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# Isoperimetric problem in $H$ -type groups and Grushin spaces

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**Abstract.** We study the isoperimetric problem in  $H$ -type groups and Grushin spaces, emphasizing a relation between them. Under a symmetry assumption that depends on the dimension, we prove existence, additional symmetry, and regularity properties of isoperimetric sets.

## 1. Introduction

Let  $M$  be a manifold,  $V$  be a volume, and  $P$  a perimeter measure on  $M$ . For a regular set  $E \subset M$ ,  $P(E)$  is the area of the boundary  $\partial E$ . The isoperimetric problem relative to  $V$  and  $P$  consists in studying existence, symmetries, regularity and, if possible, classifying the minimizers of the problem

$$(1.1) \quad \min \{P(E) : E \in \mathcal{A} \text{ such that } V(E) = v\},$$

for a given volume  $v > 0$  and for a given family of admissible sets  $\mathcal{A}$ . Minimizers of (1.1) are called isoperimetric sets.

In space forms (Euclidean space, sphere and hyperbolic space) with their natural volume and perimeter, isoperimetric sets are precisely metric balls. In  $\mathbb{R}^n$  with volume  $e^{-|x|^2} \mathcal{L}^n$  and perimeter  $e^{-|x|^2} \mathcal{H}^{n-1}$ , isoperimetric sets are half-spaces, [3]. This is the Gaussian isoperimetric problem, the model of the current research direction on isoperimetric problems with density, [21]. A different way to weight perimeter is by a surface tension, i.e., by the support function  $\tau : \mathbb{S}^{n-1} \rightarrow [0, \infty)$  of a convex body  $K \subset \mathbb{R}^n$  with  $0 \in \text{int}(K)$ ,  $\tau(\nu) = \sup_{x \in K} \langle x, \nu \rangle$ . Namely, one can consider

$$P(E) = \int_{\partial E} \tau(\nu_E) d\mathcal{H}^{n-1}, \quad \nu_E \text{ outer normal to } \partial E.$$

The isoperimetric problem for this perimeter and with  $V = \mathcal{L}^n$  is known as Wulff problem and isoperimetric sets are translates and dilates of the set  $K$ , see the quantitative version in [4].

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In a different approach, the perimeter of a Lebesgue measurable set  $E \subset \mathbb{R}^n$  is defined via a system  $X = \{X_1, \dots, X_h\}$ ,  $h \geq 2$ , of self-adjoint vector fields in  $\mathbb{R}^n$ ,

$$(1.2) \quad P_X(E) = \sup \left\{ \int_E \sum_{i=1}^h X_i \varphi_i(x) dx : \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^h), \max_{x \in \mathbb{R}^n} |\varphi(x)| \leq 1 \right\}.$$

This definition is introduced and studied systematically in [10]. The perimeter  $P_X$  is known as  $X$ -perimeter (horizontal, sub-elliptic, or sub-Riemannian perimeter). One important example is the Heisenberg perimeter, that is subject of intensive research in connection with Pansu's conjecture on the shape of isoperimetric sets (see [2], [5], [13], [15], [17], [18], [19], [20]) and in connection with the regularity problem of minimal surfaces.

In this paper, we study perimeters that are related to the Heisenberg perimeter. Namely, we study the isoperimetric problem in  $H$ -type groups and in Grushin spaces.

1) *H-type groups.* Let  $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$  be a stratified nilpotent real Lie algebra of dimension  $n \geq 3$  and step 2. Thus we have  $\mathfrak{h}_2 = [\mathfrak{h}_1, \mathfrak{h}_1]$ . We fix on  $\mathfrak{h}$  a scalar product  $\langle \cdot, \cdot \rangle$  that makes  $\mathfrak{h}_1$  and  $\mathfrak{h}_2$  orthogonal. The Kaplan mapping, introduced in [11], is the mapping  $J : \mathfrak{h}_2 \rightarrow \text{End}(\mathfrak{h}_1)$  defined via the identity

$$(1.3) \quad \langle J_Y(X), X' \rangle = \langle Y, [X, X'] \rangle,$$

holding for all  $X, X' \in \mathfrak{h}_1$  and  $Y \in \mathfrak{h}_2$ . The algebra  $\mathfrak{h}$  is called an  $H$ -type algebra if for all  $X, X' \in \mathfrak{h}_1$  and  $Y \in \mathfrak{h}_2$  there holds

$$(1.4) \quad \langle J_Y(X), J_Y(X') \rangle = |Y|^2 \langle X, X' \rangle,$$

where  $|Y| = \langle Y, Y \rangle^{1/2}$ . We can identify  $\mathfrak{h}$  with  $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ ,  $\mathfrak{h}_1$  with  $\mathbb{R}^h \times \{0\}$ , and  $\mathfrak{h}_2$  with  $\{0\} \times \mathbb{R}^k$ , where  $h \geq 2$  and  $k \geq 1$  are integers. In fact,  $h$  is an even integer. We can also assume that  $\langle \cdot, \cdot \rangle$  is the standard scalar product of  $\mathbb{R}^n$ . Using exponential coordinates, the connected and simply connected Lie group of  $\mathfrak{h}$  can be identified with  $\mathbb{R}^n$ . Denoting points of  $\mathbb{R}^n$  as  $(x, y) \in \mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ , the Lie group product  $\cdot : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is of the form  $(x, y) \cdot (x', y') = (x + x', y + y' + Q(x, x'))$ , where  $Q : \mathbb{R}^h \times \mathbb{R}^h \rightarrow \mathbb{R}^k$  is a bilinear skew-symmetric mapping. Let  $Q_{ij}^\ell \in \mathbb{R}$  be the numbers

$$Q_{ij}^\ell = \langle Q(e_i, e_j), e_\ell \rangle, \quad i, j = 1, \dots, h, \quad \ell = 1, \dots, k,$$

where  $e_i, e_j \in \mathbb{R}^h$  and  $e_\ell \in \mathbb{R}^k$  are the standard coordinate versors. An orthonormal basis of the Lie algebra of left-invariant vector fields of the  $H$ -type group  $(\mathbb{R}^n, \cdot)$  is given by

$$(1.5) \quad \begin{aligned} X_i &= \frac{\partial}{\partial x_i} - \sum_{\ell=1}^k \sum_{j=1}^h Q_{ij}^\ell x_j \frac{\partial}{\partial y_\ell}, \quad i = 1, \dots, h, \\ Y_j &= \frac{\partial}{\partial y_j}, \quad j = 1, \dots, k. \end{aligned}$$

We denote by  $P_H(E) = P_X(E)$  the perimeter of a set  $E \subset \mathbb{R}^n$  defined as in (1.2), relative to the system of vector fields  $X = \{X_1, \dots, X_h\}$ . The vector fields  $Y_1, \dots, Y_k$  are not considered.

2) *Grushin spaces*. Let  $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ , where  $h, k \geq 1$  are integers and  $n = h + k$ . For a given real number  $\alpha > 0$ , let us define the vector fields in  $\mathbb{R}^n$

$$(1.6) \quad \begin{aligned} X_i &= \frac{\partial}{\partial x_i}, \quad i = 1, \dots, h, \\ Y_j &= |x|^\alpha \frac{\partial}{\partial y_j}, \quad j = 1, \dots, k, \end{aligned}$$

where  $|x|$  is the standard norm of  $x$ . We denote by  $P_\alpha(E) = P_X(E)$  the perimeter of a set  $E \subset \mathbb{R}^n$  defined as in (1.2) relative to the system of vector fields  $X = \{X_1, \dots, X_h, Y_1, \dots, Y_k\}$ . We call  $P_\alpha(E)$  the  $\alpha$ -perimeter of  $E$ .

We study the isoperimetric problem in the class of  $x$ -spherically symmetric sets in  $H$ -type groups and Grushin spaces. These two problems are related to each other. We say that a set  $E \subset \mathbb{R}^h \times \mathbb{R}^k$  is  $x$ -spherically symmetric if there exists a set  $F \subset \mathbb{R}^+ \times \mathbb{R}^k$ , called generating set of  $E$ , such that

$$E = \{(x, y) \in \mathbb{R}^n : (|x|, y) \in F\}.$$

We denote by  $\mathcal{S}_x$  the class of  $\mathcal{L}^n$ -measurable,  $x$ -spherically symmetric sets.

Starting from the  $x$ -spherical symmetry, we can prove that the class of sets involved in the minimization (1.1) can be restricted to a smaller class of sets with more symmetries (see Section 3). Using this additional symmetry, we can implement the concentration-compactness argument in order to have the existence of isoperimetric sets. In Carnot groups, the existence is already known, see [12]. In Grushin spaces, the existence is less clear because  $x$ -translations do not preserve  $\alpha$ -perimeter.

In fact, we have existence of isoperimetric sets that are  $x$ - and  $y$ -Schwarz symmetric, i.e., of the form

$$(1.7) \quad E = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\},$$

for some function  $f : (0, r_0) \rightarrow \mathbb{R}^+$ ,  $r_0 > 0$ , which is called the profile function of  $E$ . The profile function has the necessary regularity to solve a second order ordinary differential equation expressing the fact that the boundary of  $E$  has a certain “mean curvature” that is constant. This differential equation can be partially integrated and, for the profile function of a minimizer, it can be expressed in the following equivalent way:

$$(1.8) \quad \frac{f'(r)}{\sqrt{r^{2\alpha} + f'(r)^2}} = \frac{k-1}{r^{h-1}} \int_0^r \frac{s^{2\alpha+h-1}}{f(s)\sqrt{s^{2\alpha} + f'(s)^2}} ds - \frac{\kappa}{h} r, \quad \text{for } r \in (0, r_0),$$

where  $h, k$  are the dimensional parameters,  $\alpha > 0$  is the real parameter in the Grushin vector fields (1.6) (in  $H$ -type groups we have  $\alpha = 1$ ), and  $\kappa > 0$  is a real parameter (the “mean curvature”) related to perimeter and volume.

In  $H$ -type groups, the Haar measure is the Lebesgue measure. Moreover, Lebesgue measure and  $H$ -perimeter are homogeneous with respect to the anisotropic dilations

$$(x, y) \mapsto \delta_\lambda(x, y) = (\lambda x, \lambda^2 y), \quad \lambda > 0.$$

In fact, for any measurable set  $E \subset \mathbb{R}^n$  and for all  $\lambda > 0$  we have  $\mathcal{L}^n(\delta_\lambda(E)) = \lambda^Q \mathcal{L}^n(E)$  and  $P_H(\delta_\lambda(E)) = \lambda^{Q-1} P_H(E)$ , where the number  $Q = h + 2k$  is the homogeneous dimension of the group. Then, the isoperimetric ratio

$$\mathcal{I}_H(E) = \frac{P_H(E)^Q}{\mathcal{L}^n(E)^{Q-1}}$$

is homogeneous of degree 0 and the isoperimetric problem (1.1) can be formulated in scale invariant form. In the following, by a vertical translation we mean a mapping of the form  $(x, y) \mapsto (x, y + y_0)$  for some  $y_0 \in \mathbb{R}^k$ .

**Theorem 1.1.** *In any  $H$ -type group, the isoperimetric problem*

$$(1.9) \quad \min \{ \mathcal{I}_H(E) : E \in \mathcal{S}_x \text{ with } 0 < \mathcal{L}^n(E) < \infty \}$$

*has solutions and, up to a vertical translation and a null set, any isoperimetric set is of the form (1.7) for a function  $f \in C([0, r_0]) \cap C^1([0, r_0)) \cap C^\infty(0, r_0)$ , with  $0 < r_0 < \infty$ , satisfying  $f(r_0) = 0$ ,  $f' \leq 0$  on  $(0, r_0)$ , and solving equation (1.8) with  $\alpha = 1$  and  $\kappa = \frac{QP_H(E)}{(Q-1)\mathcal{L}^n(E)}$ .*

Isoperimetric sets are, in fact,  $C^\infty$ -smooth sets away from  $y = 0$ . Removing the assumption of  $x$ -spherical symmetry is a difficult problem that is open even in the basic example of the 3-dimensional Heisenberg group.

For the special dimension  $h = 1$ , we are able to prove the  $x$ -symmetry of isoperimetric sets for  $\alpha$ -perimeter. Lebesgue measure and  $\alpha$ -perimeter are homogeneous with respect to the group of anisotropic dilations

$$(x, y) \mapsto \delta_\lambda(x, y) = (\lambda x, \lambda^{1+\alpha} y), \quad \lambda > 0.$$

In fact, for any measurable set  $E \subset \mathbb{R}^n$  and for all  $\lambda > 0$  we have  $\mathcal{L}^n(\delta_\lambda(E)) = \lambda^d \mathcal{L}^n(E)$  and  $P_\alpha(\delta_\lambda(E)) = \lambda^{d-1} P_\alpha(E)$ , where  $d = h + k(1 + \alpha)$ . Then, the isoperimetric ratio

$$\mathcal{I}_\alpha(E) = \frac{P_\alpha(E)^d}{\mathcal{L}^n(E)^{d-1}}$$

is homogeneous of degree 0.

**Theorem 1.2.** *Let  $\alpha > 0$ ,  $h = 1$ ,  $k \geq 1$  and  $n = 1 + k$ . The isoperimetric problem*

$$(1.10) \quad \min \{ \mathcal{I}_\alpha(E) : E \subset \mathbb{R}^n \text{ } \mathcal{L}^n\text{-measurable with } 0 < \mathcal{L}^n(E) < \infty \}$$

*has solutions and, up to a vertical translation and a null set, any isoperimetric set is of the form (1.7) for a function  $f \in C([0, r_0]) \cap C^1([0, r_0)) \cap C^\infty(0, r_0)$ , with  $0 < r_0 < \infty$ , satisfying  $f(r_0) = 0$ ,  $f' \leq 0$  on  $(0, r_0)$ , and solving equation (1.8) with  $h = 1$  and  $\kappa = \frac{dP_\alpha(E)}{(d-1)\mathcal{L}^n(E)}$ .*

In particular, for  $h = 1$  isoperimetric sets are  $x$ -symmetric. When  $h \geq 2$  we need to assume the  $x$ -spherical symmetry.

**Theorem 1.3.** *Let  $\alpha > 0$ ,  $h \geq 2$ ,  $k \geq 1$  and  $n = h + k$ . The isoperimetric problem*

$$(1.11) \quad \min \{ \mathcal{J}_\alpha(E) : E \in \mathcal{S}_x \text{ with } 0 < \mathcal{L}^n(E) < \infty \}$$

*has solutions and, up to a vertical translation and a null set, any isoperimetric set is of the form (1.7) for a function  $f \in C([0, r_0]) \cap C^1([0, r_0)) \cap C^\infty(0, r_0)$ , with  $0 < r_0 < \infty$ , satisfying  $f(r_0) = 0$ ,  $f' \leq 0$  on  $(0, r_0)$ , and solving equation (1.8) with  $\kappa = \frac{dP_\alpha(E)}{(d-1)\mathcal{L}^n(E)}$ .*

In the special case  $k = 1$ , equation (1.8) can be integrated and we have an explicit formula for isoperimetric sets. Namely, with the normalization  $\kappa = h$  – that implies  $r_0 = 1$ , – the profile function solving (1.8) gives the isoperimetric set

$$(1.12) \quad E = \left\{ (x, y) \in \mathbb{R}^n : |y| < \int_{\arcsin |x|}^{\pi/2} \sin^{\alpha+1}(s) ds \right\}.$$

This formula generalizes to dimensions  $h \geq 2$  the results of [16]. When  $k = 1$  and  $\alpha = 1$ , the profile function satisfying the final condition  $f(1) = 0$  is  $f(r) = \frac{1}{2}(\arccos(r) + r\sqrt{1-r^2})$ ,  $r \in [0, 1]$ . This is the profile function of the Pansu's ball in the Heisenberg group.

In Section 2, we prove various representation formulas for the perimeter of smooth and symmetric sets. In particular, we show that for  $x$ -spherically symmetric sets we have the identity  $P_H(E) = P_\alpha(E)$  with  $\alpha = 1$ . This makes Theorem 1.1 a special case of Theorem 1.3.

In Section 3, we prove the rearrangement theorems. We show that when  $h = 1$  the isoperimetric problem with no symmetry assumption can be reduced to  $x$ -symmetric sets. When  $h \geq 2$ , we show that the  $x$ -spherical symmetry can be improved to the  $x$ -Schwarz symmetry. We also study perimeter under  $y$ -Schwarz rearrangement. The equality case in this rearrangement does not imply that, before rearrangement, the set is already  $y$ -Schwarz symmetric because the centers of the  $x$ -balls may vary. However, for isoperimetric sets the centers are constant, see Proposition 5.4. To prove this, we use the regularity of the profile function (see Section 5).

The existence of isoperimetric sets is established in Section 4 by the concentration - compactness method. Here, we borrow some ideas from [9] and we also use the isoperimetric inequalities (with non-sharp constants) obtained in [10], [6], and [7].

Finally, in Section 5 we deduce the differential equation for the profile function, we use minimality to derive its equivalent version (1.8), and we establish some elementary properties of solutions.

## 2. Representation and reduction formulas

In this section, we derive some formulas for the representation of  $H$ - and  $\alpha$ -perimeter of smooth sets and of sets with symmetry. For any open set  $A \subset \mathbb{R}^n$  and  $m \in \mathbb{N}$ , let us define the family of test functions

$$\mathcal{F}_m(A) = \left\{ \varphi \in C_c^1(A; \mathbb{R}^m) : \max_{(x,y) \in A} |\varphi(x,y)| \leq 1 \right\}.$$

### 2.1. Relation between $H$ -perimeter and $\alpha$ -perimeter

Let  $X_1, \dots, X_h$  be the generators of an  $H$ -type Lie algebra, thought of as left-invariant vector fields in  $\mathbb{R}^n$  as in (1.5). For an open set  $E \subset \mathbb{R}^n$  with Lipschitz boundary, the Euclidean outer unit normal  $N^E : \partial E \rightarrow \mathbb{R}^n$  is defined at  $\mathcal{H}^{n-1}$ -a.e. point of  $\partial E$ . We define the mapping  $N_H^E : \partial E \rightarrow \mathbb{R}^h$

$$N_H^E = (\langle N^E, X_1 \rangle, \dots, \langle N^E, X_h \rangle).$$

Here,  $\langle \cdot, \cdot \rangle$  is the standard scalar product of  $\mathbb{R}^n$  and  $X_i$  is thought of as an element of  $\mathbb{R}^n$  with respect to the standard basis  $\partial_1, \dots, \partial_n$ .

**Proposition 2.1.** *If  $E \subset \mathbb{R}^n$  is a bounded open set with Lipschitz boundary then the  $H$ -perimeter of  $E$  in  $\mathbb{R}^n$  is*

$$(2.1) \quad P_H(E) = \int_{\partial E} |N_H^E(x,y)| d\mathcal{H}^{n-1},$$

where  $\mathcal{H}^{n-1}$  is the standard  $(n-1)$ -dimensional Hausdorff measure in  $\mathbb{R}^n$ .

*Proof.* The proof of (2.1) is standard and we only sketch it. The inequality

$$P_H(E) \leq \int_{\partial E} |N_H^E(x,y)| d\mathcal{H}^{n-1}$$

follows by the Cauchy-Schwarz inequality applied to the right hand side of the identity

$$\int_E \sum_{i=1}^h X_i \varphi_i dx dy = \int_{\partial E} \langle N_H^E, \varphi \rangle d\mathcal{H}^{n-1},$$

that holds for any  $\varphi \in \mathcal{F}_h(\mathbb{R}^n)$ .

The opposite inequality follows by approximating  $N_H^E/|N_H^E|$  with functions in  $\mathcal{F}_h(\mathbb{R}^n)$ . In fact, by a Lusin-type and Tietze-extension argument, for any  $\varepsilon > 0$  there exists  $\varphi \in \mathcal{F}_h(\mathbb{R}^n)$  such that

$$\int_{\partial E} \langle N_H^E, \varphi \rangle d\mathcal{H}^{n-1} \geq \int_{\partial E} |N_H^E(x,y)| d\mathcal{H}^{n-1} - \varepsilon.$$

□

The outer normal  $N^E$  can be split in the following way

$$N^E = (N_x^E, N_y^E) \quad \text{with } N_x^E \in \mathbb{R}^h \text{ and } N_y^E \in \mathbb{R}^k.$$

For any  $\alpha > 0$ , we call the mapping  $N_\alpha^E : \partial E \rightarrow \mathbb{R}^n$

$$(2.2) \quad N_\alpha^E = (N_x^E, |x|^\alpha N_y^E)$$

the  $\alpha$ -normal to  $\partial E$ . The same argument used to prove (2.1) also shows that

$$(2.3) \quad P_\alpha(E) = \int_{\partial E} |N_\alpha^E(x, y)| d\mathcal{H}^{n-1},$$

for any set  $E \subset \mathbb{R}^n$  with Lipschitz boundary.

*Remark 2.2.* Formulas (2.1) and (2.3) hold also when  $\partial E$  is  $\mathcal{H}^{n-1}$ -rectifiable.

**Proposition 2.3.** *For any  $x$ -spherically symmetric set  $E \in \mathcal{S}_x$  there holds  $P_H(E) = P_\alpha(E)$  with  $\alpha = 1$ .*

*Proof.* By a standard approximation, using the results of [8], it is sufficient to prove the claim for smooth sets, e.g., for a bounded set  $E \subset \mathbb{R}^n$  with Lipschitz boundary. By (2.1) and (2.3), the claim  $P_H(E) = P_\alpha(E)$  with  $\alpha = 1$  reads

$$(2.4) \quad P_H(E) = \int_{\partial E} \sqrt{|N_x^E|^2 + |x|^2 |N_y^E|^2} d\mathcal{H}^{n-1},$$

where  $N^E = (N_x^E, N_y^E) \in \mathbb{R}^h \times \mathbb{R}^k$  is the unit Euclidean normal to  $\partial E$ . By the representation formula (2.1), we have

$$P_H(E) = \int_{\partial E} \left( \sum_{i=1}^h \langle X_i, N^E \rangle^2 \right)^{1/2} d\mathcal{H}^{n-1},$$

where, by (1.5), for any  $i = 1, \dots, h$

$$\begin{aligned} \langle X_i, N^E \rangle^2 &= \left( N_{x_i}^E - \sum_{\ell=1}^k \sum_{j=1}^h Q_{ij}^\ell x_j N_{y_\ell}^E \right)^2 \\ &= (N_{x_i}^E)^2 - 2N_{x_i}^E \sum_{\ell=1}^k \sum_{j=1}^h Q_{ij}^\ell x_j N_{y_\ell}^E + \left( \sum_{\ell=1}^k \sum_{j=1}^h Q_{ij}^\ell x_j N_{y_\ell}^E \right)^2, \end{aligned}$$

and thus

$$(2.5) \quad \sum_{i=1}^h \langle X_i, N^E \rangle^2 = |N_x^E|^2 - 2 \sum_{\ell=1}^k \sum_{j=1}^h Q_{ij}^\ell x_j N_{x_i}^E N_{y_\ell}^E + \sum_{i=1}^h \sum_{\ell, m=1}^k \sum_{j, p=1}^h Q_{ij}^\ell Q_{ip}^m x_j x_p N_{y_\ell}^E N_{y_m}^E.$$

Since the set  $E$  is  $x$ -spherically symmetric, the component  $N_x^E$  of the normal satisfies the identity

$$(2.6) \quad N_x^E = \frac{x}{|x|} |N_x^E|.$$

The bilinear form  $Q : \mathbb{R}^h \times \mathbb{R}^h \rightarrow \mathbb{R}^k$  is skew-symmetric, i.e., we have  $Q(x, x') = -Q(x', x)$  for all  $x, x' \in \mathbb{R}^h$  or, equivalently,  $Q_{ij}^\ell = -Q_{ji}^\ell$ . Using (2.6), it follows that for any  $\ell = 1, \dots, k$  we have

$$(2.7) \quad \sum_{i,j=1}^h Q_{ij}^\ell x_j N_{x_i}^E = \frac{|N_x^E|}{|x|} \sum_{i,j=1}^h Q_{ij}^\ell x_i x_j = 0.$$

Next, we insert into identity (1.4), that defines an  $H$ -type group, the vector fields

$$X = X' = \sum_{i=1}^h x_i X_i, \quad Y = \sum_{j=1}^k N_{y_j}^E Y_j,$$

where  $x \in \mathbb{R}^h$ ,  $N_y^E = (N_{y_1}^E, \dots, N_{y_k}^E)$ , and  $X_i, Y_j$  are the orthonormal vector fields in (1.5). After some computations that are omitted, using the definition (1.3) of the Kaplan mapping, we obtain the identity

$$(2.8) \quad \sum_{\ell,m=1}^k \sum_{i,j,p=1}^h Q_{ij}^\ell Q_{ip}^m N_{y_\ell}^E N_{y_m}^E x_j x_p = |x|^2 |N_y^E|^2.$$

From (2.5), (2.7), and (2.8) we deduce that

$$\sum_{i=1}^h \langle X_i, N^E \rangle^2 = |N_x^E|^2 + |x|^2 |N_y^E|^2,$$

and formula (2.4) follows.  $\square$

## 2.2. $\alpha$ -Perimeter for symmetric sets

Thanks to Proposition 2.3, from now on we will consider only  $\alpha$ -perimeter.

We say that a set  $E \subset \mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$  is  $x$ - and  $y$ -spherically symmetric if there exists a set  $G \subset \mathbb{R}^+ \times \mathbb{R}^+$  such that

$$E = \{(x, y) \in \mathbb{R}^n : (|x|, |y|) \in G\}.$$

We call  $G$  the generating set of  $E$ . In the following we will use the constant

$$c_{hk} = hk\omega_h\omega_k,$$

where  $\omega_m = \mathcal{L}^m(\{x \in \mathbb{R}^m : |x| < 1\})$ , for  $m \in \mathbb{N}$ .

**Proposition 2.4.** *Let  $E \subset \mathbb{R}^n$  be a bounded open set with finite  $\alpha$ -perimeter that is  $x$ - and  $y$ -spherically symmetric with generating set  $G \subset \mathbb{R}^+ \times \mathbb{R}^+$ . Then we have:*

$$(2.9) \quad P_\alpha(E) = c_{hk} \sup_{\psi \in \mathcal{F}_2(\mathbb{R}^+ \times \mathbb{R}^+)} \int_G \left( s^{k-1} \partial_r (r^{h-1} \psi_1) + r^{h-1+\alpha} \partial_s (s^{k-1} \psi_2) \right) dr ds.$$

In particular, if  $E$  has Lipschitz boundary then we have:

$$(2.10) \quad P_\alpha(E) = c_{hk} \int_{\partial G} |(N_r^G, r^\alpha N_s^G)| r^{h-1} s^{k-1} d\mathcal{H}^1(r, s),$$

where  $N^G = (N_r^G, N_s^G) \in \mathbb{R}^2$  is the outer unit normal to the boundary  $\partial G \subset \mathbb{R}^+ \times \mathbb{R}^+$ .

*Proof.* We prove a preliminary version of (2.9). We claim that if  $E$  is of finite  $\alpha$ -perimeter and  $x$ -spherically symmetric with generating set  $F \subset \mathbb{R}^+ \times \mathbb{R}^k$ , then we have:

$$(2.11) \quad P_\alpha(E) = h\omega_h \sup_{\psi \in \mathcal{F}_{1+k}(\mathbb{R}^+ \times \mathbb{R}^k)} \int_F \left( \partial_r (r^{h-1} \psi_1) + r^{h-1+\alpha} \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \right) dr dy = Q(F),$$

where  $Q$  is defined via the last identity. For any test function  $\psi \in \mathcal{F}_{1+k}(\mathbb{R}^+ \times \mathbb{R}^k)$  we define the test function  $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$

$$(2.12) \quad \varphi(x, y) = \left( \frac{x}{|x|} \psi_1(|x|, y), \psi_2(|x|, y), \dots, \psi_{1+k}(|x|, y) \right) \text{ for } |x| \neq 0,$$

and  $\varphi(0, y) = 0$ . For any  $i = 1, \dots, h$ ,  $j = 1, \dots, k$ , and  $x \neq 0$ , we have the identities

$$\begin{aligned} \partial_{x_i} \varphi_i(x, y) &= \left( \frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right) \psi_1(|x|, y) + \frac{x_i^2}{|x|^2} \partial_r \psi_1(|x|, y), \\ \partial_{y_j} \varphi_{h+j}(x, y) &= \partial_{y_j} \psi_{1+j}(|x|, y), \end{aligned}$$

and thus, the  $\alpha$ -divergence defined by

$$(2.13) \quad \operatorname{div}_\alpha \varphi(x, y) = \sum_{i=1}^h \frac{\partial \varphi_i(x, y)}{\partial x_i} + |x|^\alpha \sum_{j=1}^k \frac{\partial \varphi_{h+j}(x, y)}{\partial y_j}$$

satisfies

$$(2.14) \quad \operatorname{div}_\alpha \varphi(x, y) = \frac{h-1}{|x|} \psi_1(|x|, y) + \partial_r \psi_1(|x|, y) + |x|^\alpha \sum_{j=1}^k \partial_{y_j} \psi_{1+j}(|x|, y).$$

For any  $y \in \mathbb{R}^k$  we define the section  $F^y = \{r > 0 : (r, y) \in F\}$ . Using Fubini-Tonelli theorem, spherical coordinates in  $\mathbb{R}^h$ , the symmetry of  $E$ , and (2.14) we

obtain

(2.15)

$$\begin{aligned}
\int_E \operatorname{div}_\alpha \varphi \, dx dy &= \int_{\mathbb{R}^k} \int_{Fy} \int_{|x|=r} \left( \frac{h-1}{r} \psi_1 + \partial_r \psi_1 + r^\alpha \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \right) d\mathcal{H}^{h-1}(x) dr dy \\
&= h\omega_h \int_{\mathbb{R}^k} \int_{Fy} r^{h-1} \left( \frac{h-1}{r} \psi_1 + \partial_r \psi_1 + r^\alpha \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \right) dr dy \\
&= h\omega_h \int_F \partial_r(r^{h-1} \psi_1) + r^{\alpha+h-1} \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \, dr dy.
\end{aligned}$$

Because  $\psi$  is arbitrary, this proves the inequality  $\geq$  in (2.11).

We prove the opposite inequality when  $E \subset \mathbb{R}^n$  is an  $x$ -symmetric bounded open set with smooth boundary. The unit outer normal  $N^E = (N_x^E, N_y^E)$  is continuously defined on  $\partial E$ . At points  $(0, y) \in \partial E$ , however, we have  $N_x^E(0, y) = 0$  and thus  $N_\alpha^E(0, y) = 0$ . For any  $\varepsilon > 0$  we consider the compact set  $K = \{(x, y) \in \partial E : |x| \geq \delta\}$ , where  $\delta > 0$  is such that  $P_\alpha(E; \{|x| = \delta\}) = 0$  and

$$(2.16) \quad \int_{\partial E \setminus K} |N_\alpha^E(x, y)| \, d\mathcal{H}^{n-1} < \varepsilon.$$

Let  $H \subset \mathbb{R}^+ \times \mathbb{R}^k$  be the generating set of  $K$ . By standard extension theorems, there exists  $\psi \in \mathcal{F}_{1+k}(\mathbb{R}^+ \times \mathbb{R}^k)$  such that

$$\psi(r, y) = \frac{(N_r^F(r, y), r^\alpha N_y^F(r, y))}{|(N_r^F(r, y), r^\alpha N_y^F(r, y))|} \quad \text{for } (r, y) \in H.$$

The mapping  $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$  introduced in (2.12) satisfies

$$(2.17) \quad \varphi(x, y) = \frac{N_\alpha^E(x, y)}{|N_\alpha^E(x, y)|}, \quad \text{for } (x, y) \in K.$$

Then, by identity (2.15), the divergence theorem, (2.17), (2.16), and (2.3) we have

$$\begin{aligned}
Q(F) &\geq \int_F \left( \partial_r(r^{h-1} \psi_1) + r^{h-1+\alpha} \sum_{j=1}^k \partial_{y_j} \psi_{1+j} \right) dr dy \\
&= \int_E \operatorname{div}_\alpha \varphi \, dx dy = \int_{\partial E} \langle \varphi, N_\alpha^E \rangle \, d\mathcal{H}^{n-1} \\
&= \int_K |N_\alpha^E(x, y)| \, d\mathcal{H}^{n-1} + \int_{\partial E \setminus K} \langle \varphi, N_\alpha^E \rangle \, d\mathcal{H}^{n-1} \\
&\geq P_\alpha(E) - 2\varepsilon.
\end{aligned}$$

This proves (2.11) when  $\partial E$  is smooth. The general case follows by approximation: let  $E \subset \mathbb{R}^n$  be a set of finite  $\alpha$ -perimeter and finite Lebesgue measure that is  $x$ -symmetric with generating set  $F \subset \mathbb{R}^+ \times \mathbb{R}^k$ . By [8, Theorem 2.2.2], there exists a sequence  $(E_j)_{j \in \mathbb{N}}$  such that each  $E_j$  is of class  $C^\infty$

$$\lim_{j \rightarrow \infty} \mathcal{L}^n(E_j \Delta E) = 0 \quad \text{and} \quad \lim_{j \rightarrow \infty} P_\alpha(E_j) = P_\alpha(E).$$

Each  $E_j$  can be also assumed to be  $x$ -spherically symmetric with generating set  $F_j \subset \mathbb{R}^+ \times \mathbb{R}^k$ . Then we also have

$$\lim_{j \rightarrow \infty} \mathcal{L}^{1+k}(F_j \Delta F) = 0.$$

By lower semicontinuity and (2.11) for the smooth case, we have

$$Q(F) \leq \liminf_{j \rightarrow \infty} Q(F_j) = \lim_{j \rightarrow \infty} P_\alpha(E_j) = P_\alpha(E).$$

This concludes the proof of (2.11) for any set  $E$  with finite  $\alpha$ -perimeter.

The general formula (2.9) for sets that are also  $y$ -spherically symmetric can be proved in a similar way and we can omit the details.

Formula (2.10) for sets  $E$  with Lipschitz boundary follows from (2.9) with the same argument sketched in the proof of Proposition 2.3.  $\square$

### 2.3. $\alpha$ -Perimeter in the case $h = 1$

When  $h = 1$  there exists a change of coordinates that transforms  $\alpha$ -perimeter into the standard perimeter (see [16] for the case of the plane  $h = k = 1$ ). Let  $n = 1 + k$  and consider the mappings  $\Phi, \Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\Psi(x, y) = \left( \operatorname{sgn}(x) \frac{|x|^{\alpha+1}}{\alpha+1}, y \right) \quad \text{and} \quad \Phi(\xi, \eta) = \left( \operatorname{sgn}(\xi) |(\alpha+1)\xi|^{\frac{1}{\alpha+1}}, \eta \right).$$

Then we have  $\Phi \circ \Psi = \Psi \circ \Phi = \operatorname{Id}_{\mathbb{R}^n}$ .

**Proposition 2.5.** *Let  $h = 1$  and  $n = 1 + k$ . For any measurable set  $E \subset \mathbb{R}^n$  we have*

$$(2.18) \quad P_\alpha(E) = \sup \left\{ \int_{\Psi(E)} \operatorname{div} \psi \, d\xi d\eta : \psi \in \mathcal{F}_n(\mathbb{R}^n) \right\}.$$

*Proof.* First notice that the supremum in the right hand side can be equivalently computed over all vector fields  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  in the Sobolev space  $W_0^{1,1}(\mathbb{R}^n; \mathbb{R}^n)$  such that  $\|\psi\|_\infty \leq 1$ .

For any  $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$ , let  $\psi = \varphi \circ \Phi$ . Then for any  $j = 1, \dots, k = n - 1$ , we have

$$(2.19) \quad \begin{aligned} \partial_\xi \psi_1(\xi, \eta) &= \partial_\xi (\varphi_1 \circ \Phi)(\xi, \eta) = |(\alpha+1)\xi|^{-\frac{\alpha}{\alpha+1}} \partial_x \varphi_1(\Phi(\xi, \eta)), \\ \partial_{\eta_j} \psi_{1+j}(\xi, \eta) &= \partial_{\eta_j} (\varphi_{1+j} \circ \Phi)(\xi, \eta) = \partial_{y_j} \varphi_{1+j}(\Phi(\xi, \eta)). \end{aligned}$$

In particular, we have  $\psi \in W_0^{1,1}(\mathbb{R}^n; \mathbb{R}^n)$  and  $\|\psi\|_\infty \leq 1$ . Then, the standard divergence of  $\psi$  satisfies

$$\operatorname{div} \psi(\xi, \eta) = |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}} \operatorname{div}_\alpha \phi(\Phi(\xi, \eta)).$$

The determinant Jacobian of the change of variables  $(x, y) = \Phi(\xi, \eta)$  is

$$(2.20) \quad |\det J\Phi(\xi, \eta)| = |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}}.$$

and thus we obtain

$$(2.21) \quad \begin{aligned} \int_E \operatorname{div}_\alpha \varphi(x, y) \, dx dy &= \int_{\Psi(E)} \operatorname{div}_\alpha \varphi(\Phi(\xi, \eta)) |\det J\Phi(\xi, \eta)| \, d\xi d\eta \\ &= \int_{\Psi(E)} \operatorname{div} \psi(\xi, \eta) \, d\xi d\eta. \end{aligned}$$

The claim follows. □

### 3. Rearrangements

In this section, we prove various rearrangement inequalities for  $\alpha$ -perimeter in  $\mathbb{R}^n$ . We consider first the case  $h = 1$ . In this case, there are a Steiner type rearrangement in the  $x$ -variable and a Schwarz rearrangement in the  $y$  variables that reduce the isoperimetric problem in  $\mathbb{R}^n$  to a problem for Lipschitz graphs in the first quadrant  $\mathbb{R}^+ \times \mathbb{R}^+$ . Then we consider dimensions  $h \geq 2$ , where we can rearrange sets in  $\mathbb{R}^h$  that are already  $x$ -spherically symmetric.

#### 3.1. Rearrangement in the case $h = 1$

Let  $h = 1$  and  $n = 1 + k$ . We say that a set  $E \subset \mathbb{R}^n$  is  $x$ -symmetric if  $(x, y) \in E$  implies  $(-x, y) \in E$ ; we say that  $E$  is  $x$ -convex if the section  $E^y = \{x \in \mathbb{R} : (x, y) \in E\}$  is an interval for every  $y \in \mathbb{R}^k$ ; finally, we say that  $E$  is  $y$ -Schwarz symmetric if for every  $x \in \mathbb{R}$  the section  $E^x = \{y \in \mathbb{R}^k : (x, y) \in E\}$  is an (open) Euclidean ball in  $\mathbb{R}^k$  centered at the origin.

**Theorem 3.1.** *Let  $h = 1$  and  $n = 1 + k$ . For any set  $E \subset \mathbb{R}^n$  such that  $P_\alpha(E) < \infty$  and  $0 < \mathcal{L}^n(E) < \infty$  there exists an  $x$ -symmetric,  $x$ -convex, and  $y$ -Schwarz symmetric set  $E^* \subset \mathbb{R}^n$  such that  $P_\alpha(E^*) \leq P_\alpha(E)$  and  $\mathcal{L}^n(E^*) = \mathcal{L}^n(E)$ .*

*Moreover, if  $P_\alpha(E^*) = P_\alpha(E)$  then  $E$  is  $x$ -symmetric,  $x$ -convex and there exist functions  $c : [0, \infty) \rightarrow \mathbb{R}^k$  and  $f : [0, \infty) \rightarrow [0, \infty]$  such that for  $\mathcal{L}^1$ -a.e.  $x \in \mathbb{R}$  we have*

$$(3.1) \quad E^x = \{y \in \mathbb{R}^k : |y - c(|x|)| < f(|x|)\}.$$

*Proof.* By Proposition 2.5, the set  $F = \Psi(E) \subset \mathbb{R}^n$  satisfies  $P(F) = P_\alpha(E)$ , where  $P$  stands for the standard perimeter in  $\mathbb{R}^n$ . We define the measure  $\mu$  on  $\mathbb{R}^n$

$$(3.2) \quad \mu(F) = \int_F |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}} d\xi d\eta.$$

Then, by (2.20) we also have the identity  $\mu(F) = \mathcal{L}^n(E)$ .

We rearrange the set  $F$  using Steiner symmetrization in direction  $\xi$ . Namely, we let

$$F_1 = \{(\xi, \eta) \in \mathbb{R}^n : |\xi| < \mathcal{L}^1(F^\eta)/2\},$$

where  $F^\eta = \{\xi \in \mathbb{R} : (\xi, \eta) \in F\}$ . The set  $F_1$  is  $\xi$ -symmetric and  $\xi$ -convex. By classical results on Steiner symmetrization we have  $P(F_1) \leq P(F)$  and the equality  $P(F_1) = P(F)$  implies that  $F$  is  $\xi$ -convex: namely, a.e. section  $F^\eta$  is (equivalent to) an interval.

The  $\mu$ -volume of  $F_1$  is

$$\mu(F_1) = \int_{F_1} |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}} d\xi d\eta = \int_{\mathbb{R}^k} \left( \int_{F_1^\eta} |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}} d\xi \right) d\eta.$$

For any measurable set  $I \subset \mathbb{R}$  with finite measure, the symmetrized set  $I^* = (-\mathcal{L}^1(I)/2, \mathcal{L}^1(I)/2)$  satisfies the following inequality (see [16], page 361)

$$(3.3) \quad \int_I |\xi|^{-\frac{\alpha}{\alpha+1}} d\xi \leq \int_{I^*} |\xi|^{-\frac{\alpha}{\alpha+1}} d\xi.$$

Moreover, if  $\mathcal{L}^1(I \Delta I^*) > 0$  then the inequality is strict. This implies that  $\mu(F_1) \geq \mu(F)$  and the inequality is strict if  $F$  is not equivalent to an  $\xi$ -symmetric and  $\xi$ -convex set.

We rearrange the set  $F_1$  using Schwarz symmetrization in  $\mathbb{R}^k$ , namely we let

$$F_2 = \left\{ (\xi, \eta) \in \mathbb{R}^n : |\eta| < \left( \frac{\mathcal{L}^k(F_1^\xi)}{\omega_k} \right)^{\frac{1}{k}} \right\}.$$

By classical results on Schwarz rearrangement, we have  $P(F_2) \leq P(F_1)$  and the equality  $P(F_2) = P(F_1)$  implies that a.e. section  $F_1^\xi$  is an Euclidean ball

$$(3.4) \quad F_1^\xi = \{\eta \in \mathbb{R}^k : |\eta - d(|\xi|)| < \varrho(|\xi|)\}$$

for some  $d(|\xi|) \in \mathbb{R}^k$  and  $\varrho(|\xi|) \in [0, \infty]$ . By Fubini-Tonelli theorem, the  $\mu$ -volume is preserved:

$$(3.5) \quad \mu(F_2) = \int_{\mathbb{R}} |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}} \mathcal{L}^k(F_2^\xi) d\xi = \int_{\mathbb{R}} |(\alpha + 1)\xi|^{-\frac{\alpha}{\alpha+1}} \mathcal{L}^k(F_1^\xi) d\xi = \mu(F_1).$$

Recall that  $\delta_\lambda(x, y) = (\lambda x, \lambda^{\alpha+1} y)$ . The set  $E^* = \delta_\lambda(\Phi(F_2))$ , with  $\lambda > 0$  such that  $\mathcal{L}^n(E^*) = \mathcal{L}^n(E)$ , satisfies the claims in the statement of the theorem. In fact, we have  $0 < \lambda \leq 1$  because

$$\mathcal{L}^n(\Phi(F_2)) = \mu(F_2) = \mu(F_1) \geq \mu(F) = \mathcal{L}^n(E),$$

and then, by the scaling property of  $\alpha$ -perimeter we have

$$P_\alpha(E^*) = \lambda^{d-1} P_\alpha(\Phi(F_2)) \leq P_\alpha(\Phi(F_2)) = P(F_2) \leq P(F_1) \leq P(F) = P_\alpha(E).$$

This proves the first part of the theorem.

If  $P_\alpha(E^*) = P_\alpha(E)$  then we have  $P(F_2) = P(F_1)$  and  $\lambda = 1$ . From the first equality we deduce that the sections  $F_1^\xi$  are of the form (3.4) and claim (3.1) holds with  $c(|x|) = d(|x|^{\alpha+1}/(\alpha+1))$  and  $f(|x|) = \varrho(|x|^{\alpha+1}/(\alpha+1))$ . From  $\lambda = 1$  we deduce that

$$\mu(F) = \mathcal{L}^n(E) = \mathcal{L}^n(E^*) = \mathcal{L}^n(\Phi(F_2)) = \mu(F_2) = \mu(F_1),$$

and thus  $F$  is  $\xi$ -symmetric and  $\xi$ -convex. The same holds then for  $E$ .  $\square$

### 3.2. Rearrangement in the case $h \geq 2$

We prove the analogous of Theorem 3.1 when  $h \geq 2$ . We need to start from a set  $E \subset \mathbb{R}^n$  that is  $x$ -spherically symmetric

$$E = \{(x, y) \in \mathbb{R}^n : (|x|, y) \in F\}$$

for some generating set  $F \subset \mathbb{R}^+ \times \mathbb{R}^k$ .

By the proof of Proposition 2.4, see (2.11), we have the identity  $P_\alpha(E) = Q(F)$ , where

$$(3.6) \quad Q(F) = h\omega_h \sup_{\psi \in \mathcal{F}_{1+k}(\mathbb{R}^+ \times \mathbb{R}^k)} \int_F \left( \partial_r(r^{h-1}\psi_1) + r^{h-1+\alpha} \sum_{j=1}^k \partial_{y_j}\psi_{1+j} \right) dr dy.$$

Our goal is to improve the  $x$ -spherical symmetry to the  $x$ -Schwarz symmetry. A set  $E \subset \mathbb{R}^n$  is  $x$ -Schwarz symmetric if for all  $y \in \mathbb{R}^k$  we have

$$E^y = \{x \in \mathbb{R}^h : (x, y) \in E\} = \{x \in \mathbb{R}^h : |x| < \varrho(y)\}$$

for some function  $\varrho : \mathbb{R}^k \rightarrow [0, \infty]$ . To obtain the Schwarz symmetry, we use the radial rearrangement technique introduced in [13].

**Theorem 3.2.** *Let  $h \geq 2$ ,  $k \geq 1$  and  $n = h + k$ . For any  $x$ -spherically symmetric set  $E \subset \mathbb{R}^n$  such that  $P_\alpha(E) < \infty$  and  $0 < \mathcal{L}^n(E) < \infty$  there exists an  $x$ - and  $y$ -Schwarz symmetric set  $E^* \subset \mathbb{R}^n$  such that  $P_\alpha(E^*) \leq P_\alpha(E)$  and  $\mathcal{L}^n(E^*) = \mathcal{L}^n(E)$ .*

*Moreover, if  $P_\alpha(E^*) = P_\alpha(E)$  then  $E$  is  $x$ -Schwarz symmetric and there exist functions  $c : [0, \infty) \rightarrow \mathbb{R}^k$  and  $f : [0, \infty) \rightarrow [0, \infty]$  such that, up to a negligible set, we have*

$$(3.7) \quad E = \{(x, y) \in \mathbb{R}^n : |y - c(|x|)| < f(|x|)\}.$$

*Proof.* Let  $F \subset \mathbb{R}^+ \times \mathbb{R}$  be the generating set of  $E$ . We define the volume of  $F$  via the following formula

$$V(F) = \omega_h \int_F r^{h-1} dr dy = \mathcal{L}^n(E).$$

We rearrange  $F$  in the coordinate  $r$  using the linear density  $r^{h-1+\alpha}$  that appears, in (3.6), in the part of divergence depending on the coordinates  $y$ . Namely, we define the function  $g : \mathbb{R}^k \rightarrow [0, \infty]$  via the identity

$$(3.8) \quad \frac{1}{h+\alpha} g(y)^{h+\alpha} = \int_0^{g(y)} r^{h-1+\alpha} dr = \int_{F_y} r^{h-1+\alpha} dr,$$

and we let

$$F^\sharp = \{(r, y) \in \mathbb{R}^+ \times \mathbb{R}^k : 0 < r < g(y)\}.$$

We claim that  $Q(F^\sharp) \leq Q(F)$  and  $V(F^\sharp) \geq V(F)$ , with equality  $V(F^\sharp) = V(F)$  holding if and only if  $F^\sharp = F$ , up to a negligible set.

For any open set  $A \subset \mathbb{R}^+ \times \mathbb{R}^k$ , we define

$$(3.9) \quad \begin{aligned} Q_0(F; A) &= \sup_{\psi \in \mathcal{F}_1(A)} \int_F \partial_r(r^{h-1}\psi) dr dy, \\ Q_j(F; A) &= \sup_{\psi \in \mathcal{F}_1(A)} \int_F r^{h-1+\alpha} \partial_{y_j} \psi dr dy, \quad j = 1, \dots, k. \end{aligned}$$

The open sets mappings  $A \mapsto Q_j(F; A)$ ,  $j = 0, 1, \dots, k$ , extend to Borel measures. For any Borel set  $B \subset \mathbb{R}^k$  and  $j = 0, 1, \dots, k$ , we define the measures

$$\begin{aligned} \mu_j(B) &= Q_j(F; \mathbb{R}^+ \times B), \\ \mu_j^\sharp(B) &= Q_j(F^\sharp; \mathbb{R}^+ \times B). \end{aligned}$$

By Step 1 and Step 2 of the proof of Theorem 1.5 in [13], see page 106, we have  $\mu_j^\sharp(B) \leq \mu_j(B)$  for any Borel set  $B \subset \mathbb{R}^k$  and for any  $j = 0, 1, \dots, k$ . It follows that the vector valued Borel measures  $\mu = (\mu_0, \dots, \mu_k)$  and  $\mu^\sharp = (\mu_0^\sharp, \dots, \mu_k^\sharp)$  satisfy

$$|\mu^\sharp|(\mathbb{R}^k) \leq |\mu|(\mathbb{R}^k),$$

where  $|\cdot|$  denotes the total variation. This is equivalent to  $Q(F^\sharp) \leq Q(F)$ .

We claim that for any  $y \in \mathbb{R}^k$  we have

$$(3.10) \quad \frac{1}{h} g(y)^h = \int_{F_y^\sharp} r^{h-1} dr \geq \int_{F_y} r^{h-1} dr,$$

with strict inequality unless  $F_y^\sharp = F_y$  up to a negligible set. From (3.10), by Fubini-Tonelli theorem it follows that  $V(F^\sharp) \geq V(F)$  with strict inequality unless  $F^\sharp = F$  up to a negligible set. By (3.8), claim (3.10) is equivalent to

$$(3.11) \quad \left( (h+\alpha) \int_{F_y} r^{h-1+\alpha} dr \right)^{\frac{1}{h+\alpha}} \geq \left( h \int_{F_y} r^{h-1} dr \right)^{\frac{1}{h}},$$

and this inequality holds for any measurable set  $F_y \subset \mathbb{R}^+$ , for any  $h \geq 2$ , and  $\alpha > 0$ , by Example 2.5 in [13]. Moreover, we have equality in (3.11) if and only if  $F_y = (0, g(y))$ .

Let  $E_1^\sharp \subset \mathbb{R}^n$  be the  $x$ -Schwarz symmetric set with generating set  $F^\sharp$ . Then we have

$$\mathcal{L}^n(E_1^\sharp) = V(F^\sharp) \geq V(F) = \mathcal{L}^n(E),$$

with strict inequality unless  $F^\sharp = F$ . Then there exists  $0 < \lambda \leq 1$  such that the set  $E^\sharp = \delta_\lambda(E_1^\sharp)$  satisfies  $\mathcal{L}^n(E^\sharp) = \mathcal{L}^n(E)$ . Since  $\lambda \leq 1$ , we also have

$$P_\alpha(E^\sharp) = \lambda^{d-1} P_\alpha(E_1^\sharp) \leq P_\alpha(E_1^\sharp) = Q(F^\sharp) \leq Q(F) = P_\alpha(E).$$

If  $P_\alpha(E^\sharp) = P_\alpha(E)$  then it must be  $\lambda = 1$  and thus  $F^\sharp = F$ , that in turn implies  $E^\sharp = E$ , up to a negligible set.

Now the proof of the theorem can be concluded applying to  $E^\sharp$  a Schwarz rearrangement in the variable  $y \in \mathbb{R}^k$ . This rearrangement is standard, see the general argument in [14]. The resulting set  $E^* \subset \mathbb{R}^n$  satisfies  $P_\alpha(E^*) \leq P_\alpha(E)$  and also the other claims in the theorem.  $\square$

#### 4. Existence of isoperimetric sets

In this section, we prove existence of solutions to the isoperimetric problem for  $\alpha$ -perimeter and  $H$ -perimeter. When  $h \geq 2$ , we prove the existence of solutions in the class of  $x$ -spherically symmetric sets. The proof is based on a concentration-compactness argument.

For any set  $E \subset \mathbb{R}^n$  and  $t > 0$ , we let

$$(4.1) \quad \begin{aligned} E_{t-}^x &= \{(x, y) \in E : |x| < t\} \quad \text{and} \quad E_t^x = \{(x, y) \in E : |x| = t\}, \\ E_{t-}^y &= \{(x, y) \in E : |y| < t\} \quad \text{and} \quad E_t^y = \{(x, y) \in E : |y| = t\}. \end{aligned}$$

We also define

$$(4.2) \quad v_E^x(t) = \mathcal{H}^{n-1}(E_t^x),$$

and

$$(4.3) \quad v_E^y(t) = \int_{E_t^y} |x|^\alpha d\mathcal{H}^{n-1}.$$

In the following, we use the short notation  $\{|x| < t\} = \{(x, y) \in \mathbb{R}^n : |x| < t\}$  and  $\{|y| < t\} = \{(x, y) \in \mathbb{R}^n : |y| < t\}$ .

**Proposition 4.1.** *Let  $E \subset \mathbb{R}^n$  be a set with finite measure and finite  $\alpha$ -perimeter. Then for a.e.  $t > 0$  we have*

$$(4.4) \quad P_\alpha(E_{t-}^x) = P_\alpha(E; E_{t-}^x) + v_E^x(t) \quad \text{and} \quad P_\alpha(E_{t-}^y) = P_\alpha(E; E_{t-}^y) + v_E^y(t).$$

*Proof.* We prove the claim for  $E_{t-}^y$ . Let  $\{\phi_\varepsilon\}_{\varepsilon>0}$  be a standard family of mollifiers in  $\mathbb{R}^n$  and let

$$f_\varepsilon(z) = \int_E \phi_\varepsilon(|z-w|)dw, \quad z \in \mathbb{R}^n.$$

Then  $f_\varepsilon \in C^\infty(\mathbb{R}^n)$  and  $f_\varepsilon \rightarrow \chi_E$  in  $L^1(\mathbb{R}^n)$  for  $\varepsilon \rightarrow 0$ . Therefore, by the coarea formula we also have, for a.e.  $t > 0$  and possibly for a suitable infinitesimal sequence of  $\varepsilon$ 's,

$$(4.5) \quad \lim_{\varepsilon \rightarrow 0} \int_{\{|y|=t\}} |f_\varepsilon - \chi_E| d\mathcal{H}^{n-1} = 0.$$

Since  $E$  has finite  $\alpha$ -perimeter, the set  $\{t > 0 : P_\alpha(E; \{|y|=t\}) > 0\}$  is at most countable, and thus

$$(4.6) \quad P_\alpha(E; \{|y|=t\}) = 0 \quad \text{for a.e. } t > 0.$$

We use the notation  $\nabla_\alpha f_\varepsilon = (X_1 f_\varepsilon, \dots, X_h f_\varepsilon, Y_1 f_\varepsilon, \dots, Y_k f_\varepsilon)$ , where  $X_i, Y_j$  are the vector fields (1.6). By the divergence Theorem, for any  $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$  we have

$$(4.7) \quad \begin{aligned} \int_{\{|y|<t\}} f_\varepsilon(z) \operatorname{div}_\alpha \varphi(z) dz &= \int_{\{|y|<t\}} (\operatorname{div}_\alpha(f_\varepsilon \varphi) - \langle \nabla_\alpha f_\varepsilon, \varphi \rangle) dz \\ &= - \int_{\{|y|=t\}} f_\varepsilon(z) |x|^\alpha \langle N, \varphi(z) \rangle d\mathcal{H}^{n-1} - \int_{\{|y|<t\}} \langle \nabla_\alpha f_\varepsilon, \varphi \rangle dz, \end{aligned}$$

where  $N = (0, -y/|y|)$  is the inner unit normal of  $\{|y| < t\}$ . For any  $t > 0$ , we have

$$(4.8) \quad \lim_{\varepsilon \rightarrow 0} \int_{\{|y|<t\}} f_\varepsilon(z) \operatorname{div}_\alpha \varphi(z) dz = \int_{E_{t-}^y} \operatorname{div}_\alpha \varphi(z) dz,$$

and, for any  $t > 0$  satisfying (4.5),

$$(4.9) \quad \lim_{\varepsilon \rightarrow 0} \int_{\{|y|=t\}} f_\varepsilon(z) |x|^\alpha \langle N, \varphi(z) \rangle d\mathcal{H}^{n-1} = \int_{E_t^y} |x|^\alpha \langle N, \varphi(z) \rangle d\mathcal{H}^{n-1}.$$

On the other hand, we claim that

$$(4.10) \quad \lim_{\varepsilon \rightarrow 0} \int_{\{|y|<t\}} \langle \nabla_\alpha f_\varepsilon, \varphi \rangle dz = \int_{\{|y|<t\}} \left\{ \sum_{i=1}^h \varphi_i d\mu_E^{x_i} + \sum_{\ell=1}^k \varphi_{h+\ell} |x|^\alpha d\mu_E^{y_\ell} \right\},$$

where  $\mu_E^{x_i}$  and  $\mu_E^{y_\ell}$  are the distributional partial derivatives of  $\chi_E$ , that are Borel

measures on  $\mathbb{R}^n$ , because  $E$  has finite  $\alpha$ -perimeter. For the coordinate  $y_\ell$ , we have

$$\begin{aligned} \int_{\{|y|<t\}} \varphi_{h+\ell}(z) |x|^\alpha \partial_{y_\ell} f_\varepsilon(z) dz &= \int_{\{|y|<t\}} \varphi_{h+\ell}(z) |x|^\alpha \int_E \partial_{y_\ell} \phi_\varepsilon(|z-w|) dw dz \\ &= - \int_{\{|y|<t\}} \varphi_{h+\ell}(z) |x|^\alpha \int_E \partial_{\eta_\ell} \phi_\varepsilon(|z-w|) dw dz \\ &= \int_{\{|y|<t\}} \varphi_{h+\ell}(z) |x|^\alpha \int_{\mathbb{R}^n} \phi_\varepsilon(|z-w|) d\mu_E^{y_\ell}(w) dz \\ &= \int_{\mathbb{R}^n} \int_{\{|y|<t\}} \varphi_{h+\ell}(z) |x|^\alpha \phi_\varepsilon(|z-w|) dz d\mu_E^{y_\ell}(w), \end{aligned}$$

where we let  $w = (\xi, \eta) \in \mathbb{R}^h \times \mathbb{R}^k$ . By (4.6), the measure  $\mu_E^{y_\ell}$  is concentrated on  $\{|y| \neq t\}$ . It follows that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} \int_{\{|y|<t\}} \varphi_{h+\ell}(z) |x|^\alpha \phi_\varepsilon(|z-w|) dz d\mu_E^{y_\ell}(w) = \int_{\{|\eta|<t\}} \varphi_{h+\ell}(w) |\xi|^\alpha d\mu_E^{y_\ell}(w).$$

This proves (4.10).

Now, from (4.7)–(4.10) we deduce that

$$\begin{aligned} \int_{E \cap \{|y|<t\}} \operatorname{div}_\alpha \varphi(z) dz &= - \int_{E \cap \{|y|=t\}} |x|^\alpha \langle N, \varphi(z) \rangle d\mathcal{H}^{n-1} \\ &\quad - \int_{\{|y|<t\}} \left\{ \sum_{i=1}^h \varphi_i d\mu_E^{x_i} + |x|^\alpha \sum_{\ell=1}^k \varphi_{h+\ell} d\mu_E^{y_\ell} \right\}, \end{aligned} \tag{4.11}$$

and the claim follows by optimizing the right hand side over  $\varphi \in \mathcal{F}_n(\mathbb{R}^n)$ .  $\square$

**Proposition 4.2.** *Let  $E \subset \mathbb{R}^n$  be a set with finite measure and finite  $\alpha$ -perimeter. For a.e.  $t > 0$  we have  $P_\alpha(E_{t-}^x) \leq P_\alpha(E)$  and  $P_\alpha(E_{t-}^y) \leq P_\alpha(E)$ .*

*Proof.* The proof is a calibration argument. Notice that

$$\begin{aligned} P_\alpha(E_{t-}^y) &= P_\alpha(E_{t-}^y; \{|y| < t\}) + P_\alpha(E_{t-}^y; \{|y| \geq t\}) \\ &= P_\alpha(E; \{|y| < t\}) + P_\alpha(E_{t-}^y; \{|y| = t\}). \end{aligned}$$

Let  $t > 0$  be such that  $P_\alpha(E; \{|y| = t\}) = 0$ ; a.e.  $t > 0$  has this property, see (4.6). It is sufficient to show that

$$P_\alpha(E_{t-}^y; \{|y| = t\}) \leq P_\alpha(E; \{|y| \geq t\}) = P_\alpha(E; \{|y| > t\}).$$

The function  $\varphi(x, y) = (0, -y/|y|) \in \mathbb{R}^n$ ,  $|y| \neq 0$ , has negative divergence:

$$\operatorname{div}_\alpha \varphi(x, y) = -|x|^\alpha \sum_{\ell=1}^k \left( \frac{1}{|y|} - \frac{y_\ell^2}{|y|^3} \right) = -\frac{(k-1)|x|^\alpha}{|y|} \leq 0.$$

As in the proof of (4.11), we have

$$\begin{aligned} 0 &\geq \int_{E \cap \{|y| > t\}} \operatorname{div}_\alpha \varphi \, dz = \int_{E_t^y} |x|^\alpha d\mathcal{H}^{n-1} - \int_{\{|y| > t\}} |x|^\alpha \sum_{\ell=1}^k \varphi_{h+\ell} d\mu_E^{y_\ell} \\ &\geq \int_{E \cap \{|y| = t\}} |x|^\alpha d\mathcal{H}^{n-1} - P_\alpha(E; \{|y| > t\}). \end{aligned}$$

By the representation formula (2.3), we obtain

$$P_\alpha(E_{t-}^y; \{|y| = t\}) = \int_{E_t^y} |x|^\alpha d\mathcal{H}^{n-1} \leq P_\alpha(E; \{|y| > t\}).$$

This ends the proof.  $\square$

We prove the existence of isoperimetric sets assuming the validity of the following isoperimetric inequality, holding for any  $\mathcal{L}^n$ -measurable set  $E \subset \mathbb{R}^n$  with finite measure

$$(4.12) \quad P_\alpha(E) \geq C \mathcal{L}^n(E)^{\frac{d-1}{d}}$$

for some geometric constant  $C > 0$ , see [10], [6], and [7]. By the homogeneity properties of Lebesgue measure and  $\alpha$ -perimeter, we can define the constant

$$(4.13) \quad C_I = \inf\{P_\alpha(E) : \mathcal{L}^n(E) = 1 \text{ and } E \in \mathcal{S}_x, \text{ if } h \geq 2\}.$$

Only when  $h \geq 2$  we are adding the constraint  $E \in \mathcal{S}_x$ . We have  $C_I > 0$  by the validity of (4.12) for some  $C > 0$ . Our goal is to prove that the infimum in (4.13) is attained.

**Theorem 4.3.** *Let  $h, k \geq 1$  and  $n = h + k$ . There exists an  $x$ - and  $y$ -Schwarz symmetric set  $E \subset \mathbb{R}^n$  realizing the infimum in (4.13).*

*Proof.* Let  $(E_m)_{m \in \mathbb{N}}$  be a minimizing sequence for the infimum in (4.13), with the additional assumption that the sets involved in the minimization are  $x$ -spherically symmetric when  $h \geq 2$ . Namely,

$$(4.14) \quad \mathcal{L}^n(E_m) = 1 \text{ and } P_\alpha(E_m) \leq C_I \left(1 + \frac{1}{m}\right), \quad m \in \mathbb{N}.$$

By Theorems 3.1 and 3.2, we can assume that every set  $E_m$  is  $x$ - and  $y$ -Schwarz symmetric. We claim that the minimizing sequence can be also assumed to be in a bounded region of  $\mathbb{R}^n$ .

Fix  $m \in \mathbb{N}$  and let  $E = E_m$ . For any  $t > 0$  such that (4.4) holds we consider the set  $E_{t-}^x = E \cap \{|x| < t\} \in \mathcal{S}_x$ .

We apply the isoperimetric inequality (4.12) with the constant  $C_I > 0$  in (4.13) to the sets  $E_{t-}^x$  and  $E \setminus E_{t-}^x$ , and we use Proposition 4.1:

$$(4.15) \quad \begin{aligned} C_I \mathcal{L}^n(E_{t-}^x)^{\frac{d-1}{d}} &\leq P_\alpha(E_{t-}^x) = P_\alpha(E; \{|x| < t\}) + v_E^x(t) \\ C_I (1 - \mathcal{L}^n(E_{t-}^x))^{\frac{d-1}{d}} &\leq P_\alpha(E \setminus E_{t-}^x) = P_\alpha(E; \{|x| > t\}) + v_E^x(t). \end{aligned}$$

As in (4.2), we let  $v_E^x(t) = \mathcal{H}^{n-1}(E_t^x)$ . Adding up the two inequalities we get

$$(4.16) \quad C_I(\mathcal{L}^n(E_{t-}^x)^{\frac{d-1}{d}} + (1 - \mathcal{L}^n(E_{t-}^x))^{\frac{d-1}{d}}) \leq P_\alpha(E) + 2v_E^x(t).$$

The function  $g : [0, \infty) \rightarrow \mathbb{R}$ ,  $g(t) = \mathcal{L}^n(E_{t-}^x)$  is continuous,  $(0, 1) \subset g([0, \infty)) \subset [0, 1]$ , and it is increasing. In particular,  $g$  is differentiable almost everywhere. For any  $t > 0$  such that  $P_\alpha(E; \{|x| = t\}) = 0$ , also the standard perimeter vanishes, namely  $P(E; \{|x| = t\}) = 0$ . With the vector field  $\varphi = (x/|x|, 0)$ , and for  $t < s$  satisfying  $P_\alpha(E; \{|x| = t\}) = P_\alpha(E; \{|x| = s\}) = 0$ , we have

$$\begin{aligned} \int_{E_{s-}^x \setminus E_{t-}^x} \frac{h-1}{|x|} dz &= \int_{E_{s-}^x \setminus E_{t-}^x} \operatorname{div} \varphi dz \\ &= \mathcal{H}^{n-1}(E_s^x) - \mathcal{H}^{n-1}(E_t^x) + \int_{\partial^* E \cap \{s < |x| < t\}} \langle \varphi, \nu_E \rangle d\mathcal{H}^{n-1}. \end{aligned}$$

This implies that

$$\lim_{s \rightarrow t} \mathcal{H}^{n-1}(E_s^x) = \mathcal{H}^{n-1}(E_t^x),$$

with limit restricted to  $s$  satisfying the above condition, and thus

$$(4.17) \quad g'(t) = \lim_{s \rightarrow t} \frac{1}{s-t} \int_t^s \mathcal{H}^{n-1}(E_\tau^x) d\tau = \mathcal{H}^{n-1}(E_t^x).$$

At this point, by (4.14), inequality (4.16) gives

$$(4.18) \quad C_I \left( g(t)^{\frac{d-1}{d}} + (1 - g(t))^{\frac{d-1}{d}} - 1 - \frac{1}{m} \right) \leq 2g'(t).$$

The function  $\psi : [0, 1] \rightarrow \mathbb{R}$ ,  $\psi(s) = s^{\frac{d-1}{d}} + (1-s)^{\frac{d-1}{d}} - 1$  is concave, it attains its maximum at  $s = 1/2$  with  $\psi(1/2) = 2^{\frac{1}{d}} - 1$ , and it satisfies  $\psi(s) = \psi(1-s)$ ,  $\psi(0) = \psi(1) = 0$ . By (4.18) we have

$$(4.19) \quad g'(t) \geq \frac{C_I}{2} \left( \psi(g(t)) - \frac{1}{m} \right) \geq \frac{C_I}{4} \psi(g(t)) + \frac{C_I}{4} \left( \psi(g(t)) - \frac{2}{m} \right),$$

for almost every  $t \in \mathbb{R}$  and every  $m \in \mathbb{N}$ . Provided that  $m \in \mathbb{N}$  is such that  $2/m \leq \max \psi = 2^{1/d} - 1$ , we show that there exist constants  $0 < a_m < b_m < \infty$  such that inequality (4.19) implies the following:

$$(4.20) \quad g'(t) \geq \frac{C_I}{4} \psi(g(t)) \text{ for a.e. } t \in [a_m, b_m].$$

In fact, by continuity of  $g$  and  $\psi$ , and by symmetry of  $\psi$  with respect to the line  $\{s = 1/2\}$ , for  $m$  large enough, there exist  $0 < a_m < b_m < \infty$  such that

$$0 < g(a_m) = 1 - g(b_m) < \frac{1}{2} \quad \text{and} \quad \psi(g(a_m)) = \psi(g(b_m)) = \frac{2}{m}.$$

By concavity of  $\psi$  and monotonicity of  $g$ , it follows that  $\psi(g(t)) \geq \frac{2}{m}$  for every  $t \in [a_m, b_m]$ , and (4.20) follows. As  $m \rightarrow \infty$  we have  $g(b_m) \rightarrow 1$ , that implies

$$\lim_{m \rightarrow \infty} b_m = \sup\{b > 0 : g(b) < 1\} > 0.$$

Moreover, as  $m \rightarrow \infty$  we also have  $g(a_m) \rightarrow 0$ . Since the set  $E$  is  $x$ -Schwarz symmetric, there holds  $g(a) > 0$  for all  $a > 0$ . Therefore, we deduce that  $a_m \rightarrow 0$ .

We infer that, for  $m$  large enough, we have  $a_m < b_m/2$ . Integrating inequality (4.20) on the interval  $[b_m/2, b_m]$ , we find

$$(4.21) \quad \frac{b_m}{2} \leq \frac{4}{C_I} \int_{b_m/2}^{b_m} \frac{g'(t)}{\psi(g(t))} dt \leq \frac{4}{C_I} \int_{g(b_m/2)}^{g(b_m)} \frac{1}{\psi(s)} ds \leq \frac{4}{C_I} \int_0^1 \frac{1}{\psi(s)} ds = \ell_1.$$

We consider the set  $\hat{E}_m = E_{b_m-}^x$ . By (4.21),  $\hat{E}_m$  is contained in the cylinder  $\{|x| < 2\ell_1\}$  and, by Proposition 4.2, it satisfies  $P_\alpha(\hat{E}_m) \leq P_\alpha(E_m)$ . Define the set  $E_m^\dagger = \delta_{\lambda_m}(\hat{E}_m)$ , where  $\lambda_m \geq 1$  is chosen in such a way that  $\mathcal{L}^n(\hat{E}_m^\dagger) = 1$ ; namely,  $\lambda_m$  is the number

$$\lambda_m = \left( \frac{1}{\mathcal{L}^n(\hat{E}_m)} \right)^{\frac{1}{d}},$$

where

$$(4.22) \quad \mathcal{L}^n(\hat{E}_m) = \mathcal{L}^n(E_m \cap \{|x| < b_m\}) = g(b_m) = 1 - g(a_m).$$

By concavity of  $\psi$ , for  $0 < s < 1/2$  the graph of  $\psi$  lays above the straight line through the origin passing through the maximum  $(1/2, \psi(1/2))$ , i.e.,  $\psi(s) > 2(2^{1/d} - 1)s$ . Therefore, since  $g(a_m) < 1/2$  and  $\psi(g(a_m)) = 2/m$ , then

$$g(a_m) \leq \frac{1}{m(2^{1/d} - 1)},$$

and thus

$$\lambda_m \leq \left( \frac{1}{1 - \frac{1}{m(2^{1/d} - 1)}} \right)^{1/d} = \left( \frac{m}{m - \frac{1}{2^{1/d} - 1}} \right)^{1/d}.$$

By homogeneity of  $\alpha$ -perimeter,

$$\begin{aligned} P_\alpha(E_m^\dagger) &= \lambda_m^{d-1} P_\alpha(\hat{E}_m) \leq \lambda_m^{d-1} P_\alpha(E_m) \leq \lambda_m^{d-1} C_I \left( 1 + \frac{1}{m} \right) \\ &\leq C_I \left( 1 + \frac{1}{m} \right) \left( \frac{m}{m - \frac{1}{2^{1/d} - 1}} \right)^{\frac{d-1}{d}}. \end{aligned}$$

In conclusion,  $(E_m^\dagger)_{m \in \mathbb{N}}$  is a minimizing sequence for  $C_I$  and, for  $m$  large enough, it is contained in the cylinder  $\{|x| < \ell\}$ , where  $\ell = 2^{1/d+1}\ell_1$ .

Now we consider the case of the  $y$ -variable. We start again from (4.15) for the sets  $E_t^y$  for  $t > 0$ . Now the set  $E$  can be assumed to be contained in the cylinder  $\{|x| < \ell\}$ . In this case, we have

$$v_E^y(t) = \int_{E_t^y} |x|^\alpha d\mathcal{H}^{n-1} \leq \ell^\alpha \mathcal{H}^{n-1}(E_t^y) = \ell^\alpha g'(t).$$

So inequality (4.16) reads

$$(4.23) \quad C_I \left( g(t)^{\frac{d-1}{d}} + (1 - g(t))^{\frac{d-1}{d}} - 1 - \frac{1}{m} \right) \leq 2\ell^\alpha g'(t).$$

Now the argument continues exactly as in the first case. The conclusion is that there exists a minimizing sequence  $(E_m)_{m \in \mathbb{N}}$  for (4.13) and there exists  $\ell > 0$  such that we have:

- i)  $\mathcal{L}^n(E_m) = 1$  for all  $m \in \mathbb{N}$ ;
- ii)  $P_\alpha(E_m) \leq C_I(1 + 1/m)$  for all  $m \in \mathbb{N}$ ;
- iii)  $E_m \subset \{(x, y) \in \mathbb{R}^n : |x| < \ell \text{ and } |y| < \ell\}$  for all  $m \in \mathbb{N}$ ;
- iv) Each  $E_m$  is  $x$ - and  $y$ -Schwarz symmetric.

By the compactness theorem for sets of finite  $\alpha$ -perimeter (see [10] for a general statement that covers our case), there exists a set  $E \subset \mathbb{R}^n$  of finite  $\alpha$ -perimeter which is the  $L^1$ -limit of (a subsequence of) the sequence  $(E_m)_{m \in \mathbb{N}}$ . Then we have

$$\mathcal{L}^n(E) = \lim_{m \rightarrow \infty} \mathcal{L}^n(E_m) = 1.$$

Moreover, by lower semicontinuity of  $\alpha$ -perimeter

$$P_\alpha(E) \leq \liminf_{m \rightarrow \infty} P_\alpha(E_m) = C_I.$$

The set  $E$  is  $x$ - and  $y$ -Schwarz symmetric, because these symmetries are preserved by the  $L^1$ -convergence. This concludes the proof.  $\square$

## 5. Profile of isoperimetric sets

In Theorem 4.3, we proved existence of isoperimetric sets, in fact in the class of  $x$ -spherically symmetric sets when  $h \geq 2$ . By the characterization of the equality case in Theorems 3.1 and 3.2, any isoperimetric set  $E$  is  $x$ -Schwarz symmetric and there are functions  $c : [0, \infty) \rightarrow \mathbb{R}^k$  and  $f : [0, \infty) \rightarrow [0, \infty)$  such that

$$(5.1) \quad E = \{(x, y) \in \mathbb{R}^n : |y - c(|x|)| < f(|x|)\}.$$

The function  $f$  is decreasing. We will prove in Proposition 5.4 that, for isoperimetric sets, the function  $c$  is constant.

We start with the characterization of an isoperimetric set  $E$  with constant function  $c = 0$ . Let  $F \subset \mathbb{R}^+ \times \mathbb{R}^+$  be the generating set of  $E$

$$E = \{(x, y) \in \mathbb{R}^n : (|x|, |y|) \in F\}.$$

The set  $F$  is of the form

$$(5.2) \quad F = \{(r, s) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < s < f(r), r \in (0, r_0)\},$$

where  $f : (0, r_0) \rightarrow (0, \infty)$  is a decreasing function, for some  $0 < r_0 \leq \infty$ .

By the regularity theory of  $\Lambda$ -minimizers of perimeter, the boundary  $\partial E$  is a  $C^\infty$  hypersurface where  $x \neq 0$ . We do not need the general regularity theory, and we prove this fact in our case by an elementary method that gives also the  $C^\infty$ -smoothness of the function  $f$  in (5.2).

### 5.1. Smoothness of $f$

We prove that the boundary  $\partial F \subset \mathbb{R}^+ \times \mathbb{R}^+$  is the graph of a smooth function  $s = f(r)$ .

We rotate clockwise by 45 degrees the coordinate system  $(r, s) \in \mathbb{R}^2$  and we call the new coordinates  $(\varrho, \sigma)$ ; namely, we let

$$r = \frac{\sigma + \varrho}{\sqrt{2}}, \quad s = \frac{\sigma - \varrho}{\sqrt{2}}.$$

There exist  $-\infty \leq a < 0 < b \leq \infty$  and a function  $g : (a, b) \rightarrow \mathbb{R}$  such that the boundary  $\partial F \subset \mathbb{R}^+ \times \mathbb{R}^+$  is a graph  $\sigma = g(\varrho)$ ; namely, we have

$$\partial F = \left\{ (r(\varrho), s(\varrho)) = \left( \frac{g(\varrho) + \varrho}{\sqrt{2}}, \frac{g(\varrho) - \varrho}{\sqrt{2}} \right) : \varrho \in (a, b) \right\}.$$

Since the function  $f$  is decreasing, the function  $g$  is 1-Lipschitz continuous.

By formula (2.10) and by the standard length formula for Lipschitz graphs, the  $\alpha$ -perimeter of  $E$  is

$$P_\alpha(E) = c_{hk} \int_a^b \sqrt{s'^2 + r^{2\alpha} r'^2} r^{h-1} s^{k-1} d\varrho,$$

where  $c_{hk} = hk\omega_h\omega_k$ . On the other hand, the volume of  $E$  is

$$\mathcal{L}^n(E) = c_{hk} \int_a^b \left( \int_{|\varrho|}^{g(\varrho)} \left( \frac{\sigma + \varrho}{\sqrt{2}} \right)^{h-1} \left( \frac{\sigma - \varrho}{\sqrt{2}} \right)^{k-1} d\sigma \right) d\varrho.$$

For  $\varepsilon \in \mathbb{R}$  and  $\psi \in C_c^\infty(a, b)$ , let  $g_\varepsilon = g + \varepsilon\psi$ ,  $s_\varepsilon = s + \varepsilon\frac{\psi}{\sqrt{2}}$ ,  $r_\varepsilon = r + \varepsilon\frac{\psi}{\sqrt{2}}$  and let  $F_\varepsilon \subset \mathbb{R}^+ \times \mathbb{R}^+$  be the subgraph in  $\sigma > |\varrho|$  of the function  $g_\varepsilon$ . The set  $E_\varepsilon \subset \mathbb{R}^n$  with generating set  $F_\varepsilon$  has  $\alpha$ -perimeter

$$\begin{aligned} p(\varepsilon) &= P_\alpha(E_\varepsilon) \\ &= c_{hk} \int_a^b \sqrt{\left( s' + \varepsilon \frac{\psi'}{\sqrt{2}} \right)^2 + \left( r + \varepsilon \frac{\psi}{\sqrt{2}} \right)^{2\alpha} \left( r' + \varepsilon \frac{\psi'}{\sqrt{2}} \right)^2} \left( r + \varepsilon \frac{\psi}{\sqrt{2}} \right)^{h-1} \left( s + \varepsilon \frac{\psi}{\sqrt{2}} \right)^{k-1} d\varrho, \end{aligned}$$

and volume

$$v(\varepsilon) = \mathcal{L}^n(E_\varepsilon) = c_{hk} \int_a^b \left( \int_{|\varrho|}^{g(\varrho) + \varepsilon\psi(\varrho)} \left( \frac{\sigma + \varrho}{\sqrt{2}} \right)^{h-1} \left( \frac{\sigma - \varrho}{\sqrt{2}} \right)^{k-1} d\sigma \right) d\varrho.$$

Since  $E$  is an isoperimetric set, we have

$$0 = \frac{d}{d\varepsilon} \frac{p(\varepsilon)^d}{v(\varepsilon)^{d-1}} \Big|_{\varepsilon=0} = \frac{dp^{d-1}p'v^{d-1} - p^d(d-1)v^{d-2}v'}{v^{2d-2}} \Big|_{\varepsilon=0},$$

that gives

$$(5.3) \quad p'(0) - C_{hk\alpha} v'(0) = 0, \quad \text{where} \quad C_{hk\alpha} = \frac{d-1}{d} \frac{P_\alpha(E)}{\mathcal{L}^n(E)}.$$

After some computations, we find

$$(5.4) \quad p'(0) = \frac{c_{hk}}{\sqrt{2}} \int_a^b \left\{ \frac{(r^{2\alpha} r' - s')\psi' + 2\alpha r^{2\alpha-1} r'^2 \psi}{\sqrt{s'^2 + r^{2\alpha} r'^2}} + \sqrt{s'^2 + r^{2\alpha} r'^2} \left[ \frac{h-1}{r} + \frac{k-1}{s} \right] \psi \right\} r^{h-1} s^{k-1} d\rho,$$

and

$$(5.5) \quad v'(0) = c_{hk} \int_a^b r^{h-1} s^{k-1} \psi d\rho.$$

From (5.3), (5.4), and (5.5) we deduce that  $g$  is a 1-Lipschitz function that, via the auxiliary functions  $r$  and  $s$ , solves in a weak sense the ordinary differential equation

$$(5.6) \quad \frac{d}{d\rho} \left( r^{h-1} s^{k-1} \frac{r^{2\alpha} r' - s'}{\sqrt{s'^2 + r^{2\alpha} r'^2}} \right) = r^{h-1} s^{k-1} \left\{ \frac{2\alpha r^{2\alpha-1} r'^2}{\sqrt{s'^2 + r^{2\alpha} r'^2}} + \sqrt{s'^2 + r^{2\alpha} r'^2} \left[ \frac{h-1}{r} + \frac{k-1}{s} \right] - \sqrt{2} C_{hk\alpha} \right\}.$$

By an elementary argument that is omitted, it follows that  $g \in C^\infty(a, b)$ .

We claim that for all  $\rho \in (a, b)$  there holds  $g'(\rho) \neq -1$ . By contradiction, assume that there exists  $\bar{\rho} \in (a, b)$  such that  $g'(\bar{\rho}) = -1$ , i.e.,  $r'(\bar{\rho}) = 0$  and  $s'(\bar{\rho}) = -\sqrt{2}$ . Inserting these values into the differential equation (5.6) we can compute  $g''(\bar{\rho})$  as a function of  $g(\bar{\rho})$ ; namely, we obtain

$$(5.7) \quad g''(\bar{\rho}) = 2^{\alpha+1} \frac{2(h-1) - \sqrt{2} C_{hk\alpha} r(\bar{\rho})}{r(\bar{\rho})^{2\alpha+1}}.$$

Now there are three possibilities:

- (1)  $g''(\bar{\rho}) < 0$ . In this case,  $g$  is strictly concave at  $\bar{\rho}$  and this contradicts the fact that  $E$  is  $y$ -Schwarz symmetric.
- (2)  $g''(\bar{\rho}) > 0$ . In this case,  $g$  is strictly increasing at  $\bar{\rho}$  and since  $g'(\bar{\rho}) = -1$  this contradicts the fact the  $g$  is 1-Lipschitz, equivalently, the fact that  $E$  is  $x$ -Schwarz symmetric.
- (3)  $g''(\bar{\rho}) = 0$ . In this case, the value of  $g$  at  $\bar{\rho}$  is, by (5.7),

$$(5.8) \quad g(\bar{\rho}) = -\bar{\rho} + \frac{\sqrt{2}(h-1)}{C_{hk\alpha}}.$$

The function  $\widehat{g}(\rho) = -\rho + \frac{\sqrt{2}(h-1)}{C_{hk\alpha}}$ ,  $\rho \in \mathbb{R}$ , is the unique solution to the ordinary differential equation (5.6) with initial conditions  $g(\bar{\rho})$  given by (5.8) and  $g'(\bar{\rho}) = -1$ . It follows that  $g = \widehat{g}$  and this contradicts the boundedness of the isoperimetric set; namely, the fact that isoperimetric sets have finite measure.

This proves that  $g'(\rho) \neq -1$  for all  $\rho \in (a, b)$ .

## 5.2. Differential equations for the profile function

By the discussion in the previous section, the function  $f$  appearing in the definition of the set  $F$  in (5.2) is in  $C^\infty(0, r_0)$ . The function  $f$  is decreasing,  $f' \leq 0$ . By formula (2.10), the perimeter of the set  $E$  with generating set  $F$  is

$$(5.9) \quad P_\alpha(E) = c_{hk} \int_0^{r_0} \sqrt{f'(r)^2 + r^{2\alpha}} r^{h-1} f(r)^{k-1} dr,$$

and the volume of  $E$  is

$$(5.10) \quad \mathcal{L}^n(E) = \frac{c_{hk}}{k} \int_0^{r_0} r^{h-1} f(r)^k dr.$$

As in the previous subsection, for  $\psi \in C_c^\infty(0, r_0)$  and  $\varepsilon \in \mathbb{R}$ , we consider the perturbation  $f + \varepsilon\psi$  and we define the set

$$E_\varepsilon = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|) + \varepsilon\psi(|x|)\}.$$

Then we have

$$\begin{aligned} p(\varepsilon) &= P_\alpha(E_\varepsilon) = c_{hk} \int_0^{r_0} \sqrt{(f' + \varepsilon\psi')^2 + r^{2\alpha}} (f + \varepsilon\psi)^{k-1} r^{h-1} dr, \\ v(\varepsilon) &= \mathcal{L}^n(E_\varepsilon) = \frac{c_{hk}}{k} \int_0^{r_0} (f + \varepsilon\psi)^k r^{h-1} dr, \end{aligned}$$

and from these formulas we compute the first derivatives at  $\varepsilon = 0$ :

$$\begin{aligned} p'(0) &= c_{hk} \int_0^{r_0} \left[ \frac{f^{k-1} f'}{\sqrt{f'^2 + r^{2\alpha}}} \psi' + (k-1) f^{k-2} \sqrt{f'^2 + r^{2\alpha}} \psi \right] r^{h-1} dr, \\ v'(0) &= c_{hk} \int_0^{r_0} f^{k-1} \psi r^{h-1} dr. \end{aligned}$$

The minimality equation (5.3) reads

$$(5.11) \quad \int_0^{r_0} \left( \frac{f' f^{k-1}}{\sqrt{f'^2 + r^{2\alpha}}} \psi' + \left[ (k-1) f^{k-2} \sqrt{f'^2 + r^{2\alpha}} - C_{hk\alpha} f^{k-1} \right] \psi \right) r^{h-1} dr = 0.$$

Integrating by parts the term with  $\psi'$  and using the fact that  $\psi$  is arbitrary, we deduce that  $f$  solves the following second order ordinary differential equation:

$$(5.12) \quad -\frac{d}{dr} \left( r^{h-1} \frac{f' f^{k-1}}{\sqrt{f'^2 + r^{2\alpha}}} \right) + r^{h-1} \left[ (k-1) \sqrt{f'^2 + r^{2\alpha}} f^{k-2} - C_{hk\alpha} f^{k-1} \right] = 0.$$

The normal form of this differential equation is

$$(5.13) \quad f'' = \frac{\alpha f'}{r} + (f'^2 + r^{2\alpha}) \left( \frac{k-1}{f} - (h-1) \frac{f'}{r^{2\alpha+1}} \right) - C_{hk\alpha} \frac{(f'^2 + r^{2\alpha})^{\frac{3}{2}}}{r^{2\alpha}},$$

and it can be rearranged in the following ways:

$$(5.14) \quad \begin{aligned} \frac{\partial}{\partial r} \left( \frac{f'}{r^\alpha} \right) &= (f'^2 + r^{2\alpha}) \left( \frac{k-1}{f r^\alpha} - (h-1) \frac{f'}{r^{3\alpha+1}} \right) - C_{hk\alpha} \frac{(f'^2 + r^{2\alpha})^{\frac{3}{2}}}{r^{3\alpha}} \\ &= r^\alpha \left( \left( \frac{f'}{r^\alpha} \right)^2 + 1 \right) \left( \frac{k-1}{f} - \frac{(h-1)}{r^{\alpha+1}} \frac{f'}{r^\alpha} \right) - C_{hk\alpha} \left( \left( \frac{f'}{r^\alpha} \right)^2 + 1 \right)^{\frac{3}{2}}. \end{aligned}$$

With the substitution

$$(5.15) \quad z = \sin \arctan \left( \frac{f'}{r^\alpha} \right) = \frac{f'}{\sqrt{r^{2\alpha} + f'^2}},$$

equation (5.14) transforms into the equation

$$(5.16) \quad (r^{h-1} z)' = r^{\alpha+h-1} \frac{k-1}{f} \sqrt{1-z^2} - C_{hk\alpha} r^{h-1}.$$

We integrate this equation on the interval  $(0, r)$ . When  $h > 1$  we use the fact that  $r^{h-1} z = 0$  at  $r = 0$ . When  $h = 1$  we use the fact that  $z$  has a finite limit as  $r \rightarrow 0^+$ . In both cases, we deduce that there exists a constant  $D \in \mathbb{R}$  such that

$$(5.17) \quad z(r) = r^{1-h} \int_0^r s^{\alpha+h-1} \frac{k-1}{f} \sqrt{1-z^2} ds - \frac{C_{hk\alpha}}{h} r + D r^{1-h}.$$

Inserting (5.15) into (5.17), we get

$$(5.18) \quad \frac{f'}{\sqrt{r^{2\alpha} + f'^2}} = r^{1-h} \int_0^r s^{2\alpha+h-1} \frac{k-1}{f \sqrt{s^{2\alpha} + f'^2}} ds - \frac{C_{hk\alpha}}{h} r + D r^{1-h}.$$

If  $h \geq 2$ , from (5.18) we deduce that  $D = 0$ . In fact, the left-hand side of (5.18) is bounded as  $r \rightarrow 0^+$ , while the right-hand side diverges to  $\pm\infty$  according to the sign of  $D \neq 0$ . In the next section, we prove that  $D = 0$  also when  $h = 1$ , provided that  $f$  is the profile of an isoperimetric set.

*Remark 5.1* (Computation of the solution when  $k = 1$ ). When  $k = 1$  and  $D = 0$ , equation (5.18) reads

$$\frac{f'}{\sqrt{r^{2\alpha} + f'^2}} = -\frac{C_{hk\alpha}}{h} r.$$

and this is equivalent to

$$(5.19) \quad f'(r) = -\frac{C_{hk\alpha} r^{\alpha+1}}{\sqrt{h^2 - C_{hk\alpha}^2 r^2}}, \quad r \in [0, r_0].$$

Without loss of generality we can assume that  $r_0 = 1$  and this holds if and only if  $C_{hk\alpha} = h$ . Integrating (5.19) with  $f(1) = 0$  we obtain the solution

$$f(r) = \int_r^1 \frac{s^{\alpha+1}}{\sqrt{1-s^2}} ds = \int_{\arcsin r}^{\pi/2} \sin^{\alpha+1}(s) ds.$$

This is the profile function for the isoperimetric set when  $k = 1$  in (1.12).

### 5.3. Proof that $D = 0$ in (5.18).

We prove that  $D = 0$  in the case  $h = 1$ . We assume by contradiction that  $D \neq 0$ . For a small parameter  $s > 0$ , let  $f_s : [0, r_0] \rightarrow \mathbb{R}^+$  be the function

$$f_s(r) = \begin{cases} f(s) & \text{for } 0 < r \leq s \\ f(r) & \text{for } r > s, \end{cases}$$

and define the set

$$E_s = \{(x, y) \in \mathbb{R}^n : |y| < f_s(|x|)\}.$$

Recall that the isoperimetric ratio is  $\mathcal{I}_\alpha(E) = P_\alpha(E)^d / \mathcal{L}^n(E)^{d-1}$ . We claim that for  $s > 0$  small, the difference of isoperimetric ratios

$$\begin{aligned} \mathcal{I}_\alpha(E_s) - \mathcal{I}_\alpha(E) &= \frac{P_\alpha(E_s)^d}{\mathcal{L}^n(E_s)^{d-1}} - \frac{P_\alpha(E)^d}{\mathcal{L}^n(E)^{d-1}} \\ (5.20) \quad &= \frac{P_\alpha(E_s)^d \mathcal{L}^n(E)^{d-1} - P_\alpha(E)^d \mathcal{L}^n(E_s)^{d-1}}{\mathcal{L}^n(E_s)^{d-1} \mathcal{L}^n(E)^{d-1}} \end{aligned}$$

is strictly negative.

The  $\alpha$ -perimeter of  $E_s$  is

$$\begin{aligned} P_\alpha(E_s) &= c_{hk} \int_0^\infty \sqrt{f_s'^2 + r^{2\alpha}} f_s^{k-1} r^{h-1} dr \\ &= c_{hk} \left[ f(s)^{k-1} \int_0^s r^{\alpha+h-1} dr + \int_s^\infty \sqrt{f'^2 + r^{2\alpha}} f^{k-1} r^{h-1} dr \right] \\ &= P_\alpha(E) + c_{hk} \int_0^s \left[ r^\alpha f(s)^{k-1} - \sqrt{f'^2 + r^{2\alpha}} f^{k-1} \right] r^{h-1} dr, \end{aligned}$$

and its volume is

$$\begin{aligned} \mathcal{L}^n(E_s) &= \frac{c_{hk}}{k} \int_0^\infty f_s^k r^{h-1} dr = \frac{c_{hk}}{k} \left( \int_0^s f(s)^k r^{h-1} dr + \int_s^\infty f(r)^k r^{h-1} dr \right) \\ &= \mathcal{L}^n(E) + \frac{c_{hk}}{k} \int_0^s (f(s)^k - f(r)^k) r^{h-1} dr, \end{aligned}$$

so, by elementary Taylor approximations, we find

$$\begin{aligned} \mathcal{L}^n(E)^{d-1} P_\alpha(E_s)^d &= \\ &= \mathcal{L}^n(E)^{d-1} \left\{ P_\alpha(E) + c_{hk} \int_0^s \left[ r^\alpha f(s)^{k-1} - \sqrt{f'^2 + r^{2\alpha}} f^{k-1} \right] r^{h-1} dr \right\}^d \\ &= \mathcal{L}^n(E)^{d-1} \left\{ P_\alpha(E)^d + dc_{hk} P_\alpha(E)^{d-1} \left( \int_0^s \left[ r^\alpha f(s)^{k-1} \right. \right. \right. \\ &\quad \left. \left. \left. - \sqrt{f'^2 + r^{2\alpha}} f^{k-1} \right] r^{h-1} dr \right) + R_1(s) \right\}, \end{aligned}$$

where  $R_1(s)$  is a higher order infinitesimal as  $s \rightarrow 0$ , and

$$\begin{aligned} P_\alpha(E)^d \mathcal{L}^n(E_s)^{d-1} &= P_\alpha(E)^d \left\{ \mathcal{L}^n(E) + \frac{c_{hk}}{k} \int_0^s (f(s)^k - f(r)^k) r^{h-1} dr \right\}^{d-1} \\ &= P_\alpha(E)^d \left\{ \mathcal{L}^n(E)^{d-1} + \frac{c_{hk}(d-1)}{k} \mathcal{L}^n(E)^{d-2} \int_0^s (f(s)^k - f(r)^k) r^{h-1} dr \right. \\ &\quad \left. + R_2(s) \right\}, \end{aligned}$$

where  $R_2(s)$  is a higher order infinitesimal as  $s \rightarrow 0$ . The difference is thus

$$\begin{aligned} \Delta(s) &= P(E_s)^d \mathcal{L}^n(E)^{d-1} - P_\alpha(E)^d \mathcal{L}^n(E_s)^{d-1} \\ &= c_{hk} P_\alpha(E)^d \mathcal{L}^n(E)^{d-1} \left\{ d \frac{A(s)}{P_\alpha(E)} - (d-1) \frac{B(s)}{k \mathcal{L}^n(E)} \right\}, \end{aligned}$$

where we let

$$\begin{aligned} A(s) &= \int_0^s \left[ r^\alpha f(s)^{k-1} - \sqrt{f'^2 + r^{2\alpha}} f^{k-1} \right] r^{h-1} dr + R_1(s) \\ B(s) &= \int_0^s (f(s)^k - f(r)^k) r^{h-1} dr + R_2(s). \end{aligned}$$

Now we let  $h = 1$  and we observe that the differential equation (5.17) or its equivalent version (5.18) imply that

$$\lim_{r \rightarrow 0^+} \frac{f'(r)}{r^\alpha} = D.$$

So for  $D \neq 0$  and, in fact, for  $D < 0$  (because  $f$  is decreasing) we have

$$\lim_{s \rightarrow 0^+} \frac{A(s)}{s^{\alpha+h}} = f(0)^{k-1} \frac{1 - \sqrt{D^2 + 1}}{\alpha + h} < 0,$$

and

$$\lim_{s \rightarrow 0^+} \frac{B(s)}{s^{\alpha+h}} = 0.$$

It follows that for  $s > 0$  small there holds

$$\frac{\Delta(s)}{s^{h+\alpha}} = f(0)^{k-1} \frac{1 - \sqrt{D^2 + 1}}{\alpha + h} dc_{hk} P_\alpha(E)^{d-1} \mathcal{L}^n(E)^{d-1} + o(1) < 0.$$

Then  $E$  is not an isoperimetric set. This proves that  $D = 0$ .

#### 5.4. Initial and final conditions for the profile function

In this section, we study the behavior of  $f$  at 0 and  $r_0$ .

**Proposition 5.2.** *The profile function  $f$  of an  $x$ - and  $y$ -Schwarz symmetric isoperimetric set  $E \subset \mathbb{R}^n$  satisfies  $f \in C^\infty(0, r_0) \cap C([0, r_0])$  for some  $0 < r_0 < \infty$ ,  $f' \leq 0$ ,  $f(r_0) = 0$ , it solves the differential equation (5.18) with  $D = 0$ , and*

$$\lim_{r \rightarrow r_0^-} f'(r) = -\infty \quad \text{and} \quad \lim_{r \rightarrow 0^+} \frac{f'(r)}{r^{\alpha+1}} = -\frac{C_{hk\alpha}}{h}.$$

*Proof.* By Remark 5.1, it is sufficient to prove that  $r_0 < \infty$  when  $k > 1$ . Assume by contradiction that  $r_0 = \infty$ . In this case, it must be

$$(5.21) \quad \lim_{r \rightarrow \infty} f(r) = 0,$$

otherwise the set  $E$  with profile  $f$  would have infinite volume.

For  $\varepsilon > 0$  and  $M > 0$ , let us consider the set

$$K_M = \{r \geq M : f'(r) \geq -\varepsilon\}.$$

Recall that in our case we have  $f' \leq 0$ . The set  $K_M$  is closed and nonempty for any  $M$ . If  $K_M = \emptyset$  for some  $M$ , then this would contradict (5.21).

Let  $\bar{r} \in K_M$ . From (5.13) we have

$$(5.22) \quad \begin{aligned} f''(\bar{r}) &\geq -\frac{\alpha\varepsilon}{\bar{r}} + \bar{r}^{2\alpha} \frac{k-1}{f(\bar{r})} - C_{hk\alpha} \frac{(\varepsilon^2 + \bar{r}^{2\alpha})^{3/2}}{\bar{r}^{2\alpha}} \\ &\geq \frac{1}{2} M^{2\alpha} \frac{k-1}{f(M)} > 0, \end{aligned}$$

provided that  $M$  is large enough. We deduce that there exists  $\delta > 0$  such that  $f'(r) \geq -\varepsilon$  for all  $r \in [\bar{r}, \bar{r} + \delta)$ . This proves that  $K_M$  is open to the right. It follows that it must be  $K_M = [M, \infty)$ . This proves that

$$\lim_{r \rightarrow \infty} f'(r) = 0,$$

and this in turn contradicts (5.22).

Now we have  $r_0 < \infty$  and we also have

$$L = \lim_{r \rightarrow r_0^-} f(r) = 0.$$

If it were  $L > 0$ , then the isoperimetric set would have a “vertical part”. We would get a contradiction by the argument at point (3) at the end of Section 5.1.

We claim that

$$\lim_{r \rightarrow r_0^-} f'(r) = -\infty.$$

For  $M > 0$  and  $0 < s < r_0$ , consider the set

$$K_s = \{s \leq r < r_0 : f'(r) \geq -M\}.$$

By contradiction assume that there exists  $M > 0$  such that  $K_s \neq \emptyset$  for all  $0 < s < r_0$ . If  $\bar{r} \in K_s$ , we have as above  $f''(\bar{r}) \geq \frac{1}{2}(k-1)s^{2\alpha}/f(s) > 0$ . We deduce that there exists  $s < r_0$  such that  $0 \geq f'(r) \geq -M$  for all  $r \in [s, r_0]$ . From (5.13), we deduce that there exists a constant  $C > 0$  such that

$$f''(r) \geq \frac{C}{f(r)}.$$

Multiplying by  $f' \leq 0$  and integrating the resulting inequality we find

$$f'(r)^2 \leq 2C \log |f(r)| + C_0,$$

for some constant  $C_0 \in \mathbb{R}$ . This is a contradiction because  $\lim_{r \rightarrow r_0^-} \log |f(r)| = -\infty$ .

By Section 5.2, we have  $D = 0$  in (5.18). In this case, by (5.17) we can compute the limit

$$\lim_{r \rightarrow 0^+} \frac{f'(r)}{r^{\alpha+1}} = \lim_{r \rightarrow 0^+} -\frac{C_{hk\alpha}}{h} + r^{-h} \int_0^r s^{\alpha+h-1} \frac{k-1}{f} \sqrt{1-z^2} ds = -\frac{C_{hk\alpha}}{h}.$$

This ends the proof.  $\square$

*Remark 5.3.* The Cauchy Problem for the differential equation (5.13), with the initial conditions  $f(0) = 1$  and  $f'(0) = 0$  has a unique decreasing solution on some interval  $[0, \delta]$ , with  $\delta > 0$ , in the class of functions  $f \in C^1([0, \delta]) \cap C^\infty((0, \delta])$  such that

$$\lim_{r \rightarrow 0^+} \frac{f'(r)}{r^{\alpha+1}} = -\frac{C_{hk\alpha}}{h}.$$

This can be proved using the Banach fixed point Theorem with the norm

$$\|f\| = \max_{r \in [0, \delta]} |f(r)| + \max_{r \in [0, \delta]} \frac{|f'(r)|}{r^{\alpha+1}}.$$

From Theorem 4.3 and Proposition 5.2, there exists a value of the constant  $C_{hk\alpha} > 0$  such that the maximal decreasing solution of the Cauchy Problem has a maximal interval  $[0, r_0]$  such that  $f(r_0) = 0$ .

### 5.5. Isoperimetric sets are $y$ -Schwarz symmetric

To conclude the proof of Theorems 1.1–1.3 we are left to show that for an isoperimetric set  $E$  of the type (5.1), the function  $c$  of the centers is constant.

**Proposition 5.4.** *Let  $h, k \geq 1$  and  $n = h + k$ . Let  $E \subset \mathbb{R}^n$  be a set of the form*

$$E = \{(x, y) \in \mathbb{R}^n : |y - c(|x|)| < f(|x|)\}$$

*for measurable functions  $c : [0, \infty) \rightarrow \mathbb{R}^k$  and  $f : [0, \infty) \rightarrow [0, \infty]$ . If  $E$  is an isoperimetric set for the problem (4.13) then the function  $c$  is constant.*

*Proof.* If  $E$  is isoperimetric, then also its  $y$ -Schwarz rearrangement  $E^* = \{(x, y) \in \mathbb{R}^n : |y| < f(|x|)\}$  is an isoperimetric set, see Theorems 3.1 and 3.2. Then, by Proposition 5.2, we have  $f \in C^\infty(0, r_0) \cap C([0, r_0])$  with  $f(r_0) = 0$  and  $f' \leq 0$ . In particular,  $f \in \text{Lip}_{\text{loc}}(0, r_0)$ . We claim that  $c \in \text{Lip}_{\text{loc}}(0, r_0)$ .

Since  $E$  is  $x$ -Schwarz symmetric, for any  $0 < r_1 < r_2 < r_0$  we have the inclusion

$$\{y \in \mathbb{R}^k : |y - c(r_2)| \leq f(r_2)\} \subset \{y \in \mathbb{R}^k : |y - c(r_1)| \leq f(r_1)\}.$$

Assume  $c(r_2) \neq c(r_1)$  and let  $\vartheta = c(r_2) - c(r_1)/|c(r_2) - c(r_1)|$ . Then we have

$$c(r_2) + \vartheta f(r_2) \in \{y \in \mathbb{R}^k : |y - c(r_1)| \leq f(r_1)\},$$

and therefore

$$|c(r_2) - c(r_1)| + f(r_2) = |c(r_2) + \vartheta f(r_2) - c(r_1)| \leq f(r_1).$$

This implies that  $c$  is locally Lipschitz on  $(0, r_0)$ .

Let  $F \subset \mathbb{R}^+ \times \mathbb{R}^k$  be the generating set of  $E$ :

$$E = \{(x, y) \in \mathbb{R}^n : (|x|, y) \in F\}.$$

By the discussion above, the set  $E$  and thus also the set  $F$  have locally Lipschitz boundary away from a negligible set. By the representation formula (2.11), we have

$$P_\alpha(E) = Q(F) = h\omega_h \int_{\partial F} \sqrt{N_r^2 + r^{2\alpha}|N_y|^2} r^{h-1} d\mathcal{H}^k,$$

where  $(N_r, N_y) \in \mathbb{R}^{1+k}$  is the unit normal to  $\partial F$  in  $\mathbb{R}^+ \times \mathbb{R}^k$ , that is defined  $\mathcal{H}^k$  almost everywhere on the boundary. By the coarea formula (see [1]) we also have

$$Q(F) = h\omega_h \int_0^\infty r^{h-1} \int_{\partial F_r} \frac{\sqrt{N_r^2 + r^{2\alpha}|N_y|^2}}{\sqrt{1 - N_r^2}} d\mathcal{H}^{k-1} dr,$$

where  $\partial F_r = \partial\{y \in \mathbb{R}^k : (r, y) \in F\} = \{y \in \mathbb{R}^k : |y - c(r)| = f(r)\}$ .

A defining equation for  $\partial F$  is  $|y - c(r)|^2 - f(r)^2 = 0$ . From this equation, we find

$$N_r = -\frac{\langle y - c, c' \rangle + ff'}{\sqrt{(\langle y - c, c' \rangle + ff')^2 + |y - c|^2}},$$

$$N_y = \frac{y - c}{\sqrt{(\langle y - c, c' \rangle + ff')^2 + |y - c|^2}},$$

and thus, by translation and scaling in the inner integral,

$$\begin{aligned} Q(F) &= h\omega_h \int_0^\infty r^{h-1} \int_{|y-c(r)|=f(r)} \sqrt{\left\{ \frac{\langle y - c(r), c'(r) \rangle}{f(r)} + f'(r) \right\}^2 + r^{2\alpha}} d\mathcal{H}^{k-1}(y) dr \\ &= h\omega_h \int_0^\infty r^{h-1} f(r)^{k-1} \int_{|y|=1} \sqrt{\{\langle y, c'(r) \rangle + f'(r)\}^2 + r^{2\alpha}} d\mathcal{H}^{k-1}(y) dr. \end{aligned}$$

For any  $r > 0$ , the function  $\Phi : \mathbb{R}^h \rightarrow \mathbb{R}^+$

$$\Phi(z) = \int_{|y|=1} \sqrt{(\langle y, z \rangle + f'(r))^2 + r^{2\alpha}} d\mathcal{H}^{k-1}(y)$$

is strictly convex. This follows from the strict convexity of  $t \mapsto \sqrt{r^{2\alpha} + t^2}$ . The function  $\Phi$  is also radially symmetric because the integral is invariant under orthogonal transformations. It follows that  $\Phi$  attains the minimum at the point  $z = 0$  and that this minimum point is unique.

Denoting by  $F^*$  the generating set of  $E^*$ , we deduce that if  $c'$  is not 0 a.e., then we have the strict inequality  $P_\alpha(E^*) = Q(F^*) < Q(F) = P_\alpha(E)$ , and  $E$  is not isoperimetric. Hence,  $c$  is constant and this concludes the proof.  $\square$

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