

HEIGHT ESTIMATE AND SLICING FORMULAS IN THE HEISENBERG GROUP

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ABSTRACT. We prove a height-estimate (distance from the tangent hyperplane) for Λ -minimizers of the perimeter in the sub-Riemannian Heisenberg group. The estimate is in terms of a power of the excess (L^2 -mean oscillation of the normal) and its proof is based on a new coarea formula for rectifiable sets in the Heisenberg group.

1. INTRODUCTION

In this article, we continue the research project started in [21] and [18] on the regularity of H -perimeter minimizing boundaries in the Heisenberg group \mathbb{H}^n . Our goal is to prove the so-called *height-estimate* for sets that are Λ -minimizers and have small *excess* inside suitable cylinders, see Theorem 1.3. The proof follows the scheme of the median choice for the measure of the boundary in certain half-cylinders together with a lower dimensional isoperimetric inequality on slices. For minimizing currents in \mathbb{R}^n , the principal ideas of the argument go back to Almgren's paper [1] and are carried over by Federer in his Theorem 5.3.4 in [5]. The argument can be also found in the Appendix of [22] and, for Λ -minimizers of perimeter in \mathbb{R}^n , in [13, Section 22.2]. For minimizers of H -perimeter, the decay estimate of excess of Almgren and De Giorgi is still an open problem, see [19].

Our main technical effort is the proof of a coarea formula (slicing formula) for intrinsic rectifiable sets, see Theorem 1.5. This formula is established in Section 2 and has a nontrivial character because the domain of integration and its slices need not be rectifiable in the standard sense. The relative isoperimetric inequalities that are used in the slices reduce to a single isoperimetric inequality in one slice that is relative to a family of varying domains with uniform isoperimetric constants. This uniformity can be established using the results on regular domains in Carnot groups of step 2 of [20] and the isoperimetric inequality in [9], see Section 3.1.

The $2n + 1$ -dimensional Heisenberg group is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \in \mathbb{N}$, endowed with the group product

$$(z, t) * (\zeta, \tau) = (z + \zeta, t + \tau + 2 \operatorname{Im}\langle z, \bar{\zeta} \rangle), \quad (1.1)$$

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where $t, \tau \in \mathbb{R}$, $z, \zeta \in \mathbb{C}^n$ and $\langle z, \bar{\zeta} \rangle = z_1 \bar{\zeta}_1 + \dots + z_n \bar{\zeta}_n$. The Lie algebra of left-invariant vector fields in \mathbb{H}^n is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad \text{and} \quad T = \frac{\partial}{\partial t}, \quad (1.2)$$

with $z_j = x_j + iy_j$ and $j = 1, \dots, n$. We denote by H the horizontal sub-bundle of $T\mathbb{H}^n$. Namely, for any $p = (z, t) \in \mathbb{H}^n$ we let

$$H_p = \text{span}\{X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)\}.$$

A horizontal section $\varphi \in C_c^1(\Omega; H)$, where $\Omega \subset \mathbb{H}^n$ is an open set, is a vector field of the form

$$\varphi = \sum_{j=1}^n \varphi_j X_j + \varphi_{n+j} Y_j,$$

where $\varphi_j \in C_c^1(\Omega)$, i.e., each coordinate φ_j is a continuously differentiable function with compact support contained in Ω .

Let g be the left-invariant Riemannian metric on \mathbb{H}^n that makes orthonormal the vector fields X_1, \dots, Y_n, T in (1.2). For tangent vectors $V, W \in T\mathbb{H}^n$ we let

$$\langle V, W \rangle_g = g(V, W) \quad \text{and} \quad |V|_g = g(V, V)^{1/2}.$$

The sup-norm with respect to g of a horizontal section $\varphi \in C_c^1(\Omega; H)$ is

$$\|\varphi\|_g = \max_{p \in \Omega} |\varphi(p)|_g.$$

The Riemannian divergence of φ is

$$\text{div}_g \varphi = \sum_{j=1}^n X_j \varphi_j + Y_j \varphi_{n+j}.$$

The metric g induces a volume form on \mathbb{H}^n that is left-invariant. Also the Lebesgue measure \mathcal{L}^{2n+1} on \mathbb{H}^n is left-invariant, and by the uniqueness of the Haar measure the volume induced by g is the Lebesgue measure \mathcal{L}^{2n+1} . In fact, the proportionality constant is 1.

The H -perimeter of a \mathcal{L}^{2n+1} -measurable set $E \subset \mathbb{H}^n$ in an open set $\Omega \subset \mathbb{H}^n$ is

$$\mu_E(\Omega) = \sup \left\{ \int_E \text{div}_g \varphi \, d\mathcal{L}^{2n+1} : \varphi \in C_c^1(\Omega; H), \|\varphi\|_g \leq 1 \right\}.$$

If $\mu_E(\Omega) < \infty$ we say that E has finite H -perimeter in Ω . If $\mu_E(A) < \infty$ for any open set $A \subset \subset \Omega$, we say that E has locally finite H -perimeter in Ω . In this case, the open sets mapping $A \mapsto \mu_E(A)$ extends to a Radon measure μ_E on Ω that is called H -perimeter measure induced by E . Moreover, there exists a μ_E -measurable function $\nu_E : \Omega \rightarrow H$ such that $|\nu_E|_g = 1$ μ_E -a.e. and the Gauss-Green integration by parts formula

$$\int_{\Omega} \langle \varphi, \nu_E \rangle_g \, d\mu_E = - \int_{\Omega} \text{div}_g \varphi \, d\mathcal{L}^{2n+1}$$

holds for any $\varphi \in C_c^1(\Omega; H)$. The vector ν_E is called *horizontal inner normal* of E in Ω .

The Korányi norm of $p = (z, t) \in \mathbb{H}^n$ is $\|p\|_K = (|z|^4 + t^2)^{1/4}$. For any $r > 0$ and $p \in \mathbb{H}^n$, we define the balls

$$B_r = \{q \in \mathbb{H}^n : \|q\|_K < r\} \quad \text{and} \quad B_r(p) = \{p * q \in \mathbb{H}^n : q \in B_r\}.$$

The *measure theoretic boundary* of a measurable set $E \subset \mathbb{H}^n$ is the set

$$\partial E = \{p \in \mathbb{H}^n : \mathcal{L}^{2n+1}(E \cap B_r(p)) > 0 \text{ and } \mathcal{L}^{2n+1}(B_r(p) \setminus E) > 0 \text{ for all } r > 0\}.$$

For a set E with locally finite H -perimeter, the H -perimeter measure μ_E is concentrated on ∂E and, actually, on a subset $\partial^* E$ of ∂E , see below. Moreover, up to modifying E on a Lebesgue negligible set, one can always assume that ∂E coincides with the topological boundary of E , see [23, Proposition 2.5].

Definition 1.1. Let $\Omega \subset \mathbb{H}^n$ be an open set, $\Lambda \in [0, \infty)$, and $r \in (0, \infty]$. We say that a set $E \subset \mathbb{H}^n$ with locally finite H -perimeter in Ω is a (Λ, r) -*minimizer of H -perimeter in Ω* if, for any measurable set $F \subset \mathbb{H}^n$, $p \in \Omega$, and $s < r$ such that $E\Delta F \subset\subset B_s(p) \subset\subset \Omega$, there holds

$$\mu_E(B_s(p)) \leq \mu_F(B_s(p)) + \Lambda \mathcal{L}^{2n+1}(E\Delta F),$$

where $E\Delta F = E \setminus F \cup F \setminus E$.

We say that E is *locally H -perimeter minimizing in Ω* if, for any measurable set $F \subset \mathbb{H}^n$ and any open set U such that $E\Delta F \subset\subset U \subset\subset \Omega$, there holds $\mu_E(U) \leq \mu_F(U)$.

We will often use the term Λ -*minimizer*, rather than (Λ, r) -minimizer, when the role of r is not relevant. In Appendix A, we list without proof some elementary properties of Λ -minimizers.

We introduce the notion of cylindrical excess. The *height function* $\mathfrak{h} : \mathbb{H}^n \rightarrow \mathbb{R}$ is defined by $\mathfrak{h}(p) = p_1$, where p_1 is the first coordinate of $p = (p_1, \dots, p_{2n+1}) \in \mathbb{H}^n$. The set $\mathbb{W} = \{p \in \mathbb{H}^n : \mathfrak{h}(p) = 0\}$ is the vertical hyperplane passing through $0 \in \mathbb{H}^n$ and orthogonal to the left-invariant vector field X_1 . The disk in \mathbb{W} of radius $r > 0$ centered at $0 \in \mathbb{W}$ induced by the Korányi norm is the set $D_r = \{p \in \mathbb{W} : \|p\|_K < r\}$. The intrinsic cylinder with central section D_r and height $2r$ is the set

$$C_r = D_r * (-r, r) \subset \mathbb{H}^n.$$

Here and in the sequel, we use the notation $D_r * (-r, r) = \{w * (se_1) \in \mathbb{H}^n : w \in D_r, s \in (-r, r)\}$, where $se_1 = (s, 0, \dots, 0) \in \mathbb{H}^n$. The cylinder C_r is comparable with the ball $B_r = \{\|p\|_K < r\}$. Namely, there exists a constant $k = k(n) \geq 1$ such that for any $r > 0$ we have

$$B_{r/k} \subset C_r \subset B_{kr}. \tag{1.3}$$

By a rotation of the system of coordinates, it is enough to consider excess in cylinders with basis in \mathbb{W} and axis X_1 .

Definition 1.2 (Cylindrical excess). Let $E \subset \mathbb{H}^n$ be a set with locally finite H -perimeter. The cylindrical excess of E at the point $0 \in \partial E$, at scale $r > 0$, and with respect to the direction $\nu = -X_1$ is defined as

$$\text{Exc}(E, r, \nu) = \frac{1}{2r^{2n+1}} \int_{C_r} |\nu_E - \nu|_g^2 d\mu_E,$$

where μ_E is the H -perimeter measure of E and ν_E is its horizontal inner normal.

Theorem 1.3 (Height estimate). *Let $n \geq 2$. There exist constants $\varepsilon_0 = \varepsilon_0(n) > 0$ and $c_0 = c_0(n) > 0$ with the following property. If $E \subset \mathbb{H}^n$ is a (Λ, r) -minimizer of H -perimeter in the cylinder C_{4k^2r} , $\Lambda r \leq 1$, $0 \in \partial E$, and*

$$\text{Exc}(E, 4k^2r, \nu) \leq \varepsilon_0,$$

then

$$\sup \{ |\mathcal{H}^n(p)| \in [0, \infty) : p \in \partial E \cap C_r \} \leq c_0 r \text{Exc}(E, 4k^2r, \nu)^{\frac{1}{2(2n+1)}}. \quad (1.4)$$

The constant $k = k(n)$ is the one in (1.3).

The estimate (1.4) does not hold when $n = 1$. In fact, there are sets $E \subset \mathbb{H}^1$ such that $\text{Exc}(E, r, \nu) = 0$ but ∂E is not flat in $C_{\varepsilon r}$ for any $\varepsilon > 0$. See the conclusions of Proposition 3.7 in [18]. Theorem 1.3 is proved in Section 3.

Besides local minimizers of H -perimeter, our interest in Λ -minimizers is also motivated by possible applications to isoperimetric sets. The height estimate is a first step in the regularity theory of Λ -minimizers of classical perimeter; we refer to [13, Part III] for a detailed account on the subject.

In order to state the slicing formula in its general form, we need the definition of a rectifiable set in \mathbb{H}^n of codimension 1. We follow closely [7], where this notion was first introduced.

The Riemannian and horizontal gradients of a function $f \in C^1(\mathbb{H}^n)$ are, respectively,

$$\begin{aligned} \nabla f &= (X_1 f)X_1 + \cdots + (Y_n f)Y_n + (T f)T, \\ \nabla_H f &= (X_1 f)X_1 + \cdots + (Y_n f)Y_n. \end{aligned}$$

We say that a continuous function $f \in C(\Omega)$, with $\Omega \subset \mathbb{H}^n$ open set, is of class $C_H^1(\Omega)$ if the horizontal gradient $\nabla_H f$ exists in the sense of distributions and is represented by continuous functions $X_1 f, \dots, Y_n f$ in Ω . A set $S \subset \mathbb{H}^n$ is an H -regular hypersurface if for all $p \in S$ there exist $r > 0$ and a function $f \in C_H^1(B_r(p))$ such that $S \cap B_r(p) = \{q \in B_r(p) : f(q) = 0\}$ and $\nabla_H f(p) \neq 0$. Sets with H -regular boundary have locally finite H -perimeter.

For any $p = (z, t) \in \mathbb{H}^n$, let us define the box-norm $\|p\|_\infty = \max\{|z|, |t|^{1/2}\}$ and the balls $U_r = \{q \in \mathbb{H}^n : \|q\|_\infty < r\}$ and $U_r(p) = p * U_r$, with $r > 0$. Let $E \subset \mathbb{H}^n$ be a set. For any $s \geq 0$ define the measure

$$\mathcal{S}^s(E) = \sup_{\delta > 0} \inf \left\{ c(n, s) \sum_{i \in \mathbb{N}} r_i^s : E \subset \bigcup_{i \in \mathbb{N}} U_{r_i}(p_i), r_i < \delta \right\}.$$

Above, $c(n, s) > 0$ is a normalization constant that we do not need to specify here. By Carathéodory's construction, $E \mapsto \mathcal{S}^s(E)$ is a Borel measure in \mathbb{H}^n . When $s = 2n + 2$, \mathcal{S}^{2n+2} turns out to be the Lebesgue measure \mathcal{L}^{2n+1} . Thus, the correct dimension to measure hypersurfaces is $s = 2n + 1$. In fact, if E is a set with locally finite H -perimeter in \mathbb{H}^n , then we have

$$\mu_E = \mathcal{S}^{2n+1} \llcorner \partial^* E, \quad (1.5)$$

where \llcorner denotes restriction and $\partial^* E$ is the H -reduced boundary of E , namely the set of points $p \in \mathbb{H}^n$ such that $\mu_E(U_r(p)) > 0$ for all $r > 0$, $\int_{U_r(p)} \nu_E d\mu_E \rightarrow \nu_E(p)$ as $r \rightarrow 0$ and $|\nu_E(p)|_g = 1$. The validity of formula (1.5) depends on the geometry of the balls $U_r(p)$, see [16]. We refer the reader to [7] for more details on the H -reduced boundary.

Definition 1.4. A set $R \subset \mathbb{H}^n$ is \mathcal{S}^{2n+1} -rectifiable if there exists a sequence of H -regular hypersurfaces $(S_j)_{j \in \mathbb{N}}$ in \mathbb{H}^n such that

$$\mathcal{S}^{2n+1} \left(R \setminus \bigcup_{j \in \mathbb{N}} S_j \right) = 0.$$

By the results of [7], the H -reduced boundary $\partial^* E$ is \mathcal{S}^{2n+1} -rectifiable. Definition 1.4 is generalized in [17], where the authors study the notion of an s -rectifiable set in \mathbb{H}^n for any integer $1 \leq s \leq 2n + 1$.

An H -regular surface S has a continuous horizontal normal ν_S that is locally defined up to the sign. This normal is given by the formula

$$\nu_S = \frac{\nabla_H f}{|\nabla_H f|_g}, \quad (1.6)$$

where f is a defining function for S . When $S = \partial E$ is the boundary of a smooth set, then ν_S agrees with the horizontal normal ν_E . Then, for an \mathcal{S}^{2n+1} -rectifiable set $R \subset \mathbb{H}^n$ there is a unit horizontal normal $\nu_R : R \rightarrow H$ that is Borel regular. This normal is uniquely defined \mathcal{S}^{2n+1} -a.e. on R up to the sign, see Appendix B. However, formula (1.8) below does not depend on the sign.

In the following theorem, $\Omega \subset \mathbb{H}^n$ is an open set and $u \in C^\infty(\Omega)$ is a smooth function. For any $s \in \mathbb{R}$, we denote by $\Sigma^s = \{p \in \Omega : u(p) = s\}$ the level sets of u .

Theorem 1.5. *Let $R \subset \Omega$ be an \mathcal{S}^{2n+1} -rectifiable set. Then, for a.e. $s \in \mathbb{R}$ there exists a Radon measure μ_R^s on $R \cap \Sigma^s$ such that for any Borel function $h : \Omega \rightarrow [0, \infty)$ the function*

$$s \mapsto \int_\Omega h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_R^s \quad (1.7)$$

is \mathcal{L}^1 -measurable, and we have the coarea formula

$$\int_{\mathbb{R}} \int_\Omega h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_R^s ds = \int_R h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} d\mathcal{S}^{2n+1}. \quad (1.8)$$

Theorem 1.5 is proved in Section 2. When $R \cap \Sigma^s$ is a regular subset of Σ^s , the measures μ_R^s are natural horizontal perimeters defined in Σ^s .

Coarea formulae in the Heisenberg group are known only for slicing of sets with positive Lebesgue measure, see [14, 15]. Theorem 1.5 is, to our knowledge, the first example of slicing of lower-dimensional sets in a sub-Riemannian framework. Also, Theorem 1.5 is a nontrivial extension of the Riemannian coarea formula, because the set R and the slices $R \cap \Sigma^s$ need not be rectifiable in the standard sense, see [12]. We need the coarea formula (1.8) in the proof of Theorem 1.3, see Section 3.3.

We conclude the introduction by stating a different but equivalent formulation of the coarea formula (1.8) that is closer to standard coarea formulae. This alternative formulation holds only when $n \geq 2$: when $n = 1$, the right hand side in (1.9) might not be well defined, see Remark 2.11.

Theorem 1.6. *Let $\Omega \subset \mathbb{H}^n$, $n \geq 2$, be an open set, $u \in C^\infty(\Omega)$ be a smooth function, and $R \subset \Omega$ be an \mathcal{S}^{2n+1} -rectifiable set. Then, for any Borel function $h : \Omega \rightarrow [0, \infty)$ there holds*

$$\int_{\mathbb{R}} \int_{\Omega} h d\mu_R^s ds = \int_R h |\nabla u|_g \sqrt{1 - \left\langle \nu_R, \frac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g^2} d\mathcal{S}^{2n+1}, \quad (1.9)$$

where μ_R^s are the measures given by Theorem 1.5.

2. PROOF OF THE COAREA FORMULA

2.1. Horizontal perimeter on submanifolds. Let $\Sigma \subset \mathbb{H}^n$ be a C^∞ hypersurface. We define the horizontal tangent bundle $H\Sigma$ letting, for any $p \in \Sigma$,

$$H_p\Sigma = H_p \cap T_p\Sigma.$$

In general, the rank of $H\Sigma$ is not constant. This depends on the presence of *characteristic points* on Σ , i.e., points such that $H_p = T_p\Sigma$. For points $p \in \Sigma$ such that $H_p \neq T_p\Sigma$, we have $\dim(H_p\Sigma) = 2n - 1$.

We denote by σ_Σ the surface measure on Σ induced by the Riemannian metric g restricted to the tangent bundle $T\Sigma$.

Definition 2.1. Let $F \subset \Sigma$ be a Borel set and let $\Omega \subset \Sigma$ be an open set. We define the *H-perimeter of F in Ω*

$$\mu_F^\Sigma(\Omega) = \sup \left\{ \int_F \operatorname{div}_g \varphi d\sigma_\Sigma : \varphi \in C_c^1(\Omega; H\Sigma), \|\varphi\|_g \leq 1 \right\}. \quad (2.10)$$

We say that the set $F \subset \Sigma$ has locally finite *H-perimeter* in Ω if $\mu_F^\Sigma(A) < \infty$ for any open set $A \subset \subset \Omega$.

By Riesz' theorem, if $F \subset \Sigma$ has locally finite *H-perimeter* in Ω , then the open sets mapping $A \mapsto \mu_F^\Sigma(A)$ extends to a Radon measure on Ω , called *H-perimeter measure* of F .

Remark 2.2. If $F \subset \Sigma$ is an open set with smooth boundary, then by the divergence theorem we have, for any $\varphi \in C_c^1(\Omega; H\Sigma)$,

$$\int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma = \int_{\partial F} \langle N_{\partial F}, \varphi \rangle_g \, d\lambda_{\partial F}, \quad (2.11)$$

where $N_{\partial F}$ is the Riemannian outer unit normal to ∂F and $d\lambda_{\partial F}$ is the Riemannian $(2n-1)$ -dimensional volume form on ∂F induced by g .

From the sup-definition (2.10) and from (2.11), we deduce that the H -perimeter measure of F has the following representation

$$\mu_F^\Sigma = |N_{\partial F}^{H\Sigma}|_g \lambda_{\partial F},$$

where $N_{\partial F}^{H\Sigma} \in H\Sigma$ is the g -orthogonal projection of $N_{\partial F} \in T\Sigma$ onto $H\Sigma$.

This formula can be generalized as follows. We denote by \mathcal{H}_g^{2n-1} the $(2n-1)$ -dimensional Hausdorff measure in \mathbb{H}^n induced by the metric g .

Lemma 2.3. *Let $F, \Omega \subset \Sigma$ be open sets and assume that there exists a compact set $N \subset \partial F$ such that $\mathcal{H}_g^{2n-1}(N) = 0$ and $(\partial F \setminus N) \cap \Omega$ is a smooth $(2n-1)$ -dimensional surface. Then, we have*

$$\mu_F^\Sigma \llcorner \Omega = |N_{\partial F}^{H\Sigma}|_g \lambda_{\partial F \setminus N} \llcorner \Omega. \quad (2.12)$$

Proof. For any $\varepsilon > 0$ there exist points $p_i \in \mathbb{H}^n$ and radii $r_i \in (0, 1)$, $i = 1, \dots, M$, such that

$$N \subset \bigcup_{i=1}^M B_g(p_i, r_i) \quad \text{and} \quad \sum_{i=1}^M r_i^{2n-1} < \varepsilon,$$

where $B_g(p, r)$ denotes the ball in \mathbb{H}^n with center p and radius r with respect to the metric g . By a partition-of-the-unity argument, there exist functions $f^\varepsilon, g_i^\varepsilon \in C^\infty(\Omega; [0, 1])$, $i = 1, \dots, M$, such that

- i) $f^\varepsilon + g_1^\varepsilon + \dots + g_M^\varepsilon = \chi_\Omega$;
- ii) $f^\varepsilon = 0$ on $\bigcup_{i=1}^M B_g(p_i, r_i/2)$;
- iii) $\operatorname{spt} g_i^\varepsilon \subset B_g(p_i, r_i)$ for each i ;
- iv) $|\nabla g_i^\varepsilon|_g \leq C r_i^{-1}$ for a constant $C > 0$ independent of ε .

Hence, for any horizontal section $\varphi \in C_c^1(\Omega; H\Sigma)$ we have

$$\begin{aligned} \int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma &= \int_F \operatorname{div}_g (f^\varepsilon \varphi) \, d\sigma_\Sigma + \sum_{i=1}^M \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g (g_i^\varepsilon \varphi) \, d\sigma_\Sigma \\ &= \int_{\partial F \setminus N} \langle f^\varepsilon \varphi, N_{\partial F} \rangle_g \, d\lambda_{\partial F \setminus N} + \sum_{i=1}^M \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g (g_i^\varepsilon \varphi) \, d\sigma_\Sigma, \end{aligned} \quad (2.13)$$

where, by iv),

$$\begin{aligned} \left| \sum_{i=1}^M \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g(g_i^\varepsilon \varphi) d\sigma_\Sigma \right| &\leq \sum_{i=1}^M \int_{B_g(p_i, r_i)} (\|\operatorname{div}_g \varphi\|_{L^\infty} + Cr_i^{-1}) d\sigma_\Sigma \\ &\leq C' \sum_{i=1}^M r_i^{2n-1} \leq C' \varepsilon, \end{aligned} \quad (2.14)$$

with a constant $C' > 0$ independent of ε .

Letting $\varepsilon \rightarrow 0$, we have $f^\varepsilon \rightarrow 1$ pointwise on $\partial F \setminus N$, by i) and iii). Then, from (2.13) and (2.14) we obtain

$$\int_F \operatorname{div}_g \varphi d\sigma_\Sigma = \int_{\partial F \setminus N} \langle \varphi, N_{\partial F} \rangle_g d\lambda_{\partial F \setminus N}$$

and claim (2.12) follows by standard arguments. \square

2.2. Proof of Theorem 1.5. Let $\Omega \subset \mathbb{H}^n$ be an open set and let $u \in C^\infty(\Omega)$. By Sard's theorem, for a.e. $s \in \mathbb{R}$ the level set

$$\Sigma^s = \{p \in \Omega : u(p) = s\}$$

is a smooth hypersurface and, moreover, we have $\nabla u \neq 0$ on Σ^s .

Let $E \subset \mathbb{H}^n$ be a Borel set such that $E \cap \Sigma^s$ has (locally) finite H -perimeter in $\Omega \cap \Sigma^s$, in the sense of Definition 2.1. Then on $\Omega \cap \Sigma^s$ we have the H -perimeter measure $\mu_{E \cap \Sigma^s}^s$ induced by $E \cap \Sigma^s$. We shall use the notation

$$\mu_E^s = \mu_{E \cap \Sigma^s}^s$$

to denote a measure on Ω that is supported on $\Omega \cap \Sigma^s$.

We start with the following coarea formula in the smooth case, that is deduced from the Riemannian formula.

Lemma 2.4. *Let $\Omega \subset \mathbb{H}^n$ be an open set and $u \in C^\infty(\Omega)$. Let $E \subset \mathbb{H}^n$ be an open set with C^∞ boundary in Ω such that $\mu_E(\Omega) < \infty$. Then we have*

$$\int_{\mathbb{R}} \int_{\Omega} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds = \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E, \quad (2.15)$$

where μ_E is the H -perimeter measure of E and ν_E is its horizontal normal.

Proof. The integral in the left hand side is well defined, because for a.e. $s \in \mathbb{R}$ there holds $\nabla u \neq 0$ on Σ^s . By the coarea formula for Riemannian manifolds, see e.g. [4], for any Borel function $h : \partial E \rightarrow [0, \infty]$ we have

$$\int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} h d\lambda_{\partial E \cap \Sigma^s} ds = \int_{\partial E} h |\nabla^{\partial E} u|_g d\sigma_{\partial E}, \quad (2.16)$$

where $\nabla^{\partial E} u$ is the tangential gradient of u on ∂E . Then we have

$$\nabla^{\partial E} u = \nabla u - \langle \nabla u, N_{\partial E} \rangle_g N_{\partial E} \quad \text{and} \quad |\nabla^{\partial E} u|_g = \sqrt{|\nabla u|_g^2 - \langle \nabla u, N_{\partial E} \rangle_g^2}. \quad (2.17)$$

Step 1. Let us define the set

$$C = \left\{ p \in \partial E \cap \Omega : \nabla u(p) \neq 0 \text{ and } N_{\partial E}(p) = \pm \frac{\nabla u(p)}{|\nabla u(p)|_g} \right\}.$$

If $s \in \mathbb{R}$ is such that $\nabla u \neq 0$ on Σ^s , then $C \cap \Sigma^s$ is a closed set in Σ^s . Using the coarea formula (2.16) with the function $h = \chi_C$, we get

$$\int_{\mathbb{R}} \lambda_{\partial E \cap \Sigma^s}(C) ds = \int_C |\nabla^{\partial E} u|_g d\sigma_{\partial E} = 0,$$

because we have $\nabla^{\partial E} u = 0$ on C . In particular, we deduce that

$$C \cap \Sigma^s \text{ is a closed set in } \Sigma^s \quad \text{and} \quad \lambda_{\partial E \cap \Sigma^s}(C \cap \Sigma^s) = 0 \quad \text{for a.e. } s \in \mathbb{R}. \quad (2.18)$$

If $p \in \Sigma^s$ is a point such that $\nabla u(p) \neq 0$ and $p \notin C$, then Σ^s is a smooth hypersurface in a neighbourhood of p and $E^s = E \cap \Sigma^s$ is a domain in Σ^s with smooth boundary in a neighbourhood of p . Moreover, we have $(\partial E \cap \Sigma^s) \setminus C = \partial E^s \setminus C$. Then, from (2.18) and from Lemma 2.3 we conclude that for a.e. $s \in \mathbb{R}$ we have

$$\mu_E^s = |N_{\partial E^s}^{H\Sigma^s}|_g \lambda_{\partial E^s}. \quad (2.19)$$

By (2.18) and (2.19), there holds

$$\mu_E^s(C \cap \Sigma^s) = \int_{C \cap \Sigma^s} |N_{\partial E^s}^{H\Sigma^s}|_g d\lambda_{\partial E^s} = 0 \quad \text{for a.e. } s \in \mathbb{R}. \quad (2.20)$$

Step 2. We prove (2.15) by plugging into (2.16) the Borel function $h : \partial E \rightarrow [0, \infty]$

$$h = \begin{cases} \frac{|N_{\partial E}^H|_g \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2}}{|\nabla u|_g \sqrt{1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g^2}} & \text{on } \partial E \setminus (C \cup \{\nabla u = 0\}) \\ 0 & \text{on } C \cup \{\nabla u = 0\}. \end{cases}$$

Above, $N_{\partial E}^H$ is the projection of the Riemannian normal $N_{\partial E}$ onto H and ν_E is the horizontal normal. Namely, we have

$$N_{\partial E}^H = N_{\partial E} - \langle N_{\partial E}, T \rangle_g T \quad \text{and} \quad \nu_E = \frac{N_{\partial E}^H}{|N_{\partial E}^H|_g}.$$

The H -perimeter measure of E is

$$\mu_E = |N_{\partial E}^H|_g \sigma_{\partial E}. \quad (2.21)$$

Using (2.17) and (2.21), we find

$$\begin{aligned} \int_{\partial E} h |\nabla^{\partial E} u| d\sigma_{\partial E} &= \int_{\partial E \setminus (C \cup \{\nabla u = 0\})} |N_{\partial E}^H|_g \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\sigma_{\partial E} \\ &= \int_{\partial E \setminus (C \cup \{\nabla u = 0\})} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E \\ &= \int_{\partial E} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E, \end{aligned} \quad (2.22)$$

where the last equality is justified by the fact that if $p \in C \cup \{\nabla u = 0\}$ then

$$\sqrt{|\nabla_H u(p)|_g^2 - \langle \nu_E(p), \nabla_H u(p) \rangle_g^2} = 0.$$

For a.e. $s \in \mathbb{R}$, we have $\nabla u \neq 0$ on Σ^s . Using (2.21) and the fact that $h = 0$ on $C \cup \{\nabla_H u = 0\}$, letting $\Lambda^s = (\partial E \cap \Sigma^s) \setminus (C \cup \{\nabla_H u = 0\})$ we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} h \, d\lambda_{\partial E^s} \, ds &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|N_{\partial E}^H|_g \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2}}{|\nabla u|_g \sqrt{1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g^2}} \, d\lambda_{\partial E^s} \, ds \\ &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} \vartheta^s \, d\lambda_{\partial E^s} \, ds, \end{aligned} \quad (2.23)$$

where we let

$$\vartheta^s = \frac{\sqrt{|N_{\partial E}^H|_g^2 - \langle N_{\partial E}^H, \frac{\nabla_H u}{|\nabla_H u|_g} \rangle_g^2}}{\sqrt{1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g^2}}.$$

We will prove in *Step 3* that, for any $s \in \mathbb{R}$ such that $\nabla u \neq 0$ on Σ^s , there holds

$$\vartheta^s = |N_{\partial E^s}^{H\Sigma^s}|_g \quad \text{on } \Lambda^s. \quad (2.24)$$

Using (2.24), (2.19), and (2.20) formula (2.23) becomes

$$\begin{aligned} \int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} h \, d\lambda_{\partial E \cap \Sigma^s} \, ds &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} |N_{\partial E^s}^{H\Sigma^s}|_g \, d\lambda_{\partial E^s} \, ds \\ &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \\ &= \int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds. \end{aligned} \quad (2.25)$$

The proof is complete, because (2.15) follows from (2.16), (2.22) and (2.25).

Step 3. We prove claim (2.24). Let us introduce the vector field W in $\Omega \setminus \{\nabla_H u = 0\}$

$$W = \frac{Tu}{|\nabla u|_g} \frac{\nabla_H u}{|\nabla_H u|_g} - \frac{|\nabla_H u|_g T}{|\nabla u|_g}.$$

It can be checked that $|W|_g = 1$ and $Wu = 0$. In particular, for a.e. s we have $W \in T\Sigma^s$. Moreover, W is g -orthogonal to $H\Sigma^s$ because any vector in $H\Sigma^s$ is orthogonal both to $\nabla_H u$ and to T . It follows that

$$N_{\partial E^s}^{H\Sigma^s} = N_{\partial E^s} - \langle N_{\partial E^s}, W \rangle_g$$

and, in particular,

$$|N_{\partial E^s}^{H\Sigma^s}|_g^2 = 1 - \langle N_{\partial E^s}, W \rangle_g^2.$$

Starting from the formula

$$N_{\partial E^s} = \frac{N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g \frac{\nabla u}{|\nabla u|_g}}{|N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g \frac{\nabla u}{|\nabla u|_g}|_g} = \frac{N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g \frac{\nabla u}{|\nabla u|_g}}{\sqrt{1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g^2}},$$

we find

$$|N_{\partial E^s}^{H\Sigma^s}|_g^2 = \frac{M}{1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g^2},$$

where we let

$$M = 1 - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g^2 - \left\langle N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g \frac{\nabla u}{|\nabla u|_g}, W \right\rangle_g^2.$$

We claim that on the open set $\{\nabla_H u \neq 0\}$ there holds

$$M = |N_{\partial E}^H|_g^2 - \langle N_{\partial E}^H, \frac{\nabla_H u}{|\nabla_H u|_g} \rangle_g^2, \quad (2.26)$$

and formula (2.24) follows from (2.26). Using the identity $\nabla u = \nabla_H u + (Tu)T$ and the orthogonality

$$\left\langle N_{\partial E} - \langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \rangle_g \frac{\nabla u}{|\nabla u|_g}, \nabla u \right\rangle_g = 0,$$

we find

$$\begin{aligned} M &= 1 - \left\langle N_{\partial E}, \frac{\nabla_H u + (Tu)T}{|\nabla u|_g} \right\rangle_g^2 - \left(\frac{Tu}{|\nabla u|_g} \langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g} \rangle_g - \frac{|\nabla_H u|_g}{|\nabla u|_g} \langle N_{\partial E}, T \rangle_g \right)^2 \\ &= 1 - \langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g} \rangle_g^2 \frac{|\nabla_H u|_g^2 + (Tu)^2}{|\nabla u|_g^2} - \langle N_{\partial E}, T \rangle_g^2 \frac{|\nabla_H u|_g^2 + (Tu)^2}{|\nabla u|_g^2} \\ &= 1 - \langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g} \rangle_g^2 - \langle N_{\partial E}, T \rangle_g^2 \\ &= 1 - \langle N_{\partial E}, T \rangle_g^2 - \left(\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g} \rangle_g - \langle \langle N_{\partial E}, T \rangle_g T, \frac{\nabla_H u}{|\nabla_H u|_g} \rangle_g \right)^2 \\ &= |N_{\partial E}^H|_g^2 - \langle N_{\partial E}^H, \frac{\nabla_H u}{|\nabla_H u|_g} \rangle_g^2. \end{aligned} \quad (2.27)$$

This ends the proof. \square

We prove a coarea inequality.

Proposition 2.5. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $E \subset \mathbb{H}^n$ a set with finite H -perimeter in Ω , and let $h : \partial E \rightarrow [0, \infty]$ be a Borel function. Then we have*

$$\int_{\mathbb{R}} \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_{\Omega} h \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \quad (2.28)$$

Proof. The coarea inequality (2.28) follows from the smooth case of Lemma 2.4 by an approximation and lower semicontinuity argument.

Step 1. By [6, Theorem 2.2.2], there exists a sequence of smooth sets $(E_j)_{j \in \mathbb{N}}$ in Ω such that

$$\chi_{E_j} \xrightarrow{L^1(\Omega)} \chi_E \quad \text{as } j \rightarrow \infty \quad \text{and} \quad \lim_{j \rightarrow \infty} \mu_{E_j}(\Omega) = \mu(\Omega).$$

By a straightforward adaptation of the proof of [2, Proposition 3.13], we also have that $\nu_{E_j} \mu_{E_j} \rightarrow \nu_E \mu_E$ weakly* in Ω . Namely, for any $\psi \in C_c(\Omega; H)$ there holds

$$\lim_{j \rightarrow \infty} \int_{\Omega} \langle \psi, \nu_{E_j} \rangle_g d\mu_{E_j} = \int_{\Omega} \langle \psi, \nu_E \rangle_g d\mu_E.$$

Let $A \subset\subset \Omega$ be an open set such that $\lim_{j \rightarrow \infty} \mu_{E_j}(A) = \mu_E(A)$. By Reshetnyak's continuity theorem (see e.g. [2, Theorem 2.39]), we have

$$\lim_{j \rightarrow \infty} \int_A f(p, \nu_{E_j}(p)) d\mu_{E_j} = \int_A f(p, \nu_E(p)) d\mu_E$$

for any continuous and bounded function f . In particular,

$$\lim_{j \rightarrow \infty} \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_{E_j}, \nabla_H u \rangle_g^2} d\mu_{E_j} = \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \quad (2.29)$$

Step 2. Let $(E_j)_{j \in \mathbb{N}}$ be the sequence introduced in *Step 1*. Then, for a.e. $s \in \mathbb{R}$ we have

$$\nabla u \neq 0 \text{ on } \Sigma^s \quad \text{and} \quad \chi_{E_j} \rightarrow \chi_E \text{ in } L^1(\Sigma^s, \sigma_{\Sigma^s}) \text{ as } j \rightarrow \infty.$$

In particular, for any such s and for any open set $A \subset \Sigma^s \cap \Omega$ there holds

$$\mu_E^s(A) \leq \liminf_{j \rightarrow \infty} \mu_{E_j}^s(A).$$

From Fatou's Lemma and from the continuity of $\frac{|\nabla_H u|_g}{|\nabla u|_g}$ on Σ^s , it follows that

$$\begin{aligned} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s &= \int_0^\infty \mu_E^s \left(\left\{ p \in A : \frac{|\nabla_H u|_g}{|\nabla u|_g}(p) > t \right\} \right) dt \\ &\leq \int_0^\infty \liminf_{j \rightarrow \infty} \mu_{E_j}^s \left(\left\{ p \in A : \frac{|\nabla_H u|_g}{|\nabla u|_g}(p) > t \right\} \right) dt \\ &\leq \liminf_{j \rightarrow \infty} \int_0^\infty \mu_{E_j}^s \left(\left\{ p \in A : \frac{|\nabla_H u|_g}{|\nabla u|_g}(p) > t \right\} \right) dt \\ &= \liminf_{j \rightarrow \infty} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s. \end{aligned}$$

Using again Fatou's Lemma and Lemma 2.4,

$$\begin{aligned} \int_{\mathbb{R}} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds &\leq \int_{\mathbb{R}} \liminf_{j \rightarrow \infty} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s ds \\ &\leq \liminf_{j \rightarrow \infty} \int_{\mathbb{R}} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s ds \\ &= \liminf_{j \rightarrow \infty} \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_{E_j}, \nabla_H u \rangle_g^2} d\mu_{E_j}. \end{aligned}$$

This, together with (2.29), gives

$$\int_{\mathbb{R}} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E.$$

Step 3. Any open set $A \subset \Omega$ can be approximated by a sequence $(A_k)_{k \in \mathbb{N}}$ of open sets such that

$$A_k \subset \subset \Omega, \quad A_k \subset A_{k+1}, \quad \bigcup_{k=1}^{\infty} A_k = A \quad \text{and} \quad \mu_E(\partial A_k) = 0.$$

In particular, for each $k \in \mathbb{N}$ we have

$$\begin{aligned} \liminf_{j \rightarrow \infty} \mu_{E_j}(A_k) &\leq \limsup_{j \rightarrow \infty} \mu_{E_j}(\bar{A}_k) \leq \mu_E(\bar{A}_k) \\ &= \mu_E(A_k) \leq \liminf_{j \rightarrow \infty} \mu_{E_j}(A_k). \end{aligned}$$

Hence, the inequalities are equalities, i.e., $\mu_E(A_k) = \lim_{j \rightarrow \infty} \mu_{E_j}(A_k)$. By *Step 2*, for any $k \in \mathbb{N}$ there holds

$$\int_{\mathbb{R}} \int_{A_k} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_{A_k} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E.$$

By monotone convergence, letting $k \rightarrow \infty$ we obtain for any open set $A \subset \Omega$

$$\int_{\mathbb{R}} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E.$$

By a standard approximation argument, it is enough to prove (2.28) for the characteristic function $h = \chi_B$ of a Borel set $B \subset \partial E$. Since the measure $\sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \mu_E$ is a Radon measure on ∂E , there exists a sequence of open sets B_j such that $B \subset B_j$ for any $j \in \mathbb{N}$ and

$$\lim_{j \rightarrow \infty} \int_{B_j} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E = \int_B \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E.$$

Therefore, we have

$$\begin{aligned} \int_{\mathbb{R}} \int_B \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds &\leq \liminf_{j \rightarrow \infty} \int_{\mathbb{R}} \int_{B_j} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \\ &\leq \lim_{j \rightarrow \infty} \int_{B_j} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E \\ &= \int_B \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E, \end{aligned}$$

and this concludes the proof. \square

In the next step, we prove an approximate coarea formula for sets E such that the boundary ∂E is an H -regular surface.

Lemma 2.6. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $E \subset \mathbb{H}^n$ an open set such that $\partial E \cap \Omega$ is an H -regular hypersurface, and $\bar{p} \in \partial E \cap \Omega$ a point such that*

$$\nabla_H u(\bar{p}) \neq 0 \quad \text{and} \quad \nu_E(\bar{p}) \neq \pm \frac{\nabla_H u(\bar{p})}{|\nabla_H u(\bar{p})|_g}.$$

Then, for any $\varepsilon > 0$ there exists $\bar{r} = \bar{r}(\bar{p}, \varepsilon) > 0$ such that $B_{\bar{r}}(\bar{p}) \subset \Omega$ and, for any $r \in (0, \bar{r})$,

$$\begin{aligned} (1 - \varepsilon) \int_{B_r(\bar{p})} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E \\ \leq \int_{\mathbb{R}} \int_{B_r(\bar{p})} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \\ \leq (1 + \varepsilon) \int_{B_r(\bar{p})} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \end{aligned}$$

Proof. We can without loss of generality assume that $\bar{p} = 0$ and $u(0) = 0$. We divide the proof into several steps.

Step 1: preliminary considerations. The horizontal vector field $V_{2n} = \frac{\nabla_H u}{|\nabla_H u|_g}$ is well defined in a neighbourhood $\Omega_\varepsilon \subset \mathbb{H}^n$ of 0. For any $s \in \mathbb{R}$, the hypersurface $\Sigma^s = \{p \in \Omega : u(p) = s\}$ is smooth in Ω_ε because $\nabla_H u \neq 0$ on Ω_ε .

There are horizontal vector fields V_1, \dots, V_{2n-1} on Ω_ε such that V_1, \dots, V_{2n} is a g -orthonormal frame. In particular, we have $V_j u = 0$ for all $j = 1, \dots, 2n - 1$, i.e.,

$$H_p \Sigma^s = \text{span}\{V_1(p), \dots, V_{2n-1}(p)\} \quad \text{for all } p \in \Sigma^s \cap \Omega_\varepsilon. \quad (2.30)$$

Possibly shrinking Ω_ε , reordering $\{V_j\}_{j=1, \dots, 2n-1}$, and changing the sign of V_1 , we can assume (see [24, Lemma 4.3 and Lemma 4.4]) that there exist a function $f : \Omega_\varepsilon \rightarrow \mathbb{R}$ and a number $\delta > 0$ such that:

- a) $f \in C_H^1(\Omega_\varepsilon) \cap C^\infty(\Omega_\varepsilon \setminus \partial E)$;
- b) $E \cap \Omega_\varepsilon = \{p \in \Omega_\varepsilon : f(p) > 0\}$;
- c) $V_1 f \geq \delta > 0$ on Ω_ε .

By [24, Remark 4.7], we have also $\nu_E = \frac{\nabla_H f}{|\nabla_H f|_g}$ on $\partial E \cap \Omega_\varepsilon$.

Step 2: change of coordinates. Let $S \subset \mathbb{H}^n$ be a $(2n - 1)$ -dimensional smooth submanifold such that:

- i) $0 \in S$;
- ii) $S \subset \Sigma^0 \cap \Omega_\varepsilon$; in particular, ∇u is g -orthogonal to S ;
- iii) $V_1(0)$ is g -orthogonal to S at 0;
- iv) there exists a diffeomorphism $H : U \rightarrow \mathbb{H}^n$, where $U \subset \mathbb{R}^{2n-1}$ is an open set with $0 \in U$, such that $H(0) = 0$ and $H(U) = S \cap \Omega_\varepsilon$;
- v) the area element JH of H satisfies $JH(0) = 1$. Namely, there holds

$$JH(0) = \lim_{r \rightarrow 0} \frac{\lambda_S(H(B_r^E))}{\mathcal{L}^{2n-1}(B_r^E)} = 1,$$

where $B_r^E = \{p \in \mathbb{R}^{2n-1} : |p| < r\}$ is a Euclidean ball and λ_S is the Riemannian $(2n - 1)$ -volume measure on S induced by g .

For small enough $a, b > 0$ and possibly shrinking U and Ω_ε , the mapping $G : (-a, a) \times (-b, b) \times U \rightarrow \mathbb{H}^n$

$$G(v, z, w) = \exp(vV_1) \exp\left(z \frac{\nabla u}{|\nabla u|_g}\right)(H(w))$$

is a diffeomorphism from $\tilde{\Omega}_\varepsilon = (-a, a) \times (-b, b) \times U$ onto Ω_ε . The differential of G satisfies

$$dG\left(\frac{\partial}{\partial v}\right) = V_1 \quad \text{and} \quad dG(0)\left(\frac{\partial}{\partial z}\right) = \frac{\nabla u(0)}{|\nabla u(0)|_g^2}.$$

Moreover, the tangent space $T_0S = \text{Im } dH(0)$ is g -orthogonal to $V_1(0)$ and $\frac{\nabla u(0)}{|\nabla u(0)|_g^2}$. We denote by G_z the restriction of G to $(-a, a) \times \{z\} \times U$, i.e., $G_z(v, w) = G(v, z, w)$. From the above considerations, we deduce that the area elements of G and of G_0 satisfy

$$JG(0) = \frac{1}{|\nabla u(0)|_g} \quad \text{and} \quad JG_0(0) = 1.$$

Then, possibly shrinking further $\tilde{\Omega}_\varepsilon$, we have

$$(1 - \varepsilon)JG(v, z, w) \leq \frac{JG_z(v, w)}{|\nabla u \circ G(v, z, w)|_g} \leq (1 + \varepsilon)JG(v, z, w), \quad (2.31)$$

for all $(v, z, w) \in \tilde{\Omega}_\varepsilon$.

For $j = 1, \dots, 2n$, we define on $\tilde{\Omega}_\varepsilon$ the vector fields $\tilde{V}_j = (dG)^{-1}(V_j)$. By the definition of G , we have $\tilde{V}_1 = \partial/\partial v$. We also define $\tilde{u} = u \circ G \in C^\infty(\tilde{\Omega}_\varepsilon)$, $\tilde{f} = f \circ G : \tilde{\Omega}_\varepsilon \rightarrow \mathbb{R}$, and $\tilde{E} = G^{-1}(E)$. Then:

- 1) we have $\tilde{E} = \{q \in \tilde{\Omega}_\varepsilon : \tilde{f}(q) > 0\}$;
- 2) we have $\tilde{f} \in C^\infty(\tilde{\Omega}_\varepsilon \setminus \partial\tilde{E})$;
- 3) the derivative $\tilde{V}_j \tilde{f}$ is defined in the sense of distributions with respect to the measure $\mu = JG \mathcal{L}^{2n+1}$. Namely, for all $\psi \in C_c^\infty(\tilde{\Omega}_\varepsilon)$ we have

$$\int_{\tilde{\Omega}_\varepsilon} (\tilde{V}_j \tilde{f}) \psi \, d\mu = - \int_{\tilde{\Omega}_\varepsilon} \tilde{f} \tilde{V}_j^* \psi \, d\mu,$$

where \tilde{V}_j^* is the adjoint operator of \tilde{V}_j with respect to μ . Then we have $\tilde{V}_j \tilde{f} = (V_j f) \circ G$ and so $\tilde{V}_j \tilde{f}$ is a continuous function for any $j = 1, \dots, 2n$. In particular, we have $\tilde{V}_1 \tilde{f} = \partial_v \tilde{f} \geq \delta > 0$.

Step 3: approximate coarea formula. We follow the argument of [24, Propositions 4.1 and 4.5], see also Remark 4.7 therein.

Possibly shrinking $\tilde{\Omega}_\varepsilon$ and Ω_ε , there exists a continuous function $\phi : (-b, b) \times U \rightarrow (-a, a)$ such that:

- A) $\partial\tilde{E} \cap \tilde{\Omega}_\varepsilon$ is the graph of ϕ . Namely, letting $\Phi : (-b, b) \times U \rightarrow \mathbb{R}^{2n+1}$, $\Phi(z, w) = (\phi(z, w), z, w)$, we have:

$$\partial\tilde{E} \cap \tilde{\Omega}_\varepsilon = \Phi((-b, b) \times U).$$

- B) The measure μ_E is

$$\mu_E \llcorner \Omega_\varepsilon = (G \circ \Phi)_\# \left(\left(\frac{|\tilde{V} \tilde{f}|}{\tilde{V}_1 \tilde{f}} JG \right) \circ \Phi \mathcal{L}^{2n} \llcorner ((-b, b) \times U) \right), \quad (2.32)$$

where $(G \circ \Phi)_\#$ denotes the push-forward and

$$|\tilde{V}f| = \left(\sum_{j=1}^{2n} (\tilde{V}_j f)^2 \right)^{1/2}.$$

Using $V_1 u = 0$ and $u \circ H = 0$ (this follows from $H(U) = S \cap \Omega_\varepsilon \subset \Sigma^0 \cap \Omega_\varepsilon$), we obtain

$$\begin{aligned} \tilde{u}(v, z, w) &= u(G(v, z, w)) = u\left(\exp(vV_1) \exp\left(z \frac{\nabla u}{|\nabla u|_g^2}\right)(H(w))\right) \\ &= u\left(\exp\left(z \frac{\nabla u}{|\nabla u|_g^2}\right)(H(w))\right) \\ &= z + u(H(w)) = z. \end{aligned}$$

In particular, from $\tilde{u} = u \circ G$ we deduce that

$$G^{-1}(\Sigma^s \cap \Omega_\varepsilon) = (-a, a) \times \{s\} \times U.$$

We denote by JG_s the Jacobian (area element) of G_s . We also define the restriction $\Phi_s : U \rightarrow \mathbb{R}^{2n+1}$, $\Phi_s(w) = \Phi(s, w)$, for any $s \in (-b, b)$.

By (2.30), for any $s \in \mathbb{R}$ the measure $\mu_E^s = \mu_{E \cap \Sigma_s}^{\Sigma^s}$ is the horizontal perimeter of $E \cap \Sigma^s$ with respect to the Carnot-Carathéodory structure induced by the family V_1, \dots, V_{2n-1} on Σ^s . We can repeat the argument that lead to (2.32) to obtain

$$\mu_E^s \llcorner \Omega_\varepsilon = (G \circ \Phi_s)_\# \left(\left(\frac{|\tilde{V}'f|}{\tilde{V}_1 f} JG_s \right) \circ \Phi_s \mathcal{L}^{2n-1} \llcorner U \right), \quad (2.33)$$

where $\tilde{V}'f = (\tilde{V}_1 f, \dots, \tilde{V}_{2n-1} f)$. We omit details of the proof of (2.33). The proof is a line-by-line repetition of Proposition 4.5 in [24] with the unique difference that now the horizontal perimeter is defined in a curved manifold.

Let us fix $\bar{r} > 0$ such that $B_{\bar{r}} \subset \Omega_\varepsilon$, and for any $r \in (0, \bar{r})$ let

$$\begin{aligned} A_{s,r} &= \{w \in U : G(0, s, w) \in B_r\}, \\ A_r &= \{(s, w) \in (-b, b) \times U : w \in A_{s,r}\}. \end{aligned}$$

By Fubini-Tonelli theorem and by (2.33), the function

$$s \mapsto \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s = \int_{A_{s,r}} \left(\frac{|\nabla_H u|_g}{|\nabla u|_g} \circ G \right) \left(\frac{|\tilde{V}'f|}{\tilde{V}_1 f} JG_s \right) \circ \Phi_s d\mathcal{L}^{2n-1} \quad (2.34)$$

is \mathcal{L}^1 -measurable. Here and hereafter, the composition $\circ \Phi_s$ acts on the product. Thus, from Fubini-Tonelli theorem and (2.31) we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds &= \int_{\mathbb{R}} \int_{A_{s,r}} \left(\frac{|\nabla_H u|_g}{|\nabla u|_g} \circ G \right) \left(\frac{|\tilde{V}'f|}{\tilde{V}_1 f} JG_s \right) \circ \Phi_s(w) d\mathcal{L}^{2n-1}(w) ds \\ &= \int_{A_r} (|\nabla_H u|_g \circ G) \left(\frac{|\tilde{V}'f|}{\tilde{V}_1 f} \frac{JG_s}{|\nabla u|_g \circ G} \right) \circ \Phi(s, w) d\mathcal{L}^{2n}(s, w) \\ &\leq (1 + \varepsilon) \int_{A_r} (|\nabla_H u|_g \circ G) \left(\frac{|\tilde{V}f|}{\tilde{V}_1 f} \sqrt{1 - \frac{(\tilde{V}_{2n} f)^2}{|\tilde{V}f|^2}} JG \right) \circ \Phi(s, w) d\mathcal{L}^{2n}(s, w). \end{aligned} \quad (2.35)$$

From the identity

$$\frac{\widetilde{V}_{2n}\widetilde{f}}{|\widetilde{V}\widetilde{f}|} = \frac{V_{2n}f}{|\nabla_H f|_g} \circ G = \left\langle \frac{\nabla_H u}{|\nabla_H u|_g}, \frac{\nabla_H f}{|\nabla_H f|_g} \right\rangle_g \circ G = \left\langle \frac{\nabla_H u}{|\nabla_H u|_g}, \nu_E \right\rangle_g \circ G, \quad (2.36)$$

and from (2.32) we deduce that

$$\begin{aligned} \int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds &\leq (1 + \varepsilon) \int_{B_r} |\nabla_H u|_g \sqrt{1 - \left\langle \frac{\nabla_H u}{|\nabla_H u|_g}, \nu_E \right\rangle_g^2} d\mu_E \\ &= (1 + \varepsilon) \int_{B_r} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \end{aligned} \quad (2.37)$$

In a similar way, we obtain

$$\int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \geq (1 - \varepsilon) \int_{B_r} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E.$$

This concludes the proof. \square

We can now prove the coarea formula for H -regular boundaries.

Proposition 2.7. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$, and $E \subset \mathbb{H}^n$ be an open domain such that $\partial E \cap \Omega$ is an H -regular hypersurface. Then*

$$\int_{\mathbb{R}} \int_{\Omega} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds = \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \quad (2.38)$$

Proof. Let us define the set

$$A = \left\{ p \in \partial E \cap \Omega : \nabla_H u(p) \neq 0 \text{ and } \nu_E(p) \neq \pm \frac{\nabla_H u(p)}{|\nabla_H u(p)|_g} \right\}.$$

The set A is relatively open in $\partial E \cap \Omega$. Let $\varepsilon > 0$ be fixed. Since the measure μ_E is locally doubling on $\partial E \cap \Omega$ (see e.g. [24, Corollary 4.13]), by Lemma 2.6 and Vitali covering Theorem (see e.g. [11, Theorem 1.6]) there exists a countable (or finite) collection of balls $B_{r_i}(p_i)$, $i \in \mathbb{N}$, such that:

- i) for any $i \in \mathbb{N}$ we have $p_i \in A$ and $0 < r_i < \bar{r}(p_i, \varepsilon)$, where \bar{r} is as in the statement of Lemma 2.6;
- ii) the balls $B_{r_i}(p_i)$ are contained in A and pairwise disjoint;
- iii) $\mu_E(A \setminus \bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)) = 0$.

It follows that we have:

$$\begin{aligned} \int_{\mathbb{R}} \int_{\bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds &\leq (1 + \varepsilon) \int_{\bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E \\ &= (1 + \varepsilon) \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E \\ &= (1 + \varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \end{aligned} \quad (2.39)$$

The last equality follows from the fact that $\sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} = 0$ outside A . In the same way one also obtains

$$\int_{\mathbb{R}} \int_{\bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \geq (1 - \varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \quad (2.40)$$

Moreover, by Proposition 2.5, there holds

$$\int_{\mathbb{R}} \int_{\Omega \setminus \bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_{\Omega \setminus \bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E = 0.$$

In particular, the integral on the left hand side of the last inequality is 0 and, by (2.39) and (2.40), we obtain

$$\begin{aligned} (1 - \varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E \\ \leq \int_{\mathbb{R}} \int_{\Omega} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \\ \leq (1 + \varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this concludes the proof. \square

By a standard approximation argument, we also have the following extension of the coarea formula (2.38).

Proposition 2.8. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$, and E be an open domain such that $\partial E \cap \Omega$ is an H -regular hypersurface. Then, for any Borel function $h : \partial E \rightarrow [0, \infty)$ there holds*

$$\int_{\mathbb{R}} \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds = \int_{\Omega} h \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E.$$

Our next step is to prove the coarea formula for \mathcal{S}^{2n+1} -rectifiable sets.

Lemma 2.9. *Let $R \subset \mathbb{H}^n$ be an \mathcal{S}^{2n+1} -rectifiable set. Then, there exists a Borel \mathcal{S}^{2n+1} -rectifiable set $R' \subset \mathbb{H}^n$ such that $\mathcal{S}^{2n+1}(R \Delta R') = 0$.*

Proof. By assumption, there exist a \mathcal{S}^{2n+1} -negligible set N and H -regular hypersurfaces $S_j \subset \mathbb{H}^n$, $j \in \mathbb{N}$, such that

$$R \subset N \cup \bigcup_{j=1}^{\infty} S_j.$$

It is proved in [7, 3] that (up to a localization argument), for any $j \in \mathbb{N}$, there exist an open set $U_j \subset \mathbb{R}^{2n}$, a homeomorphism $\Phi_j : U_j \rightarrow S_j$, and a continuous function $\rho_j : U_j \rightarrow [1, \infty)$ such that $\mathcal{S}^{2n+1} \llcorner S_j = \Phi_{j\#}(\rho_j \mathcal{L}^{2n} \llcorner U_j)$. Since the Lebesgue measure \mathcal{L}^{2n} is a complete Borel measure, for any $j \in \mathbb{N}$ there exists a Borel set $T_j \subset U_j$ such that

$$\mathcal{L}^{2n}(T_j \Delta \Phi_j^{-1}(R \cap S_j)) = 0.$$

In particular, the Borel set

$$R' = \bigcup_{j=1}^{\infty} \Phi_j(T_j)$$

is \mathcal{S}^{2n+1} -equivalent to R . \square

Proof of Theorem 1.5. Step 1. We prove (1.8) when R is an H -regular hypersurface. Then, R is locally the boundary of an open set $E \subset \mathbb{H}^n$ with H -regular boundary. Moreover, we have (locally) $\mu_E = \mathcal{S}^{2n+1} \llcorner R$ and $\nu_E = \nu_R$, up to the sign.

We define the measures $\mu_R^s = \mu_E^s$ for any s such that $\nabla u \neq 0$ on Σ^s . The measurability of the function in (1.7) follows from the argument (2.34). Formula (1.8) follows from Proposition 2.8.

Step 2. We prove (1.8) when R is an \mathcal{S}^{2n+1} -rectifiable Borel set. There exist a \mathcal{S}^{2n+1} -negligible set N and H -regular hypersurfaces $S_j \subset \mathbb{H}^n$, $j \in \mathbb{N}$ such that

$$R \subset N \cup \bigcup_{j=1}^{\infty} S_j.$$

Each S_j is (locally) the boundary of an open set E_j with H -regular boundary. We denote by $\mu_{E_j}^s$ the perimeter measure on $\partial E_j \cap \Sigma^s$ induced by E_j .

We define the pairwise disjoint Borel sets $R_j = (R \cap S_j) \setminus \bigcup_{h=1}^{j-1} S_h$ and we let

$$\mu_R^s = \sum_{j=1}^{\infty} \mu_{E_j}^s \llcorner R_j.$$

The definition is well posed for any s such that $\nabla u \neq 0$ on Σ^s . We have $\nu_R = \pm \nu_{E_j}$ \mathcal{S}^{2n+1} -a.e. on R_j and the sign of ν_R does not affect formula (1.8). From the *Step 1*, for each $j \in \mathbb{N}$ the function

$$s \mapsto \int_{R_j} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s$$

is \mathcal{L}^1 -measurable; here, we were allowed to utilize *Step 1* because χ_{R_j} is Borel regular. Thus also the function

$$s \mapsto \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_R^s = \sum_{j=1}^{\infty} \int_{R_j} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s$$

is measurable. Moreover, we have

$$\begin{aligned} \int_{\mathbb{R}} \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_R^s ds &= \sum_{j=1}^{\infty} \int_{\mathbb{R}} \int_{R_j} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s ds \\ &= \sum_{j=1}^{\infty} \int_{R_j} h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} d\mathcal{S}^{2n+1} \\ &= \int_R h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} d\mathcal{S}^{2n+1}. \end{aligned}$$

Step 3. Finally, if R is \mathcal{S}^{2n+1} -rectifiable but not Borel, we set $\mu_R^s = \mu_{R'}^s$, where R' is a Borel set as in Lemma 2.9. Again, this definition is well posed for a.e. $s \in \mathbb{R}$. This concludes the proof. \square

2.3. Proof of Theorem 1.6. In this subsection we assume $n \geq 2$.

Lemma 2.10. *For $n \geq 2$, let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $R \subset \Omega$ an \mathcal{S}^{2n+1} -rectifiable set. Then*

$$\mathcal{S}^{2n+1}(\{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\}) = 0.$$

Proof. It is enough to prove the lemma when R is an H -regular hypersurface. Let

$$A = \{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\}.$$

We claim that $\mathcal{S}^{2n+1}(A) = 0$.

Let $p \in A$ be a fixed point and let $\nu_R(p)$ be the horizontal normal to R at p . Since $n \geq 2$, we have

$$\dim\{V(p) \in H_p : \langle V(p), \nu_R(p) \rangle_g = 0\} = 2n - 1 \geq n + 1.$$

Thus there exist left invariant horizontal vector fields V, W such that

$$\langle V(p), \nu_R(p) \rangle_g = \langle W(p), \nu_R(p) \rangle_g = 0 \quad \text{and} \quad [V, W] = T.$$

From $\nabla_H u(p) = 0$ and $\nabla u(p) \neq 0$ we deduce that $Tu(p) \neq 0$. It follows that

$$VWu(p) - WVu(p) = Tu(p) \neq 0,$$

and, in particular, we have either $VWu(p) \neq 0$ or $WVu(p) \neq 0$. Without loss of generality, we assume that $VWu(p) \neq 0$. Then the set $S = \{q \in \Omega : Wu(q) = 0\}$ is an H -regular hypersurface near the point $p \in S$. Since we have

$$\langle V(p), \nu_R(p) \rangle_g = 0 \quad \text{and} \quad \langle V(p), \nu_S(p) \rangle_g = \frac{VWu(p)}{|\nabla_H Wu(p)|_g} \neq 0,$$

we deduce that $\nu_R(p)$ and $\nu_S(p)$ are linearly independent. Then there exists $r > 0$ such that the set $R \cap S \cap B_r(p)$ is a 2-codimensional H -regular surface (see [8]). Therefore, by [8, Corollary 4.4], the Hausdorff dimension in the Carnot-Carathéodory metric of $A \cap B_r(p) \subset R \cap S \cap B_r(p)$ is not greater than $2n$. This is enough to conclude. \square

Remark 2.11. Lemma 2.10 is not valid in the case $n = 1$. Consider the smooth surface $R = \{(x, y, t) \in \mathbb{H}^1 : x = 0\}$ and the function $u(x, y, t) = t - 2xy$. We have

$$\nabla u = -4xY + T \quad \text{and} \quad \nabla_H u = -4xY.$$

Then we have

$$\{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\} = R$$

and $\mathcal{S}^3(R) = \infty$.

If $n \geq 2$ and Ω , u , and R are as in Lemma 2.10, then the function

$$|\nabla u|_g \sqrt{1 - \left\langle \nu_E, \frac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g^2}$$

is defined \mathcal{L}^{2n+1} -almost everywhere on R . We agree that its value is 0 when $|\nabla u|_g = 0$. Notice that, in this case, $\frac{\nabla_H u}{|\nabla_H u|_g}$ is not defined.

Proof of Theorem 1.6. Let $\varepsilon > 0$ be fixed. Then (1.9) can be obtained by plugging the function $\frac{|\nabla u|_g}{\varepsilon + |\nabla_H u|_g} h$ into (1.8), letting $\varepsilon \rightarrow 0$ and using the monotone convergence theorem. \square

3. HEIGHT ESTIMATE

In this section, we prove Theorem 1.3. We discuss first a relative isoperimetric inequality on slices. Then we list some elementary properties of excess, and finally we proceed with the proof.

We assume throughout this section that $n \geq 2$.

3.1. Relative isoperimetric inequalities. For each $s \in \mathbb{R}$, we define the level sets of the height function

$$\mathbb{H}_s^n = \{p \in \mathbb{H}^n : \mathfrak{h}(p) = s\}.$$

Let H^s be the g -orthogonal projection of H onto the tangent space of \mathbb{H}_s^n . Since the vector field X_1 is orthogonal to \mathbb{H}_s^n , while the vector fields $X_2, \dots, X_n, Y_1, \dots, Y_n$ are tangent to \mathbb{H}_s^n , then at any point $p \in \mathbb{H}_s^n$ we have

$$H_p^s = \text{span}\{X_2(p), \dots, X_n(p), Y_1^s(p), Y_2(p), \dots, Y_n(p)\},$$

where $X_2, Y_2, \dots, X_n, Y_n$ are as in (1.2) and

$$Y_1^s = \frac{\partial}{\partial y_1} - 2s \frac{\partial}{\partial t}.$$

The natural volume in \mathbb{H}_s^n is the Lebesgue measure \mathcal{L}^{2n} . For any measurable set $F \subset \mathbb{H}_s^n$ and any open set $\Omega \subset \mathbb{H}_s^n$, we define

$$\mu_F^s(\Omega) = \sup \left\{ \int_F \text{div}_g^s \varphi \, d\mathcal{L}^{2n} : \varphi \in C_c^1(\Omega; H^s), \|\varphi\|_g \leq 1 \right\},$$

where $\text{div}_g^s \varphi = X_2 \varphi_2 + \dots + X_n \varphi_n + Y_1^s \varphi_{n+1} + \dots + Y_n \varphi_{2n}$. If $\mu_F^s(\Omega) < \infty$ then μ_F^s is a Radon measure in Ω .

By Theorem 1.6, for any Borel function $h : \mathbb{H}^n \rightarrow [0, \infty)$ and any set E with locally finite H -perimeter in \mathbb{H}^n , we have the following coarea formula

$$\int_{\mathbb{R}} \int_{\mathbb{H}_s^n} h \, d\mu_{E^s}^s \, ds = \int_{\mathbb{H}^n} h \sqrt{1 - \langle \nu_E, X_1 \rangle_g^2} \, d\mu_E, \quad (3.41)$$

where $E^s = E \cap \mathbb{H}_s^n$ is the section of E with \mathbb{H}_s^n . Notice that $\nabla_H \mathfrak{h} = X_1$.

In the proof of Theorem 1.3, we need a relative isoperimetric inequality in each slice \mathbb{H}_s^n for $s \in (-1, 1)$. These slices are cosets of $\mathbb{W} = \mathbb{H}_0^n$ and the isoperimetric

inequalities in \mathbb{H}_s^n can be reduced to an isoperimetric inequality in the central slice $\mathbb{W} = \mathbb{H}_0^n$ relative to a family of varying domains.

For any $s \in (-1, 1)$ let $\Omega_s \subset \mathbb{W}$ be the set $\Omega_s = (-se_1) * D_1 * (se_1)$. This is the left translation by $-se_1$ of the section $C_1 \cap \mathbb{H}_s^n$. See the introduction for the definition of D_1 and C_1 . With the coordinates $(y_1, \hat{z}, t) \in \mathbb{W} = \mathbb{R} \times \mathbb{C}^{n-1} \times \mathbb{R}$, we have

$$\Omega_s = \{(y_1, \hat{z}, t) \in \mathbb{W} : (y_1^2 + |\hat{z}|^2)^2 + (t - 4sy_1)^2 < 1\}.$$

The sets $\Omega_s \subset \mathbb{W}$ are open and convex in the standard sense. The boundary $\partial\Omega_s$ is a $(2n - 1)$ -dimensional C^∞ embedded surface with the following property. There are $4n$ open convex sets $U_1, \dots, U_{4n} \subset \mathbb{W}$ such that $\partial\Omega_s \subset \bigcup_{i=1}^{4n} U_i$ and for each i the portion of boundary $\partial\Omega_s \cap U_i$ is a graph of the form $p_j = f_i^s(\hat{p}_j)$ with $j = 2, \dots, 2n+1$ and $\hat{p}_j = (p_2, \dots, p_{j-1}, p_{j+1}, \dots, p_{2n+1}) \in V_i$, where $V_i \subset \mathbb{R}^{2n-1}$ is an open convex set and $f_i^s \in C^\infty(V_i)$ is a function such that

$$|\nabla f_i^s(\hat{p}_j) - \nabla f_i^s(\hat{q}_j)| \leq K|\hat{p}_j - \hat{q}_j| \quad \text{for all } \hat{p}_j, \hat{q}_j \in V_i, \quad (3.42)$$

where $K > 0$ is a constant independent of $i = 1, \dots, 4n$ and independent of $s \in (-1, 1)$. In other words, the boundary $\partial\Omega_s$ is of class $C^{1,1}$ uniformly in $s \in (-1, 1)$.

By Theorem 3.2 in [20], the domain $\Omega_s \subset \mathbb{W}$ is a non-tangentially accessible (NTA) domain in the metric space (\mathbb{W}, d_{CC}) where d_{CC} is the Carnot-Carathéodory metric induced by the horizontal distribution H_p^0 . In particular, Ω_s is a (weak) John domain in the sense of [10]. Namely, there exist a point $p_0 \in \Omega_s$, e.g. $p_0 = 0$, and a constant $C_J > 0$ such that for any point $p \in \Omega_s$ there exists a continuous curve $\gamma : [0, 1] \rightarrow \Omega_s$ such that $\gamma(1) = p_0$, $\gamma(0) = p$, and

$$\text{dist}_{CC}(\gamma(\sigma), \partial\Omega_s) \geq C_J d_{CC}(\gamma(\sigma), p), \quad \sigma \in [0, 1]. \quad (3.43)$$

By Theorem 3.2 in [20], the John constant C_J depends only on the constant $K > 0$ in (3.42). This claim is not stated explicitly in Theorem 3.2 of [20] but it is evident from the proof. In particular, the John constant C_J is independent of $s \in (-1, 1)$. Then, by Theorem 1.22 in [9] we have the following result

Theorem 3.1. *Let $n \geq 2$. There exists a constant $C(n) > 0$ such that for any $s \in (-1, 1)$ and for any measurable set $F \subset \mathbb{W}$ there holds*

$$\min\{\mathcal{L}^{2n}(F \cap \Omega_s), \mathcal{L}^{2n}(\Omega_s \setminus F)\}^{\frac{2n}{2n+1}} \leq C(n) \frac{\text{diam}_{CC}(\Omega_s)}{\mathcal{L}^{2n}(\Omega_s)^{\frac{1}{2n+1}}} \mu_F^0(\Omega_s). \quad (3.44)$$

An alternative proof of Theorem 3.1 can be obtained using the Sobolev-Poincaré inequalities proved in [10] in the general setting of metric spaces.

The diameter $\text{diam}_{CC}(\Omega_s)$ is bounded for $s \in (-1, 1)$ and $\mathcal{L}^{2n}(\Omega_s) > 0$ is a constant independent of s . Then we obtain the following version of (3.44).

Corollary 3.2. *Let $n \geq 2$. For any $\tau \in (0, 1)$ there exists a constant $C(n, \tau) > 0$ such that for $s \in (-1, 1)$ and for any measurable set $F \subset \mathbb{W}$ satisfying*

$$\mathcal{L}^{2n}(F \cap \Omega_s) \leq \tau \mathcal{L}^{2n}(\Omega_s)$$

there holds

$$\mu_F^0(\Omega_s) \geq C(n, \tau) \mathcal{L}^{2n}(F \cap \Omega_s)^{\frac{2n}{2n+1}}.$$

3.2. Elementary properties of the excess. We list here, without proof, the most basic properties of the cylindrical excess introduced in Definition 1.2. Their proofs are easy adaptations of those for the classical excess, see e.g. [13, Chapter 22]. Note that, except for property 3), they hold also in the case $n = 1$.

1) For all $0 < r < s$ we have

$$\text{Exc}(E, r, \nu) \leq \left(\frac{s}{r}\right)^{2n+1} \text{Exc}(E, s, \nu). \quad (3.45)$$

2) If $(E_j)_{j \in \mathbb{N}}$ is a sequence of sets with locally finite H -perimeter such that $E_j \rightarrow E$ as $j \rightarrow \infty$ in $L_{\text{loc}}^1(\mathbb{H}^n)$, then we have for any $r > 0$

$$\text{Exc}(E, r, \nu) \leq \liminf_{j \rightarrow \infty} \text{Exc}(E_j, r, \nu). \quad (3.46)$$

3) Let $n \geq 2$. If $E \subset \mathbb{H}^n$ is a set such that $\text{Exc}(E, r, \nu) = 0$ and $0 \in \partial^*E$, then

$$E \cap C_r = \{p \in C_r : \mathfrak{h}(p) < 0\}. \quad (3.47)$$

In particular, we have $\nu_E = \nu$ in $C_r \cap \partial E$. See also [18, Proposition 3.6].

4) For any $\lambda > 0$ and $r > 0$ we have

$$\text{Exc}(\lambda E, \lambda r, \nu) = \text{Exc}(E, r, \nu), \quad (3.48)$$

where $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\}$.

3.3. Proof of Theorem 1.3. The following result is a first suboptimal version of Theorem 1.3.

Lemma 3.3. *Let $n \geq 2$. For any $s \in (0, 1)$, $\Lambda \in [0, \infty)$, and $r \in (0, \infty]$ with $\Lambda r \leq 1$, there exists a constant $\omega(n, s, \Lambda, r) > 0$ such that if $E \subset \mathbb{H}^n$ is a (Λ, r) -minimizer of H -perimeter in the cylinder C_2 , $0 \in \partial E$, and $\text{Exc}(E, 2, \nu) \leq \omega(n, s, \Lambda, r)$, then*

$$\begin{aligned} |\mathfrak{h}(p)| &< s \text{ for any } p \in \partial E \cap C_1, \\ \mathcal{L}^{2n+1}(\{p \in E \cap C_1 : \mathfrak{h}(p) > s\}) &= 0, \\ \mathcal{L}^{2n+1}(\{p \in C_1 \setminus E : \mathfrak{h}(p) < -s\}) &= 0. \end{aligned}$$

Proof. By contradiction, assume that there exist $s \in (0, 1)$ and a sequence of sets $(E_j)_{j \in \mathbb{N}}$ that are (Λ, r) -minimizers in C_2 and such such that

$$\lim_{j \rightarrow \infty} \text{Exc}(E_j, 2, \nu) = 0$$

and at least one of the following facts holds:

$$\text{either} \quad \text{there exists } p \in \partial E_j \cap C_1 \text{ such that } s \leq |\mathfrak{h}(p)| \leq 1, \quad (3.49)$$

$$\text{or} \quad \mathcal{L}^{2n+1}(\{p \in E_j \cap C_1 : \mathfrak{h}(p) > s\}) > 0, \quad (3.50)$$

$$\text{or} \quad \mathcal{L}^{2n+1}(\{p \in C_1 \setminus E_j : \mathfrak{h}(p) < -s\}) > 0. \quad (3.51)$$

By Theorem 4.3 in the Appendix A, there exists a measurable set $F \subset C_{5/3}$ such that F is a (Λ, r) -minimizer in $C_{5/3}$, $0 \in \partial F$ and (possibly up to subsequences) $E_j \cap C_{5/3} \rightarrow F$ in $L^1(C_{5/3})$. By (3.46) and (3.45), we obtain

$$\text{Exc}(F, 4/3, \nu) \leq \liminf_{j \rightarrow \infty} \text{Exc}(E_j, 4/3, \nu) \leq \left(\frac{3}{2}\right)^{2n+1} \lim_{j \rightarrow \infty} \text{Exc}(E_j, 2, \nu) = 0.$$

Since $0 \in \partial F$, by (3.47) the set $F \cap C_{4/3}$ is (equivalent to) a halfspace with horizontal inner normal $\nu = -X_1$, and, namely,

$$F \cap C_{4/3} = \{p \in C_{4/3} : \mathfrak{h}(p) < 0\}.$$

Assume that (3.49) holds for infinitely many j . Then, up to a subsequence, there are points $(p_j)_{j \in \mathbb{N}}$ and p_0 such that

$$p_j \in \partial E_j \cap C_1, \quad |\mathfrak{h}(p_j)| \in (s, 1] \quad \text{and} \quad p_j \rightarrow p_0 \in \partial F \cap \bar{C}_1.$$

We used again Theorem 4.3 in the Appendix A. This is a contradiction because $\partial F \cap \bar{C}_1 = \{p \in \bar{C}_1 : \mathfrak{h}(p) = 0\}$. Here, we used $n \geq 2$. Therefore, there exists $j_0 \in \mathbb{N}$ such that

$$\{p \in \partial E_j \cap C_1 : s \leq |\mathfrak{h}(p)| \leq 1\} = \emptyset \quad \text{for all } j \geq j_0,$$

and hence

$$\mu_{E_j}(C_1 \setminus \{p \in \mathbb{H}^n : |\mathfrak{h}(p)| \leq s\}) = 0.$$

This implies that, for $j \geq j_0$, χ_{E_j} is constant on the two connected components $C_1 \cap \{p : \mathfrak{h}(p) > s\}$ and $C_1 \cap \{p : \mathfrak{h}(p) < -s\}$. Since the sequence $(E_j)_{j \in \mathbb{N}}$ converges in $L^1(C_1)$ to the halfspace F , then for any $j \geq j_0$ we have

$$\begin{aligned} \chi_{E_j} &= 0 \quad \mathcal{L}^{2n+1}\text{-a.e. on } C_1 \cap \{p : \mathfrak{h}(p) > s\}, \quad \text{and} \\ \chi_{E_j} &= 1 \quad \mathcal{L}^{2n+1}\text{-a.e. on } C_1 \cap \{p : \mathfrak{h}(p) < -s\}. \end{aligned}$$

This contradicts both (3.50) and (3.51) and concludes the proof. \square

Let $\pi : \mathbb{H}^n \rightarrow \mathbb{W}$ be the group projection defined, for any $p \in \mathbb{H}^n$, by the formula

$$p = \pi(p) * (\mathfrak{h}(p)e_1).$$

For any set $E \subset \mathbb{H}^n$ and for any $s \in \mathbb{R}$, we let $E^s = E \cap \mathbb{H}_s^n$ and we define the projection

$$E_s = \pi(E^s) = \{w \in \mathbb{W} : w * (se_1) \in E\}.$$

Lemma 3.4. *Let $n \geq 2$, let $E \subset \mathbb{H}^n$ be a set with locally finite H -perimeter and $0 \in \partial E$, and let $s_0 \in (0, 1)$ be such that*

$$|\mathfrak{h}(p)| < s_0 \text{ for any } p \in \partial E \cap C_1, \tag{3.52}$$

$$\mathcal{L}^{2n+1}(\{p \in E \cap C_1 : \mathfrak{h}(p) > s_0\}) = 0, \tag{3.53}$$

$$\mathcal{L}^{2n+1}(\{p \in C_1 \setminus E : \mathfrak{h}(p) < -s_0\}) = 0. \tag{3.54}$$

Then, for a.e. $s \in (-1, 1)$ and for any continuous function $\varphi \in C_c(D_1)$ we have, with $M = \partial^*E \cap C_1$ and $M_s = M \cap \{\mathfrak{h} > s\}$,

$$\int_{E_s \cap D_1} \varphi d\mathcal{L}^{2n} = - \int_{M_s} \varphi \circ \pi \langle \nu_E, X_1 \rangle_g d\mathcal{S}^{2n+1}. \quad (3.55)$$

In particular, for any Borel set $G \subset D_1$, we have

$$\mathcal{L}^{2n}(G) = - \int_{M \cap \pi^{-1}(G)} \langle \nu_E, X_1 \rangle_g d\mathcal{S}^{2n+1}, \quad (3.56)$$

$$\mathcal{L}^{2n}(G) \leq \mathcal{S}^{2n+1}(M \cap \pi^{-1}(G)). \quad (3.57)$$

Proof. It is enough to prove (3.55). Indeed, taking $s < -s_0$ in (3.55) and recalling (3.52) and (3.54), we obtain

$$\int_{D_1} \varphi d\mathcal{L}^{2n} = - \int_M \varphi \circ \pi \langle \nu_E, X_1 \rangle_g d\mathcal{S}^{2n+1}. \quad (3.58)$$

Formula (3.56) follows from (3.58) by considering smooth approximations of χ_G . Formula (3.57) is immediate from (3.56) and $|\langle \nu_E, X_1 \rangle_g| \leq 1$.

We prove (3.55) for a.e. $s \in (-1, 1)$ and, namely, for those s satisfying the property (3.61) below. Up to an approximation argument, we may assume that $\varphi \in C_c^1(D_1)$. Let $r \in (0, 1)$ and $\sigma \in (\max\{s_0, s\}, 1)$ be fixed. We define

$$F = E \cap (D_r * (s, \sigma)) = E \cap \{w * (\varrho e_1) \in \mathbb{H}^n : w \in D_r, \varrho \in (s, \sigma)\}.$$

We claim that for a.e. $r \in (0, 1)$ and any s satisfying (3.61) we have

$$\langle \nu_F, X_1 \rangle_g \mu_F = \langle \nu_E, X_1 \rangle_g \mathcal{S}^{2n+1} \llcorner \partial^*E \cap (D_r * (s, \sigma)) + \mathcal{L}^{2n} \llcorner E \cap D_r^s. \quad (3.59)$$

Above, we let $D_r^s = \{w * (s e_1) \in \mathbb{H}^n : w \in D_r\}$. We postpone the proof of (3.59). Let Z be a horizontal vector field of the form $Z = (\varphi \circ \pi) X_1$. We have $\operatorname{div}_g Z = 0$ because $X_1(\varphi \circ \pi) = 0$. Hence, we obtain

$$0 = \int_F \operatorname{div}_g Z d\mathcal{L}^{2n+1} = - \int_{\mathbb{H}^n} \varphi \circ \pi \langle \nu_F, X_1 \rangle_g d\mu_F,$$

i.e., by Fubini-Tonelli theorem and by (3.59),

$$- \int_{E_s \cap D_r} \varphi d\mathcal{L}^{2n} = - \int_{E \cap D_r^s} \varphi \circ \pi d\mathcal{L}^{2n} = \int_{\partial^*E \cap (D_r * (s, \sigma))} \varphi \circ \pi \langle \nu_E, X_1 \rangle_g d\mathcal{S}^{2n+1}.$$

Formula (3.55) follows on letting first $r \nearrow 1$ and then $\sigma \nearrow 1$.

We are left with the proof of (3.59). Let $\psi \in C_c^1(\mathbb{H}^n)$ be a test function. For any $w \in \mathbb{W}$ we let

$$E_w = \{\varrho \in \mathbb{R} : w * (\varrho e_1) \in E\}, \quad \psi_w(\varrho) = \psi(w * (\varrho e_1)).$$

Then we have $\psi_w \in C_c^1(\mathbb{R})$ and, by Fubini-Tonelli theorem,

$$\begin{aligned}
-\int_F X_1 \psi d\mathcal{L}^{2n+1} &= -\int_{D_r} \int_s^\sigma \chi_E(w * (\varrho e_1)) X_1 \psi(w * (\varrho e_1)) d\varrho d\mathcal{L}^{2n}(w) \\
&= -\int_{D_r} \int_s^\sigma \chi_{E_w}(\varrho) \psi'_w(\varrho) d\varrho d\mathcal{L}^{2n}(w) \\
&= \int_{D_r} \left[\int_s^\sigma \psi_w dD\chi_{E_w} - \psi_w(\sigma) \chi_{E_w}(\sigma^-) + \psi_w(s) \chi_{E_w}(s^+) \right] d\mathcal{L}^{2n}(w),
\end{aligned} \tag{3.60}$$

where $D\chi_{E_w}$ is the derivative of χ_{E_w} in the sense of distributions and $\chi_{E_w}(\sigma^-), \chi_{E_w}(s^+)$ are the classical trace values of χ_{E_w} at the endpoints of the interval (s, σ) . We used the fact that the function χ_{E_w} is of bounded variation for \mathcal{L}^{2n} -a.e. $w \in \mathbb{W}$, which in turn is a consequence of the fact that $X_1 \chi_E$ is a signed Radon measure. For any such w , the trace of χ_{E_w} satisfies

$$\chi_{E_w}(s^+) = \chi_{E_w}(s) = \chi_E(w * (s e_1)) \quad \text{for a.e. } s,$$

so that, by Fubini's Theorem, for a.e. $s \in \mathbb{R}$ there holds

$$\chi_{E_w}(s^+) = \chi_E(w * (s e_1)) \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } w \in D_1. \tag{3.61}$$

With a similar argument, using (3.53) and the fact that $\sigma > s_0$ one can see that

$$\chi_{E_w}(\sigma^-) = \chi_E(w * (\sigma e_1)) = 0 \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } w \in D_1. \tag{3.62}$$

We refer the reader to [2] for an extensive account on BV functions and traces. By (3.60), (3.61) and (3.62) we obtain

$$\begin{aligned}
-\int_F X_1 \psi d\mathcal{L}^{2n+1} &= \int_{D_r} \int_s^\sigma \psi_w dD\chi_{E_w} d\mathcal{L}^{2n}(w) + \int_{D_r} \psi_w(s) \chi_{E_w}(s) d\mathcal{L}^{2n}(w) \\
&= \int_{D_{r^*}(s, \sigma)} \psi \langle \nu_E, X_1 \rangle_g d\mu_E + \int_{E \cap D_r^*} \psi d\mathcal{L}^{2n} \\
&= \int_{\partial^* E \cap (D_{r^*}(s, \sigma))} \psi \langle \nu_E, X_1 \rangle_g d\mathcal{S}^{2n+1} + \int_{E \cap D_r^*} \psi d\mathcal{L}^{2n},
\end{aligned}$$

and (3.59) follows. \square

Corollary 3.5. *Under the same assumptions and notation of Lemma 3.4, for a.e. $s \in (-1, 1)$ there holds*

$$0 \leq \mathcal{S}^{2n+1}(M_s) - \mathcal{L}^{2n}(E_s \cap D_1) \leq \text{Exc}(E, 1, \nu). \tag{3.63}$$

Moreover, we have

$$\mathcal{S}^{2n+1}(M) - \mathcal{L}^{2n}(D_1) = \text{Exc}(E, 1, \nu). \tag{3.64}$$

Proof. On approximating χ_{D_1} with functions $\varphi \in C_c(D_1)$, by (3.55) we get

$$\mathcal{L}^{2n}(E_s \cap D_1) = - \int_{M_s} \langle \nu_E, X_1 \rangle_g d\mathcal{S}^{2n+1},$$

and the first inequality in (3.63) follows. The second inequality follows from

$$\begin{aligned} \mathcal{S}^{2n+1}(M_s) - \mathcal{L}^{2n}(E_s \cap D_1) &= \int_{M_s} (1 + \langle \nu_E, X_1 \rangle_g) d\mathcal{S}^{2n+1} \\ &= \int_{M_s} \frac{|\nu_E - \nu|_g^2}{2} d\mathcal{S}^{2n+1} \\ &\leq \text{Exc}(E, 1, \nu). \end{aligned} \quad (3.65)$$

Notice that $\nu = -X_1$. Finally, (3.64) follows on choosing a suitable $s < -s_0$ and recalling (3.52) and (3.54). In this case, the inequality in (3.65) becomes an equality and the proof is concluded. \square

Proof of Theorem 1.3. Step 1. Up to replacing E with the rescaled set $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\}$ with $\lambda = 1/2k^2r$ and recalling (3.48), we can without loss of generality assume that E is a $(\Lambda', \frac{1}{2k^2})$ -minimizer of H -perimeter in C_2 with

$$\frac{\Lambda'}{2k^2} \leq 1, \quad 0 \in \partial E, \quad \text{Exc}(E, 2, \nu) \leq \varepsilon_0(n). \quad (3.66)$$

Our goal is to find $\varepsilon_0(n)$ and $c_1(n) > 0$ such that, if (3.66) holds, then

$$\sup \{ |\mathfrak{h}(p)| : p \in \partial E \cap C_{1/2k^2} \} \leq c_1(n) \text{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}}. \quad (3.67)$$

We require

$$\varepsilon_0(n) \leq \omega\left(n, \frac{1}{4k}, 2k^2, \frac{1}{2k^2}\right), \quad (3.68)$$

where ω is given by Lemma 3.3. Two further assumptions on $\varepsilon_0(n)$ will be made later in (3.80) and (3.85). By (3.66), E is a $(2k^2, \frac{1}{2k^2})$ -minimizer in C_2 . Letting $M = \partial E \cap C_1$, by Lemma 3.3 and (3.68) we have

$$|\mathfrak{h}(p)| < \frac{1}{4k} \text{ for any } p \in M, \quad (3.69)$$

$$\mathcal{L}^{2n+1}(\{p \in E \cap C_1 : \mathfrak{h}(p) > \frac{1}{4k}\}) = 0, \quad (3.70)$$

$$\mathcal{L}^{2n+1}(\{p \in C_1 \setminus E : \mathfrak{h}(p) < -\frac{1}{4k}\}) = 0. \quad (3.71)$$

By (3.64) and (3.45) we get

$$0 \leq \mathcal{S}^{2n+1}(M) - \mathcal{L}^{2n}(D_1) \leq \text{Exc}(E, 1, \nu) \leq 2^{2n+1} \text{Exc}(E, 2, \nu). \quad (3.72)$$

Corollary 3.5 implies that, for a.e. $s \in (-1, 1)$,

$$0 \leq \mathcal{S}^{2n+1}(M_s) - \mathcal{L}^{2n}(E_s \cap D_1) \leq \text{Exc}(E, 1, \nu) \leq 2^{2n+1} \text{Exc}(E, 2, \nu) \quad (3.73)$$

where, as before, $M_s = M \cap \{\mathfrak{h} > s\}$.

Step 2. Consider the function $f : (-1, 1) \rightarrow [0, \mathcal{S}^{2n+1}(M)]$ defined by

$$f(s) = \mathcal{S}^{2n+1}(M_s), \quad s \in (-1, 1).$$

The function f is nonincreasing, right-continuous and, by (3.69), it satisfies

$$\begin{aligned} f(s) &= \mathcal{S}^{2n+1}(M) \text{ for any } s \in (-1, -\frac{1}{4k}], \\ f(s) &= 0 \text{ for any } s \in (\frac{1}{4k}, 1]. \end{aligned}$$

In particular, there exists $s_0 \in (-\frac{1}{4k}, \frac{1}{4k})$ such that

$$\begin{aligned} f(s) &\geq \mathcal{S}^{2n+1}(M)/2 \text{ for any } s < s_0, \\ f(s) &\leq \mathcal{S}^{2n+1}(M)/2 \text{ for any } s \geq s_0. \end{aligned} \quad (3.74)$$

Let $s_1 \in (s_0, \frac{1}{4k})$ be such that

$$\begin{aligned} f(s) &\geq \sqrt{\text{Exc}(E, 2, \nu)} \text{ for any } s < s_1, \\ f(s) &= \mathcal{S}^{2n+1}(M_s) \leq \sqrt{\text{Exc}(E, 2, \nu)} \text{ for any } s \geq s_1. \end{aligned} \quad (3.75)$$

We claim that there exists $c_2(n) > 0$ such that

$$\mathfrak{h}(p) \leq s_1 + c_2(n) \text{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}} \text{ for any } p \in \partial E \cap C_{1/2k^2}. \quad (3.76)$$

Inequality (3.76) is trivial for any $p \in \partial E \cap C_{1/2k^2}$ with $\mathfrak{h}(p) \leq s_1$. If $p \in \partial E \cap C_{1/2k^2}$ is such that $\mathfrak{h}(p) > s_1$, then

$$B_{\mathfrak{h}(p)-s_1}(p) \subset B_{1/2k}(p) \subset B_{1/k} \subset C_1.$$

We used the fact that $\|p\|_K \leq \frac{1}{2k}$ whenever $p \in C_{1/2k^2}$, see (1.3). Therefore

$$B_{\mathfrak{h}(p)-s_1}(p) \subset C_1 \cap \{\mathfrak{h} > s_1\}$$

and, by the density estimate (4.91) of Theorem 4.1 in Appendix A,

$$\begin{aligned} k_3(n)(\mathfrak{h}(p) - s_1)^{2n+1} &\leq \mu_E(B_{\mathfrak{h}(p)-s_1}(p)) \leq \mu_E(C_1 \cap \{\mathfrak{h} > s_1\}) \\ &= \mathcal{S}^{2n+1}(M_{s_1}) = f(s_1) \leq \sqrt{\text{Exc}(E, 2, \nu)}. \end{aligned}$$

This proves (3.76).

Step 3. We claim that there exists $c_3(n) > 0$ such that

$$s_1 - s_0 \leq c_3(n) \text{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}}. \quad (3.77)$$

By the coarea formula (3.41) with $h = \chi_{C_1}$, $D_1^s = \{p \in C_1 : \mathfrak{h}(p) = s\}$, and $E^s = \{p \in E : \mathfrak{h}(p) = s\}$, we have

$$\int_{-1}^1 \int_{D_1^s} d\mu_{E^s}^s ds = \int_{C_1} \sqrt{1 - \langle \nu_E, X_1 \rangle_g^2} d\mu_E \leq \sqrt{2} \int_M \sqrt{1 + \langle \nu_E, X_1 \rangle_g} d\mathcal{S}^{2n+1}.$$

By Hölder inequality, (4.91), (3.56), and (3.72), we deduce that

$$\begin{aligned} \int_{-1}^1 \int_{D_1^s} d\mu_{E^s}^s ds &\leq \sqrt{2\mathcal{S}^{2n+1}(M)} \left(\int_M (1 + \langle \nu_E, X_1 \rangle_g) d\mathcal{S}^{2n+1} \right)^{1/2} \\ &\leq c_4(n) (\mathcal{S}^{2n+1}(M) - \mathcal{L}^{2n}(D_1))^{1/2} \\ &\leq c_5(n) \sqrt{\text{Exc}(E, 2, \nu)}. \end{aligned} \quad (3.78)$$

By Corollary 3.5 and (3.72), we obtain, for a.e. $s \in [s_0, s_1]$,

$$\begin{aligned} \mathcal{L}^{2n}(E_s \cap D_1) &\leq \mathcal{S}^{2n+1}(M_s) = f(s) \leq f(s_0) \\ &\leq \frac{\mathcal{S}^{2n+1}(M)}{2} \leq \frac{\mathcal{L}^{2n}(D_1) + 2^{2n+1} \text{Exc}(E, 2, \nu)}{2} \leq \frac{3}{4} \mathcal{L}^{2n}(D_1). \end{aligned} \quad (3.79)$$

The last inequality holds provided that

$$2^{2n+1}\varepsilon_0(n) \leq \frac{\mathcal{L}^{2n}(D_1)}{4}. \quad (3.80)$$

Let $\Omega_s = (-se_1) * D_1^s = (-se_1) * D_1 * (se_1)$ and $F_s = (-se_1) * E^s$. We have

$$\mathcal{L}^{2n}(\Omega_s) = \mathcal{L}^{2n}(D_1^s) = \mathcal{L}^{2n}(D_1), \quad (3.81)$$

and, by (3.79),

$$\mathcal{L}^{2n}(F_s \cap \Omega_s) = \mathcal{L}^{2n}(E^s \cap D_1^s) = \mathcal{L}^{2n}(E_s \cap D_1) \leq \frac{3}{4}\mathcal{L}^{2n}(D_1). \quad (3.82)$$

Moreover, by left invariance we also have

$$\mu_{E^s}^s(D_1^s) = \mu_{F_s}^0(\Omega_s). \quad (3.83)$$

By (3.81)–(3.83) and Corollary 3.2, there exists a constant $k(n) > 0$ independent of $s \in (-1, 1)$ such that

$$\mu_{E^s}(D_1^s) = \mu_{F_s}^0(\Omega_s) \geq k(n) \mathcal{L}^{2n}(F_s \cap \Omega_s)^{\frac{2n}{2n+1}} = k(n) \mathcal{L}^{2n}(E^s \cap D_1^s)^{\frac{2n}{2n+1}}. \quad (3.84)$$

This, together with (3.78), gives

$$\begin{aligned} c_6(n) \sqrt{\text{Exc}(E, 2, \nu)} &\geq \int_{s_0}^{s_1} \mathcal{L}^{2n}(E^s \cap D_1^s)^{\frac{2n}{2n+1}} ds \\ &\stackrel{(3.73)}{\geq} \int_{s_0}^{s_1} \left(\mathcal{L}^{2n+1}(M_s) - 2^{2n+1} \text{Exc}(E, 2, \nu) \right)^{\frac{2n}{2n+1}} ds \\ &\stackrel{(3.75)}{\geq} \int_{s_0}^{s_1} \left(\sqrt{\text{Exc}(E, 2, \nu)} - 2^{2n+1} \text{Exc}(E, 2, \nu) \right)^{\frac{2n}{2n+1}} ds \\ &\geq \frac{1}{2} \int_{s_0}^{s_1} \text{Exc}(E, 2, \nu)^{\frac{n}{2n+1}} ds. \end{aligned}$$

In the last inequality, we require that $\varepsilon_0(n)$ satisfies

$$\sqrt{z} - 2^{2n+1}z \geq \frac{1}{2}\sqrt{z} \quad \text{for all } z \in [0, \varepsilon_0(n)]. \quad (3.85)$$

It follows that

$$c_6(n) \sqrt{\text{Exc}(E, 2, \nu)} \geq \frac{1}{2} \text{Exc}(E, 2, \nu)^{\frac{n}{2n+1}} (s_1 - s_0),$$

and (3.77) follows.

Step 4. Recalling (3.76) and (3.77), we proved that there exist $\varepsilon_0(n)$ and $c_6(n)$ such that the following holds. If E is a $(2k^2, \frac{1}{2k^2})$ -minimizer of H -perimeter in C_2 such that

$$0 \in \partial E, \quad \text{Exc}(E, 2, \nu) \leq \varepsilon_0(n)$$

and $s_0 = s_0(E)$ satisfies (3.74), then

$$\frac{1}{2}(p) - s_0 \leq c_7(n) \text{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \quad (3.86)$$

Let us introduce the mapping $\Psi : \mathbb{H}^n \rightarrow \mathbb{H}^n$

$$\Psi(x_1, x_2, \dots, x_n, y_1, \dots, y_n, t) = (-x_1, -x_2, \dots, -x_n, y_1, \dots, y_n, -t).$$

Then we have $\Psi^{-1} = \Psi$, $\Psi(C_2) = C_2$, $\langle X_j, \nu_{\Psi(F)} \rangle_g = -\langle X_j, \nu_F \rangle_g \circ \Psi$, $\langle Y_j, \nu_{\Psi(F)} \rangle_g = \langle Y_j, \nu_F \rangle_g \circ \Psi$, and $\mu_{\Psi(F)} = \Psi_{\#} \mu_F$, for any set F with locally finite H -perimeter; here, $\Psi_{\#}$ denotes the standard push-forward of measures. Therefore, the set $\tilde{E} = \Psi(\mathbb{H}^n \setminus E)$ satisfies the following properties:

- i) \tilde{E} is a $(2k^2, \frac{1}{2k^2})$ -minimizer of H -perimeter in C_2 ;
- ii) $0 \in \partial \tilde{E}$ and

$$\text{Exc}(\tilde{E}, 2, \nu) = \frac{1}{2^Q} \int_{\partial^* \tilde{E} \cap C_2} |\nu_{\tilde{E}} - \nu|_g^2 d\mathcal{S}^{2n+1} = \text{Exc}(E, 2, \nu) \leq \varepsilon_0(n);$$

- iii) setting $\tilde{M} = \partial^* \tilde{E} \cap C_1 = \Psi(M)$ and $\tilde{f}(s) = \mathcal{S}^{2n+1}(\tilde{M} \cap \{\mathfrak{h} > s\})$, we have

$$\begin{aligned} \tilde{f}(s) &\geq \mathcal{S}^{2n+1}(\tilde{M})/2 = \mathcal{S}^{2n+1}(M)/2 \text{ for any } s < -s_0, \\ \tilde{f}(s) &\leq \mathcal{S}^{2n+1}(M)/2 \text{ for any } s \geq -s_0. \end{aligned}$$

Formula (3.86) for the set \tilde{E} gives

$$\mathfrak{h}(p) + s_0 \leq c_7(n) \text{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}} \quad \text{for any } p \in \partial \tilde{E} \cap C_{1/2k^2}.$$

Notice that we have $p \in \partial \tilde{E}$ if and only if $\Psi(p) \in \partial E$ and, moreover, $\mathfrak{h}(\Psi(p)) = -\mathfrak{h}(p)$. Hence, we have

$$-\mathfrak{h}(p) + s_0 \leq c_7(n) \text{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \quad (3.87)$$

By (3.86) and (3.87) we obtain

$$|\mathfrak{h}(p) - s_0| \leq c_7(n) \text{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}, \quad (3.88)$$

and, in particular,

$$|s_0| \leq c_7(n) \text{Exc}(E, 2, \nu)^{\frac{1}{2(2n+1)}}, \quad (3.89)$$

because $0 \in \partial E \cap C_{1/2k^2}$. Inequalities (3.88) and (3.89) give (3.67). This completes the proof. \square

4. APPENDIX A

We list some basic properties of Λ -minimizers of H -perimeter in \mathbb{H}^n . The proofs are straightforward adaptations of the proofs for Λ -minimizers of perimeter in \mathbb{R}^n .

Theorem 4.1 (Density estimates). *There exist constants $k_1(n), k_2(n), k_3(n), k_4(n) > 0$ with the following property. If E is a (Λ, r) -minimizer of H -perimeter in $\Omega \subset \mathbb{H}^n$, $p \in \partial E \cap \Omega$, $B_r(p) \subset \Omega$ and $s < r$, then*

$$k_1(n) \leq \frac{\mathcal{L}^{2n+1}(E \cap B_s(p))}{s^{2n+2}} \leq k_2(n) \quad (4.90)$$

$$k_3(n) \leq \frac{\mu_E(B_s(p))}{s^{2n+1}} \leq k_4(n). \quad (4.91)$$

For a proof see [13, Theorem 21.11]. By standard arguments Theorem 4.1 implies the following corollary.

Corollary 4.2. *If E is a (Λ, r) -minimizer of H -perimeter in $\Omega \subset \mathbb{H}^n$, then*

$$\mathcal{S}^{2n+1}((\partial E \setminus \partial^* E) \cap \Omega) = 0.$$

Theorem 4.3. *Let $(E_j)_{j \in \mathbb{N}}$ be a sequence of (Λ, r) -minimizers of H -perimeter in an open set $\Omega \subset \mathbb{H}^n$, $\Lambda r \leq 1$. Then there exists a (Λ, r) -minimizer E of H -perimeter in Ω and a subsequence $(E_{j_k})_{k \in \mathbb{N}}$ such that*

$$E_{j_k} \rightarrow E \text{ in } L^1_{\text{loc}}(\Omega) \quad \text{and} \quad \nu_{E_{j_k}} \mu_{E_{j_k}} \xrightarrow{*} \nu_E \mu_E$$

as $k \rightarrow \infty$. Moreover, the measure theoretic boundaries ∂E_{j_k} converge to ∂E in the sense of Kuratowski, i.e.,

- i) if $p_{j_k} \in \partial E_{j_k} \cap \Omega$ and $p_{j_k} \rightarrow p \in \Omega$, then $p \in \partial E$;
- ii) if $p \in \partial E \cap \Omega$, then there exists a sequence $(p_{j_k})_{k \in \mathbb{N}}$ such that $p_{j_k} \in \partial E_{j_k} \cap \Omega$ and $p_{j_k} \rightarrow p$.

For a proof in the case of the perimeter in \mathbb{R}^n , see [13, Chapter 21].

5. APPENDIX B

We define a Borel unit normal ν_R to an \mathcal{S}^{2n+1} -rectifiable set $R \subset \mathbb{H}^n$ and we show that the definition is well posed \mathcal{S}^{2n+1} -almost everywhere, up to the sign. The normal ν_S to an H -regular hypersurface $S \subset \mathbb{H}^n$ is defined in (1.6).

Definition 5.1. Let $R \subset \mathbb{H}^n$ be an \mathcal{S}^{2n+1} -rectifiable set such that

$$\mathcal{S}^{2n+1}\left(R \setminus \bigcup_{j \in \mathbb{N}} S_j\right) = 0 \tag{5.92}$$

for a sequence of H -regular hypersurfaces $(S_j)_{j \in \mathbb{N}}$ in \mathbb{H}^n . For any $p \in R \cap \bigcup_{j \in \mathbb{N}} S_j$ we define

$$\nu_R(p) = \nu_{S_{\bar{j}}}(p),$$

where \bar{j} is the unique integer such that $p \in S_{\bar{j}} \setminus \bigcup_{j < \bar{j}} S_j$.

We show that Definition 5.1 is well posed, up to a sign, for \mathcal{S}^{2n+1} -a.e. p . Namely, let $(S_j^1)_{j \in \mathbb{N}}$ and $(S_j^2)_{j \in \mathbb{N}}$ be two sequences of H -regular hypersurfaces in \mathbb{H}^n for which (5.92) holds and denote by ν_R^1 and ν_R^2 , respectively, the associated normals to R according to Definition 5.1. We show that $\nu_R^1 = \nu_R^2$ \mathcal{S}^{2n+1} -a.e. on R , up to the sign.

Let $A \subset R$ be the set of points such that either $\nu_R^1(p)$ is not defined, or $\nu_R^2(p)$ is not defined, or they are both defined and $\nu_R^1(p) \neq \pm \nu_R^2(p)$. It is enough to show that $\mathcal{S}^{2n+1}(A) = 0$. This is a consequence of the following lemma.

Lemma 5.2. *Let S_1, S_2 be two H -regular hypersurfaces in \mathbb{H}^n and let*

$$A = \{p \in S_1 \cap S_2 : \nu_{S_1}(p) \neq \pm \nu_{S_2}(p)\}.$$

Then, the Hausdorff dimension of A in the Carnot-Carathéodory metric is at most $2n$, $\dim_{CC}(A) \leq 2n$, and, in particular, $\mathcal{S}^{2n+1}(A) = 0$.

Proof. The blow-up of S_i , $i = 1, 2$, at a point $p \in A$ is a vertical hyperplane $\Pi_i \times \mathbb{R} \subset \mathbb{R}^{2n} \times \mathbb{R} \equiv \mathbb{H}^n$, see e.g. [7], where:

i) by blow-up of S_i at p we mean the limit

$$\lim_{\lambda \rightarrow \infty} \lambda(p^{-1} * S_i)$$

in the Gromov-Hausdorff sense. Recall that, for $E \subset \mathbb{H}^n$, we define $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\}$.

ii) For $i = 1, 2$, $\Pi_i \subset \mathbb{R}^{2n}$ is the normal hyperplane to $\nu_{S_i}(p) \in H_p \equiv \mathbb{R}^{2n}$.

It follows that the blow-up of A at p is contained in the blow-up of $S_1 \cap S_2$ at p , i.e., in $(\Pi_1 \cap \Pi_2) \times \mathbb{R}$. Since $\nu_{S_1}(p) \neq \pm \nu_{S_2}(p)$, $\Pi_1 \cap \Pi_2$ is a $(2n - 2)$ -dimensional plane in \mathbb{R}^{2n} , and we conclude thanks to the following lemma. \square

Lemma 5.3. *Let $k = 0, 1, \dots, 2n$ and $A \subset \mathbb{H}^n$ be such that for any $p \in A$, the blow-up of A at p is contained in $\Pi_p \times \mathbb{R}$ for some plane $\Pi_p \subset \mathbb{R}^{2n}$ of dimension k . Then we have $\dim_{CC}(A) \leq k + 2$.*

Proof. We claim that for any $\eta > 0$ we have

$$\mathcal{S}^{k+2+\eta}(A) = 0. \quad (5.93)$$

Let $\varepsilon \in (0, 1/2)$ be such that $C\varepsilon^\eta \leq 1/2$, where $C = C(n)$ is a constant that will be fixed later in the proof. By the definition of blow-up, for any $p \in A$ there exists $r_p > 0$ such that for all $r \in (0, r_p)$ we have

$$(p^{-1} * A) \cap U_r \subset (\Pi_p)_{\varepsilon r} \times \mathbb{R},$$

where $(\Pi_p)_{\varepsilon r}$ denotes the (εr) -neighbourhood of Π_p in \mathbb{R}^{2n} . For any $j \in \mathbb{N}$ set

$$A_j = \{p \in A \cap B_j : r_p > 1/j\}.$$

To prove (5.93), it is enough to prove that

$$\mathcal{S}^{k+2+\eta}(A_j) = 0$$

for any fixed $j \geq 1$. This, in turn, will follow if we show that, for any fixed $\delta \in (0, \frac{1}{2j})$, one has

$$\begin{aligned} \inf \left\{ \sum_{i \in \mathbb{N}} r_i^{k+2+\eta} : A_j \subset \bigcup_{i \in \mathbb{N}} U_{r_i}(p_i), r_i < 2\varepsilon\delta \right\} &\leq \\ &\leq \frac{1}{2} \inf \left\{ \sum_{i \in \mathbb{N}} r_i^{k+2+\eta} : A_j \subset \bigcup_{i \in \mathbb{N}} U_{r_i}(p_i), r_i < \delta \right\}. \end{aligned} \quad (5.94)$$

Let $(U_{r_i}(p_i))_{i \in \mathbb{N}}$ be a covering of A_j with balls of radius smaller than δ . There exist points $\bar{p}_i \in A_j$ such that $(U_{2r_i}(\bar{p}_i))_{i \in \mathbb{N}}$ is a covering of A_j with balls of radius smaller than $2\delta < 1/j$. By definition of A_j , we have

$$(\bar{p}_i^{-1} * A_j) \cap U_{2r_i} \subset ((\Pi_{\bar{p}_i})_{\varepsilon r_i} \times \mathbb{R}) \cap U_{2r_i}.$$

The set $((\Pi_{\bar{p}_i})_{\varepsilon r_i} \times \mathbb{R}) \cap U_{2r_i}$ can be covered by a family of balls $(U_{\varepsilon r_i}(p_h^i))_{h \in H_i}$ of radius $\varepsilon r_i < 2\varepsilon\delta$ in such a way that the cardinality of H_i is bounded by $C\varepsilon^{-k-2}$,

where the constant C depends only on n and not on ε . In particular, the family of balls $(U_{\varepsilon r_i}(\bar{p}_i * p_h^i))_{i \in \mathbb{N}, h \in H_i}$ is a covering of A_j and

$$\begin{aligned} \sum_{i \in \mathbb{N}} \sum_{h \in H_i} (\text{radius } U_{\varepsilon r_i}(\bar{p}_i * p_h^i))^{k+2+\eta} &= \sum_{i \in \mathbb{N}} \sum_{h \in H_i} (\varepsilon r_i)^{k+2+\eta} \leq C \varepsilon^{-k-2} \sum_{i \in \mathbb{N}} (\varepsilon r_i)^{k+2+\eta} \\ &= C \varepsilon^\eta \sum_{i \in \mathbb{N}} r_i^{k+2+\eta} \leq \frac{1}{2} \sum_i r_i^{k+2+\eta}. \end{aligned}$$

This proves (5.94) and concludes the proof. \square

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