. *Let $(\mathbb{R}^n, d)$, X, Ω and Q be as in Corollary 4.1.8 case (i), (ii), (iii) or (iv). Then the embedding $\mathrm{W}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^q(\Omega)$ is compact for all $1 \le p < Q$ and $1 \le q < p^*$. Moreover, the embedding $\mathrm{W}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ is compact for all $p \ge 1$.*

2. Functions with bounded $X-$variation

In this section we introduce the space of functions with bounded variation with respect to a family of vector fields and study some of their basic properties.

Such functions have been introduced in [**89**] where an existence theorem of surfaces with minimal $X-$perimeter is proved and then in [**80**] in connection with relaxation of functionals depending on vector fields (see also [**26**]). The theory of BV_X functions has been subsequently used in the study of rectifiability in the Heisenberg group (see [**82**]) and in the study of $\Gamma-$convergence properties of families of functionals involving degenerate energies (see [**148**] and [**149**]). Functions with bounded variation in settings different from the Euclidean or Riemannian one have been also introduced in Euclidean spaces with suitable weights [**14**] and in Finsler spaces [**22**].

The definition of BV_X we are going to introduce turns out to be equivalent to a definition of BV function in a general metric spaces endowed with a doubling measure and supporting a Poincaré inequality for balls (see [**138**] and [**7**]). We shall briefly discuss the issue in Remark 4.2.7 below.

2.1. Introduction. Given $Y \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ we shall denote by Y^* the operator formally adjoint to Y in $\mathrm{L}^2(\mathbb{R}^n)$, that is the operator which for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \psi Y\varphi \, dx = \int_{\mathbb{R}^n} \varphi Y^* \psi \, dx.$$

More explicitly,

$$\text{if} \quad Y\varphi(x) = \sum_{i=1}^n a_i(x)\partial_i\varphi(x) \quad \text{then} \quad Y^*\psi(x) = -\sum_{i=1}^n \partial_i(a_i(x)\psi(x)).$$

If $X = (X_1, \dots, X_m)$ is a family of locally Lipschitz vector fields and $\varphi \in C^1(\mathbb{R}^n;\mathbb{R}^m)$ is a $m-$vector valued function, the $X-$*divergence* of φ is

$$\mathrm{div}_X(\varphi) = -\sum_{j=1}^m X_j^* \varphi_j.$$

Definition 4.2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. The space $\mathrm{BV}_X(\Omega)$ of the functions with bounded $X-$variation is the set of all $u \in \mathrm{L}^1(\Omega)$ such that there exists a $m-$vector valued Radon measure $\mu = (\mu_1, ..., \mu_m)$ on Ω such that for all $\varphi \in C_0^1(\Omega;\mathbb{R}^m)$

$$\int_\Omega u \, \mathrm{div}_X(\varphi) \, dx = -\int_\Omega \langle \varphi, d\mu \rangle$$

where $\langle \varphi, d\mu \rangle = \sum_{j=1}^m \varphi_j \, d\mu_j$. By $\mathrm{BV}_{X,\mathrm{loc}}(\Omega)$ we denote the set of the functions belonging to $\mathrm{BV}_X(U)$ for any $U \Subset \Omega$.

Next we introduce the $X-$variation of a function. For any open set $\Omega \subset \mathbb{R}^n$ introduce the test functions

$$F(\Omega;\mathbb{R}^m) := \{\varphi \in C_0^1(\Omega;\mathbb{R}^m) : |\varphi(x)| \le 1 \ \text{ for all } x \in \Omega\}.$$

The $X-$*variation* in Ω of a function $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ is

$$|Xu|(\Omega) = \sup_{\varphi \in F(\Omega;\mathbb{R}^m)} \int_\Omega u \, \mathrm{div}_X(\varphi) \, dx. \tag{4.2.14}$$

By means of Riesz duality Theorem the following Proposition can be easily proved (see [**8**, Proposition 3.6]).

Proposition 4.2.2. *Let $u \in \mathrm{L}^1(\Omega)$. Then $u \in \mathrm{BV}_X(\Omega)$ if and only if $|Xu|(\Omega) < +\infty$. Moreover, $|Xu|(\Omega) = |\mu|(\Omega)$, where μ is the vector valued Radon measure in Definition 4.2.1.*

By Proposition 4.2.2 it follows that if $u \in \mathrm{W}_X^{1,1}(\Omega)$ then $u \in \mathrm{BV}_X(\Omega)$ and

$$|Xu|(\Omega) = \int_\Omega |Xu(x)| \, dx.$$

As in the Euclidean case a simple but important property of BV_X is the lower semicontinuity of the variation with respect to the $\mathrm{L}^1_{\mathrm{loc}}$ convergence.

Proposition 4.2.3. *Let $u, u_k \in \mathrm{L}^1(\Omega)$, $k \in \mathbb{N}$, be such that $u_k \to u$ in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$, then*

$$\liminf_{k\to\infty} |Xu_k|(\Omega) \geq |Xu|(\Omega).$$

Proof. If $\varphi \in F(\Omega;\mathbb{R}^m)$

$$\int_\Omega u \operatorname{div}_X(\varphi)\,dx = \lim_{k\to\infty}\int_\Omega u_k \operatorname{div}_X(\varphi)\,dx \leq \liminf_{k\to\infty} |Xu_k|(\Omega),$$

and taking the sup the claim follows. □

We now going to define the space of functions with bounded variation with respect to a symmetric, non negative matrix. This space has been introduced in [**80**] and used in [**149**] in connection with the study of $\Gamma-$convergence of functionals with degenerate energies (see also [**23**] for some general motivations in the case when the matrix is positive definite).

Let $A(x)$ be a symmetric, non negative $n\times n$ matrix defined for $x \in \Omega$. Let $V_x \subset \mathbb{R}^n$ be the range of $A(x)$, i.e. $V_x = \{A(x)\xi : \xi \in \mathbb{R}^n\}$, and denote by $L_x : V_x \to V_x$ the linear map associated with $A(x)$, i.e. $L_x(\xi) = A(x)\xi$ for all $x \in \Omega$ and $\xi \in V_x$. The map L_x is invertible and it can be easily checked that

$$|v|_x := \langle v, L_x^{-1} v\rangle^{1/2}, \quad v \in V_x$$

is a norm on V_x. Let

$$F_A(\Omega) = \{\psi \in \mathrm{Lip}_0(\Omega;\mathbb{R}^n) : \psi(x) \in V_x \text{ and } |\psi(x)|_x \leq 1 \text{ for all } x \in \Omega\}, \tag{4.2.15}$$

and define

$$|Du|_A(\Omega) = \sup_{\varphi\in F_A(\Omega)} \int_\Omega u \operatorname{div}(\psi)\,dx, \tag{4.2.16}$$

and

$$\mathrm{BV}_A(\Omega) = \big\{u \in \mathrm{L}^1(\Omega) :\ |Du|_A(\Omega) < +\infty\big\}. \tag{4.2.17}$$

If $A(x) = C(x)^T C(x)$ for all $x \in \Omega$ for some $m\times n-$matrix C with locally Lipschitz continuous entries (see [**167**, Theorems 5.2.2 and 5.2.3] for a sufficient condition that ensures this factorization) let $X = (X_1, ..., X_m)$ be the system of vector fields such that $X_{ji} = C_{ji}$. An interesting relation between the spaces $\mathrm{BV}_X(\Omega)$ and $\mathrm{BV}_A(\Omega)$ is given by the following result (see [**80**, Proposition 2.1.7 and Remark 2.1.8]).

Proposition 4.2.4. *Let A and X be as above. Then $\mathrm{BV}_X(\Omega) = \mathrm{BV}_A(\Omega)$ and for any $u \in \mathrm{BV}_X(\Omega)$ we have $|Du|(\Omega) = |Xu|(\Omega)$.*

Remark 4.2.5. If $A(x) = C(x)^T C(x)$ definition (4.2.16) can be equivalently given as

$$|Du|_A(\Omega) = \sup\Big\{\int_\Omega u \operatorname{div}(C^T\psi)\,dx : \psi = (\psi_1, ..., \psi_m) \text{ is such that}$$
$$C^T\psi \in \mathrm{Lip}_0(\Omega;\mathbb{R}^n),\ |\psi| \leq 1\Big\}.$$

Moreover, if A is positive definite on Ω, i.e. there exists a constant $\lambda > 0$ such that

$$\langle A(x)\xi, \xi\rangle \geq \lambda|\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

then $\mathrm{BV}_A(\Omega) = \mathrm{BV}(\Omega)$ (see [**81**]).

2.2. Approximation theorem. The linear space $\mathrm{BV}_X(\Omega)$ is a Banach space endowed with the norm $\|u\|_{\mathrm{BV}_X(\Omega)} = \|u\|_{\mathrm{L}^1(\Omega)} + |Xu|(\Omega)$. Anyway, smooth functions are dense in $\mathrm{BV}_X(\Omega)$ only in the following weak sense (see [**12**] for the classical result).

THEOREM 4.2.6. *Let $X = (X_1, ..., X_m)$ be a system of locally Lipschitz vector fields and let $\Omega \subset \mathbb{R}^n$ be an open set. If $u \in \mathrm{BV}_X(\Omega)$ then there exists a sequence $(u_h)_{h\in\mathbb{N}} \subset C^\infty(\Omega) \cap \mathrm{BV}_X(\Omega)$ such that*

$$\lim_{h\to\infty} \|u_h - u\|_{\mathrm{L}^1(\Omega)} = 0 \quad \text{and} \quad \lim_{h\to\infty} \int_\Omega |Xu_h|\, dx = |Xu|(\Omega).$$

The proof of Theorem (4.2.6) uses standard Fridrichs regularization (see [**80**, Theorem 2.2.2] and [**89**, Theorem 1.14]).

REMARK 4.2.7. In view of Theorem 4.2.6 and Proposition 4.2.3 the total variation $|Xu|(\Omega)$ of a function $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ could have also been defined as

$$|Xu|(\Omega) = \inf \Big\{ \liminf_{h\to\infty} \int_\Omega |Xu_h|\, dx : \ (u_h)_{h\in\mathbb{N}} \subset C^1(\Omega),\ u_h \to u \text{ in } \mathrm{L}^1_{\mathrm{loc}}(\Omega) \Big\}. \tag{4.2.18}$$

Till now no metric structure on $\mathbb{R}^n$ was needed. Assume that $X_1, ..., X_m$ induce on $\mathbb{R}^n$ a C-C metric d which is continuous in the Euclidean topology. If $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, d)$ then by Theorem 2.2.1 the weak derivatives $X_1u, ..., X_mu$ exist almost everywhere and belong to $\mathrm{L}^\infty_{\mathrm{loc}}(\Omega)$. In (4.2.18) we could also have required $u_h \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, d)$ instead of $u_h \in C^1(\Omega)$. Indeed, by Proposition 1.1.4 we have $C^1(\Omega) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega, d)$. Notice the if $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ then $|Xu|$ is a minimum upper gradient of u in (Ω, d) (see [**100**, Theorem 11.7]).

If we have a metric space endowed with a Borel measure and such that locally Lipschitz functions have minimum upper gradient then the total variation of a locally integrable function u can be defined by the relaxation argument in (4.2.18) taking sequences of locally Lipschitz functions converging to u and considering the integral of their minimum upper gradients (see [**138**]).

2.3. Compactness and Sobolev-Poincaré inequality. Thanks to Theorem 4.2.6 many properties of anisotropic Sobolev spaces with $p = 1$ remain true for $\mathrm{BV}_X(\Omega)$ functions. Let $X = (X_1, ..., X_m)$ be a system of Hörmander or Grushin type vector fields on $\mathbb{R}^n$ and let $(\mathbb{R}^n, d)$ be the associated C-C space. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $Q \geq n$ be the homogeneous dimension of the space relatively to balls with center in Ω and small radius as in Definitions 1.6.3 and 1.6.4.

THEOREM 4.2.8. *Let X, Ω and Q be as above. If Ω is a John domain (with small diameter) then:*

(i) *There exists $C > 0$ such that*

$$\Big(\fint_\Omega |u - u_\Omega|^{\frac{Q}{Q-1}}\, dx \Big)^{\frac{Q-1}{Q}} \leq C \frac{\mathrm{diam}(\Omega)}{|\Omega|} |Xu|(\Omega) \tag{4.2.19}$$

for all $u \in \mathrm{BV}_X(\Omega)$.

(ii) *The embedding $\mathrm{BV}_X(\Omega) \hookrightarrow \mathrm{L}^q(\Omega)$ is compact for any $1 \leq q < Q/(Q-1)$.*

For the proof of Theorem 4.2.8 see [**89**] (but see also [**73**]). The Poincaré inequality 4.2.19 is the main tool to get isoperimetric inequalities in C-C spaces.

Theorem 4.2.9. *Let X, Ω and Q be as above. If Ω is a John domain there exists $C > 0$ such that for all measurable set $E \subset \mathbb{R}^n$*

$$\min\{|E \cap \Omega|, |(\mathbb{R}^n \setminus E) \cap \Omega|\}^{\frac{Q-1}{Q}} \leq C \frac{\operatorname{diam}(\Omega)}{|\Omega|^{1/Q}} |X\chi_E|_X(\Omega). \tag{4.2.20}$$

CHAPTER 5

Measures of surface type in C-C spaces

In this chapter we study several surface measures that can be defined in C-C spaces: the perimeter measure, the Minkowski content and the Hausdorff measures defined with the C-C metric. Sets of finite $X-$perimeter, which are the natural generalization to the context of C-C spaces of the sets with finite perimeter introduced by Caccioppoli [**35**] and De Giorgi [**58**] and [**59**], have been introduced in [**89**] and [**80**] (see also [**26**]). The definition of $X-$perimeter does not require any metric structure but when this structure is available the definition turns out to be a special case of a general definition of sets with finite perimeter in metric spaces (see [**138**] and [**7**]). In Theorem 5.2.1 we prove that if an open set has regular boundary then its perimeter equals the Minkowski content of the boundary. This result has been proved in [**148**]. Finally, in section 3 the interplay with the Hausdorff measures will be discussed. In the special case of the Heisenberg group perimeter also equals spherical Hausdorff measure of codimension 1 (see [**82**]).

1. Sets of finite $X-$perimeter

1.1. Introduction. We begin with some preliminary notation. Given a system $X = (X_1, ..., X_m)$ of locally Lipschitz vector fields in $\mathbb{R}^n$ we write for $j = 1, ..., m$

$$X_j(x) = \sum_{i=1}^{n} c_{ji}(x)\partial_i, \quad \text{and} \quad C = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \dots & c_{mn} \end{pmatrix}. \tag{5.1.1}$$

The adjoint operators X_j^*, the divergence div_X and $F(\Omega; \mathbb{R}^m)$ with $\Omega \subset \mathbb{R}^n$ open set have been introduced in chapter 4 section 2.

Definition 5.1.1. The *total $X-$variation* (or *$X-$perimeter*) of a measurable set $E \subset \mathbb{R}^n$ in an open set $\Omega \subset \mathbb{R}^n$ is

$$|\partial E|_X(\Omega) = \sup_{\varphi \in F(\Omega;\mathbb{R}^m)} \int_E \mathrm{div}_X(\varphi)\, dx.$$

The set E is *of finite $X-$perimeter* (or a *$X-$Caccioppoli set*) in Ω if $|\partial E|_X(\Omega) < +\infty$. The set E is *of locally finite $X-$perimeter* in Ω if $|\partial E|_X(U) < +\infty$ for any open set $U \Subset \Omega$.

Remark 5.1.2. Let $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, and let $E \subset \mathbb{R}^n$ be a $X-$Caccioppoli set in Ω. If ∂E is the topological boundary of E it can be easily checked that $|\partial E|_X(\Omega \setminus \partial E) = 0$ and that $|\partial E|_X = |\partial(\mathbb{R}^n \setminus E)|_X$.

When E is an open set with Lipschitz boundary its $X-$ perimeter has the following integral representation.

THEOREM 5.1.3. *Let $E \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $\Omega \subset \mathbb{R}^n$ be an open set. Then*

$$|\partial E|_X(\Omega) = \int_{\partial E\cap\Omega} |Cn| d\mathcal{H}^{n-1}, \tag{5.1.2}$$

where $n(x)$ is the Euclidean normal to ∂E at x and C is the matrix (5.1.1).

Proof. First notice that

$$\mathrm{div}_X(\varphi) = -\sum_{j=1}^m X_j^*\varphi_j = \sum_{j=1}^m\sum_{i=1}^n \partial_i(c_{ji}\varphi_j) = \sum_{i=1}^n \partial_i\Big(\sum_{j=1}^m c_{ji}\varphi_j\Big),$$

and then by the divergence Theorem

$$\begin{aligned}\int_E \mathrm{div}_X(\varphi)\,dx &= \int_E \sum_{i=1}^n \partial_i\Big(\sum_{j=1}^m c_{ji}\varphi_j\Big)\,dx = \int_{\partial E}\sum_{i=1}^n n_i \sum_{j=1}^m c_{ji}\varphi_j\,d\mathcal{H}^{n-1}\\ &= \int_{\partial E}\langle\varphi, Cn\rangle\,d\mathcal{H}^{n-1}.\end{aligned}$$

Thus

$$|\partial E|_X(\Omega) = \sup_{\varphi\in F(\Omega;\mathbb{R}^m)}\int_{\partial E}\langle\varphi, Cn\rangle\,d\mathcal{H}^{n-1} \le \int_{\partial E\cap\Omega}|Cn|\,d\mathcal{H}^{n-1}.$$

We have to prove the converse inequality. The set

$$H = \{x\in\partial E\cap\Omega : n(x) \text{ exists and } Cn(x)\neq 0\}$$

is $\mathcal{H}^{n-1}-$measurable and since ∂E is Lipschitz Cn is a $\mathcal{H}^{n-1}-$measurable function on H. Fix $\varepsilon > 0$. By Lusin Theorem there exists a compact set $K\subset H$ such that $\mathcal{H}^{n-1}(H\setminus K)\le\varepsilon$ and Cn is continuous on K. By Tietze-Urysohn Theorem there exists $\psi\in C_0(\Omega)$ such that

$$\psi(x) = \frac{Cn(x)}{|Cn(x)|}\quad\text{for all } x\in K,\quad\text{and}\quad |\psi(x)|\le 1\quad\text{for all } x\in\Omega.$$

Finally, by Friedrichs regularization there exists $\varphi\in C_0^1(\Omega;\mathbb{R}^m)$ such that $\|\varphi\|_\infty\le 1$ and $\|\varphi-\psi\|_\infty\le\varepsilon$. Thus

$$|\partial E|_X(\Omega)\ge\int_{\partial E}\langle\varphi, Cn\rangle\,d\mathcal{H}^{n-1} = \int_{\partial E}\langle\varphi-\psi, Cn\rangle\,d\mathcal{H}^{n-1} + \int_{\partial E}\langle\psi, Cn\rangle\,d\mathcal{H}^{n-1}.$$

But

$$\int_{\partial E}\langle\varphi-\psi, Cn\rangle\,d\mathcal{H}^{n-1}\ge -\varepsilon\mathcal{H}^{n-1}(\partial E)\max_{x\in\partial E}\|C(x)\|,$$

where $\|C\| = \max_{|n|\le 1}|Cn|$, and

$$\begin{aligned}\int_{\partial E}\langle\psi, Cn\rangle\,d\mathcal{H}^{n-1} &= \int_K\langle\psi, Cn\rangle\,d\mathcal{H}^{n-1} + \int_{H\setminus K}\langle\psi, Cn\rangle\,d\mathcal{H}^{n-1}\\ &= \int_H|Cn|\,d\mathcal{H}^{n-1} - \int_{H\setminus K}|Cn|\,d\mathcal{H}^{n-1} + \int_{H\setminus K}\langle\psi, Cn\rangle\,d\mathcal{H}^{n-1},\end{aligned}$$

where

$$\int_{H\setminus K}|Cn|\,d\mathcal{H}^{n-1}\le\mathcal{H}^{n-1}(H\setminus K)\max_{x\in\partial E}\|C(x)\|\le\varepsilon\max_{x\in\partial E}\|C(x)\|$$

and analogously

$$\int_{H\setminus K} \langle \psi, Cn\rangle \, d\mathcal{H}^{n-1} \geq -\varepsilon \max_{x\in\partial E} \|C(x)\|.$$

Putting together all the estimates we eventually find

$$|\partial E|_X(\Omega) \geq \int_{\partial E\cap\Omega} |Cn| \, d\mathcal{H}^{n-1} - \varepsilon \max_{x\in\partial E} \|C(x)\|(2+\mathcal{H}^{n-1}(\partial E)),$$

and since $\varepsilon > 0$ is arbitrary the claim follows. □

If $\chi_E \in \mathrm{L}^1(\Omega)$ then E is of finite X−perimeter if and only if $\chi_E \in \mathrm{BV}_X(\Omega)$ and moreover $|\partial E|_X(\Omega) = |X\chi_E|(\Omega)$. If E is measurable then it is of locally finite X−perimeter in Ω if and only if $\chi_E \in \mathrm{BV}_{X,\mathrm{loc}}(\Omega)$.

If $E \subset \Omega$ is a set of locally finite X−perimeter the distributional derivative $\mu = X\chi_E$ is a m−vector valued Radon measure (Definition 4.2.1) and $|\mu|(U) = |X\chi_E|(U) = |\partial E|_X(U)$ for any open set $U \Subset \Omega$ (Proposition 4.2.2). By the Polar decomposition Theorem (see [**8**, Corollary 1.29]) there exists a $|\mu|$−measurable function $\nu_E : \Omega \to \mathbb{R}^m$ such that $\mu = \nu_E|\mu|$ and $|\nu_E| = 1$ $|\mu|$−almost everywhere. We shall write $|\mu| = |\partial E|_X$.

Theorem 5.1.4. *Let $E \subset \Omega$ be a set with locally finite X−perimeter. The following generalized Gauss-Green formula holds*

$$\int_\Omega \mathrm{div}_X(\varphi)\, dx = -\int_\Omega \langle \varphi, \nu_E\rangle \, d|\partial E|_X,$$

for all $\varphi \in C^1_0(\Omega; \mathbb{R}^m)$.

The vector ν_E will be called X−*generalized inner normal* of E.

Consider now a Carnot group $(\mathbb{R}^n, \cdot, \delta_\lambda, d)$ with canonical generating vector fields $X = (X_1, ..., X_m)$. If $h \in \mathbb{R}^n$ we denote by $\tau_h : \mathbb{R}^n \to \mathbb{R}^n$ the left translation $\tau_h(x) = h \cdot x$. The integer $Q \geq n$ is the homogeneous dimension of the group defined in (1.7.87). The following proposition describes the invariance properties of the perimeter in Carnot groups.

Proposition 5.1.5. *If $E \subset \mathbb{R}^n$ is a measurable set then for any Borel set $B \subset \mathbb{R}^n$, for all $h \in \mathbb{R}^n$ and $\lambda > 0$:*

(i) $|\partial\tau_h(E)|_X(\tau_h(B)) = |\partial E|_X(B)$;
(ii) $|\partial\delta_\lambda(E)|_X(\delta_\lambda(B)) = \lambda^{Q-1}|\partial E|_X(B)$.

Proof. We shall prove (ii). First notice that, if $\psi \in C^1(\mathbb{R}^n)$, then

$$X_j(\psi \circ \delta_\lambda)(x) = \lambda(X_j\psi)(\delta_\lambda(x)), \tag{5.1.3}$$

for $j = 1, ..., m$ and $\lambda > 0$. Indeed, recall that $\delta_\lambda(x) = (\lambda^{\alpha_1}x_1, ..., \lambda^{\alpha_n}x_n)$ where $\alpha_1 = ... = \alpha_m = 1$ and $\alpha_{m+1}, ..., \alpha_n$ are integers greater or equal than 2. The vector fields are of the form $X_j(x) = \partial_j + \sum_{i=m+1}^n a_{ij}(x)\partial_i$, where $a_{ij}(\delta_\lambda(x)) = \lambda^{\alpha_i-1}a_{ij}(x)$

(see (1.7.84)). Thus

$$\begin{aligned}X_j(\psi\circ\delta_\lambda)(x) &= \partial_j(\psi\circ\delta_\lambda)(x) + \sum_{i=m+1}^{n} a_{ij}(x)\partial_i(\psi\circ\delta_\lambda)(x)\\ &= \lambda\Big[\partial_j\psi(\delta_\lambda(x)) + \sum_{i=m+1}^{n}\lambda^{\alpha_i-1}a_{ij}(x)\partial_i\psi(\delta_\lambda(x))\Big]\\ &= \lambda(X_j\psi)(\delta_\lambda(x)).\end{aligned}$$

Without loss of generality we can assume that $B=\Omega$ is an open set. Take $\varphi\in F(\delta_\lambda(\Omega);\mathbb{R}^m)$. Since the determinant of the Jacobian of $\delta_\lambda(x)$ is λ^Q and $X_j^*=-X_j$, we can write

$$\begin{aligned}\int_{\delta_\lambda(E\cap\Omega)}\mathrm{div}_X(\varphi)\,dx &= \lambda^Q\int_{E\cap\Omega}\mathrm{div}_X(\varphi)(\delta_\lambda(x))\,dx\\ &= \lambda^Q\int_{E\cap\Omega}\sum_{j=1}^{m}(X_j\varphi_j)(\delta_\lambda(x))\,dx = \lambda^{Q-1}\int_{E\cap\Omega}\sum_{j=1}^{m}X_j(\varphi_j\circ\delta_\lambda)\,dx.\end{aligned}$$

Since $\varphi\circ\delta_\lambda\in F(\Omega;\mathbb{R}^m)$ it immediately follows that

$$|\partial\delta_\lambda(E)|_X(\delta_\lambda(\Omega))\le\lambda^{Q-1}|\partial E|_X(A).$$

The converse inequality can be proved in the same way. □

1.2. Coarea formula. In this section we study the coarea formula for vector fields which has been proved in [**89**], [**80**] and [**148**]. A similar coarea formula in the setting of metric spaces has been recently proved also in [**138**]. In the coarea formula a solid integral is expressed as a superposition of surface integrals and the integration measure is the perimeter of the boundary of the level sets of a Lipschitz function. The problem of replacing the perimeter with Hausdorff measures has been recently studied in [**130**] and [**131**].

THEOREM 5.1.6. *Let $X_1,\dots,X_m\in\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ and let $\Omega\subset\mathbb{R}^n$ be an open set. If $f\in\mathrm{BV}_X(\Omega)$ then*

$$|Xf|(\Omega)=\int_{-\infty}^{+\infty}|\partial E_t|_X(\Omega)\,dt, \tag{5.1.4}$$

where $E_t=\{x\in\Omega: f(x)>t\}$.

Moreover, if $X_1,\dots,X_m$ induce on $\mathbb{R}^n$ a continuous metric d and $f\in\mathrm{Lip}(\Omega,d)$ and $u\in\mathrm{L}^1(\Omega)$, then

$$\int_\Omega u\,|Xf|\,dx=\int_{-\infty}^{+\infty}\Big(\int_{\{x\in\Omega: f(x)=t\}}u\,d|\partial E_t|_X\Big)\,dt. \tag{5.1.5}$$

Proof. We begin with the proof of (5.1.4). First notice that the function

$$t\to|\partial E_t|_X(\Omega)=\sup_{\varphi\in F(\Omega;\mathbb{R}^m)}\int_{E_t}\mathrm{div}_X(\varphi)\,dx$$

is measurable being (countable) supremum of measurable functions. Indeed $C_0^1(\Omega;\mathbb{R}^m)$ is separable.

Without loss of generality assume that $f \geq 0$. Then by Fubini-Tonelli Theorem

$$\begin{aligned}|Xf|(\Omega) &= \sup_{\varphi\in F(\Omega;\mathbb{R}^m)} \int_\Omega f \operatorname{div}_X(\varphi)\,dx \\ &= \sup_{\varphi\in F(\Omega;\mathbb{R}^m)} \int_0^{+\infty}\int_{E_t} \operatorname{div}_X(\varphi)\,dx\,dt \\ &\leq \int_0^{+\infty} \sup_{\varphi\in F(\Omega;\mathbb{R}^m)} \int_{E_t} \operatorname{div}_X(\varphi)\,dx\,dt = \int_0^{+\infty} |\partial E_t|_X(\Omega)\,dt.\end{aligned}$$

In order to prove the converse inequality

$$|Xf|(\Omega) \geq \int_{-\infty}^{+\infty} |\partial E_t|_X(\Omega)\,dt \tag{5.1.6}$$

we begin with the case $f \in \mathrm{BV}_X(\Omega)\cap C^1(\Omega)$. The function

$$m(t) = \int_{\Omega\setminus E_t} |Xf|\,dx$$

is differentiable almost everywhere because it is non decreasing. For $h > 0$ let

$$\eta_h(s) = \begin{cases} 1 & \text{if } s \geq t+h \\ \frac{s-t}{h} & \text{if } t < s < t+h \\ 0 & \text{if } s \leq t, \end{cases}$$

and notice that

$$\frac{m(t+h)-m(t)}{h} = \frac{1}{h}\int_{E_t\setminus E_{t+h}} |Xf|\,dx = \int_\Omega |X(\eta_h\circ f)|\,dx.$$

Since $\eta_h \circ f \to \chi_{E_t}$ in $\mathrm{L}^1(\Omega)$ as $h \downarrow 0$, by Proposition 4.2.3

$$m'(t) \geq \liminf_{h\downarrow 0} \int_\Omega |X(\eta_h\circ f)|\,dx \geq |\partial E_t|_X(\Omega).$$

This result holds for almost every $t \in \mathbb{R}$ and integrating

$$\int_\Omega |Xf|\,dx = \int_{-\infty}^{+\infty} m'(t)\,dt \geq \int_{-\infty}^{+\infty} |\partial E_t|(\Omega)\,dt.$$

This proves 5.1.6 if $f \in \mathrm{BV}_X(\Omega)\cap C^1(\Omega)$.

Let now $f \in \mathrm{BV}_X(\Omega)$. By Theorem 4.2.6 there exists a sequence $(f_k)_{k\in\mathbb{N}} \in \mathrm{BV}_X(\Omega)\cap C^1(\Omega)$ such that $f_k \to f$ in $\mathrm{L}^1(\Omega)$ and

$$\lim_{k\to\infty} \int_\Omega |Xf_k|\,dx = |Xf|(\Omega).$$

Let $E_t^k = \{x \in \Omega : f_k(x) > t\}$ and notice that

$$\lim_{k\to\infty} \int_\Omega \int_{-\infty}^{+\infty} |\chi_{E_t^k}(x) - \chi_{E_t}(x)|\,dt\,dx = \lim_{k\to\infty}\int_\Omega |f_k(x) - f(x)|\,dx = 0,$$

and thus $|\chi_{E_t^k}(x) - \chi_{E_t}(x)| \to 0$ in $\mathrm{L}^1(\Omega)$ for almost every $t \in \mathbb{R}$. Again by Proposition 4.2.3

$$\liminf_{k\to\infty} |\partial E_t^k|_X(\Omega) \geq |\partial E_t|(\Omega),$$

and finally, by Fatou Lemma

$$
\begin{aligned}
|Xf|(\Omega) &= \lim_{k\to\infty}\int_\Omega |Xf_k|\,dx = \lim_{k\to\infty}\int_{-\infty}^{+\infty}|\partial E_t^k|_X(\Omega)\,dt \\
&\geq \int_{-\infty}^{+\infty}\liminf_{k\to\infty}|\partial E_t^k|_X(\Omega)\,dt \geq \int_{-\infty}^{+\infty}|\partial E_t|_X(\Omega)\,dt.
\end{aligned}
$$

This ends the proof of (5.1.4)

The next step is to prove that (5.1.4) holds for any Borel set $B\subset\Omega$. Note first that (5.1.4) holds when Ω is replaced by a closed set $F\subset\Omega$. Indeed, the function $t\to|\partial E_t|_X(F)=|\partial E_t|_X(\Omega)-|\partial E_t|_X(\Omega\setminus F)$ is measurable and

$$
\begin{aligned}
|Xf|(F) &= |Xf|(\Omega)-|Xf|(\Omega\setminus F) \\
&= \int_{-\infty}^{+\infty}|\partial E_t|_X(\Omega)\,dt-\int_{-\infty}^{+\infty}|\partial E_t|_X(\Omega\setminus F)\,dt=\int_{-\infty}^{+\infty}|\partial E_t|_X(F)\,dt.
\end{aligned}
$$

Let $B\subset\Omega$ be a Borel set. Since $|Xf|$ is a finite Radon measure on Ω, by [**63**, Theorem 2.2.2] there exist a decreasing sequence of open sets $A_k\subset\Omega$, $k\in\mathbb{N}$, and an increasing sequence of closed sets $F_k\subset\Omega$ such that $F_k\subset B\subset A_k$ for all $k\in\mathbb{N}$ and

$$
\sup_{k\in\mathbb{N}}|Xf|(F_k)=|Xf|(B)=\inf_{k\in\mathbb{N}}|Xf|(A_k).
$$

Define $F=\bigcup_{k=1}^{\infty}F_k$ and $A=\bigcap_{k=1}^{+\infty}A_k$. The functions $t\to|\partial E_t|_X(F),|\partial E_t|_X(A)$ are measurable, being upper and lower envelopes of a countable family of measurable functions. Moreover, by monotone convergence

$$
\begin{aligned}
|Xf|(A) &= \lim_{k\to\infty}|Xf|(A_k)=\lim_{k\to\infty}\int_{-\infty}^{+\infty}|\partial E_t|_X(A_k)\,dt \\
&= \int_{-\infty}^{+\infty}\lim_{k\to\infty}|\partial E_t|_X(A_k)\,dt=\int_{-\infty}^{+\infty}|\partial E_t|_X(A)\,dt,
\end{aligned}
$$

and analogously (5.1.4) holds for F. Since $|Xf|(F)=|Xf|(B)=|Xf|(A)$, it follows that

$$
\int_{-\infty}^{+\infty}|\partial E_t|_X(A\setminus F)\,dt=0,
$$

and $|\partial E_t|_X(A\setminus F)=0$ and a fortiori $|\partial E_t|_X(A\setminus B)=0$ for a.e. $t\in\mathbb{R}$. Hence, $t\to|\partial E_t|_X(B)=|\partial E_t|_X(A)-|\partial E_t|_X(A\setminus B)$ is measurable. Finally

$$
\begin{aligned}
|Xf|(B) &= \inf_{k\in\mathbb{N}}\int_{-\infty}^{+\infty}|\partial E_t|_X(A_k)dt \\
&\geq \int_{-\infty}^{+\infty}\inf_{k\in\mathbb{N}}|\partial E_t|_X(A_k)dt\geq\int_{-\infty}^{+\infty}|\partial E_t|_X(B)dt,
\end{aligned}
$$

and

$$
\begin{aligned}
|Xf|(B) &= \sup_{k\in\mathbb{N}}\int_{-\infty}^{+\infty}|\partial E_t|_X(F_k)dt \\
&\leq \int_{-\infty}^{+\infty}\sup_{k\in\mathbb{N}}|\partial E_t|_X(F_k)dt\leq\int_{-\infty}^{+\infty}|\partial E_t|_X(B)dt.
\end{aligned}
$$

This ends the proof of (5.1.4) for Borel sets.

We notice that if f locally belongs to $\mathrm{W}^{1,1}_X$ then the measure $|Xf|$ is absolutely continuous with respect to the Lebesgue measure and (5.1.4) holds for any measurable set.

We prove (5.1.5). If $f \in \mathrm{Lip}(\Omega, d)$ then by Theorem 2.2.1 $|Xf| \in \mathrm{L}^\infty(\Omega)$ and f locally belongs to $\mathrm{W}^{1,1}_X$. Let $u \in \mathrm{L}^1(\Omega)$ be a non negative function and write $u = \sum_{k=1}^\infty 1/k \chi_{A_k}$ with $A_k \subset \Omega$ measurable with finite measure (see [**61**, Theorem 1.1.7]). By the monotone convergence theorem

$$\begin{aligned}\int_\Omega u\,|Xf|\,dx &= \sum_{k=1}^\infty \frac{1}{k}\int_\Omega \chi_{A_k}|Xf|\,dx = \sum_{k=1}^\infty \frac{1}{k}|Xf|(A_k)\\ &= \sum_{k=1}^\infty \frac{1}{k}\int_{-\infty}^{+\infty} |\partial E_t|_X(A_k)dt = \sum_{k=1}^\infty \frac{1}{k}\int_{-\infty}^{+\infty}\int_\Omega \chi_{A_k} d|\partial E_t|_X\,dt\\ &= \int_{-\infty}^{+\infty}\int_\Omega u\,d|\partial E_t|_X\,dt.\end{aligned}$$

In the general case write $u = u^+ - u^-$ and apply the argument to u^+ and u^-. Since $f \in \mathrm{Lip}(\mathbb{R}^n, d)$ and d is continuous with respect to the Euclidean topology, then f is continuous. It follows that $\partial\{x \in \Omega : f(x) > t\} \subset \{x \in \Omega : f(x) = t\}$. Thus by Remark 5.1.2 the support of the measure $|\partial E_t|_X$ is contained in $\{x \in \Omega : f(x) = t\}$. □

The Hypotheses (H1) and (H2), and the Cases (C1), (C2) and (C3) have been introduced in chapter 2 section 6.

Corollary 5.1.7. *Let $(\mathbb{R}^n, d)$ be the C-C space induced by a family of vector fields $X_1, ..., X_m \in \mathrm{Lip_{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ that satisfy (C1), (C2) or (C3). Assume (H1) and (H2). If $u \in \mathrm{L}^1(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n} u(x)\,dx = \int_0^{+\infty}\Big(\int_{\partial B(0,r)} u(x)\,d\mu_r\Big)dr, \tag{5.1.7}$$

where $\partial B(0, r) = \{x \in \mathbb{R}^n : d(x, 0) = r\}$ and $\mu_r = |\partial B(0, r)|_X$.

Proof. By Theorem 2.6.1 we have $|Xd(x, 0)| = 1$ for a.e. $x \in \mathbb{R}^n$ and by Remark 5.1.2 we can apply formula (5.1.5). □

Corollary 5.1.8. *Let $(\mathbb{R}^n, \cdot, \delta_\lambda, d)$ be a Carnot group with canonical generating vector fields $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and homogeneous dimension Q. If $u \in \mathrm{L}^1(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n} u(x)\,dx = \int_0^{+\infty}\Big(\int_{\partial B(0,1)} u(\delta_r(x))r^{Q-1}d\mu\Big)dr, \tag{5.1.8}$$

where $\mu = |B(0, 1)|_X$.

Remark 5.1.9. Formula (5.1.8) gives an explicit representation of the (unique) surface measure whose existence for Carnot groups was proved in [**67**, Proposition 1.15].

2. Minkowski content

In this section we prove that the perimeter of a $X-$Caccioppoli set is equal to the Minkowski content of its boundary. This result has been established in [**148**]. Let $(\mathbb{R}^n, d)$ be a C-C space associated with the vector fields $X_1, ..., X_m$. The metric d will be assumed to be continuous. Let $K \subset \mathbb{R}^n$ be a closed set and define $d_K(x) = \min_{y\in K} d(x,y)$. If $r > 0$, the $r-$tubular neighborhood of K is

$$I_r(K) = \{x \in \mathbb{R}^n : d_K(x) < r\}.$$

The *upper* and *lower Minkowski content* of K in an open set $\Omega \subset \mathbb{R}^n$ are respectively

$$M^+(K)(\Omega) := \limsup_{r\downarrow 0} \frac{|I_r(K) \cap \Omega|}{2r},$$

$$M^-(K)(\Omega) := \liminf_{r\downarrow 0} \frac{|I_r(K) \cap \Omega|}{2r}.$$

If $M^+(K)(\Omega) = M^-(K)(\Omega)$ this common value will be called the *Minkowski content* of K in Ω and denoted by $M(K)(\Omega)$.

We shall prove that if $K = \partial E$ with $E \subset \mathbb{R}^n$ bounded open set with C^2 boundary the Minkowski content of ∂E equals the perimeter of E. Our proof will work in the following three cases:

(i) $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$.
(ii) $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n) \cap \mathrm{L}^\infty(\mathbb{R}^n)$.
(iii) $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$ and there exists a bounded open set $\Omega_0 \subset \mathbb{R}^n$ such that $E \Subset \Omega_0$ and

$$(1 + \sup_{x\in\Omega_0} \|\mathcal{A}(x)\|)\mathrm{diam}(E) < \min_{x\in E, y\in\partial\Omega_0} |x - y|, \tag{5.2.9}$$

where $\mathcal{A} = C^T$ is the matrix of the vector fields (according to (1.1.1) and (5.1.1)).

The key property ensured by (i), (ii) and (iii) is $d^{(k)} \le d$ for all $k \in \mathbb{N}$, being $d^{(k)}$ the Riemmanian distances approximating d which have been constructed in chapter 1 section 2 (see Theorem 1.2.1 for case (iii), Remark 1.2.3 for case (ii) and Remark 1.2.2 for case (i)).

THEOREM 5.2.1. *Assume (i), (ii) or (iii). Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary and such that $\mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0$. Then $M(\partial E)(\Omega)$ exists and*

$$M(\partial E)(\Omega) = |\partial E|_X(\Omega). \tag{5.2.10}$$

Proof. The proof will be written for case (iii). We prove separately that

$$M^-(\partial E)(\Omega) \ge |\partial E|_X(\Omega), \tag{5.2.11}$$

$$M^+(\partial E)(\Omega) \le |\partial E|_X(\Omega). \tag{5.2.12}$$

The former inequality follows from the lower semicontinuity of the perimeter. The latter one requires the Riemannian approximation.

Define the signed distance

$$\varrho(x) = \begin{cases} d_{\partial E}(x) & \text{if } x \in E \\ -d_{\partial E}(x) & \text{if } x \in \mathbb{R}^n \setminus E, \end{cases}$$

and if $\varepsilon > 0$ let for $x \in \mathbb{R}^n$

$$\varphi_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon}\varrho(x) + \frac{1}{2} & \text{if } |\varrho(x)| < \varepsilon \\ 1 & \text{if } \varrho(x) \geq \varepsilon \\ 0 & \text{if } \varrho(x) \leq -\varepsilon. \end{cases}$$

The function $\varrho : (\mathbb{R}^n, d) \to [0, +\infty)$ is $1-$Lipschitz and by Theorem 2.2.1 (d is continuous) $|X\varrho(x)| \leq 1$ for a.e. $x \in \mathbb{R}^n$. Then

$$\begin{aligned} |X\varphi_\varepsilon|(\Omega) &= \frac{1}{2\varepsilon}\int_{\Omega\cap\{|\varrho|<\varepsilon\}} |X\varrho|\, dx \\ &\leq \frac{1}{2\varepsilon}|\{x \in \Omega : |\varrho(x)| < \varepsilon\}| = \frac{|I_\varepsilon(\partial E) \cap \Omega|}{2\varepsilon}. \end{aligned}$$

As $\varphi_\varepsilon \to \chi_E$ in $\mathrm{L}^1(\Omega)$, by Proposition 4.2.3

$$|\partial E|_X(\Omega) \leq \liminf_{\varepsilon\downarrow 0} |X\varphi_\varepsilon|(\Omega) \leq M^-(\partial E)(\Omega).$$

This proves (5.2.11).

We turn to (5.2.12). Let $X^{(k)}$, $k \in \mathbb{N}$, be the family of $m + n$ vector fields defined in (1.2.15) which generates a metric $d^{(k)}$ of Riemannian type. Let C_k be the matrix of the coefficients of $X^{(k)}$ as in (5.1.1). By (5.2.9) and Theorem 1.2.1

$$d(x, y) = \sup_{k\in\mathbb{N}} d^{(k)}(x, y) \tag{5.2.13}$$

for all x, y belonging to a neighborhood of E. We notice here that only the inequality $d(x, y) \geq d^{(k)}(x, y)$ will be needed. Such an inequality holds in cases (i) and (ii) by Remarks 1.2.2 and 1.2.3.

Let $d^{(k)}_{\partial E}(x) = \min_{y\in\partial E} d^{(k)}(x, y)$ and define

$$\varrho_k(x) = \begin{cases} d^{(k)}_{\partial E}(x) & \text{if } x \in E \\ -d^{(k)}_{\partial E}(x) & \text{if } x \in \mathbb{R}^n \setminus E. \end{cases}$$

Since ∂E is of class C^2 the function ϱ_k is of class C^1 in a neighborhood of ∂E. This is a classical result in Riemannian Geometry. Moreover, by Theorem 2.6.1 $|X^{(k)}\varrho_k(x)| = 1$ in this neighborhood.

Now define the upper and lower Minkowski contents

$$M^+_k(\partial E)(\Omega) := \limsup_{r\downarrow 0} \frac{|\{x \in \Omega : |\varrho_k(x)| < r\}}{2r},$$

$$M^-_k(\partial E)(\Omega) := \liminf_{r\downarrow 0} \frac{|\{x \in \Omega : |\varrho_k(x)| < r\}}{2r}.$$

By (5.2.13) $|\varrho_k| \leq |\varrho|$ and thus $\{x \in \Omega : |\varrho(x)| < r\} \subset \{x \in \Omega : |\varrho_k(x)| < r\}$. It follows that

$$M^+(\partial E)(\Omega) \leq M^+_k(\partial E)(\Omega). \tag{5.2.14}$$

We shall soon prove that

$$M^+_k(\partial E)(\Omega) = M^-_k(\partial E)(\Omega) = |\partial E|_k(\Omega). \tag{5.2.15}$$

Here and in the sequel we write $|\partial E|_k(\Omega) := |\partial E|_{X^{(k)}}(\Omega)$.

By Proposition 5.1.3

$$\begin{aligned}\lim_{k\to\infty}|\partial E|_k(\Omega) &= \lim_{k\to\infty}\int_{\Omega\cap\partial E}|C_k n|\,d\mathcal{H}^{n-1}\\ &= \int_{\Omega\cap\partial E}|Cn|\,d\mathcal{H}^{n-1} = |\partial E|_X(\Omega).\end{aligned} \tag{5.2.16}$$

In fact, $C_k(x)\to C(x)$ for all $x\in\mathbb{R}^n$, C being the matrix of the vector fields $X_1,...,X_m$ as in (5.1.1). Thus, by (5.2.14) and (5.2.15)

$$M^+(\partial E)(\Omega)\le\lim_{k\to\infty}M_k^+(\partial E)(\Omega)=\lim_{k\to\infty}|\partial E|_k(\Omega)=|\partial E|_X(\Omega).$$

This completes the proof of the Theorem if we prove (5.2.15).

Now k is fixed. Let $E_s=\{x\in\mathbb{R}^n:\varrho_k(x)>s\}$. Since $|X^{(k)}\varrho_k|=1$ in a neighborhood of ∂E by the Coarea formula (5.1.5)

$$\frac{|\{x\in\Omega:|\varrho_k(x)|<t\}|}{2t}=\frac{1}{2t}\int_{\{|\varrho_k|<t\}\cap\Omega}|X^{(k)}\varrho_k|\,dx=\frac{1}{2t}\int_{-t}^{+t}|\partial E_s|_k(\Omega)\,ds.$$

If we show that

$$\lim_{t\to 0}|\partial E_t|_k(\Omega)=|\partial E|_k(\Omega) \tag{5.2.17}$$

then (5.2.15) is proved.

We consider first the case $\Omega=\mathbb{R}^n$ and $t>0$. By Theorem 5.1.4

$$\int_{E_t\setminus E}\mathrm{div}_{X^{(k)}}(X^{(k)}\varrho_k)\,dx=\int_{\mathbb{R}^n}\langle X^{(k)}\varrho_k,\nu_{E_t}\rangle\,d|\partial E_t|_k-\int_{\mathbb{R}^n}\langle X^{(k)}\varrho_k,\nu_E\rangle d|\partial E|_k,$$

and by (5.1.2)

$$\nu_E=\frac{C_k n_E}{|C_k n_E|}=\frac{X^{(k)}\varrho_k}{|X^{(k)}\varrho_k|},$$

where $n_E(x)=\frac{\nabla\varrho_k(x)}{|\nabla\varrho_k(x)|}$ is the Euclidean normal to ∂E at x. An analogous representation formula holds for ν_{E_t}. Thus, since $|X^{(k)}\varrho_k|=1$ in a neighborhood of ∂E

$$\begin{aligned}\int_{E_t\setminus E}\mathrm{div}_{X^{(k)}}(X^{(k)}\varrho_k)\,dx&=\int_{\mathbb{R}^n}\langle X^{(k)}\varrho_k,\frac{X^{(k)}\varrho_k}{|X^{(k)}\varrho_k|}\rangle\,d|\partial E_t|_k-\int_{\mathbb{R}^n}\langle X^{(k)}\varrho_k,\frac{X^{(k)}\varrho_k}{|X^{(k)}\varrho_k|}\rangle\,d|\partial E|_k\\&=|\partial E_t|_k(\mathbb{R}^n)-|\partial E|_k(\mathbb{R}^n).\end{aligned}$$

Since $\mathrm{div}_{X^{(k)}}(X^{(k)}\varrho_k)\in\mathrm{L}^1$ in a neighborhood of ∂E, the first term converges to zero when $t\downarrow 0$, and we deduce that $|\partial E_t|_k(\mathbb{R}^n)\to|\partial E|_k(\mathbb{R}^n)$ as $t\downarrow 0$. Then (5.2.17) is proved and this concludes the proof if $\Omega=\mathbb{R}^n$.

We finally consider an arbitrary open set $\Omega\subset\mathbb{R}^n$. Since $\chi_{E_t}\to\chi_E$ both in $\mathrm{L}^1(\Omega)$ and in $\mathrm{L}^1(\mathbb{R}^n\setminus\overline{\Omega})$, by Proposition 4.2.3

$$\begin{aligned}|\partial E|_k(\Omega)&\le\liminf_{t\downarrow 0}|\partial E_t|_k(\Omega),\\ |\partial E|_k(\mathbb{R}^n\setminus\overline{\Omega})&\le\liminf_{t\downarrow 0}|\partial E_t|_k(\mathbb{R}^n\setminus\overline{\Omega}).\end{aligned} \tag{5.2.18}$$

From

$$|\partial E_t|_k(\Omega)\le|\partial E_t|_k(\overline{\Omega})=|\partial E_t|_k(\mathbb{R}^n)-|\partial E_t|_k(\mathbb{R}^n\setminus\overline{\Omega}),$$

using the second inequality (5.2.18) and the convergence in $\mathbb{R}^n$ established above we find

$$\begin{aligned}\limsup_{t\downarrow 0}|\partial E_t|_k(\Omega) &\le |\partial E|_k(\mathbb{R}^n) - \liminf_{t\downarrow 0}|\partial E_t|_k(\mathbb{R}^n\setminus\overline{\Omega})\\ &\le |\partial E|_k(\mathbb{R}^n) - |\partial E|_k(\mathbb{R}^n\setminus\overline{\Omega})\\ &\le |\partial E|_k(\Omega) + |\partial E|_k(\partial\Omega)\\ &\le |\partial E|_k(\Omega) + \int_{\partial E\cap\partial\Omega}|C_k n_E|\,d\mathcal{H}^{n-1} = |\partial E|_k(\Omega).\end{aligned}$$

Here we used $\mathcal{H}^{n-1}(\partial E\cap\partial\Omega)=0$. Together with the first inequality in (5.2.18) this proves that $|\partial E_t|_k(\Omega)\to|\partial E|_k(\Omega)$ as $t\downarrow 0$. The case $t\to 0^-$ is quite similar and the theorem is completely proved. □

Remark 5.2.2. The approximation technique used in the proof of Theorem 5.2.1 is "Riemannian" only from the metric point of view. The metric $d^{(k)}$ is Riemannian but the measure of the $k-$tubular neighborhood of ∂E and the surface area of ∂E have been computed respectively by Lebesgue measure and perimeter instead of using the Riemannian volume and area. The reason is that these latter diverge.

The Riemannian quadratic form inducing on $\mathbb{R}^n$ the metric $d^{(k)}$ is given by $g_k(x) = (C_k(x)^T C_k(x))^{-1}$ and if $E\subset\mathbb{R}^n$ is a bounded open set with regular boundary, the Riemannian volume and area of ∂E are respectively

$$\begin{aligned}\mathrm{Vol}_k(E) &= \int_E\sqrt{\det g_k(x)}\,dx = \int_E\frac{1}{\sqrt{\det(C_k(x)^T C_k(x))}}\,dx,\\ \mathrm{Area}_k(\partial E) &= \int_{\partial E}\langle g_k^{-1}n(x),n(x)\rangle^{1/2}\sqrt{\det g_k(x)}\,d\mathcal{H}^{n-1}\\ &= \int_{\partial E}\frac{|C_k n(x)|}{\sqrt{\det(C_k(x)^T C_k(x))}}\,d\mathcal{H}^{n-1},\end{aligned}$$

where $n(x)$ is the *Euclidean* normal to ∂E at x.

Consider, for instance, in $\mathbb{R}^3$ the Heisenberg vector fields $X=\partial_x+2y\partial_t$ and $Y=\partial_y-2x\partial_t$. It can be easily checked that $\det(C_k^T C_k) = 1/k^2(1+1/k^2)[4(x^2+y^2)+1+1/k^2]$ and

$$\lim_{k\to\infty}\mathrm{Vol}_k(E) = \lim_{k\to\infty}\mathrm{Area}_k(\partial E) = +\infty.$$

Remark 5.2.3. The proof of Theorem 5.2.1 shows that $M_k(\partial E)(\mathbb{R}^n) = |\partial E|_k(\mathbb{R}^n)$ for all $k\in\mathbb{N}$ and that

$$\lim_{k\to\infty}M_k(\partial E)(\mathbb{R}^n) = |\partial E|_X(\mathbb{R}^n)$$

for *any* family of vector fields $X=(X_1,...,X_m)$.

3. Hausdorff measures, regular surfaces and rectifiability

In this section we study the relationship between perimeter and Hausdorff measures defined with the C-C metric. Only few results concerning this problem are known, and mainly in the setting of Carnot groups [**83**] and in particular in the Heisenberg group [**82**]. One of the problems is the lack of a geometric covering theorem: even in the Heisenberg group a covering theorem of Besicovitch type does not seem to hold (see the counterexamples constructed in [**158**] and [**120**]). Anyway, the asymptotic doubling formula for perimeters in metric spaces proved in [**7**] makes possible the spherical differentiation which yields the exact relation between perimeter and spherical Hausdorff measures.

3.1. Hausdorff measures. Let $(\mathbb{R}^n, d)$ be a Carnot group with homogeneous dimension $Q \geq n$. Using the left invariant metric d the following Hausdorff measures can be defined in $\mathbb{R}^n$ (see [**63**, 2.10] and [**133**, chapter 4]). For any $0 \leq s \leq Q$, $\delta > 0$ and for any $A \subset \mathbb{R}^n$ let

$$\mathcal{H}^s_{d,\delta}(A) = \inf\Big\{\gamma(s)\sum_{j=1}^{+\infty}(\mathrm{diam}(E_j))^s : A \subset \bigcup_{j=1}^{+\infty} E_j,\ \mathrm{diam}(E_j) \leq \delta\Big\},$$

$$\mathcal{S}^s_{d,\delta}(A) = \inf\Big\{\gamma(s)\sum_{j=1}^{+\infty}(\mathrm{diam}(B_j))^s : A \subset \bigcup_{j=1}^{+\infty} B_j,\ \mathrm{diam}(B_j) \leq \delta,\ B_j \text{ closed balls}\Big\},$$

and then define

$$\mathcal{H}^s_d(A) = \sup_{\delta>0} \mathcal{H}^s_{d,\delta}(A), \quad \text{and} \quad \mathcal{S}^s_d(A) = \sup_{\delta>0} \mathcal{S}^s_{d,\delta}(A).$$

Here, $\mathrm{diam}(E)$ is the diameter of $E \subset \mathbb{R}^n$ in the metric d and $\gamma(s)$ is a suitable normalization constant. The measure $\mathcal{S}^s_d$ is usually called $s-$dimensional *spherical* Hausdorff measure. Since a set $E \subset \mathbb{R}^n$ is contained in a closed ball B with radius $\mathrm{diam}(E)$ and since $\mathrm{diam}(B)$ is twice the radius of B (this is true in all Carnot groups) it easily follows that $\mathcal{H}^s_d(A) \leq \mathcal{S}^s_d(A) \leq 2^s \mathcal{H}^s_d(A)$.

Let $X = (X_1, ..., X_m)$ be a system of generators of the Lie algebra of the group and if $E \subset \mathbb{R}^n$ is a measurable set denote by $|\partial E|_X(\mathbb{R}^n)$ its $X-$perimeter in $\mathbb{R}^n$. Sets having the corkscrew property have been defined in Definition 3.1.4.

PROPOSITION 5.3.1. *Let $E \subset \mathbb{R}^n$ be an open set with the corkscrew property. There exists $C > 0$ depending on the homogeneous dimension Q and on the corkscrew constant such that $\mathcal{H}^{Q-1}_d(\partial E) \leq C|\partial E|_X(\mathbb{R}^n)$.*

Proof. By Proposition 3.1.5 there exists $C > 0$ such that for all $x \in \partial E$ and $0 < r \leq r_0$

$$|B(x,r)| \leq C \min\{|B(x,r) \cap E|, |B(x,r) \cap (\mathbb{R}^n \setminus E)|\}. \tag{5.3.19}$$

Fix $0 < r \leq r_0/5$. By Vitali covering Theorem there exists a disjoint sequence of balls $\{B_i = B(x_i, r) : i \in \mathbb{N}\}$ with $x_i \in \partial E$ such that the enlarged family $\{5B_i\}$ covers ∂E. By compactness there exists $N \in \mathbb{N}$ (depending on r) such that

$$\partial E \subset \bigcup_{i=1}^{N} B(x_i, 5r).$$

Thus by (5.3.19)

$$\begin{aligned}\mathcal{H}^{Q-1}_{d,10r}(\partial E) &\leq \gamma(Q-1)\sum_{i=1}^{N}\mathrm{diam}(B(x_i,5r))^{Q-1} \leq \frac{C}{r}\sum_{i=1}^{N}|B(x_i,r)| \\ &\leq \frac{C}{r}\sum_{i=1}^{N}|B(x_i,r)|^{1/Q}\min\{|B(x_i,r)\cap E|, |B(x_i,r)\cap(\mathbb{R}^n\setminus E)|\}^{(Q-1)/Q}.\end{aligned}$$

By the isoperimetric inequality (4.2.20) with $\Omega = B(x_i, r)$ (which is a John domain)

$$\frac{1}{r}|B(x_i,r)|^{1/Q}\min\{|B(x_i,r)\cap E|, |B(x_i,r)\cap(\mathbb{R}^n\setminus E)|\}^{(Q-1)/Q} \le C|\partial E|_X(B(x_i,r)),$$

and hence, since the balls are disjoint

$$\mathcal{H}^{Q-1}_{d,10r}(\partial E) \le C\sum_{i=1}^{N}|\partial E|_X(B(x_i,r)) \le C|\partial E|_X(\mathbb{R}^n).$$

The claim follows letting $r \downarrow 0$. $\square$

REMARK 5.3.2. The proof of Proposition 5.3.1 works in any C-C space provided that the relative isoperimetric inequality for balls holds.

3.2. Regular surfaces. We introduce regular surfaces in Carnot groups. The implicit function Theorem stated in this subsection actually holds for vector fields of "Carnot type" (see [**83**]).

DEFINITION 5.3.3. Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $f : \Omega \to \mathbb{R}$ is said to belong to $C^1_X(\Omega)$ if it is continuous in Ω and the derivatives $X_1 f, ..., X_m f$ exist in distributional sense and are continuous functions.

DEFINITION 5.3.4. A set $S \subset \mathbb{R}^n$ is a $X-$*regular hypersurface* if for all $x \in S$ there exist an open neighborhood $\mathcal{U}$ of x and $f \in C^1_X(\mathcal{U})$ such that
 (i) $|Xf(x)| \ne 0$;
 (ii) $S \cap \mathcal{U} = \{y \in \mathcal{U} : f(y) = 0\}$.

REMARK 5.3.5. If $S \subset \mathbb{R}^n$ is a C^1 hypersurface then it is $X-$regular if and only if it does not contain points which are characteristic with respect to the vector fields X. On the other hand, there are $X-$regular hypersurfaces that are not of class C^1 and not even locally Lipschitz in the Euclidean sense (see [**82**, Remarks 5.9 and 6.6]).

The implicit function theorem we are going to state has been proved in [**83**] and in [**82**] for the special case of the Heisenberg group. We refer to these papers for the proof.

Let $\Omega \subset \mathbb{R}^n$ be a fixed open set such that $0 \in \Omega$ and let $f \in C^1_X(\Omega)$ be such that $f(0) = 0$. Define

$$E = \{x \in \Omega : f(x) < 0\}, \quad S = \{x \in \Omega : f(x) = 0\}.$$

If S is a $X-$regular hypersurface we can without loss of generality assume that $X_1 f(0) > 0$.

THEOREM 5.3.6 (Implicit Function Theorem). *There exists an open neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^n$ such that E is of finite $X-$perimeter in $\mathcal{U}$, $\partial E \cap \mathcal{U} = S \cap \mathcal{U}$ and*

$$\nu_E(x) = -\frac{Xf(x)}{|Xf(x)|} \text{ for all } x \in S \cap \mathcal{U},$$

where ν_E is the generalized inner unit normal given by Theorem 5.1.4. Moreover, there exist an open neighborhood $\mathcal{V} \subset \mathbb{R}^{n-1}$ of 0 and a continuous function $\Phi : \mathcal{V} \to \mathbb{R}^n$ such that $S \cap \mathcal{U} = \{\Phi(\xi) \in \mathcal{U} : \xi \in \mathcal{V}\}$ and the $X-$perimeter has the integral representation

$$|\partial E|_X(\mathcal{U}) = \int_{\mathcal{V}} \frac{|Xf(\Phi(\xi))|}{X_1 f(\Phi(\xi))}\, d\xi.$$

REMARK 5.3.7. The function Φ is continuous. The problem of determining what kind of additional regularity Φ could have seems to be an open problem even in the Heisenberg group (see [**82**]).

3.3. Rectifiability in the Heisenberg groups. In the Heisenberg group the link between perimeter and spherical Heusdorff measures has been investigated in [**82**]. Consider $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ endowed with the algebraic and metric Heisenberg structure. In this subsection we shall denote by X the system of the Heisenberg vector fields (1.8.91), by $x \cdot y$ the product (1.8.89), by δ_λ the dilations (1.8.90) and by $Q = 2n+2$ the homogeneous dimension.

If $E \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $r > 0$ define

$$E_{r,x} = \{y \in \mathbb{R}^n : x \cdot \delta_r(y) \in E\} = \delta_{1/r}(x^{-1} \cdot E)\}.$$

The projection $\pi : \mathbb{R}^n \to \mathbb{R}^{2n}$ is defined by $\pi(x_1, ..., x_{2n+1}) = (x_1, ..., x_{2n})$. If $v \in \mathbb{R}^{2n}$ let

$$\begin{aligned} S^+(v) &= \{x \in \mathbb{R}^n : \langle \pi(x), v\rangle \geq 0\} \\ S^-(v) &= \{x \in \mathbb{R}^n : \langle \pi(x), v\rangle \leq 0\}. \end{aligned} \tag{5.3.20}$$

It can be checked that for any $v \in \mathbb{R}^{2n}$

$$T(v) = S^+(v) \cap S^-(v) = \{x \in \mathbb{R}^n : \langle \pi(x), v\rangle = 0\}$$

is a subgroup of $\mathbb{H}^n$.

DEFINITION 5.3.8. Let $E \subset \mathbb{R}^n$ be a $X-$Caccioppoli set. A point $x \in \mathbb{R}^n$ is said to belong to the *reduced boundary* of E, $x \in \partial^* E$, if

(i) $|\partial E|_X(B(x,r)) > 0$ for all $r > 0$;
(ii) if $\nu_E \in \mathbb{R}^{2n}$ denotes the generalized inward normal given in Theorem 5.1.4 then

$$\nu_E(x) = \lim_{r\downarrow 0} \fint_{B(x,r)} \nu_E \, d|\partial E|_X,$$

and moreover $|\nu_E(x)| = 1$.

The following blow up theorem for sets of finite perimeter in the Heisenberg group at points of the reduced boundary has been proved in [**81**] along the way of De Giorgi classical result. Actually, in [**81**] homogeneous cylinders have been used instead of C-C balls, but these can be used as well.

THEOREM 5.3.9. *If E is a X-Caccioppoli set, $x \in \partial^* E$ and $\nu_E \in \mathbb{R}^{2n}$ is the generalized inward normal, then the characteristic function of $E_{r,x}$ converges in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ as $r \downarrow 0$ to the characteristic function of $S^+(\nu_E(x))$. In addition, for all $R > 0$*

$$\lim_{r\downarrow 0} |\partial E_{r,x}|_X(B(0,R)) = |\partial S^+(\nu_E(x))|_X(B(0,R)) = cR^{2n+1}, \tag{5.3.21}$$

where $c > 0$ is a geometric constant.

DEFINITION 5.3.10. A set $K \subset \mathbb{R}^n$ is X-*rectifiable* if there exists a sequence of X-regular hypersurfaces $(S_i)_{i\in\mathbb{N}}$ such that

$$\mathcal{H}_d^{Q-1}\Big(K \setminus \bigcup_{i\in\mathbb{N}} S_i\Big) = 0. \tag{5.3.22}$$

THEOREM 5.3.11. *If $E \subset \mathbb{R}^n$ is a X-Caccioppoli set then*

(i) *$\partial^* E$ is X-rectifiable, that is $\partial^* E = N \cup \bigcup_{i=1}^{\infty} K_i$, where $\mathcal{H}_d^{Q-1}(N) = 0$ and K_i is a compact subset of a X-regular hypersurface S_i;*

(ii) *$|\partial E|_X = \mathcal{S}_d^{Q-1} \llcorner \partial^* E$ with a suitable choice of $\gamma(Q-1)$.*

Balogh has recently proved in [**17**] that if E is an open set of class C^1 and $C(\partial E)$ denotes the set of characteristic points of ∂E then $\mathcal{H}_d^{Q-1}(C(\partial E)) = 0$. Since points in $\partial E \setminus C(\partial E)$ are in the reduced boundary then from Theorem 5.3.11 the following Corollary immediately follows

COROLLARY 5.3.12. *If $E \subset \mathbb{R}^{2n+1}$ is an open set of class C^1 then $|\partial E|_X(\mathbb{R}^{2n+1}) = \mathcal{S}_d^{Q-1}(\partial E)$.*

The main technical problem in the proof of Theorem 5.3.11 given in [**82**] is proving that the measure $|\partial E|_X$ has support in $\partial^* E$. In order to establish such property the following result, which has been established within the theory of perimeters in metric spaces in [**7**], plays a crucial role. We shall state it in the context of the Heisenberg group.

THEOREM 5.3.13. *Let $E \subset \mathbb{R}^{2n+1}$ be a X-Caccioppoli set. There exist $\tau > 0$ and $k > 0$ such that for $|\partial E|_X$−a.e. $x \in \mathbb{R}^{2n+1}$*

$$\tau < \liminf_{r \downarrow 0} \frac{|\partial E|_X(B(x,r))}{r^{Q-1}} \leq \limsup_{r \downarrow 0} \frac{|\partial E|_X(B(x,r))}{r^{Q-1}} < +\infty$$

and

$$\liminf_{r \downarrow 0} \min \left\{ \frac{|B(x,r) \cap E|}{|B(x,r)|}, \frac{|B(x,r) \cap (\mathbb{R}^{2n+1} \setminus E)|}{|B(x,r)|} \right\} \geq k.$$

CHAPTER 6

An application to a phase transitions model

1. Introduction

In this chapter we apply several results obtained in the previous chapters to the study of a problem of the Calculus of Variations connected to phase transitions models. Consider the family of functionals

$$Q_\varepsilon(u) = \varepsilon \int_\Omega q(x, Du)\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx, \quad \varepsilon > 0, \tag{6.1.1}$$

where Ω is a smooth, bounded open set of $\mathbb{R}^n$, $u : \Omega \to \mathbb{R}$, and $W : \mathbb{R} \to [0, +\infty)$ is a double-well potential that supports two phases of the model (i.e. W has two isolated global minimum points). For the sake of simplicity we assume here $W(u) = u^2(1-u)^2$ but W can be more general (see section 3). The integral perturbation with integrand function $q : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a term that penalizes the formation of interfaces in the model and it may degenerate in the sense that q could vanish on big parts of $\Omega \times \mathbb{R}^n$.

Functionals of type (6.1.1) have arisen in a variety of applications as, for instance, in the study of stable configurations in the context of Van der Waals-Cahn-Hilliard theory of phase transitions (see [**36**], [**96**]). This model can be described by a fluid under isothermal conditions which is confined in a bounded container Ω and whose Gibbs free energy per unit volume is a prescribed non convex function W of the density function u. The space of admissible smooth densities is the class

$$\mathcal{A} = \Big\{ u : \Omega \to [0, 1] : u \in C^1(\Omega),\ \int_\Omega u\, dx = V \Big\},$$

where $0 < V < |\Omega|$ is the given total mass of the fluid in Ω.

In the classic isotropic model to every density u one can associate the energy $\mathcal{E}_\varepsilon(u) = \varepsilon Q_\varepsilon(u)$ where

$$q(x, \xi) = |\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n, \tag{6.1.2}$$

and $\varepsilon > 0$ is a small parameter (see [**96**] for a physical motivation and also [**2**] for a simple nice introduction to the subject). The problem of determining the stable configurations is the study of the variational problem $\inf\{\mathcal{E}_\varepsilon(u) : u \in \mathcal{A}\}$ and the mathematical problem is then to study the asymptotic behaviour as $\varepsilon \downarrow 0$ of the solutions u_ε of these problems or equivalently, as the sets of the solutions agree, the ones of the rescaled problems

$$\inf\{Q_\varepsilon(u) :\ u \in \mathcal{A}\}. \tag{P_ε}$$

A relevant variational convergence which turned out to be very useful to this goal is the $\Gamma-$convergence introduced by De Giorgi (see [**53**] for an introduction to this topic). More precisely, the functional $Q_\varepsilon : \mathcal{A} \to [0, +\infty]$ can be extended, with a slight abuse of notation, to a functional $Q_\varepsilon : \mathrm{L}^1(\Omega) \to [0,+\infty]$ defined $+\infty$ outside $\mathcal{A}$, and now the variational problem is the characterization of $Q = \Gamma(\mathrm{L}^1(\Omega)) - \lim_{\varepsilon\downarrow 0} Q_\varepsilon$.

In the isotropic scalar case, i.e. when q is as in (6.1.2), this variational problem was studied by Gurtin ([**96**]) in some particular situations, who also proposed several conjectures (see also [**97**]). Following a Gurtin's conjecture and using previous $\Gamma-$convergence arguments contained in [**140**] Modica proved in [**139**] that

$$Q(u) = \begin{cases} 2\alpha|\partial E|(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}(\Omega),\ |E \cap \Omega| = V \\ +\infty & \text{otherwise} \end{cases} \tag{6.1.3}$$

where $|\partial E|(\Omega)$ is the *perimeter* of E in Ω, $\mathrm{BV}(\Omega)$ is the classical space of functions with bounded variation in Ω and

$$\alpha = \int_0^1 \sqrt{W(s)}\, ds. \tag{6.1.4}$$

Moreover, Modica also proved the existence of a sequence $(u_{\varepsilon_h})_{h\in\mathbb{N}}$ of solutions of the relaxed problems (P_{ε_h}) strongly converging in $\mathrm{L}^1(\Omega)$ as $\varepsilon_h \downarrow 0$ to a function $u_0 = \chi_E$ solution of the geometric problem

$$\inf\{2\alpha\mathcal{H}^{n-1}(\partial^* E \cap \Omega) :\ \chi_E \in \mathrm{BV}(\Omega),\ |E \cap \Omega| = V\}. \tag{6.1.5}$$

Here $\partial^* E$ is the (Euclidean) essential boundary of E. In particular, this result yields a "selection criterion" singling out a solution u_0 among the infinite collection of the ones of the imperturbated real physical problem

$$\min\Big\{ \int_\Omega W(u)\, dx :\ u \in \mathrm{L}^1(\Omega),\ \int_\Omega u\, dx = V \Big\} \tag{6.1.6}$$

(see [**96**] for a discussion of the physical meaning of this problem).

These results were generalized by Bouchitté ([**29**]) and Owen-Sternberg ([**152**]) to anisotropic functionals Q_ε allowing the function q to be very general but always assuming at least a coercivity property which, in the case when q is a positive quadratic form, i.e.

$$q(x,\xi) = \langle A(x)\xi, \xi\rangle \quad x \in \Omega \ \text{ and } \ \xi \in \mathbb{R}^n, \tag{6.1.7}$$

with $A(x)$ symmetric $n \times n$ matrix, amounts to the existence of a constant $\lambda_0 > 0$ such that

$$\langle A(x)\xi, \xi\rangle \geq \lambda_0|\xi|^2 \quad \text{ for all } x \in \Omega \ \text{ and } \ \xi \in \mathbb{R}^n. \tag{6.1.8}$$

Under this hypothesis Bouchitté proved in [**29**] that there exists a limit solution $u_0 = \chi_E$ which solves the following geometric problem

$$\inf\Big\{2\alpha \int_{\Omega\cap\partial^* E} \langle A(x)\nu_E(x), \nu_E(x)\rangle^{1/2} d\mathcal{H}^{n-1} :\ \chi_E \in \mathrm{BV}(\Omega),\ |E\cap\Omega| = V\Big\} \tag{6.1.9}$$

where ν_E denotes the *generalized outward normal* to E (see [**8**]) and α is the constant (6.1.4).

The isotropic vector valued-case, i.e. if $u : \Omega \to \mathbb{R}^p$ and $q : \Omega \times \mathbb{R}^{pn} \to [0,+\infty)$ is as in (6.1.2), was studied by Sternberg ([**166**]), by Kohn and Sternberg ([**118**]), by Baldo [**16**] and by Fonseca and Tartar ([**69**]). The anisotropic vector-valued case was also studied by Barroso and Fonseca ([**19**]). Moreover, other variations of the functionals Q_ε in (6.1.1) have been studied by Alberti and Bellettini ([**3**] and [**4**]), Alberti, Bouchitté and Seppecher ([**5**]) and Fonseca and Mantegazza ([**68**]). Finally, Baldi and Franchi have recently proved in [**15**] a $\Gamma-$convergence result for the family of functionals $(Q_\varepsilon)_\varepsilon$ in the special case when $q(x,\xi)=|\xi|^2\omega(x)^{1-2/n}$ and ω is a strong $A_\infty-$weight on $\mathbb{R}^n$.

In this chapter we prove $\Gamma-$convergence results in the case when $q:\Omega\times\mathbb{R}^n\to[0,+\infty)$ is a non negative quadratic form, i.e. q is as in (6.1.7) but the matrix $A(x)$ is only non negative definite on Ω; in particular (6.1.8) may fail. More precisely, suppose that there exists a $m\times n$ matrix $C(x)=[c_{ji}(x)]$ with Lipschitz continuous entries on $\mathbb{R}^n$ such that

$$A(x)=C(x)^TC(x)\quad\text{for all }\ x\in\Omega, \tag{6.1.10}$$

where C^T denotes the transposed matrix of C. Clearly, according to (5.1.1) the rows of the matrix C defines a family of vector fields which, after a Riemmanian approximation, will be the key tool in our proofs. In chapter 4 section 2 the space $\mathrm{BV}_A(\Omega)$ has been defined for any non negative definite matrix A (see (4.2.17)). In a natural way the $A-$*perimeter measure* in Ω of a measurable set $E\subset\mathbb{R}^n$ is

$$|\partial E|_A(\Omega)=|D\chi_E|_A(\Omega). \tag{6.1.11}$$

Now, let $Q:\mathrm{L}^1(\Omega)\to[0,+\infty]$ be the functional

$$Q(u)=\begin{cases}2\alpha|\partial E|_A(\Omega) & \text{if } u=\chi_E\in\mathrm{BV}_A(\Omega),\ |E\cap\Omega|=V\\ +\infty & \text{otherwise,}\end{cases} \tag{6.1.12}$$

where α is the constant (6.1.4).

Then, if Q_ε are the functionals (6.1.1) with q of the form (6.1.7) with A satisfying (6.1.10) we prove that

$$Q=\Gamma(\mathrm{L}^1(\Omega))-\lim_{\varepsilon\downarrow 0}Q_\varepsilon \tag{6.1.13}$$

for every bounded open set $\Omega\subset\mathbb{R}^n$ with boundary of class C^2 (see Theorem 6.3.3 and Remark 6.3.4).

Under the weak assumption (6.1.10) only, the result (6.1.13) does not provide a meaningful selection criterion to single out preferred solutions among the ones of the limit geometric problem

$$\inf\{2\alpha|\partial E|_A(\Omega):E\subset\mathbb{R}^n,\ |E\cap\Omega|=V\} \tag{6.1.14}$$

because a minimizing sequence $(u_{\varepsilon_h})_{h\in\mathbb{N}}$ of the problems (P_{ε_h}) need not be relatively compact in $\mathrm{L}^1(\Omega)$ if A vanishes on big parts of Ω.

Under the hypotheses

(1) X is a family of *Hörmander* or *Grushin's type* vector fields, and
(2) Ω is a bounded open set of class C^2 and a *John domain* in the C-C space $(\mathbb{R}^n,d)$ induced by the vector fields X (see Definition 3.1.1 in chapter 3)

we prove that the relaxed problem of (P_ε) has a solution u_ε in the anisotropic Sobolev space $\mathrm{H}^1_X(\Omega)$, (see (6.3.46) and Theorem 6.4.3). Moreover, a sequence of solutions $(u_{\varepsilon_h})_{h\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^1(\Omega)$, and using the $\Gamma-$convergence result (6.1.13) we show that, up to a subsequence, it strongly converges in $\mathrm{L}^1(\Omega)$ to a solution $u_0=\chi_E$ of problem (6.1.14) (see Theorem 6.5.2).

In section 5 several examples will be given in which all previous hypotheses are satisfied.

2. Preliminary results

First of all we recall the definition of $\Gamma-$convergence. We refer to [**53**] for a general introduction to the subject.

DEFINITION 6.2.1. Let (M, d) be a metric space, and let $F, F_h : M \to [-\infty, +\infty]$, $h \in \mathbb{N}$. F is said to be the $\Gamma-$limit of the sequence $(F_h)_{h\in\mathbb{N}}$, and we shall write $F = \Gamma(M) - \lim_{h\to\infty} F_h$, if the following conditions hold

$$\text{if } x \in M \text{ and } x_h \to x \quad \text{then} \quad F(x) \le \liminf_{h\to\infty} F_h(x_h), \tag{6.2.15}$$

$$\forall x \in M\ \exists (x_h)_{h\in\mathbb{N}} \quad \text{such that } x_h \to x \quad \text{and } F(x) \ge \limsup_{h\to\infty} F_h(x_h). \tag{6.2.16}$$

The proof of the following "Reduction Lemma" can be found in [**140**].

LEMMA 6.2.2. *Let (M, d) be a metric space, $F, F_h : M \to [-\infty, +\infty]$, $h \in \mathbb{N}$, $D \subset M$ and $x \in M$. Suppose that:*

(i) *for every $y \in D$ there exists a sequence $(y_h)_{h\in\mathbb{N}} \subset M$ such that $y_h \to y$ in M and $\limsup_{h\to\infty} F_h(y_h) \le F(y)$;*
(ii) *there exists $(x_h)_{h\in\mathbb{N}} \subset D$ such that $x_h \to x$ and $\limsup_{h\to\infty} F(x_h) \le F(x)$.*

Then there exists $(\bar{x}_h)_{h\in\mathbb{N}} \subset M$ such that $\limsup_{h\to\infty} F_h(\bar{x}_h) \le F(x)$.

Next, we state an approximation theorem for BV_X functions, or better for sets of finite $X-$perimeter, which is necessary in order to bypass the following technical difficulty. In the Euclidean setting one of the main tools in the approximation of a set of finite perimeter in Ω by means of sets with regular boundary in $\mathbb{R}^n$ (not only in Ω) is the property of a function $u \in \mathrm{BV}(\Omega) \cap \mathrm{L}^\infty(\Omega)$ to be extendible to a function $\widetilde{u} \in \mathrm{BV}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\mathbb{R}^n)$ with $|D\widetilde{u}|(\partial\Omega) = 0$, if Ω has Lipschitz boundary (see [**139**, Lemma 1] and [**166**, Lemma 1]). It is not known if such a property does hold for $\mathrm{BV}_X(\Omega)$ functions. Nevertheless, the following Proposition can be proved (see [**149**]). $X = (X_1, ..., X_m)$ is a given system of locally Lipschitz vector fields.

PROPOSITION 6.2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary, and let $E \subset \Omega$ be a measurable set such that $|\partial E|_X(\Omega) < +\infty$ and $0 < |E| < |\Omega|$. Then there exists a sequence $(E_h)_{h\in\mathbb{N}}$ of open sets of $\mathbb{R}^n$ such that*

(i) *E_h is bounded and ∂E_h is of class C^∞ for all $h \in \mathbb{N}$;*
(ii) *$E_h \to E$ in $\mathrm{L}^1(\Omega)$;*
(iii) *$|\partial E_h|_X(\Omega) \to |\partial E|_X(\Omega)$;*
(iv) *$\mathcal{H}^{n-1}(\partial E_h \cap \partial\Omega) = 0$ for all $h \in \mathbb{N}$;*
(v) *$|E_h \cap \Omega| = |E|$ for all $h \in \mathbb{N}$.*

Since we are working in a bounded region the vector fields may be assumed globally bounded and Lipschitz continuous. Precisely, we assume that there exists $L > 0$ such that

$$|X_j(x)| \le L \quad \text{and} \quad |X_j(x) - X_j(y)| \le L|x - y| \tag{6.2.17}$$

for all $x, y \in \mathbb{R}^n$ and $j = 1, ..., m$.

Let $\sigma > 0$ and consider the family of vector fields $X_{\sigma,\eta} = (X_1^\eta, ..., X_m^\eta, \sigma\partial_1, ..., \sigma\partial_n)$ where $X_j^\eta = J_\eta * X_j$ and and $(J_\eta)_{\eta>0}$ is a family of mollifiers. Under assumptions

(6.2.17) we proved in chapter 1 section 2 that for any $\sigma > 0$ there exists $\eta_\sigma > 0$ such that

$$\sum_{j=1}^{m} \langle X_j(x), \xi\rangle^2 \le \sigma^2 |\xi|^2 + \sum_{j=1}^{m} \langle X_j^{\eta_\sigma}(x), \xi\rangle^2 \tag{6.2.18}$$

for all $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$. We shall write

$$X_\sigma = X_{\sigma,\eta_\sigma}. \tag{6.2.19}$$

The coefficients of the vector fields X_σ are of class C^∞ and if d_σ is the C-C metric induced by them then the C-C space $(\mathbb{R}^n, d_\sigma)$ is a complete Riemannian manifold (see chapter 1 section 2 and Theorem 1.4.2).

3. The results of Γ−convergence

This section deals with the Γ−convergence results. First, we introduce the functionals involved. Let $W \in C^2(\mathbb{R})$ be a function with two "wells" of equal depth

$$W(0) = W(1) = 0, \quad W(s) > 0 \ \text{ if } \ s \neq 0, 1, \quad W''(0) > 0, \ W''(1) > 0. \tag{6.3.20}$$

Let X be a given system of locally Lipschitz continuous vector fields in $\mathbb{R}^n$. Fix a bounded open set $\Omega \subset \mathbb{R}^n$ and for $\varepsilon > 0$ define the functionals $F_\varepsilon, F : \mathrm{L}^1(\Omega) \to [0, +\infty]$

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \Big(\varepsilon |Xu|^2 + \frac{1}{\varepsilon} W(u)\Big)\, dx & \text{if } u \in \mathrm{H}^1_X(\Omega) \\ +\infty & \text{if } u \in \mathrm{L}^1(\Omega) \setminus \mathrm{H}^1_X(\Omega), \end{cases}$$

and

$$F(u) = \begin{cases} 2\alpha \|\partial E\|_X(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_X(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

where $\alpha = \int_0^1 \sqrt{W(s)}\, ds$.

Let $0 < V < |\Omega|$, introduce the set of admissible functions

$$\mathcal{A}_V = \Big\{ u \in \mathrm{L}^1(\Omega) : \int_\Omega u\, dx = V, \ u \ge 0 \text{ a.e. in } \Omega \Big\}, \tag{6.3.21}$$

and let I_V be the *indicator function* of $\mathcal{A}_V$, i.e. the function which takes the value 0 on $\mathcal{A}_V$ and $+\infty$ outside. Finally, define

$$G_\varepsilon = F_\varepsilon + I_V \quad \text{and} \quad G = F + I_V. \tag{6.3.22}$$

Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$ and let $G_h = G_{\varepsilon_h}$, $F_h = F_{\varepsilon_h}$.

Theorem 6.3.1. *Suppose that $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, let $W \in C^2(\mathbb{R})$ be as in* (6.3.20) *and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Then*

$$G = \Gamma(\mathrm{L}^1(\Omega)) - \lim_{h\to\infty} G_h,$$

i.e. by definition

$$\forall u \in \mathrm{L}^1(\Omega) \text{ and } \forall (u_h) \subset \mathrm{L}^1(\Omega) \text{ if } u_h \to u \text{ in } \mathrm{L}^1(\Omega) \text{ then } G(u) \le \liminf_{h\to\infty} G_h(u_h), \tag{6.3.23}$$

$$\forall u \in \mathrm{L}^1(\Omega)\, \exists (u_h) \subset \mathrm{L}^1(\Omega) \text{ such that } u_h \to u \text{ in } \mathrm{L}^1(\Omega) \text{ and } G(u) \ge \limsup_{h\to\infty} G_h(u_h). \tag{6.3.24}$$

Proof of Theorem 6.3.1. We divide the proof in two steps.

Step 1. Assume that $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, and that the system X induces on $\mathbb{R}^n$ a finite C-C metric d which is continuous in the Euclidean topology. We also assume the following eikonal equation:

(Ek) Let $K \subset \mathbb{R}^n$ be a closed set. If $d_K(x) := \inf_{y\in K} d(x,y)$ then $Xd_K(x) = (X_1 d_K(x), ..., X_m d_K(x)) \in \mathbb{R}^m$ exists and $|Xd_K(x)| = 1$ for a.e. $x \in \mathbb{R}^n \setminus K$.

Under such hypotheses we shall prove the thesis. We begin with (6.3.23). Let $u_h \to u$ in $\mathrm{L}^1(\Omega)$ and assume without loss of generality that $\liminf_{h\to\infty} G_h(u_h) < +\infty$. Possibly extracting a subsequence we can also assume that $u_h(x) \to u(x)$ for a.e. $x \in \Omega$. By Fatou Lemma

$$\int_\Omega W(u(x))\,dx \le \liminf_{h\to\infty} \int_\Omega W(u_h(x))\,dx \le \liminf_{h\to\infty} \varepsilon_h G_h(u_h) = 0.$$

We deduce that $u(x) \in \{0,1\}$ for a.e. $x \in \Omega$ and we can write $u = \chi_E$ where $E := \{x \in \Omega : u(x) = 1\}$. Moreover $u = \chi_E \in \mathcal{A}_V$.

Define the increasing function $\varphi \in C^1(\mathbb{R})$ by $\varphi(t) = \int_0^t \sqrt{W(s)}\,ds$ and put $w(x) = \varphi(u(x))$ and $w_h(x) = \varphi(u_h(x))$. By the coarea formula (5.1.4)

$$\begin{aligned}
|Xw|(\Omega) &= \int_{-\infty}^{+\infty} |\partial\{x \in \Omega : \varphi(u(x)) > t\}|_X(\Omega)\,dt \\
&= \int_0^1 |\partial\{x \in \Omega : u(x) > s\}|_X(\Omega)\varphi'(s)\,ds \\
&= |\partial E|_X(\Omega) \int_0^1 \sqrt{W(s)}\,ds = \frac{1}{2} G(u).
\end{aligned}$$

We can assume that $0 \le u_h(x) \le 1$ and from

$$\int_\Omega |w_h(x) - w(x)|\,dx \le \sup_{t\in[0,1]} |\varphi'(t)| \int_\Omega |u_h(x) - u(x)|\,dx$$

we deduce that $w_h \to w$ in $\mathrm{L}^1(\Omega)$. By Proposition 4.2.3

$$\begin{aligned}
G(u) = 2|Xw|(\Omega) &\le 2\liminf_{h\to\infty} \int_\Omega |Xw_h(x)|\,dx \\
&\le 2\liminf_{h\to\infty} \int_\Omega |Xu_h(x)||\varphi'(u_h(x))|\,dx \\
&\le \liminf_{h\to\infty} \int_\Omega \Big(\varepsilon_h |Xu_h(x)|^2 + \frac{1}{\varepsilon_h} W(u_h(x))\Big)\,dx \\
&\le \liminf_{h\to\infty} G_h(u_h),
\end{aligned}$$

and (6.3.23) follws.

We now turn to the upper bound estimate (6.3.24). By Proposition 6.2.3 and by Lemma 6.2.2 we can reduce to prove (6.3.24) for $u = \chi_E$, $E \subset \mathbb{R}^n$ bounded open set with C^∞ boundary such that $|E \cap \Omega| = V$ and $\mathcal{H}^{n-1}(\partial\Omega \cap \partial E) = 0$.

Define $\varrho : \mathbb{R}^n \to [0, +\infty)$

$$\varrho(x) = \begin{cases} \min_{y\in\partial E} d(x,y) & x \in E \\ -\min_{y\in\partial E} d(x,y) & x \in \mathbb{R}^n \setminus E, \end{cases}$$

and write $\chi_0(t) = \chi_{(0,+\infty)}(t)$. Then $u(x) = \chi_0(\varrho(x))$ for all $x \in \mathbb{R}^n$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be the maximal solution of the Cauchy problem

$$\begin{cases} \chi'(t) = \sqrt{W(\chi(t))} \\ \chi(0) = \frac{1}{2}. \end{cases}$$

It is easy to see that, as $W(0) = W(1) = 0$, χ is a strictly increasing C^2 function such that $\lim_{t\to+\infty} \chi(t) = 1$ and $\lim_{t\to-\infty} \chi(t) = 0$. Moreover there exist $\bar{t} \in \mathbb{R}$, $c_1, c_2 > 0$ such that (see [**166**, (1.21)])

$$1 - \chi(t) \le c_1 e^{-c_2 t}, \quad \text{for all } t \ge \bar{t}. \tag{6.3.25}$$

Fix $\varepsilon > 0$ and write $t_\varepsilon = \vartheta\varepsilon \log 1/\varepsilon$ where $\vartheta \ge 3$ is a constant that will be determined later. Define the function $\Lambda_\varepsilon : \mathbb{R} \to \mathbb{R}$ in the following way

$$\Lambda_\varepsilon(t) = \begin{cases} \chi(t) & \text{if } 0 \le t < \frac{t_\varepsilon}{\varepsilon} \\ p_\varepsilon(t) & \text{if } \frac{t_\varepsilon}{\varepsilon} \le t < \frac{2t_\varepsilon}{\varepsilon} \\ 1 & \text{if } t \ge \frac{2t_\varepsilon}{\varepsilon} \\ 1 - \Lambda_\varepsilon(-t) & \text{if } t < 0. \end{cases}$$

where $p_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the uniquely determined polynomial of degree 3 for which $\Lambda_\varepsilon \in C^{1,1}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{\pm t_\varepsilon/\varepsilon, \pm 2t_\varepsilon/\varepsilon\})$ (see [**24**] for the construction of p_ε) satisfying

$$\|p_\varepsilon - 1\|_{\mathrm{L}^\infty(t_\varepsilon/\varepsilon, 2t_\varepsilon/\varepsilon)} = O(\varepsilon^{2\vartheta-1}) \quad \text{and} \quad \|p'_\varepsilon\|_{\mathrm{L}^\infty(t_\varepsilon/\varepsilon, 2t_\varepsilon/\varepsilon)} = O(\varepsilon^{2\vartheta}). \tag{6.3.26}$$

Now define $\chi_\varepsilon(t) = \Lambda_\varepsilon(t/\varepsilon)$ for $t \in \mathbb{R}$ and $v_\varepsilon(x) = \chi_\varepsilon(\varrho(x))$. It is easy to see that $v_\varepsilon \in \mathrm{H}^{1,\infty}_X(\Omega)$ and $Xv_\varepsilon(x) = \chi'_\varepsilon(\varrho(x)) X\varrho(x)$ a.e. It can be easily checked that (see for instance [**148**, Theorem 9])

$$\lim_{\varepsilon\downarrow 0} \int_\Omega |v_\varepsilon - u| dx = 0, \tag{6.3.27}$$

$$\limsup_{\varepsilon\downarrow 0} F_\varepsilon(v_\varepsilon) \le F(u) = G(u). \tag{6.3.28}$$

The functions v_ε will be now perturbated so as to satisfy the integral constraint without disturbing inequality (6.3.28). Let us begin to show that if $\delta_\varepsilon = \int_\Omega v_\varepsilon\, dx - V$, then $\delta_\varepsilon = O(\varepsilon)$ (see also [**166**, Theorem 1]). Notice that

$$\begin{aligned} \delta_\varepsilon &= \int_\Omega (v_\varepsilon - u)\, dx \\ &= \int_{\{x\in\Omega: 0<\varrho(x)<t_\varepsilon\}} (\chi(\varrho(x)/\varepsilon) - 1)\, dx + \int_{\{x\in\Omega: t_\varepsilon \le \varrho(x) \le 2t_\varepsilon\}} (p_\varepsilon(\varrho(x)/\varepsilon) - 1)\, dx \\ &\quad + \int_{\{x\in\Omega: -t_\varepsilon<\varrho(x)<0\}} (1 - \chi(-\varrho(x)/\varepsilon))\, dx + \int_{\{x\in\Omega: -2t_\varepsilon \le \varrho(x) \le t_\varepsilon\}} (1 - p_\varepsilon(-\varrho(x)/\varepsilon))\, dx. \end{aligned}$$

Because of (6.3.26) if $\vartheta \ge 1$ the second and fourth integrals are $O(\varepsilon)$.

We estimate the first one. By hypothesis (Ek) $|X\varrho| = 1$ a.e. on $\mathbb{R}^n$ and using the coarea formula (5.1.4) we get for $t \geq 0$

$$V^+(t) := |\{x \in \Omega : 0 < \varrho(x) \leq t\}| = \int_0^t |\partial E_s|_X(\Omega)\, ds,$$

where $E_s := \{x \in \mathbb{R}^n : \varrho(x) > s\}$. By the coarea formula (5.1.5) and integrating by parts

$$\begin{aligned}\int_{\{x\in\Omega:0<\varrho(x)<t_\varepsilon\}} (1-\chi(\varrho(x)/\varepsilon))\, dx &= \int_0^{t_\varepsilon} (1-\chi(s/\varepsilon))\|\partial E_s\|_X(\Omega)\, ds \\ &= V^+(t_\varepsilon)(1-\chi(\vartheta\log(1/\varepsilon))) + \frac{1}{\varepsilon}\int_0^{t_\varepsilon} \chi'(s/\varepsilon)V^+(s)ds.\end{aligned}$$

By Theorem 5.2.1 (see also [**11**]) $V^+(t) = Lt + t\delta^+(t)$, where $L = |\partial E|_X(\Omega)$ and $\delta^+ : [0, +\infty) \to \mathbb{R}$ is a function such that

$$\lim_{\varepsilon\downarrow 0} \sup_{s\in[0,t_\varepsilon]} |\delta^+(s)| = 0.$$

By (6.3.25) it follows that $V^+(t_\varepsilon)(1-\chi(\vartheta\log(1/\varepsilon))) = O(\varepsilon)$ if $\vartheta c_2 \geq 1$. Moreover

$$\begin{aligned}\left|\frac{1}{\varepsilon}\int_0^{t_\varepsilon} \chi'(s/\varepsilon)V^+(s)\, ds\right| &\leq \frac{1}{\varepsilon}\int_0^{t_\varepsilon} \sqrt{W(\chi(s/\varepsilon))}V^+(s)\, ds \\ &\leq (L + \sup_{s\in[0,t_\varepsilon]}|\delta^+(s)|)\frac{1}{\varepsilon}\int_0^{t_\varepsilon} s\sqrt{W(\chi(s/\varepsilon))}\, ds \\ &\leq \varepsilon(L + \sup_{s\in[0,t_\varepsilon]}|\delta^+(s)|)\int_0^{+\infty} s\sqrt{W(\chi(s))}\, ds,\end{aligned}$$

and the integral in the last expression is bounded because of (6.3.25). In conclusion if we choose $\vartheta \geq \max\{3, 1/c_2\}$ this ends the proof of $\delta_\varepsilon = O(\varepsilon)$.

Consider now the family of functions $u_\varepsilon = (1+\eta_\varepsilon)v_\varepsilon$ with $\eta_\varepsilon = -\delta_\varepsilon/\int_\Omega v_\varepsilon dx$. Of course, $u_\varepsilon \in \mathrm{H}^{1,\infty}_X(\Omega)$ and $u_\varepsilon \in \mathcal{A}_V$ since $1+\eta_\varepsilon > 0$ and $\int_\Omega u_\varepsilon dx = V$. If we show that

$$\limsup_{\varepsilon\downarrow 0} G_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon\downarrow 0} F_\varepsilon(v_\varepsilon), \tag{6.3.29}$$

statement (6.3.24) will be proved.

Notice that

$$\begin{aligned}G(u_\varepsilon) = &\int_{\{x\in\Omega:|\varrho(x)|\leq 2t_\varepsilon\}} \Big(\varepsilon(1+\eta_\varepsilon)^2|Xv_\varepsilon|^2 + \frac{1}{\varepsilon}W(v_\varepsilon+\eta_\varepsilon v_\varepsilon)\Big)\, dx \\ &+ \frac{1}{\varepsilon}W(1+\eta_\varepsilon)|\{x\in\Omega : \varrho(x) > 2t_\varepsilon\}| \\ \leq\ &\varepsilon\int_\Omega |Xv_\varepsilon|^2\, dx + \frac{\eta_\varepsilon(2+\eta_\varepsilon)}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|\leq 2t_\varepsilon\}} |\Lambda'_\varepsilon(\varrho/\varepsilon)|^2\, dx \\ &+ \frac{1}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|\leq 2t_\varepsilon\}} W(v_\varepsilon+\eta_\varepsilon v_\varepsilon)\, dx + \frac{1}{\varepsilon}W(1+\eta_\varepsilon)|\{x\in\Omega : \varrho(x) > 2t_\varepsilon\}|.\end{aligned}$$

By (6.3.20) and by Taylor's formula

$$\frac{1}{\varepsilon}W(1+\eta_\varepsilon)|\{x\in\Omega : \varrho(x) > 2t_\varepsilon\}| \leq \frac{|\Omega|}{2\varepsilon}W''(\xi_\varepsilon)\eta_\varepsilon^2$$

for some $\xi_\varepsilon \in (1-\eta_\varepsilon, 1+\eta_\varepsilon)$ and hence this term is $O(\varepsilon)$. Moreover, since

$$\int_{\{x\in\Omega:|\varrho(x)|\le 2t_\varepsilon\}} |\Lambda'_\varepsilon(\varrho/\varepsilon)|^2\,dx \le \sup|\chi'|^2|\{x\in\Omega : |\varrho(x)|\le t_\varepsilon\}| \\ + \|p'_\varepsilon\|^2_{\mathrm{L}^\infty(t_\varepsilon/\varepsilon, 2t_\varepsilon/\varepsilon)}|\{x\in\Omega : t_\varepsilon < |\varrho(x)|\le 2t_\varepsilon\}|,$$

and by (6.3.26) we get

$$\lim_{\varepsilon\downarrow 0}\frac{\eta_\varepsilon(2+\eta_\varepsilon)}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|\le 2t_\varepsilon\}} |\Lambda'_\varepsilon(\varrho/\varepsilon)|^2\,dx = 0.$$

In order to prove (6.3.29) it suffices to show that

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|< 2t_\varepsilon\}} (W(u_\varepsilon)-W(v_\varepsilon))\,dx = 0.$$

Indeed, by the Mean Value Theorem there exists $\tau > 0$ such that

$$\frac{1}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|< 2t_\varepsilon\}} |W(u_\varepsilon)-W(v_\varepsilon)|\,dx \le \frac{|\eta_\varepsilon|}{\varepsilon}|\{x\in\Omega : |\varrho(x)| < 2t_\varepsilon\}|\sup_{s\in[0,1+\tau]}|W'(s)|,$$

and the last quantity approaches to zero as $\varepsilon\downarrow 0$.

Step 2. We prove the thesis under the only assumption $X_1,...,X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. Indeed $X = (X_1,...,X_m)$ may be assumed to satisfy (6.2.17). For $\sigma > 0$ let X_σ be the family of vector fields defined in (6.2.19), i.e.

$$X_\sigma = (X_1^{\eta_\sigma},...,X_m^{\eta_\sigma},\sigma\partial_1,...,\sigma\partial_n) \equiv (X_1^\sigma,...,X_{m+n}^\sigma).$$

Now, $X_j^\sigma \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ for all $j = 1,...,m+n$, these vector fields are bounded on $\mathbb{R}^n$ and by (6.2.18)

$$\sum_{j=1}^m \langle X_j(x),\xi\rangle^2 \le \sum_{j=1}^{m+n}\langle X_j^\sigma(x),\xi\rangle^2 \quad \text{for all } x,\xi\in\mathbb{R}^n. \tag{6.3.30}$$

The C-C distance d_σ induced on $\mathbb{R}^n$ by X_σ is a Riemannian metric and since the vector fields are bounded $(\mathbb{R}^n, d_\sigma)$ is a complete metric space. We notice that by Theorem 2.6.1 the family X_σ satisfies the eikonal hypothesis (Ek).

Therefore the first step of the proof does apply to the functionals $G_\varepsilon^\sigma : \mathrm{L}^1(\Omega) \to [0,+\infty]$

$$G_\varepsilon^\sigma(u) = \begin{cases} \varepsilon\displaystyle\int_\Omega |X_\sigma u|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx & \text{if } u\in \mathrm{H}^1_{X_\sigma}(\Omega)\cap\mathcal{A}_V \\ +\infty & \text{otherwise.}\end{cases} \tag{6.3.31}$$

Precisely, for all $\sigma > 0$

$$\Gamma(\mathrm{L}^1(\Omega)) - \lim_{\varepsilon\downarrow 0} G_\varepsilon^\sigma = G^\sigma, \tag{6.3.32}$$

where $G^\sigma : \mathrm{L}^1(\Omega) \to [0,+\infty]$ is the functional

$$G^\sigma(u) = \begin{cases} 2\alpha|\partial E|_{X_\sigma}(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_{X_\sigma}(\Omega)\cap\mathcal{A}_V \\ +\infty & \text{otherwise.}\end{cases} \tag{6.3.33}$$

By the vector fields' form

$$\mathrm{H}^1_{X_\sigma}(\Omega) = \mathrm{H}^1(\Omega) \subset \mathrm{H}^1_X(\Omega), \quad \text{for all } \sigma > 0,$$

and then by (6.3.30)

$$G_\varepsilon(u) \le G_\varepsilon^\sigma(x), \quad \text{for all } u \in \mathrm{L}^1(\Omega) \text{ and for all } \varepsilon, \sigma > 0. \tag{6.3.34}$$

Let $G', G'' : \mathrm{L}^1(\Omega) \to [0, +\infty]$ be respectively the lower and upper $\Gamma-$limits of $(G_\varepsilon)_{\varepsilon>0}$ (see [**53**, chapter 4]), i.e. if $u \in \mathrm{L}^1(\Omega)$

$$G'(u) = \Gamma(\mathrm{L}^1(\Omega)) - \liminf_{\varepsilon \downarrow 0} G_\varepsilon(u),$$
$$G''(u) = \Gamma(\mathrm{L}^1(\Omega)) - \limsup_{\varepsilon \downarrow 0} G_\varepsilon(u).$$

Then, from [**53**, Proposition 6.7], (6.3.34) and (6.3.32)

$$G'(u) \le G''(u) \le G^\sigma(u) \quad \text{for all } u \in \mathrm{L}^1(\Omega) \text{ and for all } \sigma > 0. \tag{6.3.35}$$

We claim that

$$G(u) \le G'(u) \quad \text{for all } u \in \mathrm{L}^1(\Omega). \tag{6.3.36}$$

Indeed, by [**53**, Proposition 8.1] we have to prove that for every $u \in \mathrm{L}^1(\Omega)$, for every sequence $(u_h)_{h\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ strongly converging to u in $\mathrm{L}^1(\Omega)$ and for every sequence $(\varepsilon_h)_{h\in\mathbb{N}}$ of real numbers such that $\varepsilon_h \downarrow 0$

$$G(u) \le \liminf_{h\to\infty} G_{\varepsilon_h}(u_h),$$

and this can be done exactly as in the first step of the proof where only the coarea formula (5.1.4) is involved.

Define

$$\mathcal{D} = \{\chi_E : E \subset \mathbb{R}^n \text{ bounded open set}, \partial E \in C^\infty, |E \cap \Omega| = V, \mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0\},$$

and notice that $\mathcal{D} \subset \mathrm{BV}_{X_\sigma}(\Omega)$ for all $\sigma > 0$. If $u = \chi_E \in \mathcal{D}$ then from (5.1.2)

$$G^\sigma(u) = 2\alpha |\partial E|_{X_\sigma}(\Omega) = 2\alpha \int_{\partial E \cap \Omega} |C^\sigma n| \, d\mathcal{H}^{n-1}, \tag{6.3.37}$$

where $C^\sigma(x)$ is the $(m+n) \times n$ matrix of the coefficients of the vector fields X_j^σ's as in (5.1.1), and n is the Euclidean normal to ∂E.

In particular, from (6.3.37) we get for all $u = \chi_E \in \mathcal{D}$

$$\lim_{\sigma\downarrow 0} G^\sigma(u) = 2\alpha \int_{\partial E\cap\Omega} |Cn| \, d\mathcal{H}^{n-1} = G(u), \tag{6.3.38}$$

being $C(x)$ the matrix of the coefficients of the vector fields X_j's. On the other hand, from (6.3.36), (6.3.35) and (6.3.38)

$$G(u) \le G'(u) \le G''(u) \le G(u) \quad \text{for all } u \in \mathcal{D},$$

whence

$$G(u) = \Gamma(\mathrm{L}^1(\Omega)) - \lim_{\varepsilon\downarrow 0} G_\varepsilon(u) \quad \text{for all } u \in \mathcal{D}. \tag{6.3.39}$$

Applying (6.3.36), (6.3.39), Proposition 6.2.3 and Lemma 6.2.2 we finally find

$$G = \Gamma(\mathrm{L}^1(\Omega)) - \lim_{\varepsilon\downarrow 0} G_\varepsilon.$$

$\square$

The last result in this section deals with the Γ−convergence of functionals defined with degenerate quadratic forms. Let $A(x)$ be a symmetric, semidefinite positive matrix and consider the functionals $Q, Q_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$ defined as

$$Q_\varepsilon(u) = \begin{cases} \varepsilon \int_\Omega \langle ADu, Du\rangle \, dx + \frac{1}{\varepsilon} \int_\Omega W(u)\, dx & \text{if } u \in C^1(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases} \tag{6.3.40}$$

and

$$Q(u) = \begin{cases} 2\alpha |\partial E|_A(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_A(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases} \tag{6.3.41}$$

where V, $\mathcal{A}_V$, W and α are as in Theorem 6.3.1.

The following Lemma gives a sufficient condition for the factorization property (6.1.10). Its proof can be found in [**167**, Theorem 5.2.3].

LEMMA 6.3.2. *Let $A(x)$ be a symmetric, non negative $n \times n-$matrix with entries of class $C^2(\mathbb{R}^n)$ and assume there exists $\Lambda_0 > 0$ such that*

$$|\langle \frac{\partial^2 A}{\partial x_i^2}(x)\xi, \xi\rangle| \le \Lambda_0 |\xi|^2 \quad \textit{for all } x, \xi \in \mathbb{R}^n \textit{ and } i = 1, ..., n. \tag{6.3.42}$$

Then there exists a symmetric $n \times n-$matrix $C(x)$ with Lipschitz continuous entries such that $A(x) = C(x)^T C(x)$ for all $x \in \mathbb{R}^n$.

THEOREM 6.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary and let $A(x)$ be a symmetric, positive semidefinite $n \times n-$matrix , i.e. $\langle A(x)\xi, \xi\rangle \ge 0$ for all $x, \xi \in \mathbb{R}^n$. Suppose that A has C^2 entries and satisfies* (6.3.42). *Moreover, assume that there exist $C \ge 1$, $u_0 > 0$ and $p \ge 1$ such that*

$$C^{-1}|u|^p \le W(u) \le C|u|^p \quad \textit{for all } |u| \ge u_0. \tag{6.3.43}$$

Then

$$Q = \Gamma(\mathrm{L}^1(\Omega)) - \lim_{\varepsilon \downarrow 0} Q_\varepsilon. \tag{6.3.44}$$

REMARK 6.3.4. When the matrix A is *positive definite* on Ω, i.e. there exists $\lambda_0 > 0$ such that $\langle A(x)\xi, \xi\rangle \ge \lambda_0|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$ Theorem 6.3.3 is well known under the only hypothesis of continuity of the matrix entries (see [**29**] and [**23**]).

Proof of Theorem 6.3.3.By Lemma 6.3.2 there exists a $n \times n$ matrix $C(x)$ with Lipschitz continuous entries such that $A(x) = C(x)^T C(x)$ for all $x \in \mathbb{R}^n$. Let $X_1, ..., X_n$ be the family of vector fields whose coefficients are the rows of the matrix $C(x)$ as in (5.1.1). By Propostion 4.2.4 we can write the functionals Q_ε and Q as follows

$$Q_\varepsilon(u) = \begin{cases} \varepsilon \int_\Omega |Xu|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega W(u)\, dx & \text{if } u \in C^1(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$Q(u) = \begin{cases} 2\alpha |\partial E|_X(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_X(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise.} \end{cases}$$

By a general $\Gamma-$convergence result (see [**53**, Proposition 6.11]) (6.3.44) holds if and only if

$$Q = \Gamma(\mathrm{L}^1(\Omega)) - \lim_{\varepsilon \downarrow 0} \mathrm{sc}^-(\mathrm{L}^1(\Omega)) Q_\varepsilon, \tag{6.3.45}$$

where $\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$ is the *relaxed functional* of Q_ε with respect to the topology of $\mathrm{L}^1(\Omega)$.

Recalling Theorem 6.3.1 we only have to prove that for every $\varepsilon > 0$

$$\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) = G_\varepsilon(u) = \begin{cases} \varepsilon \displaystyle\int_\Omega |Xu|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx & \text{if } u \in \mathrm{H}^1_X(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise.} \end{cases} \tag{6.3.46}$$

The inequality $\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) \geq G_\varepsilon(u)$ follows at once by a well known characterization of the relaxed functional (see, for instance, [**53**, Proposition 3.6]) and by the lower semicontinuity of G_ε with respect to the topology of $\mathrm{L}^1(\Omega)$. We claim that

$$\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) \leq G_\varepsilon(u) \quad \text{for all } u \in \mathrm{L}^1(\Omega). \tag{6.3.47}$$

If $G_\varepsilon(u) = +\infty$ there is nothing to prove. Let $u \in \mathrm{H}^1_X(\Omega) \cap \mathcal{A}_V$ be such that $G_\varepsilon(u) < +\infty$. The growth condition (6.3.43) implies $u \in \mathrm{L}^p(\Omega)$. Since $u \in \mathrm{H}^1_X(\Omega)$ by Theorem 4.1.2 there exists a sequence $(v_h)_{h\in\mathbb{N}} \subset C^1(\Omega) \cap \mathrm{H}^1_X(\Omega)$ such that $v_h \to u$ in $\mathrm{H}^1_X(\Omega)$. Moreover, as $u \in \mathrm{L}^p(\Omega)$ and the technique of approximation by convolution is involved, it is not restrictive to assume that $v_h \to u$ in $\mathrm{L}^p(\Omega)$. Let $c_h = \int_\Omega u\, dx / \int_\Omega v_h\, dx$ and define $u_h = c_h v_h$. Then $u_h \in \mathrm{H}^1_X(\Omega) \cap \mathcal{A}_V$, $u_h \to u$ in $\mathrm{H}^1_X(\Omega)$ and

$$u_h \to u \quad \text{in } \mathrm{L}^p(\Omega). \tag{6.3.48}$$

By (6.3.43), (6.3.48) and Carathéodory continuity Theorem (see [**53**, Example 1.22])

$$\lim_{h\to\infty} \int_\Omega W(u_h)\, dx = \int_\Omega W(u)\, dx.$$

Eventually

$$\begin{aligned} \mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) &\leq \liminf_{h\to\infty} \left(\varepsilon \int_\Omega |Xu_h|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(u_h)\, dx \right) \\ &\leq \varepsilon \int_\Omega |Xu|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx = G_\varepsilon(u). \end{aligned}$$

This proves (6.3.47). As a consequence, (6.3.46) and (6.3.45) do hold. □

4. Convergence of minima and minimizers

In this section we study existence and asymptotic behavior of minima and minimizers of the functionals G_ε and Q_ε defined in (6.3.22) and (6.3.40). To this purpose we recall the following fundamental variational property of $\Gamma-$convergence (see [**53**, Corollary 7.20]).

THEOREM 6.4.1. *Let (M, ϱ) be a metric space and let $F, F_h : M \to [0, +\infty]$ be such that $F = \Gamma(M) - \lim_{h\to\infty} F_h$. Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$, and let $(u_h)_{h\in\mathbb{N}} \subset M$ be a relatively compact sequence of ε_h-minimizers, i.e. $F_h(u_h) \leq \inf_M F_h + \varepsilon_h$ for all $h \in \mathbb{N}$. Then*

(i) $\displaystyle\min_{u\in M} F(u) = \lim_{h\to\infty} \inf_{u\in M} F_h(u)$;

(ii) *every cluster point* $u \in M$ *of* $(u_h)_{h\in\mathbb{N}}$ *is a minimum of* F, *i.e.* $F(u) = \min_{v\in M} F(v)$.

In order to apply Theorem 6.4.1 a fundamental tool will be the compact embedding of $\mathrm{H}^{1,p}_X(\Omega)$ in $\mathrm{L}^p(\Omega)$ which has been discussed in chapter 4. Here we shall proceed somehow axiomatically. An open set $\Omega \subset \mathbb{R}^n$ will be said to support the $\mathrm{H}^{1,p}_X(\Omega)-$compact embedding, $1 \le p \le +\infty$, if

$(\mathcal{C})_p$ the embedding $\mathrm{H}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ is compact.

In the Euclidean case the compact embedding is known to imply a Poincaré inequality. Following the same proof an analogous result for vector fields can be obtained. Anyway, we notice that assumptions ensuring $(\mathcal{C})_p$) usually also ensure the Poincaré inequality.

Proposition 6.4.2. *Let* $X = (X_1, ..., X_m)$ *be a family of Lipschitz vector fields on* $\mathbb{R}^n$ *that connect the space. Let* $\Omega \subset \mathbb{R}^n$ *be a connected bounded open set. If* $(\mathcal{C})_p$ *holds for* $1 \le p < +\infty$ *then there exists* $C > 0$ *such that*

$$\int_\Omega |u - u_\Omega|^p \, dx \le C \int_\Omega |Xu|^p \, dx \tag{6.4.49}$$

for all $u \in \mathrm{H}^{1,p}_X(\Omega)$, *where* $u_\Omega := \fint_\Omega u \, dx$.

Let G_ε be as in (6.3.22). The first result of this section is the existence of minima for the functionals G_ε and the compactness of the family of such minima.

Theorem 6.4.3. *Let* $X = (X_1, ..., X_m)$ *be a family of Lipschitz vector fields on* $\mathbb{R}^n$ *that connect the space, let* $\Omega \subset \mathbb{R}^n$ *be a connected, bounded open set such that the compact embedding* $(\mathcal{C})_2$ *holds, and finally let* $W : \mathbb{R} \to \mathbb{R}$ *be a function satisfying* (6.3.43) *for some* $p > 2$. *Then for all* $\varepsilon > 0$ *there exists* $u_\varepsilon \in \mathcal{A}_V$ *such that*

$$G_\varepsilon(u_\varepsilon) = \min_{u\in \mathrm{L}^1(\Omega)} G_\varepsilon(u). \tag{6.4.50}$$

If, in addition, Ω *supports the compact embedding* $(\mathcal{C})_1$, *then the family* $\{u_\varepsilon : \varepsilon > 0\}$ *is relatively compact in* $\mathrm{L}^1(\Omega)$.

Let G be the functional defined in (6.3.22). Choosing $M = \mathrm{L}^1(\Omega)$, $F_h = G_{\varepsilon_h}$ and $F = G$ in Theorem 6.4.1 and taking into account Theorem 6.3.1 and Theorem 6.4.3 we get the following Corollary.

Corollary 6.4.4. *Let* X, Ω *and* W *be as in Theorem 6.4.3. Moreover, assume that* Ω *is of class* C^2 *and* W *satisfies* (6.3.20). *Let* $(\varepsilon_h)_{h\in\mathbb{N}}$ *be a sequence of real numbers such that* $\varepsilon_h \downarrow 0$. *Then:*

(i) *there exists* $\min_{u\in\mathrm{L}^1(\Omega)} G(u) = \lim_{h\to\infty} \min_{u\in\mathrm{L}^1(\Omega)} G_{\varepsilon_h}(u)$;

(ii) *if* $(u_h)_{h\in\mathbb{N}}$ *is a sequence of minimizers of* $(G_{\varepsilon_h})_{h\in\mathbb{N}}$ $(G_{\varepsilon_h}(u_h) = \min_{u\in\mathrm{L}^1(\Omega)} G_{\varepsilon_h}(u))$ *then there exist a subsequence* $(u_{h_j})_{j\in\mathbb{N}}$ *and a function* $u_0 = \chi_E \in \mathrm{BV}_X(\Omega)$ *such that* $u_{h_j} \to u_0$ *in* $\mathrm{L}^1(\Omega)$ *and* $G(u_0) = \min_{u\in\mathrm{L}^1(\Omega)} G(u)$.

Proof of Theorem 6.4.1.The proof can be essentially carried out as in [**139**] and we shall only sketch the main steps.

The existence of $u_\varepsilon \in \mathcal{A}_V$ such that (6.4.50) holds can be proved by the direct method of Calculus of Variations. To this aim we have to check that $G_\varepsilon : \mathrm{L}^1(\Omega) \to$

$[0,+\infty]$ is lower semicontinuous and coercive (see, for instance, [**53**, Theorem 1.15]). The lower semicontinuity and the coercivity follow as in the classic case by the compact embedding $(\mathcal{C})_2$, by the Poincaré inequality (6.4.49) and by Fatou Lemma.

Let us prove that the family of minima $\{u_\varepsilon : \varepsilon > 0\}$ is relatively compact in $\mathrm{L}^1(\Omega)$. Define $\varphi \in C^1(\mathbb{R})$ by $\varphi(t) = \int_0^t \sqrt{W(s)}\,ds$, and let $v_\varepsilon(x) := \varphi(u_\varepsilon(x)) \in \mathrm{H}^1_X(\Omega)$. By (6.3.43) and arguing as in [**139**, Proposition 3, proof] we get the existence of two positive constants c_3, c_4 such that

$$\int_\Omega v_\varepsilon\,dx \le c_3|\Omega| + c_4 G_\varepsilon(u_\varepsilon) \quad \text{for all } \varepsilon \in (0,1),$$

and moreover

$$\int_\Omega |Xv_\varepsilon|\,dx = \int_\Omega \varphi'(u_\varepsilon)|Xu_\varepsilon|\,dx \le \frac{1}{2}\int_\Omega \Big(\varepsilon|Xu_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\,dx = \frac{1}{2}G_\varepsilon(u_\varepsilon).$$

If we show that $G_\varepsilon(u_\varepsilon) \le C < +\infty$ for all $\varepsilon > 0$ and for some $C > 0$, then the set $\{v_\varepsilon : \varepsilon > 0\}$ is bounded in $\mathrm{H}^{1,1}_X(\Omega)$ und hence relatively compact in $\mathrm{L}^1(\Omega)$ by the compact embedding $(\mathcal{C})_1$. The function

$$w_\varepsilon(x) = \begin{cases} 1 & \text{if } x_1 \le \delta_\varepsilon - \varepsilon \\ \frac{1}{2} + \frac{1}{2\varepsilon}(x_1 - \delta_\varepsilon) & \text{if } \delta_\varepsilon - \varepsilon < x_1 < \delta_\varepsilon + \varepsilon \\ 0 & \text{if } x_1 \ge \delta_\varepsilon + \varepsilon \end{cases}$$

belongs to $\mathrm{H}^1_X(\Omega)$ for all $\varepsilon > 0$ and for all $\delta_\varepsilon \in \mathbb{R}$. Since $0 < V < |\Omega|$, $\delta_\varepsilon \in \mathbb{R}$ can be chosen in such a way that $w_\varepsilon \in \mathcal{A}_V$. If $x \in (\delta_\varepsilon - \varepsilon, \delta_\varepsilon + \varepsilon) \times \mathbb{R}^{n-1} \cap \Omega$ then

$$|Xw_\varepsilon(x)|^2 = \sum_{j=1}^m (X_j w_\varepsilon(x))^2 = \frac{1}{4\varepsilon^2}\sum_{j=1}^m (c_{j1}(x))^2 \le C/\varepsilon^2$$

Moreover $W(w_\varepsilon) \le \sup_{t\in[0,1]} W(t)$ and thus

$$\begin{aligned} G_\varepsilon(w_\varepsilon) &= \int_{\Omega\cap\{\delta_\varepsilon-\varepsilon<x_1<\delta_\varepsilon+\varepsilon\}} \Big(\varepsilon|Xw_\varepsilon|^2 + \frac{1}{\varepsilon}W(w_\varepsilon)\Big)dx \\ &\le \frac{C}{\varepsilon}|\Omega \cap \{\delta_\varepsilon - \varepsilon < x_1 < \delta_\varepsilon + \varepsilon\}| \le C < +\infty. \end{aligned}$$

This proves that $G_\varepsilon(u_\varepsilon) \le C < +\infty$ for all $\varepsilon > 0$.

Since the set $\{v_\varepsilon \in \mathrm{L}^1(\Omega) : \varepsilon > 0\}$ is relatively compact there exist $v \in \mathrm{L}^1(\Omega)$ and $\varepsilon_h \downarrow 0$ such that $v_{\varepsilon_h} \to v$ in $\mathrm{L}^1(\Omega)$. The function φ is strictly increasing and thus there exists $\psi = \varphi^{-1} \in C^1(\mathbb{R})$. Define $u(x) := \psi(v(x))$ and notice that $u_{\varepsilon_h} = \psi(v_{\varepsilon_h})$. Arguing as in [**139**] we finally get $u_{\varepsilon_h} \to u$ in $\mathrm{L}^1(\Omega)$. □

Let V and $\mathcal{A}_V$ be as in (6.3.21) and let Q_ε be the functionals defined in (6.3.40). The second result of this section deals with the compactness of Q_ε's minimizers.

THEOREM 6.4.5. *Let Ω be a connected, bounded open set, let $A(x)$ be a symmetric matrix of functions on $\mathbb{R}^n$ and let $Y = (Y_1, ..., Y_r)$ be a family of Lipschitz continuous vector fields on $\mathbb{R}^n$ that connect the space. Assume that:*

(i) *$A(x)$ has entries of class $C^2(\mathbb{R}^n)$ and satisfies* (6.3.42)*;*
(ii) *$\langle A(x)\xi, \xi\rangle \ge \sum_{j=1}^r \langle Y_j(x), \xi\rangle^2$ for all $x, \xi \in \mathbb{R}^n$;*
(iii) *the compact embeddings $(\mathcal{C})_1$ and $(\mathcal{C})_2$ hold with $X \equiv Y$ relatively to Ω;*
(iv) *the function W in the functional Q_ε satisfies* (6.3.20) *and* (6.3.43).

Let $(\varepsilon_h)_{h\in\mathbb{N}}$ *be a sequence of real numbers such that* $\varepsilon_h \downarrow 0$. *Then every sequence* $(u_h)_{h\in\mathbb{N}}$ *of* ε_h-*minimizers of* Q_ε *(i.e.* $Q_{\varepsilon_h}(u_h) \le \inf_{u\in\mathcal{A}_V} Q_{\varepsilon_h}(u) + \varepsilon_h$*) is relatively compact in* $\mathrm{L}^1(\Omega)$.

Let Q be the functional defined in (6.3.41). Choosing $M = \mathrm{L}^1(\Omega)$, $F_h = Q_{\varepsilon_h}$ and $F = Q$ from Theorem 6.4.1 and Theorem 6.4.5 we get the following Corollary.

COROLLARY 6.4.6. *Let* Ω, A *and* Y *be as in Theorem 6.4.5. Assume that* Ω *has* C^2 *boundary and that* W *satisfies* (6.3.20) *and* (6.3.43). *Let* $(\varepsilon_h)_{h\in\mathbb{N}}$ *be a sequence of real numbers such that* $\varepsilon_h \downarrow 0$. *Then:*

(i) *there exists* $\displaystyle\min_{u\in\mathrm{L}^1(\Omega)} Q(u) = \lim_{h\to\infty} \inf_{u\in\mathrm{L}^1(\Omega)} Q_{\varepsilon_h}(u)$;

(ii) *if* $(u_h)_{h\in\mathbb{N}}$ *is a sequence of* ε_h-*minimizers of* $(Q_{\varepsilon_h})_{h\in\mathbb{N}}$ *then there exist a subsequence* $(u_{h_j})_{j\in\mathbb{N}}$ *and a function* $u_0 = \chi_E \in \mathrm{BV}_A(\Omega)$ *such that* $u_{h_j} \to u_0$ *in* $\mathrm{L}^1(\Omega)$ *and* $Q(u_0) = \min_{u\in\mathrm{L}^1(\Omega)} Q(u)$.

Proof of Theorem 6.4.5.By assumption (i) Lemma 6.3.2 can be applied and arguing as in the proof of Theorem 6.3.3 we conclude that

$$Q_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \Big(\varepsilon|Xu|^2 + \frac{1}{\varepsilon}W(u)\Big)\,dx & \text{if } u \in C^1(\Omega)\cap\mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases}$$

for a suitable family $X = (X_1, ..., X_n)$ of Lipschitz continuous vector fields. Moreover, for every $\varepsilon > 0$ and for all $u \in \mathrm{L}^1(\Omega)$

$$\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) = G_\varepsilon(u),$$

being $\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon$ the relaxed functional of Q_ε with respect to the $\mathrm{L}^1(\Omega)$ topology and G_ε the functional defined in (6.3.22).

On the other hand by assumptions (ii) X can be assumed to satisfy (Xc), and by (iii) $(\mathcal{C})_1$ and $(\mathcal{C})_2$ can be assumed to hold relatively to X and Ω. Theorem 6.4.3 can be applied. As pointed out in the first part of the proof of Theorem 6.4.3 G_ε is coercive with respect to the $\mathrm{L}^1(\Omega)$ topology and from a well-known result of relaxation theory (see, for instance, [**53**, Theorem 3.8]) there exists

$$\min_{u\in\mathrm{L}^1(\Omega)} G_{\varepsilon_h}(u) = \inf_{u\in\mathrm{L}^1(\Omega)} Q_{\varepsilon_h}(u).$$

The thesis follows. □

5. Examples

The compact embedding $(\mathcal{C})_p$ is known to hold when Ω is a John domain in the metric space $(\mathbb{R}^n, d)$, being d the C-C metric induced by the vector fields (see Theorem 4.1.12 in chapter 4). A particular case of Corollary 4.1.13 is the following result.

COROLLARY 6.5.1. *Let* $(\mathbb{R}^n, d)$ *be the C-C space induced by the vector fields* X *and let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set. Assume we are in one of the cases (i), (ii), (iii) or (iv) in Corollary 4.1.8. Then the embedding* $\mathrm{W}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ *is compact for all* $1 \le p < +\infty$.

From Corollary 6.5.1, Theorem 6.4.5 and Corollary 6.4.6 we get the following result. Let Q_ε, Q be as in (6.3.40) and (6.3.41) and let W be a function which satisfies (6.3.20) and (6.3.43).

THEOREM 6.5.2. *Let* $Y \equiv X$, $(\mathbb{R}^n, d)$ *and* Ω *connected, bounded open set of class* C^2 *be one of the cases (i), (ii), (iii) or (iv) in Corollary 4.1.8. Let* $A(x)$ *be a matrix of functions on* $\mathbb{R}^n$. *Assume that:*

(i) $A(x) = C^T(x)C(x)$ *for all* $x \in \Omega$ *where* $C(x)$ *is a* $m \times n$ *matrix with Lipschitz continuous entries on* $\mathbb{R}^n$*;*

(ii) $\langle A(x)\xi, \xi\rangle \geq \sum_{j=1}^{r} \langle Y_j(x), \xi\rangle^2$ *for all* $x, \xi \in \mathbb{R}^n$*;*

Then, if $(u_h)_{h\in\mathbb{N}}$ *is a sequence of* ε_h*−minimizers of* Q_{ε_h} $(Q_{\varepsilon_h}(u_h) \leq \inf_{u\in\mathcal{A}_V} Q_{\varepsilon_h}(u) + \varepsilon_h$ *with* $\varepsilon_h \downarrow 0)$ *then there exists a subsequence* $(u_{h_j})_{j\in\mathbb{N}}$ *and a function* $u_0 = \chi_E \in \mathrm{BV}_A(\Omega)$ *such that* $u_{h_j} \to u_0$ *in* $\mathrm{L}^1(\Omega)$ *and* $Q(u_0) = \min_{u\in\mathrm{L}^1(\Omega)} Q(u)$.

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