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Trace theorems for vector fields

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Abstract. In the setting of Carnot-Carathéodory spaces we prove some trace theorems for Sobolev functions. We consider the trace on a non characteristic surface for Hörmander vector fields of step $r \geq 1$ and the trace on the boundary of a class of domains in the Grushin plane.

1 Introduction

In the last years the theory of the functional spaces related to vector fields has been deeply developed in several directions. Sobolev-Poincaré-type inequalities have been widely studied and applied to the analysis of solutions of second order Partial Differential Equations. An important tool in the study of boundary value problems is the estimate of the trace on the boundary $\partial\Omega$ of a Sobolev function u defined in an open set Ω. Only few results concerning this problem are known in the degenerate setting of vector fields. In this paper we give a contribution to the research in this direction.

In order to introduce our discussion let us recall the following classical result: if $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set with regular boundary $\partial\Omega$, then there exists a constant $C > 0$ such that for any $u \in W^{1,p}(\Omega)$

$$\int_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n-1+ps}}\, d\mathcal{H}^{n-1}(x) d\mathcal{H}^{n-1}(y) \leq C \int_\Omega |\nabla u(x)|^p\, dx, \quad (1)$$

where $s = 1 - \frac{1}{p}$ is the fractional order of differentiability of the trace $u = u_{|\partial\Omega}$.

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We are interested in generalizations of (1) of the form

$$[u]_{s,p}(\partial\Omega) \le C \sum_{j=1}^{m} \Big(\int_{\Omega} |X_j u(x)|^p dx \Big)^{1/p}, \tag{2}$$

for a family of vector fields $X_j = \sum_{k=1}^{n} a_{jk}(x)\partial_k$, $j = 1, \dots, m$, $x \in \mathbb{R}^n$. The left hand side of (2) will be some kind of fractional semi-norm, whose form will be discussed later.

Let us begin with some considerations about the history of the problem. In the paper [F] Franchi studies the trace problem for anisotropic Sobolev spaces related to diagonal vector fields $X_j = \lambda_j(x)\partial_j$, $j = 1, ..., n$, and proves optimal trace estimates on corners of cubes using a semi-norm constructed by a sum of one dimensional fractional derivatives with suitable weights. The proof relies on representation formulas modeled on the geometry of the vector fields and does not seem to work in non diagonal situations.

Berhanu and Pesenson [BP] prove a trace and lifting theorem for a family of Hörmander vector fields of step two. Actually, their result holds only for the trace on a non characteristic surface with one transversal vector field and the "projections" of the other ones satisfying the Hörmander condition of step two relatively to the surface. The approach of this paper is not completely satisfactory since the definition of the semi-norm, which involves increments along the integral curves of the projected vector fields, does not apply, for instance, to simple situations when the projected vector fields vanish on the surface.

Bahouri, Chemin and Xu [BCX] using the Weyl Hörmander calculus prove a lifting theorem and a trace theorem on non characteristic surfaces for Sobolev spaces associated with a system of vector fields of step 2. The case of isolated characteristic points in the framework of the Heisenberg group is also studied.

In the recent remarkable paper [DGN2], which continues the project started in [DGN1], Danielli, Garofalo and Nhieu prove the following trace theorem for Hörmander vector fields: if Ω is a $(\varepsilon, \delta)-$domain and μ is a Borel measure supported in $\partial\Omega$ such that $\mu(B(x, r)) \le C|B(x, r)|/r$ for any Carnot-Carathéodory ball centered at $x \in \partial\Omega$ with radius $0 < r < r_0$, then for any $p > 1$ the space $W^{1,p}_X(\Omega)$ is continuously embedded in $B^{1-\frac{1}{p},p}(\partial\Omega, d\mu)$, where the last Besov space is defined by the semi–norm (4) letting $s = 1-1/p$. Conversely, if Ω is a bounded open set, $|\partial\Omega| = 0$ and μ is a Borel measure supported in $\partial\Omega$ such that $\mu(B(x, r)) \ge C|B(x, r)|/r$ for all $x \in \partial\Omega$ and $0 < r < r_0$, then, given a function $u \in B^{1-1/p,p}(\partial\Omega)$, there exists a function $\mathcal{E}u \in W^{1,p}_X(\Omega)$ which extends u and such that the operator $\mathcal{E}$ is continuous between the expected spaces. The trace result is proved by extending a function $u \in W^{1,p}_X(\Omega)$ to a Sobolev function on the the whole space and then by a restriction technique. The fact that Ω is an extension domain is guaranteed by the subelliptic version of Jones' extension theorem for (ε, δ) domains [GN]. The notion of (ε, δ) or uniform domain in $\mathbb{R}^n$ has been introduced by Martio and Sarvas [MS] and Jones [J]. Several properties of (ε, δ) domains are studied by Väisälä [V].

Finally, although not strictly related to our work, we would like to mention the paper by Hajłasz and Martio [HM] where the trace problem on general subsets of $\mathbb{R}^n$ is treated in the Euclidean setting from a "metric" point of view.

Before stating our results we introduce the basic definitions. Given a family $X = (X_1, ..., X_m)$ of vector fields with $X_j(x) = \sum_{i=1}^n a_{ij}(x)\partial_i$ ($j = 1, ..., m$) and $a_{ij} \in Lip(\mathbb{R}^n)$ ($j = 1, ..., m$, $i = 1, ..., n$), we call *subunit* a Lipschitz continuous curve $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ such that

$$\dot\gamma(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)), \quad \text{and} \quad \sum_{j=1}^m h_j^2(t) \le 1 \quad \text{for a.e. } t \in [0, T],$$

with $h_1, ..., h_m$ measurable coefficients. Define the Carnot-Carathéodory (briefly C-C) distance between the points $x, y \in \mathbb{R}^n$

$$\begin{aligned} d(x, y) = \inf\{T \ge 0 : \ & \text{there exists a subunit path } \gamma : [0, T] \to \mathbb{R}^n \\ & \text{such that } \gamma(0) = x \text{ and } \gamma(T) = y\}. \end{aligned}$$

We define $B(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$, for $x \in \mathbb{R}^n$ and $r > 0$. The function d is finite and is a distance in the following two cases, that are the object of our study:

(1) The family $X_1, \dots, X_m \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the Hörmander condition

$$\operatorname{rank} \mathcal{L}(X_1, ..., X_m)(x) = n \quad \text{for every } x \in \mathbb{R}^n,$$

where $\mathcal{L}(X_1, ..., X_m)$ is the Lie algebra generated by the vector fields.

(2) The vector fields are of the form

$$X_1 = \partial_x, \quad \text{and} \quad X_2 = |x|^\alpha \partial_y \quad \text{in } \mathbb{R}^2, \tag{3}$$

with $\alpha > 0$.

Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary $\partial\Omega$ of class C^1. Denote $\nu(x)$ the Euclidean unit normal to $\partial\Omega$ at $x \in \partial\Omega$ and define

$$|X\nu(x)| = \Big(\sum_{j=1}^m \langle X_j(x), \nu(x)\rangle^2\Big)^{1/2}.$$

A point $x \in \partial\Omega$ is said to be characteristic if $|X\nu(x)| = 0$. The natural surface measure that takes into account characteristic points in the boundary is

$$\mu := |X\nu|\mathcal{H}^{n-1} \llcorner \partial\Omega,$$

that is the $(n-1)-$dimensional Hausdorff measure restricted to the boundary with the weight $|X\nu|$. The measure μ is the variational surface measure associated with the vector fields $X_1, ..., X_m$ and $\mu(\partial\Omega)$ equals the Minkowski content of $\partial\Omega$ in the metric space $(\mathbb{R}^n, d)$ (see [MSC]). Define the fractional semi-norm

$$[u]_{s,p}(\partial\Omega) = \Big(\int_{\partial\Omega\times\partial\Omega} \frac{|u(x)-u(y)|^p}{d(x,y)^{ps}\mu(B(x,d(x,y)))}\, d\mu(x)d\mu(y)\Big)^{1/p}. \tag{4}$$

The fact that the semi–norm (4) gives the correct left hand side in the trace inequality has been discovered in [DGN2]. A “solid” version of (4) has been studied in [M].

In this note we study the trace inequality (2) using the semi-norm (4). We shall first consider the case when the boundary is non characteristic with respect to a family of Hörmander vector fields. We develop a technique inspired by the original paper of Gagliardo [G] which relies upon the possibility of connecting points on the boundary $\partial\Omega$ by means of sub-unit curves lying in Ω. The construction of such paths is a byproduct of a structure theorem for the restriction of C-C balls to non characteristic surfaces, theorem that seems to be of independent interest (see Sect. 2). When the boundary contains characteristic points the analysis is more difficult. Nonetheless, our technique still works in some situations and we focused our attention on the simple but significant case of the Grushin plane. Our main results can be summarized in the following way. Let $1 < p < \infty$ and $s = 1 - \frac{1}{p}$. If

(1) $X_1, \ldots, X_m$ are Hörmander vector fields in $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set whose boundary is of class C^∞ and does not contain characteristic points, or alternatively
(2) X_1, X_2 are of the form (3) and $\Omega \subset \mathbb{R}^2$ is a C^1 bounded open set which is α-admissible (see Definition 1),

then there exist constants $C, \delta_0 > 0$ such that

$$\int\limits_{\partial\Omega\times\partial\Omega\cap\{d(x,y)<\delta_0\}} \frac{|u(x)-u(y)|^p\, d\mu(x)d\mu(y)}{d(x,y)^{ps}\mu(B(x,d(x,y)))} \le C \int\limits_{\Omega} |Xu(x)|^p\, dx \tag{5}$$

for all $u \in C^1(\Omega) \cap C(\overline{\Omega})$.

These trace estimates are optimal. This follows from the extension Theorem 4.1 in [DGN2] whose hypotheses are verified in Corollary 1 and Lemma

6 below. In case (1) if Ω is any smooth bounded set, inequality (5) continues to hold provided the integration in the left hand side takes place on a fixed compact set $K \subset \partial\Omega$ which does not contain characteristic points.

It seems reasonable that both non characteristic domains for Hörmander vector fields and our admissible domains in the Grushin plane enjoy the $(\varepsilon, \delta)-$property. If this were the case our trace theorems could be obtained using the results by Danielli, Garofalo and Nhieu [DGN2]. However, the study of this property probably involves difficulties comparable to those one has to face attempting a direct proof of the trace theorem.

In Sect. 2 we prove the mentioned structure theorem for C-C balls. Section 3 deals with Case (1). Our results are the natural generalization to vector fields of arbitrary step $r \geq 2$ of the trace results obtained in [BP] and [BCX] for vector fields of step 2. Section 4 is devoted to Case (2). The condition of α-admissibility identifies a large class of domains for which the trace theorem holds in relation with the "flatness" of their boundary at characteristic points. At the end of Sect. 4 this condition will be shown to be necessary.

In the paper we will denote by C a generic constant which may change even in a single string of estimates. We write $u \simeq v$ to state that there exist two positive constants c_1 and c_2 such that $c_1 u \leq v \leq c_2 u$. A vector field $X = \sum_{k=1}^{n} a_k(x)\partial_k$ will be identified with the vector function $(a_1(x), \ldots, a_n(x))$.

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2 Structure of balls restricted to non characteristic surfaces

Following the basic ideas contained in the classic paper [NSW] and the generalization in [M], we shall represent (in Theorem 2) C-C balls restricted to non characteristic surfaces by means of suitable exponential maps which are "small perturbations" of the exponential of the commutators of the vector fields (see Lemma 2). These maps enjoy a "factorization property" (see Lemma 1) which is crucial in the proof of the trace theorem.

First we recall that a non characteristic surface can be made flat by a diffeomorphism. A resulting transversal vector field can be orthogonalized and the other ones can be made lie on the surface.

Lemma 1. *Let $\mathcal{U} \subset \mathbb{R}^n$ be a neighborhood of $0 \in \mathbb{R}^n$ and let $X \in C^\infty(\mathcal{U};\mathbb{R}^n)$ be a vector field such that $\langle X(0), e_n\rangle \neq 0$. Let $x_n = g(x_1, \ldots, x_{n-1}) = g(x')$ be a function of class C^∞ such that $g(0) = 0$ and $\nabla g(0) = 0$. Possibly shrinking $\mathcal{U}$, there exists a diffeomorphism $\Phi \in C^\infty(\mathcal{U};\mathbb{R}^n)$ such* *that* $d\Phi(x)X(x) = e_n$ *for all* $x \in \mathcal{U}$ *and* $\Phi(x', g(x')) = (x', 0)$ *for all* $(x', g(x')) \in \mathcal{U}$.

The proof of Lemma 1 can be essentially found in [FW, p. 83] where even less regularity is required.

Remark 1. Let $\widetilde{X}_1, ..., \widetilde{X}_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ satisfy the Hörmander condition and induce the C-C metric $\widetilde{d}$. Write $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and assume the vector fields are of the form

$$\widetilde{X}_j = b_j(x,t)\partial_t + \sum_{i=1}^{n-1} a_{ij}(x,t)\partial_i, \; j = 1, ..., m-1, \quad \widetilde{X}_m = \partial_t.$$

The new family of vector fields

$$X_j = \sum_{i=1}^{n-1} a_{ij}(x,t)\partial_i, \; j = 1, ..., m-1, \quad X_m = \partial_t. \tag{6}$$

still satisfies the Hörmander condition. Moreover, if d is the corresponding C-C metric and $K \subset \mathbb{R}^n$ is a compact set, there exist c_1 and c_2 such that

$$c_1\widetilde{d} \le d \le c_2\widetilde{d} \quad \text{and} \quad c_1|\widetilde{X}u| \le |Xu| \le c_2|\widetilde{X}u|$$

for all $u \in C^1$. A proof of the equivalence between d and $\widetilde{d}$ can be found in [FW, p.87]. Actually, it can be proved that each of the previous equivalences implies the other one (see [HK, Theorem 11.11]).

Consider m vector fields $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ of the form (6) and satisfying the Hörmander condition. We shall write $X_m = T$. For any multi-index $I = (i_1, \dots, i_k)$, $1 \le i_j \le m$ and $k \in \mathbb{N}$, let

$$X_{[I]} = [X_{i_1}, [X_{i_2}, \cdots [X_{i_{k-1}}, X_{i_k}] \cdots]],$$

where $[X, Y]$ denotes the commutator of the vector fields X and Y. If $I = (i_1, \dots, i_k)$ we set $|I| = k$ and we say that the commutator $X_{[I]}$ has *length* or *degree* $d(X_{[I]}) = k$.

For any commutator $Y \neq T$ and for small $s \in \mathbb{R}$ we shall define a map $\exp_T(sY) : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$. We proceed by induction on $d(Y)$. If $d(Y) = 1$ and $Y = X_j$ with $j \in \{1, ..., m-1\}$ define for $x \in \mathbb{R}^{n-1}$

$$\exp_T(sY)(x) = \begin{cases} \exp(-sT)\exp(s(X_j + T))(x) & \text{if } s \ge 0, \\ \exp(s(X_j + T))\exp(-sT)(x) & \\ \quad = \exp_T(|s|Y)^{-1}(x) & \text{if } s < 0. \end{cases} \tag{7}$$

The map is well defined provided x belongs to a compact set and s is small. We also set $\exp_T(sT) = \exp(sT)$. Suppose now $d(Y) = k$, $Y = X_{[J]}$ with $|J| = k$, and $J = (j_1, ..., j_k)$. Set $J' = (j_2, ..., j_k)$ and define

$$\exp_T(sY)(x) = \begin{cases} \exp_T(s^{\frac{k-1}{k}} X_{[J']})^{-1} \exp_T(s^{\frac{1}{k}} X_{j_1})^{-1} & \\ \qquad \cdot \exp_T(s^{\frac{k-1}{k}} X_{[J']}) \exp_T(s^{\frac{1}{k}} X_{j_1})(x) & \text{if } s \geq 0, \\ \exp_T(|s|Y)^{-1}(x) & \text{if } s < 0. \end{cases} \tag{8}$$

Some useful features of the maps $\exp_T$ are described in the following two lemmas. In Lemma 2, which is a generalization of [NSW, Lemma 2.21], we shall use the Campbell-Hausdorff formula

$$\exp(u) \exp(v) = \exp\left(u + v - \frac{1}{2}[u, v] + S(u, v)\right),$$

where u and v are non commuting indeterminates and S is a formal series of commutators of u and v of length at least 3. We refer to the Appendix of [NSW] for a discussion and for the related references.

Lemma 2. *For any commutator $X_{[J]}$, $J = (j_1, \dots, j_k)$, of length $k \geq 1$*

$$\exp_T(sX_{[J]}) = \exp\left(sX_{[J]} + \operatorname{sgn}(s) \sum_{|I|>k} c_{J,I} |s|^{|I|/k} X_{[I]}\right), \tag{9}$$

where the $c_{J,I}$ are suitable constants.

The formal equality (9) means that, if x belongs to a compact set K and $p > k$ is an integer, then

$$\left| \exp_T(sX_{[J]})(x) - \exp\left(sX_{[J]} + \operatorname{sgn}(s) \sum_{k<|I|\leq p} c_{J,I} s^{|I|/k} X_{[I]}\right)(x) \right| \\ \leq C s^{(p+1)/k}.$$

Proof. We proceed by induction. Consider first a commutator of length 1, i.e. a vector field X_j, $j = 1, \dots, m$. Applying the Campbell-Hausdorff formula to (7) we get for $s > 0$

$$\begin{aligned} \exp_T(sX_j) &= \exp(-sT) \exp(s(X_j + T)) \\ &= \exp\Big(- sT + s(X_j + T) + \frac{1}{2} s^2 [T, X_j + T] + \cdots \Big) \\ &= \exp\Big(sX_j + \sum_{|I|>1} c_{(j),I} s^{|I|} X_{[I]}\Big). \end{aligned}$$

For $s<0$ note that

$$\exp_T(sX_j) = \exp_T(|s|X_j)^{-1} = \exp\Big(-|s|X_j - \sum_{|I|>1} c_{(j),I}|s|^{|I|}X_{[I]}\Big).$$

We prove now the inductive step. Recall first that an application of the Campbell-Hausdorff formula asserts that, if u and v are non commuting indeterminates, then

$$\exp(v)^{-1}\exp(u)^{-1}\exp(v)\exp(u) = \exp([u,v]+R),$$

where $R=R(u,v)$ denotes a formal series containing commutators (of u and v) of length at least 3. Let $k>1$, $J=(j_1,...,j_k)$, $J'=(j_2,...,j_k)$ and $s\geq 0$. Let also

$$\begin{aligned} u &= s^{1/k}X_{j_1} + \sum_{|I|>1} c_{(j_1),I}s^{|I|/k}X_{[I]} \qquad \text{and} \\ v &= s^{(k-1)/k}X_{[J']} + \sum_{|I|>k-1} C_{J',I}s^{|I|/k}X_{[I]}. \end{aligned}$$

Note that $[u,v]=sX_{[J]}+\widetilde{R}$, where $\widetilde{R}$ is a series containing commutators of order at least $k+1$ of the original fields. Thus, by the inductive hypothesis

$$\begin{aligned} \exp_T(sX_{[J]}) &= \exp_T(s^{\frac{k-1}{k}}X_{[J']})^{-1}\exp_T(s^{\frac{1}{k}}X_{j_1})^{-1} \\ &\qquad\qquad \cdot \exp_T(s^{\frac{k-1}{k}}X_{[J']})\exp_T(s^{\frac{1}{k}}X_{j_1}) \\ &= \exp(v)^{-1}\exp(u)^{-1}\exp(v)\exp(u) \\ &= \exp([u,v]+R)) \\ &= \exp(sX_{[J]}+\widetilde{R}+R)) \\ &= \exp\Big(sX_{[J]} + \sum_{|I|>k} c_{J,I}s^{|I|/k}X_{[I]}\Big), \end{aligned}$$

for suitable constants $c_{J,I}$. We used the fact that the series R is actually a series of commutators of length at least $k+1$ of the original fields. If $s<0$, formula (9) follows analogously. $\square$

Define for $\lambda>0$ and for any vector field X

$$\begin{aligned} S_1(\lambda,X) &= \exp(\lambda(X-T))\exp(\lambda T), \\ S_2(\lambda,X) &= \exp(-\lambda T)\exp(\lambda(X+T)). \end{aligned} \tag{10}$$

Theorem 1 (Factorization). *Let $Y=X_{[J]}$, $J=(j_1,\dots,j_k)$. The map $\exp_T(sY)$, $s\in\mathbb{R}$, can be factorized as the composition of a finite number of factors of the form $S_1(h|s|^{\frac{1}{k}},\tau X_j)$ and $S_2(h|s|^{\frac{1}{k}},\tau X_j)$, where $\tau\in\{-1,1\}$, $j=1,...,m$ and $1\leq h\leq k$. Moreover, the number of factors depends only on k.*

Proof. Since $S_2(h|s|^{\frac{1}{k}}, \tau X_j) = S_1(h|s|^{\frac{1}{k}}, -\tau X_j)^{-1}$, if we prove the claim for $s > 0$ it will also follows for $s < 0$. Without loss of generality we can suppose $s = 1$. First notice that

$$\begin{aligned} S_1(h, \tau X)\exp(T) &= \exp(h(\tau X - T))\exp(hT)\exp(T) \\ &= \exp(T)\exp(T)^{-1}\exp(-\tau X + T)S_1(h+1, \tau X) \\ &= \exp(T)S_2(1, -\tau X)S_1(h+1, \tau X) \end{aligned} \tag{11}$$

and

$$\begin{aligned} S_2(h, \tau X)\exp(T) &= \exp(hT)^{-1}\exp(h(\tau X + T))\exp(T) \\ &= \exp(T)S_2(h+1, \tau X)\exp(-\tau X - T)\exp(T) \\ &= \exp(T)S_2(h+1, \tau X)S_1(1, -\tau X). \end{aligned} \tag{12}$$

The proof is by induction on $k = d(Y)$. If $k = 1$ the claim follows directly from definition (7) with $h = 1$. Let $k = d(Y) > 1$ and let $Y = X_{[J]}$ with $J = (j_1, ..., j_k)$. If $j_1 \neq m$ the claim follows directly from (8) and the inductive hypothesis on $X_{[J']}$, $J' = (j_2, ..., j_k)$. Suppose $j_1 = m$ and by the inductive hypothesis write

$$\exp_T(X_{[J']}) = \prod_{i=1}^{p} S_{\sigma_i}(h_i, \tau_i X_{j_i})$$

with $\sigma_i \in \{1, 2\}$, $\tau_i \in \{-1, 1\}$, $p \in \mathbb{N}$ less than a constant depending on k, and $1 \leq h_i \leq k - 1$. Write

$$\begin{aligned} \exp_T(X_{[J]}) &= \exp_T(X_{[J']})^{-1}\exp(T)^{-1}\exp_T(X_{[J']})\exp(T) \\ &= \exp_T(X_{[J']})^{-1}\exp(T)^{-1}\prod_{i=1}^{p} S_{\sigma_i}(h_i, \tau_i X_{j_i})\exp(T). \end{aligned}$$

By (11) and (12) $\exp(T)$ can be shifted p times from right to left cancelling $\exp(T)^{-1}$ and the claim follows. $\square$

From now on fix a bounded open set $\Omega_0 \subset \mathbb{R}^n$ and let $\{Y_1, ..., Y_q\}$ be a fixed enumeration of the commutators of length $\leq k$, where k is large enough to ensure that span$\{X_{[I]}(x,t) : |I| \leq k\}$ has dimension n at each point $(x, t) \in \Omega_0$. Assume also that $Y_q = T$.

Introduce the family of multi-indices $\mathcal{I} = \{I = (i_1, \ldots, i_{n-1}) : 1 \leq i_j \leq q - 1\}$. Given a multi-index $I \in \mathcal{I}$, set $d(I) = d(Y_{i_1}) + \cdots + d(Y_{i_{n-1}})$ and for $\widetilde{h} = (h, h_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ "small enough" define

$$\begin{aligned} \Phi_{I,x}(h) &= \exp_T(h_{n-1}Y_{i_{n-1}}) \cdots \exp_T(h_1 Y_{i_1})(x, 0), \\ \widetilde{\Phi}_{I,x}(\widetilde{h}) &= \exp(h_n T)\exp_T(h_{n-1}Y_{i_{n-1}}) \cdots \exp_T(h_1 Y_{i_1})(x, 0) \\ &= (\Phi_{I,x}(h), h_n). \end{aligned} \tag{13}$$

The form of the fields (6) guarantees that $\Phi_{I,x}(h) \in \{(x,t) \in \mathbb{R}^n : t = 0\}$ for $h \in \mathbb{R}^{n-1}$. Let also

$$\|h\|_I = \max_{l=1,\dots,n-1} |h_l|^{1/d(Y_{i_l})} \quad \text{and}$$
$$\lambda_I(x) = \det(Y_{i_1}(x,0), \dots, Y_{i_{n-1}}(x,0)),$$

where the vectors Y_{i_l} are thought of as vectors in $\mathbb{R}^{n-1}$.

If $I \in \mathcal{I}$ define $\widetilde{I} = (I, q)$ and set $d(\widetilde{I}) = d(I) + 1$. If $\widetilde{h} = (h, h_n)$ and $(x,t) \in \Omega_0$ define

$$\|\widetilde{h}\|_{\widetilde{I}} = \max\{\|h\|_I, |h_n|\} \quad \text{and}$$
$$\widetilde{\lambda}_{\widetilde{I}}(x,t) = \det(Y_{i_1}(x,t), \dots, Y_{i_{n-1}}(x,t), Y_n(x,t))$$

where $Y_n = T$ and the vectors are thought of as vectors in $\mathbb{R}^n$.

Let d be the C-C metric induced by the vector fields (6) on $\mathbb{R}^n$ and consider the balls $B((x,0),r) = \{(y,t) \in \mathbb{R}^n : d((x,0),(y,t)) < r\}$ and $\overline{B}(x,r) = \{y \in \mathbb{R}^{n-1} : d((x,0),(y,0)) < r\}$. We now state and prove the structure theorem for the restricted balls $\overline{B}$.

Theorem 2. *Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded open set. There exist $r_0 > 0$ and $0 < a < b < 1$ such that for any $(x,0) \in \Omega_0$, $I \in \mathcal{I}$ and $0 < r < r_0$ such that the inequality*

$$|\lambda_I(x)| r^{d(I)} \geq \frac{1}{2} \max_{J \in \mathcal{I}} |\lambda_J(x)| r^{d(J)} \tag{14}$$

is satisfied, we have

(i) $\frac{1}{4}|\lambda_I(x)| \leq |J_h \Phi_{I,x}(h)| = |J_{\widetilde{h}} \widetilde{\Phi}_{I,x}(\widetilde{h})| \leq 4|\lambda_I(x)|$ *for every* $\|\widetilde{h}\|_{\widetilde{I}} < br$*, where* $J_h \Phi_{I,x}(h) = \det \frac{\partial}{\partial h} \Phi_{I,x}(h)$.
(ii) $B((x,0), ar) \subset \widetilde{\Phi}_{I,x}(\{\|\widetilde{h}\|_{\widetilde{I}} < br\}) \subset B((x,0), r)$.
(iii) $\overline{B}(x, ar) \subset \Phi_{I,x}(\{\|h\|_I < br\}) \subset \overline{B}(x,r)$.
(iv) *The map* $\widetilde{\Phi}_{I,x}$ *is one to one on* $\{\|\widetilde{h}\|_{\widetilde{I}} < br\}$.

Remark 2. Inclusions (iii) for the restricted balls are immediate consequence of (ii) and of the structure (13) of the map $\widetilde{\Phi}$. Indeed, starting from (ii) we get

$$\overline{B}(x, ar) \subset \widetilde{\Phi}_{I,x}(\{\|\widetilde{h}\|_{\widetilde{I}} < br\}) \cap \{t = 0\} = \Phi_{I,x}(\{\|h\|_I < br\}).$$

The opposite inclusion is analogous.

Proof of Theorem 2. Since $\lambda_I(x) = \widetilde{\lambda}_{\widetilde{I}}(x,0)$, if (14) is verified for some $(n-1)-$tuple $I \in \mathcal{I}$ then the $n-$tuple $\widetilde{I} = (I, q)$ satisfies

$$|\widetilde{\lambda}_{\widetilde{I}}(x,0)| r^{d(\widetilde{I})} \geq \frac{1}{2} \max_{J \in \mathcal{I}} |\widetilde{\lambda}_{\widetilde{J}}(x,0)| r^{d(\widetilde{J})}. \tag{15}$$

In [NSW, Theorem 7] it is proved that if $Y_{j_1}, \ldots, Y_{j_n}$ are commutators of degrees $d_1, \ldots, d_n$ which satisfy (15), then the map $\widetilde{\Phi}^*_{I,x}$ defined by $\widetilde{\Phi}^*_{I,x}(\widetilde{h}) = \exp(h_1 Y_{j_1} + \cdots + h_n Y_{j_n})(x,0)$ satisfies (i), (ii) and (iv). Moreover in [M, Lemmas 3.2-3.6] the following is proved. Assume that the exponential of any commutator Y_j can be approximated by a map $E(sY_j)$ in the sense that

$$E(sY_j) = \exp\Big(sY_j + \operatorname{sgn}(s) \sum_{|I|>d(Y_j)} k_{(j),I} |s|^{|I|/d(Y_j)} X_{[I]}\Big),$$

where the $k_{(j),I}$ are constants and assume also that for a n-tuple of commutators $Y_{j_1}, \ldots, Y_{j_n}$ (15) holds at a point $(x,0)$ and for a radius r. Then the map

$$\widetilde{\Phi}_{I,x}(\widetilde{h}) = E(h_n Y_{j_n}) \cdots E(h_1 Y_{j_1})(x,0)$$

satisfies (i), (ii) and (iv). In view of Lemma 2 this assertion can be applied to the map $E = \exp_T$ and the Theorem is proved. We also note that the estimate

$$\mu(B((x,0),r)) \simeq \sum_{I \in \mathcal{I}} |\lambda_I(x)| r^{d(I)} \tag{16}$$

holds. $\square$

Corollary 1. *Let $\Omega \subset \mathbb{R}^n$ be an open set with C^∞ boundary. Let $K \subset \partial\Omega$ be a compact set of non characteristic points with respect to the vector fields $X_1, ..., X_m \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ satisfying the Hörmander condition. If $\mu = \mathcal{H}^{n-1} \llcorner \partial\Omega$, then there exist $r_0 > 0$, $0 < m_1 < m_2$ such that*

$$m_1 \frac{|B(x,r)|}{r} \le \mu(B(x,r)) \le m_2 \frac{|B(x,r)|}{r} \tag{17}$$

for all $x \in K$ and for all $0 < r < r_0$.

Proof. In view of Lemma 1 and Remark 1 $X_1, ..., X_m$ can be assumed to be of the form (6) and $K \subset \partial\Omega \subset \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t = 0\}$. The Lemma follows from (ii) and (iii) in Theorem 2. $\square$

3 Trace for Hörmander vector fields

Let $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be a family of vector fields satisfying the Hörmander condition and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\partial\Omega$ of class C^∞ and non characteristic. In this section we prove that Ω supports a trace theorem.

Recall first the Hardy inequality. Let $0 < r \leq +\infty$. If $1 < p < \infty$ and if $f \in L^p(0,r)$ then

$$\int_0^r \Big(\frac{1}{t}\int_0^t |f(x)|\,dx\Big)^p dt \leq \Big(\frac{p}{p-1}\Big)^p \int_0^r |f(x)|^p\,dx. \tag{18}$$

Next, we shall need the following formula for integration of "radial functions".

Lemma 3. *Let $d : \mathbb{R}^n \to [0,\infty)$ be a Lipschitz function such that $|\{x \in \mathbb{R}^n : d(x) < \lambda\}| = \sigma\lambda^Q$ for some $Q > 0$, $\sigma > 0$, for all $\lambda > 0$, and $|\nabla d(x)| \neq 0$ for a.e. $x \in \mathbb{R}^n$. Then*

$$\int_{\{d(x)<r\}} \varphi(d(x))\,dx = \sigma Q \int_0^r \varphi(\lambda)\lambda^{Q-1}\,d\lambda \tag{19}$$

for all measurable functions $\varphi \geq 0$, $r > 0$.

Proof. For $\varepsilon > 0$ let $g_\varepsilon(x) = \chi_{\{|\nabla d|>\varepsilon\}}(x)$ and by the coarea formula write

$$\int_{\{d(x)<\lambda\}} g_\varepsilon(x)\varphi(d(x))\,dx = \int_0^\lambda \varphi(r)\int_{\{d(x)=r\}} \frac{g_\varepsilon(x)}{|\nabla d(x)|} d\mathcal{H}^{n-1}(x)\,dr.$$

Since $\mathcal{H}^{n-1}(\{d(x)=r\}\cap\{\nabla d(x)=0\}) = 0$ for a.e. $r > 0$, by monotone convergence we get

$$\int_{\{d(x)<\lambda\}} \varphi(d(x))\,dx = \int_0^\lambda \varphi(r)\int_{\{d(x)=r\}} \frac{1}{|\nabla d(x)|} d\mathcal{H}^{n-1}(x)\,dr,$$

for all $\lambda > 0$. Choosing $\varphi = 1$ we find

$$\sigma\lambda^Q = |\{x \in \mathbb{R}^n : d(x) < \lambda\}| = \int_0^\lambda \int_{\{d(x)=r\}} \frac{1}{|\nabla d(x)|} d\mathcal{H}^{n-1}(x)\,dr,$$

and taking the derivative we obtain for a.e. $\lambda > 0$

$$\sigma Q\lambda^{Q-1} = \int_{\{d(x)=\lambda\}} \frac{1}{|\nabla d(x)|} d\mathcal{H}^{n-1}(x),$$

which gives the proof. □

We now prove the basic trace theorem. Let $X_1, ..., X_m$ be a family of smooth vector fields on $\mathbb{R}^n$ satisfying the Hörmander condition, assume they are of the form (6) and let d be the C-C metric induced by them. We shall write $(x,t) \in \mathbb{R}^{n-1}\times\mathbb{R}$ and for the sake of simplicity we contract the notation by writing $x = (x,0)$. Let $\mu = \mathcal{H}^{n-1}\llcorner\{t = 0\}$ be the Lebesgue measure on $\mathbb{R}^{n-1}$.

Theorem 3. *Let $1 < p < \infty$, $s = 1 - \frac{1}{p}$ and let $\mathcal{U} \subset \mathbb{R}^{n-1}$ be a bounded open set. If $\lambda > 0$ and $t_0 > 0$ there exist $C > 0$ and $\delta_0 > 0$, such that*

$$\int\limits_{\mathcal{U}\times\mathcal{U}\cap\{d(x,y)<\delta_0\}} \frac{|u(x,0)-u(y,0)|^p\,dxdy}{d(x,y)^{ps}\mu(B(x,d(x,y)))} \leq C \int\limits_{\mathcal{U}_\lambda\times(0,t_0)} |Xu(x,t)|^p\,dxdt \tag{20}$$

for all $u \in C^1(\mathcal{U}_\lambda \times (0,t_0)) \cap C(\mathcal{U}_\lambda \times [0,t_0))$, where $\mathcal{U}_\lambda = \{y \in \mathbb{R}^{n-1} :$ $\mathrm{dist}(y,\mathcal{U}) < \lambda\}$.

Proof. Let $\mathcal{U} \subset \Omega_0$ for some bounded open set $\Omega_0 \subset \mathbb{R}^n$ and let $k \in \mathbb{N}$ be the minimal length of the commutators which ensures the Hörmander condition on Ω_0. Fix $r_0 > 0$ and $0 < a < b$ by Theorem 2. Define

$$N(p,\delta_0;\mathcal{U}) = \int_{\mathcal{U}\times\mathcal{U}\cap\{d(x,y)<\delta_0\}} \frac{|u(x,0)-u(y,0)|^p}{d(x,y)^{ps}\mu(B(x,d(x,y)))}\,dxdy.$$

Let $\mathcal{I}$ be the set of the multi-indices I defined in Sect. 2 and write

$$\begin{aligned} N(p,\delta_0;\mathcal{U}) &= \int\limits_{\mathcal{U}} dx \int\limits_{\mathcal{U}\cap\{d(x,y)<\delta_0\}} \frac{|u(x,0)-u(y,0)|^p}{d(x,y)^{ps}\mu(B(x,d(x,y)))}\,dy \\ &\leq \sum_{I\in\mathcal{I}} \int\limits_{\mathcal{U}} dx \int\limits_{\mathcal{U}\cap A_I(x)\cap\{d(x,y)<\delta_0\}} \frac{|u(x,0)-u(y,0)|^p\,dy}{d(x,y)^{ps}\mu(B(x,d(x,y)))} \\ &= \sum_{I\in\mathcal{I}} \int\limits_{\mathcal{U}} f_I(x)\,dx, \end{aligned} \tag{21}$$

where f_I is defined by the last equality and we introduced the annulus

$$\begin{aligned} A_I(x) := \Big\{ y \in \mathbb{R}^{n-1} : |\lambda_I(x)|(2d(x,y)/a)^{d(I)} \\ \geq \frac{1}{2}\max_{J\in\mathcal{I}} |\lambda_J(x)|(2d(x,y)/a)^{d(J)} \Big\}. \end{aligned}$$

Fix $\delta_0 \leq ar_0/2$. By Theorem 2 the map $y = \Phi_{I,x}(h)$ is one-to-one on the set $\{h \in \mathbb{R}^{n-1} : \|h\|_I < (2b/a)d(x,\bar{y})\}$ where $\bar{y} \in A_I(x)$ is such that $d(x,\bar{y}) = \min\{\delta_0, \max_{y\in A_I(x)} d(x,y)\}$ (the condition $d(x,\bar{y}) \leq \delta_0$ amounts to $2d(x,\bar{y})/a < r_0$ and ensures that Theorem 2 can be applied), and moreover $\Phi_{I,x}(\{h \in \mathbb{R}^{n-1} : \|h\|_I < (2b/a)d(x,\bar{y})\}) \supset \overline{B}(x,2d(x,\bar{y})) \supset A_I(x)$. By the same theorem statement (iii)

$$\begin{aligned} \overline{B}(x,2d(x,y)) &\subset \Phi_{I,x}\big(\{h \in \mathbb{R}^{n-1} : \|h\|_I < (2b/a)d(x,y)\}\big) \\ &\subset \overline{B}(x,2d(x,y)/a) \end{aligned}$$

for all $y \in A_I(x)$ and $d(x,y) < \delta_0$, i.e. $2d(x,y)/a < r_0$. Thus

$$\|h\|_I < \frac{2b}{a} d(x, \Phi_{I,x}(h)) \leq \frac{2b}{a}\delta_0 \leq br_0. \tag{22}$$

Set $H_{I,\delta_0}(x) = \Phi_{I,x}^{-1}(\mathcal{U} \cap A_I(x) \cap \{d(x,y) < \delta_0\})$. Thus, by the first inequality of (22),

$$f_I(x) \leq C \int_{H_{I,\delta_0}(x)} \frac{|u(x,0) - u(\Phi_{I,x}(h),0)|^p \, |J_h \Phi_{I,x}(h)|}{\|h\|_I^{ps} \mu(B(x, C\|h\|_I))} \, dh. \tag{23}$$

Note that (16) furnishes the estimate $\mu(B(x, C\|h\|_I)) \geq C|\lambda_I(x)| \|h\|^{d(I)}$. Letting $\eta = 2b\delta_0/a$ and recalling that $|J_h \Phi_{I,x}(h)| \simeq |\lambda_I(x)|$ from (21) and (23) we get

$$\begin{aligned} N(p;\delta_0;\mathcal{U}) &\leq C \sum_{I \in \mathcal{I}} \int_{\mathcal{U}} dx \int_{\{\|h\|_I < \eta\}} \frac{|u(x,0) - u(\Phi_{I,x}(h),0)|^p}{\|h\|_I^{ps+d(I)}} \, dh \\ &= C \sum_{I \in \mathcal{I}} \int_{\{\|h\|_I < \eta\}} \frac{dh}{\|h\|_I^{ps+d(I)}} \int_{\mathcal{U}} |u(x,0) - u(\Phi_{I,x}(h),0)|^p \, dx. \end{aligned} \tag{24}$$

If $I = (i_1, ..., i_{n-1})$ and $\|h\|_I < \eta$ set $z_0(x) = x$ and define $z_l(x) = \prod_{j=1}^{l} \exp_T(h_j Y_{i_j})(x)$ for $l = 1, ..., n-1$, in such a way that $z_{n-1}(x) = \Phi_{I,x}(h)$. Thus, fixed a constant λ', $0 < \lambda' < \lambda$

$$\begin{aligned} &\int_{\mathcal{U}} |u(x,0) - u(\Phi_{I,x}(h),0)|^p \, dx \\ &\quad \leq C \sum_{l=1}^{n-1} \int_{\mathcal{U}} |u(z_{l-1}(x),0) - u(z_l(x),0)|^p \, dx \\ &\quad \leq C \sum_{l=1}^{n-1} \int_{\mathcal{U}} \Big| u\Big(\prod_{j=1}^{l-1} \exp_T(h_j Y_{i_j})(x), 0\Big) \\ &\qquad\qquad - u\Big(\exp_T(h_l Y_{i_l}) \prod_{j=1}^{l-1} \exp_T(h_j Y_{i_j})(x), 0\Big)\Big|^p \, dx \\ &\quad \leq C \sum_{l=1}^{n-1} \int_{\mathcal{U}_{\lambda'}} |u(\xi,0) - u(\exp_T(h_l Y_{i_l})(\xi),0)|^p \, d\xi, \end{aligned} \tag{25}$$

where in each integral we performed the change of variable $\xi = z_{l-1}(x)$ which has Jacobian greater than a positive constant. Moreover, $\xi \in \mathcal{U}_{\lambda'}$ if

δ_0 is small enough. Then, we have to estimate a finite number of integrals of the form

$$\int_{\mathcal{U}_{\lambda'}} |u(x,0) - u(\exp_T(t(h)Y)(x),0)|^p \, dx,$$

with $d(Y) \le k$ and $|t(h)|^{1/d(Y)} \le \|h\|_I$. By Lemma 1 we can write $\exp_T(tY) = \prod_{i=1}^p S_{\sigma_i}(q_i|t|^{1/d(Y)}, \tau_i X_{j_i})$ with $\sigma_i \in \{1,2\}$, $\tau_i \in \{-1,1\}$, $1 \le q_i \le k$, p less than an absolute constant and S_1, S_2 as in (10). With triangle inequalities and changes of variable quite similar to the ones in (25) we are led to the estimate of integrals of one of the two types

$$\begin{aligned} &\int_{\mathcal{U}_{\lambda'}} |u(\exp(q|t(h)|^{1/d(Y)}T)(x),0) - u(x,0)|^p dx \qquad \text{or} \\ &\int_{\mathcal{U}_{\lambda'}} |u(\exp(q|t(h)|^{1/d(Y)}(\tau X_j + T))(x),0) - u(x,0)|^p dx, \end{aligned} \tag{26}$$

with $j = 1, ..., m-1$, $1 \le q \le k$ and $|t(h)|^{1/d(Y)} \le \|h\|_I$. If we consider, for instance, an integral of the second type with $\tau = 1$ the computation in (24) can be concluded in the following way (recall that $ps + d(I) = p - 1 + d(I)$):

$$\int_{\{\|h\|_I<\eta\}} \frac{dh}{\|h\|_I^{ps+d(I)}} \int_{\mathcal{U}_{\lambda'}} |u(\exp(q|t(h)|^{1/d(Y)}(X_j + T)(x,0))) - u(x,0)|^p \, dx$$

$$\begin{aligned} &\le C \int_{\{\|h\|_I<\eta\}} \frac{dh}{\|h\|_I^{ps+d(I)}} \int_{\mathcal{U}_{\lambda'}} \Big(\int_0^{k\|h\|_I} |Xu(\exp(t(X_j+T))(x,0)| \, dt \Big)^p dx \\ &\le C \int_{\{\|h\|_I<\eta\}} \frac{dh}{\|h\|_I^{ps+d(I)}} \Big(\int_0^{k\|h\|_I} \Big(\int_{\mathcal{U}_{\lambda'}} |Xu(\exp(t(X_j+T))(x,0)|^p dx \Big)^{\frac{1}{p}} dt \Big)^p \\ &= C \int_0^{k\eta} \frac{dr}{r^p} \Big(\int_0^r \Big(\int_{\mathcal{U}_{\lambda'}} |Xu(\exp(t(X_j+T))(x,0)|^p \, dx \Big)^{\frac{1}{p}} dt \Big)^p \\ &\le C \int_0^{k\eta} \int_{\mathcal{U}_{\lambda'}} |Xu(\exp(t(X_j+T))(x,0))|^p \, dx \, dt. \end{aligned}$$

We used the Minkowski inequality, formula (19) and the Hardy inequality (18).

Finally, write $\exp(t(X_j + T))(x,0) = \Theta(x,t)$ and perform the change of variable $(\xi, \tau) = \Theta(x,t)$. Since $\Theta(x,0) = (x,0)$ then

$$\frac{\partial \Theta(x,t)}{\partial x \partial t}\Big|_{t=0} = \begin{pmatrix} I_{n-1} & X_j(x,0) \\ 0 & 1 \end{pmatrix}$$

and thus Θ is a change of variable on the rectangle $\mathcal{U}_{\lambda'} \times (0, \varrho_0)$, where ϱ_0 is suitably small. Choosing δ_0 small we obtain $k\eta \le \varrho_0$ and $\Theta(x,t) \in \mathcal{U}_\lambda \times (0,t_0)$ for all $(x,t) \in \mathcal{U}_{\lambda'} \times (0,k\eta)$. Then

$$\int_{\mathcal{U}_{\lambda'}\times(0,k\eta)} |Xu(\Theta(x,t))|^p \, dxdt \le C \int_{\mathcal{U}_\lambda\times(0,t_0)} |Xu(\xi,\tau)|^p \, d\xi d\tau.$$

Integrals of the first type in (26) can be treated in the same way and the proof of the Theorem is concluded. □

Corollary 2. *Let $X_1, ..., X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ satisfy the Hörmander condition and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\partial\Omega$ of class C^∞ and non characteristic. Let $1 < p < \infty$ and $s = 1 - \frac{1}{p}$. There exist constants $C, \delta_0 > 0$ such that*

$$\int\limits_{\partial\Omega\times\partial\Omega\cap\{d(x,y)<\delta_0\}} \frac{|u(x)-u(y)|^p \, d\mu(x)d\mu(y)}{d(x,y)^{ps}\mu(B(x,d(x,y)))} \le C \int\limits_{\Omega} |Xu(x)|^p \, dx$$

for all $u \in C^1(\Omega) \cap C(\overline{\Omega})$, where $\mu = \mathcal{H}^{n-1} \llcorner \partial\Omega$.

Proof. The proof follows from Theorem 3 using a standard covering argument, Lemma 1 and Remark 1. □

Example 1 (Trace on subgroups of $\mathbb{H}^n$). Consider the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \ge 1$, whose elements are $(z,t) \in \mathbb{H}^n$ with $z = x + iy \in \mathbb{C}^n$, $x, y \in \mathbb{R}^n$, and $t \in \mathbb{R}$, and whose group law is $(\zeta,\tau)\cdot(z,t) = (\zeta + z, \tau + t + 2\,\mathrm{Im}(\zeta\bar{z}))$. The Lie algebra of the group is generated by the vector fields

$$X_j = \partial_{x_j} + 2y_j\partial_t \quad \text{and} \quad Y_j = \partial_{y_j} - 2x_j\partial_t, \qquad j = 1, ..., n,$$

which satisfy the Hörmander condition. The homogeneous norm $\|(z,t)\| := (|z|^4 + t^2)^{1/4}$ is equivalent to the C-C metric d, i.e. $c_1 d((z,t),(\zeta,\tau)) \le \|(\zeta,\tau)^{-1}\cdot(z,t)\| \le c_2 d((z,t),(\zeta,\tau))$ for all $(z,t),(\zeta,\tau) \in \mathbb{H}^n$ and for some $0 < c_1 < c_2$. The integer $Q = 2n+2$ is the “dimension” of $\mathbb{H}^n$ and $|B((z,t),r)| = cr^Q$ for some $c > 0$ and for all $(z,t) \in \mathbb{H}^n$ and $r \ge 0$.

Consider the half space $\Omega = \{(x,y,t) \in \mathbb{H}^n : x_j > 0\}$ for some $j = 1, ..., n$ with boundary $\partial\Omega = \{(x,y,t) \in \mathbb{H}^n : x_j = 0\}$. Actually, the hyperplane $\partial\Omega$ is a subgroup of $\mathbb{H}^n$ and all its points are non characteristic. If $\mu = \mathcal{H}^{2n} \llcorner \partial\Omega$ then $\mu(B((z,t),r)) = mr^{Q-1}$ for some $m > 0$ and for all $(z,t) \in \partial\Omega$. Using the technique developed in this section it can be proved that there exists a constant $C > 0$ such that ($1 < p < \infty$ and $s = 1 - \frac{1}{p}$)

$$\int\limits_{\partial\Omega\times\partial\Omega} \frac{|u(z,t)-u(\zeta,\tau)|^p \, d\mu(z,t)d\mu(\zeta,\tau)}{\|(\zeta,\tau)^{-1}\cdot(z,t)\|^{ps+Q-1}} \le C \int\limits_{\Omega} |\nabla_{\mathbb{H}^n} u(z,t)|^p \, dzdt$$

for all $u \in C^1(\Omega) \cap C(\overline{\Omega})$, where $\nabla_{\mathbb{H}^n} = (X_1, ..., X_n, Y_1, ..., Y_n)$.

4 Trace theorem in the Grushin plane

4.1 Trace theorem

In this section we begin the study of the trace theorem when the boundary contains characteristic points. We focus our attention on the Grushin plane where we prove that the trace estimate holds for domains which are sufficiently "flat" at characteristic points.

Let d be the C-C metric induced on $\mathbb{R}^2$ by the vector fields

$$X_1 = \partial_x \quad \text{and} \quad X_2 = |x|^\alpha \partial_y, \ \ \alpha > 0.$$

If $(x, y) \in \mathbb{R}^2$ and $r \geq 0$ let $B((x, y), r) = \{(\xi, \eta) \in \mathbb{R}^2 : d((x, y), (\xi, \eta)) < r\}$. Moreover, define the "box"

$$\mathrm{Box}((x, y), r) = [x - r, x + r] \times [y - r(|x| + r)^\alpha, y + r(|x| + r)^\alpha].$$

Such boxes are equivalent to C-C balls and the metric d can be evaluated rather explicitly. This is stated in the following Lemmas, whose proof is a consequence of the results in [FL].

Lemma 4. *There exist constants $0 < c_1 < c_2$ such that for all $(x, y) \in \mathbb{R}^2$ and $r \geq 0$*

$$\mathrm{Box}((x, y), c_1 r) \subset B((x, y), r) \subset \mathrm{Box}((x, y), c_2 r). \tag{27}$$

Lemma 5. *Let $\lambda > 0$. For all (x, y), $(\xi, \eta) \in \mathbb{R}^2$ with $|x| \geq |\xi|$*

$$d((x, y), (\xi, \eta)) \simeq |x - \xi| + \frac{|y - \eta|}{|x|^\alpha} \quad \text{if} \quad |x|^{\alpha+1} \geq \lambda |y - \eta|, \tag{28}$$

$$d((x, y), (\xi, \eta)) \simeq |x - \xi| + |y - \eta|^{\frac{1}{\alpha+1}} \ \ \text{if} \ \ |x|^{\alpha+1} < \lambda |y - \eta|, \tag{29}$$

where the equivalence constants depend on λ.

Definition 1. *Let $\Omega \subset \mathbb{R}^2$ be an open set with $\partial\Omega$ of class C^1. A point $(0, y_0) \in \partial\Omega$ is said to be $\alpha-$admissible, $\alpha > 0$, if one of the following two conditions holds:*

(i) *(Non characteristic case). There exist $\delta > 0$ and $\psi \in C^1(y_0 - \delta, y_0 + \delta)$ such that $\psi(y_0) = 0$ and*

$$\partial\Omega \cap (-\delta, \delta) \times (y_0 - \delta, y_0 + \delta) = \{(\psi(y), y) : |y - y_0|, |\psi(y)| < \delta\}.$$

(ii) *(Characteristic case). There exist $\delta > 0$ and $c > 0$ such that*

$$\partial\Omega \cap (-\delta, \delta) \times (y_0 - \delta, y_0 + \delta) = \{(x, \varphi(x)) \in \mathbb{R}^2 : |x| < \delta\},$$

where $\varphi \in C^1(-\delta, \delta)$ and $|\varphi'(x)| \leq c|x|^\alpha$ for all $x \in (-\delta, \delta)$.

Finally, Ω is said to be $\alpha-$admissible if all the points of $\partial\Omega \cap \{x = 0\}$ are $\alpha-$admissible.

Let $\Omega \subset \mathbb{R}^2$ be an open set of class C^1 and let $\nu(x, y)$ be the unit normal to $\partial\Omega$ at $(x, y) \in \partial\Omega$. Consider the modulus of the "projected" normal

$$|X\nu(x,y)| = \Big(\langle X_1(x,y), \nu(x,y)\rangle^2 + \langle X_2(x,y), \nu(x,y)\rangle^2\Big)^{\frac{1}{2}}$$
$$= \Big(\nu_1(x,y)^2 + |x|^{2\alpha}\nu_2(x,y)^2\Big)^{\frac{1}{2}},$$

and define the measure $\mu = |X\nu|\mathcal{H}^1 \llcorner \partial\Omega$. The measure μ is the one that appears in the left hand side of the trace estimates.

In the sequel we shall use the equivalence

$$\int_I |\xi|^\alpha d\xi \simeq |I| \max_{\xi\in I} |\xi|^\alpha \tag{30}$$

for any interval $I \subset \mathbb{R}$, where the equivalence constants depend only on $\alpha > 0$.

Lemma 6. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with $\partial\Omega$ of class C^1 and suppose it is $\alpha-$*admissible. *Then there exist $0 < m_1 < m_2$ and $r_0 > 0$ such that*

$$m_1 \frac{|B((x,y),r)|}{r} \le \mu(B(x,y),r) \le m_2 \frac{|B((x,y),r)|}{r} \tag{31}$$

for all $(x, y) \in \partial\Omega$ and for all $0 < r < r_0$,

Proof. Since away from the set $\{x = 0\}$ we are essentially in a Euclidean situation it suffices to prove (31) for $(x, y) \in \partial\Omega$ belonging to a neighborhood of an $\alpha-$admissible point.

Suppose first that $(0, 0) \in \partial\Omega$ is an $\alpha-$admissible point of type (i) (non characteristic). In a neighborhood of the origin $\partial\Omega$ is the graph of a function $\psi \in C^1(-\delta, \delta)$ in the variable y. If $\delta > 0$ and $r > 0$ are small, then the graph of ψ meets $\partial\mathrm{Box}((\psi(y), y), r)$ on its horizontal edges. This is ensured by $|\psi(y) - \psi(y - r(|\psi(y)| + r)^\alpha)| < r$, which holds true provided y and r are small enough. Now

$$\mu(\mathrm{Box}((\psi(y), y), r)) \simeq \int_{y-r(|\psi(y)|+r)^\alpha}^{y+r(|\psi(y)|+r)^\alpha} d\eta$$
$$= 2r(|\psi(y)| + r)^\alpha = \frac{|\mathrm{Box}((\psi(y), y), r)|}{2r},$$

and (4) gives the proof of the required estimate.

Suppose now that $(0, 0) \in \partial\Omega$ is an $\alpha-$admissible point of type (ii). Let $\varphi \in C^1(-\delta, \delta)$ be the function whose graph represents $\partial\Omega$ and such that

$|\varphi'(x)| \leq c|x|^\alpha$ for all $|x| < \delta$ and for some $c \geq 0$. Then, if $y = \varphi(x)$ and $|x| \leq \delta/2$

$$\nu(x,y) = \frac{(\varphi'(x), -1)}{\sqrt{1+\varphi'(x)^2}}, \quad \text{and} \quad |X\nu(x,y)| = \frac{\sqrt{|x|^{2\alpha} + \varphi'(x)^2}}{\sqrt{1+\varphi'(x)^2}} \simeq |x|^\alpha.$$

By Lemma 4 $\mu(\mathrm{Box}((x,y),c_1 r)) \leq \mu(B((x,y),r)) \leq \mu(\mathrm{Box}((x,y),c_2 r))$, and, supposing for instance $0 \leq x \leq \delta/2$ and $0 < r < \delta/(2c_2)$

$$\begin{aligned}\mu(\mathrm{Box}((x,y),c_2 r)) &= \int_{\mathrm{Box}((x,y),c_2 r)\cap\partial\Omega} |X\nu| d\mathcal{H}^1 \leq C\int_{x-c_2 r}^{x+c_2 r} |\xi|^\alpha \, d\xi \\ &\leq 2Cc_2 r(x+c_2 r)^\alpha \simeq \frac{|\mathrm{Box}((x,y),c_2 r)|}{r}.\end{aligned}$$

The estimate from above in (31) follows by Lemma 4. In order to prove the opposite inequality assume without loss of generality that the constant c relative to φ is greater than 1 and that $x \geq 0$. Introduce the new box

$$\begin{aligned}\overline{\mathrm{Box}}((x,y),c_1 r) := &\left[x - \frac{c_1}{c}r, x + \frac{c_1}{c}r\right] \\ &\times [y - c_1 r(x+c_1 r)^\alpha, y + c_1 r(x + c_1 r)^\alpha] \\ \subset &\ \mathrm{Box}((x,y),c_1 r).\end{aligned}$$

Since $|\varphi(x + \frac{c_1}{c}r) - \varphi(x)| \leq c_1 r(x+c_1 r)^\alpha$, the graph of φ meets $\partial\overline{\mathrm{Box}}((x,y),c_1 r)$ on its left and right vertical edges. Thus

$$\begin{aligned}\mu(B((x,y)),r)) \geq \mu(\overline{\mathrm{Box}}((x,y),c_1 r) &= \int_{\overline{\mathrm{Box}}((x,y),c_1 r)\cap\partial\Omega} |X\nu| d\mathcal{H}^1 \\ &\geq C\int_{x-\frac{c_1}{c}r}^{x+\frac{c_1}{c}r} |\xi|^\alpha \, d\xi \simeq C\frac{c_1}{c}r(x+\frac{c_1}{c}r)^\alpha \\ &\simeq \frac{|\mathrm{Box}((x,y),c_1 r)|}{r},\end{aligned}$$

which is the required estimate. We also used (30). □

Theorem 4. *Let $X_1 = \partial_x$ and $X_2 = |x|^\alpha \partial_y$, $\alpha > 0$. Let $1 < p < \infty$ and $s = 1 - \frac{1}{p}$. If $\Omega \subset \mathbb{R}^2$ is a bounded open set of class C^1 which is $\alpha-$admissible, then there exist $C > 0$ and $\delta_0 > 0$ such that*

$$\int_{\partial\Omega\times\partial\Omega\cap\{d(z,\zeta)<\delta_0\}} \frac{|u(z)-u(\zeta)|^p \ d\mu(z)d\mu(\zeta)}{d(z,\zeta)^{ps}\mu(B(z,d(z,\zeta)))} \leq C\int_\Omega |Xu(x,y)|^p \, dxdy$$

for all $u \in C^1(\Omega) \cap C(\overline{\Omega})$.

Proof. Since away from the set $\{x = 0\}$ we are essentially in the Euclidean case, it suffices to prove the estimate in a neighborhood of an $\alpha-$admissible point which may assumed to be the origin. Denote by $\mathcal{U}$ the intersection of $\partial\Omega$ with a small fixed neighborhood of $(0, 0)$. Recalling that, by Lemma 6, $d(z,\zeta)^{ps}\mu(B(z, d(z,\zeta))) \simeq d(z,\zeta)^{ps-1}|B(z, d(z,\zeta))|$, we have to prove that

$$\begin{aligned} N(p;\mathcal{U}) &:= \int_{\mathcal{U}\times\mathcal{U}} \frac{|u(z) - u(\zeta)|^p}{d(z,\zeta)^{ps-1}|B(z, d(z,\zeta))|} d\mu(z)d\mu(\zeta) \\ &\leq C\int_\Omega |Xu(x, y)|^p\, dxdy. \end{aligned}$$

The $\alpha-$admissible point can be of type (i) or of type (ii).

Type (i). We may assume that $\mathcal{U} = \{(\psi(y), y) : |y| < \delta\}$ for some $\delta > 0$ and $\psi \in C^1(-\delta, \delta)$ with $\psi(0) = 0$, and that Ω lies in the region $\{x > \psi(y)\}$. Write $z = (\psi(y), y)$ and $\zeta = (\psi(\eta), \eta)$, and notice that, by the doubling property of the Lebesgue measure, which follows from Lemma 4, $|B(z, d(z,\zeta))| \simeq |B(\zeta, d(z,\zeta))|$. Thus the kernel is essentially symmetric and the integration can be performed without loss of generality on the set $\{|\psi(\eta)| < |\psi(y)|\}$

$$\begin{aligned} N(p;\mathcal{U}) &\simeq \int_{\{|y|<\delta,\, |\eta|<\delta,\, |\psi(\eta)|<|\psi(y)|\}} \frac{|u(z) - u(\zeta)|^p}{d(z,\zeta)^{ps-1}|B(z, d(z,\zeta))|} dyd\eta \\ &= \int_A \frac{|u(z) - u(\zeta)|^p dyd\eta}{d(z,\zeta)^{ps-1}|B(z, d(z,\zeta))|} + \int_B \frac{|u(z) - u(\zeta)|^p dyd\eta}{d(z,\zeta)^{ps-1}|B(z, d(z,\zeta))|} \\ &:= I_A + I_B, \end{aligned}$$

where we let

$$\begin{aligned} A &= \{(y, \eta) : |y| < \delta,\ |\eta| < \delta,\ |\psi(\eta)| < |\psi(y)|, |\psi(y)|^{\alpha+1} \geq |y - \eta|\}, \\ B &= \{(y, \eta) : |y| < \delta,\ |\eta| < \delta,\ |\psi(\eta)| < |\psi(y)|, |\psi(y)|^{\alpha+1} < |y - \eta|\}. \end{aligned}$$

We begin with the estimate of I_A. If $(y, \eta) \in A$ then

$$\begin{aligned} d(z,\zeta) &\simeq |\psi(y) - \psi(\eta)| + \frac{|y - \eta|}{|\psi(y)|^\alpha} \\ &= \frac{|y - \eta|}{|\psi(y)|^\alpha}\Big(1 + |\psi(y)|^\alpha \frac{|\psi(y) - \psi(\eta)|}{|y - \eta|}\Big) \simeq \frac{|y - \eta|}{|\psi(y)|^\alpha} \end{aligned}$$

and

$$|B(z, d(z,\zeta))| \simeq d(z,\zeta)^2\big(|\psi(y)| + d(z,\zeta)\big)^\alpha \simeq d(z,\zeta)^2|\psi(y)|^\alpha.$$

Without loss of generality assume $y > \eta$. Let $\eta = y - h$ and write (recall that $1 + ps = p$)

$$I_A \simeq \int\limits_A \frac{|u(\psi(y), y) - u(\psi(\eta), \eta)|^p}{|y-\eta|^p} |\psi(y)|^{p\alpha-\alpha} dy d\eta$$
$$\leq \int\limits_0^{2\delta} \frac{dh}{|h|^p} \int\limits_{\{|\psi(y)|^{\alpha+1}>|h|, |y|<\delta\}} |u(\psi(y), y) - u(\psi(y-h), y-h)|^p |\psi(y)|^{p\alpha-\alpha} dy.$$

We shall connect the points $(\psi(y), y)$ and $(\psi(y-h), y-h)$ by the curves

$$\begin{aligned}\gamma_1(t) &:= \exp(t(X_1 - bX_2))(\psi(y), y) \\ &= \Big(\psi(y) + t, y - b\int_0^t |\psi(y) + \tau|^\alpha d\tau\Big) := \Psi_1(t, y),\end{aligned}$$

where $b = \min\{1, 1/L\}$, $L := \sup_{|y|<\delta} |\psi'(y)|$, and

$$\begin{aligned}\gamma_2(t) :&= \exp(tX_1)(\psi(y-h), y-h) \\ &= (\psi(y-h) + t, y-h) := \Psi_2(t, y-h).\end{aligned}$$

In order to reach the height $y - h$, the curve γ_1 needs a time t_1 such that

$$\int_0^{t_1} |\psi(y) + \tau|^\alpha d\tau = \frac{|h|}{b}. \tag{32}$$

By (30) the left hand side is greater than $Ct_1|\psi(y)|^\alpha$ and then $t_1 \leq C|h|/|\psi(y)|^\alpha$. The time t_2 such that $\gamma_2(t_2) = \gamma_1(t_1)$ can also be estimated by $|h|/|\psi(y)|^\alpha$. Indeed

$$t_2 = |\psi(y) + t_1 - \psi(y-h)| \leq L|h| + t_1 \leq C\frac{|h|}{|\psi(y)|^\alpha}.$$

The choice of the parameter b guarantees that $\gamma_1(t) \in \Omega$ for all $|y| < \delta$ and $0 < t \leq t_1$. In fact this happens if and only if

$$\psi\Big(y - b\int_0^t |\psi(y) + \tau|^\alpha d\tau\Big) < \psi(y) + t. \tag{33}$$

This last inequality is a consequence of the following

$$\Big|\psi\Big(y - b\int_0^t |\psi(y) + \tau|^\alpha d\tau\Big) - \psi(y)\Big| \leq Lb\int_0^t |\psi(y) + \tau|^\alpha d\tau < t.$$

Since $\Psi_1(t_1, y) = \Psi_2(t_2, y-h)$ then $|u(\psi(y), y) - u(\psi(y-h), y-h)|$ is less than

$$\begin{aligned}&|u(\psi(y), y) - u(\Psi_1(t_1, y))| + |u(\psi(y-h), y-h)) - u(\Psi_2(t_2, y-h))| \\ &\qquad \leq C\Big(\int_0^{t_1} |Xu(\Psi_1(t, y))| dt + \int_0^{t_2} |Xu(\Psi_2(t, y-h))| dt\Big),\end{aligned}$$

and we find

$$\begin{aligned} I_A \le C\Big[&\int_0^{2\delta} \frac{dh}{|h|^p} \int_{(-\delta,\delta)\cap\{|\psi(y)|^{\alpha+1}\ge|h|\}} |\psi(y)|^{p\alpha-\alpha}\Big(\int_0^{t_1} |Xu(\Psi_1(t,y))|dt\Big)^p dy \\ &+ \int_0^{2\delta} \frac{dh}{|h|^p} \int_{(-\delta,\delta)\cap\{|\psi(y)|^{\alpha+1}\ge|h|\}} |\psi(y)|^{p\alpha-\alpha}\Big(\int_0^{t_2} |Xu(\Psi_2(t,y-h))|dt\Big)^p dy\Big] \\ &:= C[I_A^{(1)} + I_A^{(2)}]. \end{aligned}$$

We shall estimate $I_A^{(1)}$ and $I_A^{(2)}$ by the same technique and we begin with $I_A^{(1)}$. Letting in the inner integral $\tau = |\psi(y)|^\alpha t$, recalling that $t_1 \le C|h|/|\psi(y)|^\alpha$ and using the Minkowski inequality we find

$$\begin{aligned} I_A^{(1)} &\le \int_0^{2\delta} \frac{dh}{|h|^p} \int_{(-\delta,\delta)\cap\{|\psi(y)|^{\alpha+1}\ge|h|\}} \frac{dy}{|\psi(y)|^\alpha}\Big(\int_0^{C|h|} |Xu(\Psi_1(\tau/|\psi(y)|^\alpha, y))|d\tau\Big)^p \\ &\le \int_0^{2\delta} \Big(\frac{dh}{|h|} \int_0^{C|h|} \Big(\int_{(-\delta,\delta)\cap\{|\psi(y)|^{\alpha+1}\ge|h|\}} \frac{|Xu(\Psi_1(\tau/|\psi(y)|^\alpha, y))|^p dy}{|\psi(y)|^\alpha}\Big)^{\frac{1}{p}} d\tau\Big)^p. \end{aligned}$$

Since $\{|\psi(y)|^{\alpha+1} \ge |h|\} \subset \{C|\psi(y)|^{\alpha+1} \ge \tau\}$ the last integral is estimated by an integral of the form $\int_0^{2\delta}\Big(\frac{1}{|h|}\int_0^{C|h|}|f(\tau)|d\tau\Big)^p dh$ with f not depending on h. So we can apply the Hardy inequality to get

$$\begin{aligned} I_A^{(1)} &\le C\int_0^{2\delta}\int_{(-\delta,\delta)\cap\{C|\psi(y)|^{\alpha+1}\ge\tau\}} \frac{|Xu(\Psi_1(\tau/|\psi(y)|^\alpha, y))|^p}{|\psi(y)|^\alpha} dy\, d\tau \\ &\le C\int_{-\delta}^{\delta}\int_0^{C\delta} |Xu(\Psi_1(t,y))|^p dt\, dy. \end{aligned}$$

We let $\tau/|\psi(y)|^\alpha = t$ and we used $\tau/|\psi(y)|^\alpha \le C|\psi(y)| \le C|y| \le C\delta$. The Jacobian matrix of Ψ_1 is

$$\frac{\partial\Psi_1(y,t)}{\partial y\partial t} = \begin{pmatrix} 1 & \psi'(y) \\ -b|\psi(y)+t|^\alpha & 1 - b(|\psi(y)+t|^\alpha - |\psi(y)|^\alpha)\psi'(y) \end{pmatrix}.$$

By the same argument used in the proof of (33) we can see that if $\delta > 0$ is small, then $\Psi_1((0,C\delta)\times(-\delta,\delta)) \subset \Omega$. Moreover $|J\Psi_1(t,y)| = |1 + b\psi'(y)|\psi(y)|^\alpha| \simeq 1$. Then

$$I_A^{(1)} \le C\int_\Omega |Xu(x,y)|^p dxdy.$$

We estimate now $I_A^{(2)}$. Note first that if $\delta > 0$ is small and $(y,\eta) \in A$, we have

$$|\psi(y)| \le 2|\psi(\eta)|. \tag{34}$$

Indeed $L|\psi(y)|^{\alpha+1} \ge L|y-\eta| \ge |\psi(y)-\psi(\eta)| \ge |\psi(y)| - |\psi(\eta)|$, and thus $|\psi(\eta)| \ge |\psi(y)| - L|\psi(y)|^{\alpha+1} \ge 1/2|\psi(y)|$ if $\delta > 0$ is small. Taking (34) into account with $\eta = y - h$, recalling that $t_2 \le C|h|/|\psi(y)|^\alpha \le C|h|/|\psi(y-h)|^\alpha$ and letting $\tau = |\psi(y-h)|^\alpha t$ in the inner integral we find that $I_A^{(2)}$ is smaller than

$$\begin{aligned}
&\int_0^{2\delta} \frac{dh}{|h|^p} \int_{(-\delta,\delta)\cap\{C|\psi(y-h)|^{\alpha+1}\ge|h|\}} \frac{dy}{|\psi(y-h)|^\alpha} \\
&\times \Big(\int_0^{C|h|} \Big| Xu\Big(\Psi_2\Big(\frac{\tau}{|\psi(y-h)|^\alpha}, y-h\Big)\Big)\Big| d\tau\Big)^p \\
&\le \int_0^{2\delta} \Big(\frac{dh}{|h|} \int_0^{C|h|} \Big(\int_{(-\delta,\delta)\cap\{C|\psi(y-h)|^{\alpha+1}\ge|h|\}} \frac{|Xu(\Psi_2(\frac{\tau}{|\psi(y-h)|^\alpha}, y-h))|^p}{|\psi(y-h)|^\alpha} dy\Big)^{1/p} d\tau\Big)^p \\
&\le \int_0^{2\delta} \Big(\frac{dh}{|h|} \int_0^{C|h|} \Big(\int_{(-3\delta,\delta)\cap\{C|\psi(y)|^{\alpha+1}\ge|h|\}} \frac{|Xu(\Psi_2(\tau/|\psi(y)|^\alpha, y))|^p}{|\psi(y)|^\alpha} dy\Big)^{1/p} d\tau\Big)^p.
\end{aligned}$$

Since $\{C|\psi(y)|^{\alpha+1} \ge |h|\} \subset \{C|\psi(y)|^{\alpha+1} \ge \tau\}$, we can apply the Hardy inequality to get

$$\begin{aligned}
I_A^{(2)} &\le C \int_0^{2\delta} \int_{(-3\delta,\delta)\cap\{C|\psi(y)|^{\alpha+1}\ge\tau\}} \frac{|Xu(\Psi_2(\tau/|\psi(y)|^\alpha, y))|^p}{|\psi(y)|^\alpha} dy\, d\tau \\
&\le C \int_{-3\delta}^{\delta} \int_0^{C\delta} |Xu(\Psi_2(t,y))|^p dt\, dy.
\end{aligned}$$

Since $|J\Psi_2(t,y)| = 1$ the estimate for $I_A^{(2)}$ follows.

We now turn to the estimate of I_B. Writing again $z = (\psi(y), y)$ and $\zeta = (\psi(\eta), \eta)$, if $(y,\eta) \in B$ then

$$d(z,\zeta) \simeq |\psi(y) - \psi(\eta)| + |y-\eta|^{1/(\alpha+1)} \simeq |y-\eta|^{1/(\alpha+1)}$$

because $\psi \in C^1$ and $|y - \eta| \le 2\delta$. Moreover starting from the inequality $|\psi(y)| \le |y-\eta|^{1/(\alpha+1)}$ which defines B, we find

$$\begin{aligned}
|B(z, d(z,\zeta))| &\simeq d(z,\zeta)^2 (|\psi(y)| + d(z,\zeta))^\alpha \\
&\simeq |y-\eta|^{2/(\alpha+1)} (|\psi(y)| + |y-\eta|^{1/(\alpha+1)})^\alpha \\
&\simeq |y-\eta|^{(\alpha+2)/(\alpha+1)}.
\end{aligned}$$

Assume $\eta < y$, let $\eta = y - h$ and write

$$I_B \simeq \int_B \frac{|u(\psi(y), y) - u(\psi(\eta), \eta)|^p}{|y-\eta|^{1+\frac{ps}{\alpha+1}}} dy d\eta$$
$$\leq C \int_0^{2\delta} \frac{dh}{|h|^{1+\frac{ps}{\alpha+1}}} \int_{\{|\psi(y)|^{\alpha+1}<|h|,\, |y|<\delta\}} |u(\psi(y), y) - u(\psi(y-h), y-h)|^p dy.$$

The points $(\psi(y), y)$ and $(\psi(y-h), y-h)$ can be connected by the curves $\gamma_1(t) := \exp(t(X_1 - bX_2))(\psi(y), y) = \Psi_1(t, y)$ and $\gamma_2(t) := \exp(tX_1)(\psi(y-h), y-h) = \Psi_2(t, y-h)$. In order to reach the height $y - h$, the curve γ_1 needs a time t_1 such that (32) holds. By (30)

$$\int_0^{t_1} |\psi(y) + \tau|^\alpha d\tau \simeq t_1 \max_{\tau \in [\psi(y), \psi(y)+t_1]} |\tau|^\alpha \geq t_1 \Big(\frac{t_1}{2}\Big)^\alpha.$$

This yields $t_1 \leq C|h|^{1/(\alpha+1)}$. The time t_2 such that $\gamma_2(t_2) = \gamma_1(t_1)$ can also be estimated by $|h|^{1/(\alpha+1)}$. By the triangle inequality we get

$$I_B \leq C\Big[\int_0^{2\delta} \frac{dh}{|h|^{1+\frac{ps}{\alpha+1}}} \int_{-\delta}^{\delta} \Big(\int_0^{t_1} |Xu(\Psi_1(t, y))| dt\Big)^p dy$$
$$+ \int_0^{2\delta} \frac{dh}{|h|^{1+\frac{ps}{\alpha+1}}} \int_{-\delta}^{\delta} \Big(\int_0^{t_2} |Xu(\Psi_2(t, y+h))| dt\Big)^p dy\Big]$$
$$:= C[I_B^{(1)} + I_B^{(2)}].$$

Now, by the Minkowski inequality

$$I_B^{(1)} \leq \int_0^{2\delta} \frac{dh}{|h|^{1+\frac{ps}{\alpha+1}}} \Big(\int_0^{C|h|^{1/(\alpha+1)}} \Big(\int_{-\delta}^{\delta} |Xu(\Psi_1(t, y))|^p dy\Big)^{1/p} dt\Big)^p$$
$$\leq C \int_0^{(2\delta)^{1/(\alpha+1)}} \frac{dr}{r^p} \Big(\int_0^{Cr} \Big(\int_{-\delta}^{\delta} |Xu(\Psi_1(t, y))|^p dy\Big)^{1/p} dt\Big)^p$$
$$\leq \int_{(0,(2\delta)^{1/(\alpha+1)})\times(-\delta,\delta)} |Xu(\Psi_1(t, y))|^p dt dy.$$

We used $s = 1 - 1/p$, the change of variable $r = h^{1/(\alpha+1)}$ and the Hardy inequality.

The estimate of $I_B^{(2)}$ is analogous to the one of $I_A^{(2)}$. This ends the trace estimates for $\alpha-$admissible points of type (i).

Type (ii). Write $\mathcal{U} = \{(x, \varphi(x)) \in \partial\Omega : |x| < \delta\}$ for some $\varphi \in C^1(-\delta, \delta)$ such that $|\varphi'(x)| \leq c|x|^\alpha$ for some $c \geq 0$ and for all $x \in (-\delta, \delta)$. Write $z = (x, \varphi(x))$, $\zeta = (y, \varphi(y))$, and observe that

$$N(p; \mathcal{U}) \simeq \int_{|x|<\delta,\, |y|<\delta} \frac{|u(z) - u(\zeta)|^p \, |xy|^\alpha}{d(z, \zeta)^{ps-1} |B(z, d(z, \zeta))|} dx dy.$$

Since the integrand is symmetric up to equivalence constants, the integration may take place on the set $\{|x| < |y| < \delta\}$. Since $|\varphi'(y)| \leq c|y|^\alpha$ we have $|\varphi(y) - \varphi(x)| \leq c|y-x|\,|y|^\alpha \leq 2c|y|^{\alpha+1}$. Then on the mentioned set the C-C metric behaves as

$$d(z,\zeta) \simeq |y-x| + \frac{|\varphi(y)-\varphi(x)|}{|y|^\alpha} \simeq |y-x|.$$

By Lemma 4

$$|B(z, d(z,\zeta)| \simeq |y-x|^2(|x| + |y-x|)^\alpha \simeq |y-x|^2\,|y|^\alpha,$$

and, since $ps - 1 = p - 2$, we get

$$N(p;\mathcal{U}) \simeq \int_{\{|x|<|y|<\delta\}} \frac{|u(x,\varphi(x)) - u(y,\varphi(y))|^p\,|x|^\alpha}{|y-x|^p}\,dxdy.$$

By symmetry it suffices to consider the integration on $A_1 := \{0 < x < y < \delta\}$ and $A_2 := \{x > 0,\ -\delta < y < -x\}$. Set $h = y - x$ and write

$$\begin{aligned} I_{A_1} &= \int_{\{0<x<y<\delta\}} \frac{|u(x,\varphi(x)) - u(y,\varphi(y))|^p\,|x|^\alpha}{|y-x|^p}\,dxdy \\ &\leq \int_0^\delta \frac{dh}{|h|^p} \int_0^\delta |u(x,\varphi(x)) - u(x+h,\varphi(x+h))|^p\,|x|^\alpha\,dx. \end{aligned}$$

We shall connect the points $(x,\varphi(x))$ and $(x+h,\varphi(x)+h)$ by the paths

$$\begin{aligned} \gamma_1(t) &:= \exp(t(bX_1 + X_2))(x,\varphi(x)) \\ &= \left(x + bt, \varphi(x) + \int_0^t |x+b\tau|^\alpha d\tau\right) := \Phi_1(x,t), \end{aligned}$$

for $0 \leq t \leq t_1 := |h|/b$ (here $b \in (0,1)$ is a fixed number such that $2^{\alpha+1}cb < 1$), and

$$\begin{aligned} \gamma_2(t) :&= \exp(t(X_2))(x+h,\varphi(x+h)) \\ &= (x+h, \varphi(x+h) + (x+h)^\alpha t) := \Phi_2(x+h,t). \end{aligned}$$

If $t = t_1$, γ_1 reaches the height $\varphi(x) + \int_0^{|h|/b}(x+b\tau)^\alpha d\tau$. Thus the curve γ_2 needs the time $t_2 = \frac{1}{(x+h)^\alpha}|\varphi(x) - \varphi(x+h) + \int_0^{|h|/b}(x+b\tau)^\alpha d\tau|$ to reach the same height. The hypothesis on φ and (30) give the estimate $t_2 \leq C|h|$.

The choice of b ensures that $\gamma_1(t) \in \Omega$ for all $t \in (0,t_1]$. In fact this amounts to

$$\varphi(x+bt) < \varphi(x) + \int_0^t |x+b\tau|^\alpha d\tau.$$

In view of $|\varphi(x+bt)-\varphi(x)| \leq cbt(x+bt)^\alpha$ and $\int_0^t (x+b\tau)^\alpha d\tau \geq \int_0^{t/2}(x+b\tau)^\alpha d\tau \geq t/2(x+bt/2)^\alpha$ the inequality is implied by $cb(x+bt)^\alpha < 1/2(x+bt/2)^\alpha$ which holds true if $2^{\alpha+1}cb<1$.

By the triangle inequality

$$\begin{aligned} I_{A_1} \leq C\Big[\int_0^\delta \frac{dh}{|h|^p} \int_0^\delta \Big(\int_0^{C|h|} |Xu(\Phi_1(x,t))|\, dt\Big)^p |x|^\alpha dx + \\ + \int_0^\delta \frac{dh}{|h|^p} \int_0^\delta \Big(\int_0^{C|h|} |Xu(\Phi_2(x+h,t))|\, dt\Big)^p |x|^\alpha dx\Big] \\ := C[I_{A_1}^{(1)} + I_{A_1}^{(2)}]. \end{aligned}$$

Now, by Minkowski and Hardy

$$\begin{aligned} I_{A_1}^{(1)} &\leq \int_0^\delta \Big(\frac{1}{|h|}\int_0^{C|h|} \Big(\int_0^\delta |Xu(\Phi_1(x,t))|^p |x|^\alpha dx\Big)^{1/p} dt\Big)^p dh \\ &\leq C\int_{(0,\delta)\times(0,\delta)} |Xu(\Phi_1(x,t))|^p |x|^\alpha dx dt \leq C\int_\Omega |Xu(x,y)|^p\, dxdy. \end{aligned}$$

The last inequality follows from the fact that if $\delta>0$ is small then Φ_1 is one-to-one, $\Phi_1((0,\delta)\times(0,\delta)) \subset \Omega$ and

$$\frac{\partial \Phi_1(x,t)}{\partial t \partial x} = \begin{pmatrix} 1 & b \\ \varphi'(x) + \frac{1}{b}[(x+bt)^\alpha - x^\alpha] & (x+bt)^\alpha \end{pmatrix}.$$

Thus $|J\Phi_1(x,t)| = |x^\alpha - b\varphi'(x)| \geq |x|^\alpha - b|\varphi'(x)| \geq (1-bc)|x|^\alpha \geq (1-2^{-(\alpha+1)})|x|^\alpha$, and the estimate for $I_{A_1}^{(1)}$ follows.

Analogously, recalling that $t_2 \leq C|h|$ and $|x| \leq |x+h|$

$$\begin{aligned} I_{A_1}^{(2)} &\leq \int_0^\delta \Big(\frac{1}{h}\int_0^{C|h|}\Big(\int_0^\delta |Xu(\Phi_2(x+h,t))|^p |x+h|^\alpha dx\Big)^{1/p} dt\Big)^p dh \\ &\leq \int_0^\delta \Big(\frac{1}{|h|}\int_0^{C|h|}\Big(\int_0^{2\delta} |Xu(\Phi_2(x,t))|^p |x|^\alpha dx\Big)^{1/p} dt\Big)^p dh \\ &\leq C\int_{(0,2\delta)\times(0,\delta)} |Xu(\Phi_2(x,t))|^p |x|^\alpha dx dt. \end{aligned}$$

Since $|J\Phi_2(x,t)| = |x|^\alpha$, the change of variable $(\xi,\tau) = \Phi_2(x,t)$ ends the estimate for $I_{A_1}^{(2)}$.

The integral on the set $A_2 = \{0<x<\delta, -\delta<y<-x\}$ can be treated in the same way of I_{A_1}, letting $y = x+h$ and using the curves

$$\begin{aligned} \gamma_1(t) &= \exp(t(-bX_1+X_2))(x,\varphi(x)), \\ \gamma_2(t) &= \exp(tX_2)(x+h,\varphi(x+h)). \end{aligned}$$

□

4.2 Analysis of a counterexample

The hypothesis of $\alpha-$admissibility for the domain Ω in Theorem 4 is necessary. More precisely, there exist domains of class C^1 that are not $\alpha-$admissible for which the trace estimate (4) fails.

Let $\alpha > 0$, fix $\beta \in (0, \alpha+1)$ and consider the domain

$$\Omega = \{(x,y) \in \mathbb{R}^2 : |x|^\beta < y < 1\}.$$

Except that at the points $(\pm 1, 1)$ the boundary $\partial\Omega$ is of class C^1. These points are not important, problems stem from the boundary point $(0,0)$ which is not $\alpha-$admissible.

We shall consider the case $p = 2$. As usual write $z = (x,y)$ and $\zeta = (\xi, \eta)$.

Proposition 1. *Let $\alpha > 0$ and $\beta \in (0, \alpha+1)$. There exists $\gamma > 0$ such that the function $u(x,y) = y^{-\gamma}$ satisfies*

$$I := \int_\Omega |Xu|^2\, dxdy < +\infty$$

and

$$N := \int_{\partial\Omega\times\partial\Omega} \frac{|u(z)-u(\zeta)|^2}{d(z,\zeta)\mu(B(z,d(z,\zeta)))} d\mu(z)d\mu(\zeta) = +\infty.$$

Proof. We compute first I. Indeed

$$\begin{aligned} I &= \gamma^2 \int_0^1 y^{-2\gamma-2}\Big(\int_{-y^{1/\beta}}^{y^{1/\beta}} |x|^{2\alpha}\, dx\Big) dy \\ &= \frac{2\gamma^2}{2\alpha+1}\int_0^1 y^{-2\gamma-2+(2\alpha+1)/\beta}\, dy, \end{aligned}$$

and

$$I < +\infty \;\Leftrightarrow\; -2\gamma - 2 + (2\alpha+1)/\beta > -1 \;\Leftrightarrow\; \gamma < \frac{2\alpha+1-\beta}{2\beta}. \tag{35}$$

Now we shall estimate N but first some remarks on $d(z,\zeta)$ and $\mu(B(z,d(z,\zeta)))$ are in order. Let $z = (x, x^\beta) \in \partial\Omega$ with $0 < x < 1$ and let $r > 0$. Assume that

$$r \geq x^{\beta/(\alpha+1)}. \tag{36}$$

From (36) it follows that $x^\beta \leq r^{\alpha+1} \leq r(x+r)^\alpha$ and thus $x^\beta - r(x+r)^\alpha \leq 0$. This means that

$$\mathrm{Box}(z,r) \cap \{y \leq 0\} \neq \emptyset, \tag{37}$$

i.e. the box $\mathrm{Box}(z,r)$ meets the lower half plane.

Analogously, since $\beta < \alpha + 1$ we find $x \le x^{\beta/(\alpha+1)} \le r$ and thus $x - r \le 0$. This means that

$$\mathrm{Box}(z,r) \cap \{x \le 0\} \neq \emptyset, \tag{38}$$

i.e. the box $\mathrm{Box}(z,r)$ meets the left half plane.

We now claim that, for r and x sufficiently small the right part $\{(t,t^\beta) : 0 < t < 1\}$ of the boundary of Ω meets $\partial\mathrm{Box}(z,r)$ at its upper horizontal edge. This is equivalent to show that $(x+r)^\beta \ge x^\beta + r(x+r)^\alpha$, which holds because

$$(x+r)^\beta - x^\beta \ge Cr(x+r)^{\beta-1} \ge r(x+r)^\alpha$$

for $x, r \le \sigma_0$, where σ_0 is a suitable constant (we have used $\beta < \alpha + 1$). We also note that the x-coordinate of the intersection point $\{(t,t^\beta) : 0 < t < 1\} \cap \partial\mathrm{Box}(z,r)$ is $(x^\beta + r(x+r)^\alpha)^{1/\beta}$. Then from (37) and (38)

$$\begin{aligned}\mu(\mathrm{Box}(z,r)) &\simeq \mu(\mathrm{Box}(z,r) \cap \{(\xi,\eta) : \xi \ge 0\}) \\ &\simeq \int_0^{(x^\beta + r(x+r)^\alpha)^{1/\beta}} |\xi|^{\beta-1}\, d\xi \simeq x^\beta + r(x+r)^\alpha.\end{aligned}$$

Since $r \le x + r \le 2r$ then $x + r \simeq r$ and $\mu(\mathrm{Box}(z,r)) \simeq x^\beta + r^{\alpha+1}$. But $r^{\alpha+1} \le x^\beta + r^{\alpha+1} \le 2r^{\alpha+1}$ and this proves that if (36) holds then

$$\mu(\mathrm{Box}(z,r)) \simeq r^{\alpha+1}. \tag{39}$$

We shall now briefly discuss $d(z,\zeta)$ where $z = (x, x^\beta)$ and $\zeta = (\xi, \xi^\beta)$. Assume that $0 < x < \xi$ and that

$$\xi^{\alpha+1} \le \xi^\beta - x^\beta. \tag{40}$$

From (29)

$$d(z,\zeta) \simeq (\xi - x) + (\xi^\beta - x^\beta)^{1/(\alpha+1)},$$

and using the equivalence $\xi^\beta - x^\beta \simeq (\xi - x)\xi^{\beta-1}$ we get

$$\begin{aligned}d(z,\zeta) &\simeq (\xi - x)^{1/(\alpha+1)}\big((\xi - x)^{\alpha/(\alpha+1)} + \xi^{(\beta-1)/(\alpha+1)}\big) \\ &\simeq (\xi - x)^{1/(\alpha+1)}\xi^{(\beta-1)/(\alpha+1)}.\end{aligned} \tag{41}$$

In the last equivalence we used again $\beta < \alpha + 1$.

Recalling (36) and (40) we define

$$\begin{aligned}D = \{(z,\zeta) \in \partial\Omega \times \partial\Omega : 0 < x < \xi < \sigma_0,\ \xi^{\alpha+1} \le \xi^\beta - x^\beta,\\ \sigma_0 \ge d(z,\zeta) \ge x^{\beta/(\alpha+1)}\}.\end{aligned}$$

Then, by (39) and Lemma 4

$$N \geq \int_D \frac{|u(z)-u(\zeta)|^2}{d(z,\zeta)\mu(B(z,d(z,\zeta)))} d\mu d\mu \simeq \int_D \frac{|u(z)-u(\zeta)|^2}{d(z,\zeta)^{\alpha+2}} d\mu d\mu := M.$$

By (41) there is a positive constant $k > 0$ such that

$$d(z,\zeta) \geq \Big(\frac{(\xi-x)\xi^{\beta-1}}{k}\Big)^{1/(\alpha+1)}$$

and thus $\{(z,\zeta) \in \partial\Omega \times \partial\Omega : 0 < x < \xi < \sigma_0,\ \xi^{\alpha+1} \leq \xi^\beta - x^\beta,\ (\xi - x)\xi^{\beta-1} \geq kx^\beta\} \subset D$. Then, letting

$$E = \{(x,\xi) : 0 < x < \xi < \sigma_0,\ \xi^{\alpha+1} \leq \xi^\beta - x^\beta,\ (\xi-x)\xi^{\beta-1} \geq kx^\beta\}.$$

we have

$$\begin{aligned} M &\simeq \int_E \frac{|x^{-\beta\gamma} - \xi^{-\beta\gamma}|^2 |x\xi|^{\beta-1}}{\big((\xi-x)^{1/(\alpha+1)}\xi^{(\beta-1)/(\alpha+1)}\big)^{\alpha+2}}\, dx d\xi \\ &\simeq \int_E \frac{(\xi^{\beta\gamma} - x^{\beta\gamma})^2}{x^{2\beta\gamma-\beta+1}\xi^{\varphi(\alpha,\beta,\gamma)}(\xi-x)^{(\alpha+2)/(\alpha+1)}}\, dx d\xi, \end{aligned}$$

where $\varphi(\alpha,\beta,\gamma) = 2\beta\gamma - \beta + 1 + (\alpha+2)(\beta-1)/(\alpha+1)$.

In order to separate the integration variables we perform in the last integral the change of variable $x = \xi t$. The integration domain E changes in the following way. The relation $0 < x < \xi < \sigma_0$ gives $0 < t < 1$, the relation $(\xi - x)\xi^{\beta-1} \geq kx^\beta$ gives $(1-t) \geq kt^\beta$, and finally the relation $\xi^{\alpha+1} \leq \xi^\beta - x^\beta$ gives $t^\beta \leq 1 + \xi^{\alpha-\beta+1}$ which is implied by the first one. This shows that in the new integral we may integrate on the square $\{(t,\xi) : 0 < t, \xi < \delta\}$ where $\delta > 0$ is a small but positive constant. Thus we find

$$M \geq \int_0^\delta \frac{d\xi}{\xi^{\varphi(\alpha,\beta,\gamma)-\beta+(\alpha+2)/(\alpha+1)}} \int_0^\delta \frac{(1-t^{\beta\gamma})^2}{t^{2\beta\gamma-\beta+1}(1-t)^{(\alpha+2)/(\alpha+1)}}\, dt.$$

If $\psi(\alpha,\beta,\gamma) := \varphi(\alpha,\beta,\gamma) - \beta + (\alpha+2)/(\alpha+1) \geq 1$ then $M = +\infty$, which implies $N = +\infty$. Now, $\psi(\alpha,\beta,\gamma) = 2\beta\gamma - 2\beta + \beta(\alpha+2)/(\alpha+1) + 1$, and hence $\psi(\alpha,\beta,\gamma) \geq 1$ if and only if $\gamma \geq \alpha/(2\alpha+2)$. Finally

$$\gamma \geq \frac{\alpha}{2(\alpha+1)} \quad \Rightarrow \quad N = +\infty. \tag{42}$$

Notice that if $\beta \in (0, \alpha+1)$ then

$$\frac{\alpha}{2(\alpha+1)} < \frac{2\alpha+1-\beta}{2\beta},$$

and we can therefore choose

$$\gamma \in \Big[\frac{\alpha}{2(\alpha+1)}, \frac{2\alpha+1-\beta}{2\beta}\Big).$$

The interval becomes empty when $\beta = \alpha+1$, i.e. exactly when the domain Ω becomes $\alpha-$admissible. With such a choice $I < +\infty$ by (35) and $N = +\infty$ by (42). □

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