MINIMAL SURFACES AND HARMONIC FUNCTIONS IN THE HEISENBERG GROUP

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ABSTRACT. We study the blow-up of H-perimeter minimizing sets in the Heisenberg group $\mathbb{H}^n$, $n \geq 2$. We show that the Lipschitz approximations rescaled by the square root of excess converge to a limit function. Assuming a stronger notion of local minimality, we prove that this limit function is harmonic for the Kohn Laplacian in a lower dimensional Heisenberg group.

CONTENTS

1. INTRODUCTION

One of the central facts in the regularity theory of minimal currents and of minimal boundaries in $\mathbb{R}^n$ is the existence of a harmonic function in the blow-up of the Lipschitz approximation of the current rescaled by excess. The heuristic idea behind this phenomenon is the fact that if a function $f : D \to \mathbb{R}$, with $D \subset \mathbb{R}^{n-1}$, is a local minimizer of the area functional

$$A(f) = \int_D \sqrt{1 + |\nabla f(x)|^2} dx$$

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and f is almost flat, i.e., $|\nabla f(x)|$ is almost 0, then f is almost a minimizer of the Dirichlet functional

$$D(f) = \frac{1}{2}\int_D |\nabla f(x)|^2 dx,$$

that is the first order term in the Taylor development of the area functional. For this reason, a function f in the blow-up of a minimal boundary is harmonic

$$\Delta f(x) = 0, \quad x \in D \subset \mathbb{R}^{n-1}.$$

The L^2 estimates on the derivatives of harmonic functions give the decay estimate of excess, that in turn implies the $C^{1,\alpha}$ regularity of the minimal boundary.

In this paper, we investigate the existence of a similar phenomenon in the case of a nonelliptic perimeter, as the horizontal perimeter in the Heisenberg group. Our results are not satisfactory because they hold for sets that are H-perimeter minimizing in a stronger sense, that is not the natural one. However, they are the first example of "harmonic approximation" of minimal boundaries for a nonelliptic perimeter and they suggest an interesting research direction in the regularity theory. So far, the regularity theory for H-minimal surfaces always starts from some initial regularity (see [4], [5], [6], [7], [18]). See, however, the Lipschitz approximation [14] and the height estimate proved in [17].

The $2n+1$-dimensional Heisenberg group is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \in \mathbb{N}$, endowed with the group product

$$(z,t) * (\zeta,\tau) = \big(z+\zeta, t+\tau+2\,\mathrm{Im}\langle z,\bar\zeta\rangle\big), \tag{1.1}$$

where $t,\tau \in \mathbb{R}$, $z,\zeta \in \mathbb{C}^n$ and $\langle z,\bar\zeta\rangle = z_1\bar\zeta_1+\ldots+z_n\bar\zeta_n$. The Lie algebra of left-invariant vector fields in $\mathbb{H}^n$ is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad \text{and} \quad T = \frac{\partial}{\partial t}, \tag{1.2}$$

with $z_j = x_j + iy_j$ and $j = 1,\ldots,n$. We denote by H the horizontal subbundle of $T\mathbb{H}^n$. Namely, for any $p = (z,t) \in \mathbb{H}^n$ we let

$$H_p = \mathrm{span}\big\{X_1(p),\ldots,X_n(p),Y_1(p),\ldots,Y_n(p)\big\}.$$

The *H-perimeter* of a $\mathscr{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$ in an open set $\Omega \subset \mathbb{H}^n$ is

$$P_H(E;\Omega) = \sup\left\{\int_E \mathrm{div}_H V\, dzdt : V \in C^1_c(\Omega;\mathbb{R}^{2n}), \|V\|_\infty \le 1\right\},$$

where $V : \Omega \to \mathbb{R}^{2n}$ is naturally identified with the horizontal vector field $V = \sum_{j=1}^n V_jX_j + V_{n+j}Y_j$ and the horizontal divergence of V is

$$\mathrm{div}_H V = \sum_{j=1}^n X_jV_j + Y_jV_{n+j}.$$

We use the notation $\mu_E(\Omega) = P_H(E;\Omega)$. If $\mu_E(\Omega) < \infty$ we say that E has finite H-perimeter in Ω. If $\mu_E(A) < \infty$ for any open set $A \subset\subset \Omega$, we say that E has locally finite H-perimeter in Ω. In this case, the open sets mapping $A \mapsto \mu_E(A)$ extends to a Radon measure μ_E on Ω that is called *H-perimeter measure* induced by E. Moreover, there exists a μ_E-measurable function $\nu_E : \Omega \to H$ such that $|\nu_E| = 1$ μ_E-a.e. and the Gauss-Green integration by parts formula

$$\int_\Omega \langle V, \nu_E\rangle\, d\mu_E = -\int_\Omega \mathrm{div}_H V\, dzdt$$

holds for any $V \in C_c^1(\Omega;\mathbb{R}^{2n})$. Here and hereafter, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^{2n}$. The vector ν_E is called *horizontal inner normal* of E in Ω.

We consider a set $E \subset \mathbb{H}^n$ with $0 \in \partial^* E$, the H-reduced boundary of E, that is a local minimizer of H-perimeter in a neighborhood of 0 and we rescale E to a unitary scale to have infinitesimal excess. In this way, we have a sequence of sets E_h that are H-perimeter minimizing and have infinitesimal excess η_h, $h \in \mathbb{N}$.

In Section 2, we use the Lipschitz approximation proved in [14] to obtain a sequence of instrinsic Lipschitz functions $(\varphi_h)_{h\in\mathbb{N}}$ whose graphs cover in measure a large part of the boundary of the rescaled sets E_h. By the Poincaré inequality recently proved in [8], we can show that there is subsequence of $(\varphi_h/\eta_h)_{h\in\mathbb{N}}$ converging to a function φ in a suitable Sobolev space. To have this limit function, only the density estimates for minimal boundaries are in fact used and so the result extends to Λ-minima. The Poincaré inequality mentioned above is for functions in domains of $\mathbb{R}\times\mathbb{H}^{n-1}$ and it holds only when $n \geq 2$. This is one of the reasons why our discussion is limited to dimensions $n \geq 2$.

The area functional of an intrinsic Lipschitz function $\varphi : D \to \mathbb{R}$, where now $D \subset \mathbb{R}\times\mathbb{H}^{n-1}$, is of the form

$$A(\varphi) = \int_D \sqrt{1 + |\nabla^\varphi \varphi|^2}\, dw, \tag{1.3}$$

where dw is the Lebesgue measure on $W = \mathbb{R}\times\mathbb{H}^{n-1}$ and $\nabla^\varphi\varphi$ is a nonlinear gradient that is defined in the sense of distributions, known as "intrinsic gradient", see Definition 2.3. The area formula (1.3) is obtained in [9] Theorem 6.5 part (vi) and in [2] Proposition 2.22. However, the Dirichlet functional

$$D(\varphi) = \frac{1}{2}\int_D |\nabla^\varphi\varphi|^2 dw$$

does not catch the correct regularity of the limit function, because in the blow-up there is a linearization of the nonlinear gradient $\nabla^\varphi\varphi$, see Theorem 2.5. After this linearization, the relevant Dirichlet functional turns out to be

$$D_H(\varphi) = \frac{1}{2}\int_D \Big\{\Big(\frac{\partial\varphi}{\partial y_1}\Big)^2 + \sum_{j=2}^n (X_j\varphi)^2 + (Y_j\varphi)^2\Big\} dw, \tag{1.4}$$

where $y_1 \in \mathbb{R}$ is the variable of the factor $\mathbb{R}$ in the Cartesian product $\mathbb{R} \times \mathbb{H}^{n-1}$. The Dirichlet form (1.4) identifies the differentiability class where the limit of the (sub)sequence $(\varphi_h/\eta_h)_{h\in\mathbb{N}}$ lies.

In Section 3, we deduce from the minimality of E further properties of the limit function φ. We use the first order Taylor expansion of H-perimeter (3.41), that holds for any set with finite H-perimeter (these sets may be unrectifiable in the standard sense). We obtain two results. First, we prove that if E is a set that locally minimizes H-perimeter then the function $\varphi : D \subset \mathbb{R} \times \mathbb{H}^{n-1} \to \mathbb{R}$ is independent of the variable y_1 of the factor $\mathbb{R}$, see the first claim of Theorem 3.2. This fact seems to have no counterpart in the classical theory.

The second result holds for a stronger notion of minimality. The homogeneous cube centered at $0 \in \mathbb{H}^n$ and with radius $r > 0$ is

$$Q_r = \big\{(z,t) \in \mathbb{H}^n : |x_i| < r, |y_i| < r, |t| < r^2,\ i = 1,\dots,n\big\}. \tag{1.5}$$

Definition 1.1. We say that a set $E \subset \mathbb{H}^n$ is H-perimeter minimizing in Q_r if

$$P_H(E; Q_r) \leq P_H(F, Q_r)$$

for any set $F \subset \mathbb{H}^n$ such that $E\Delta F$ is a compact subset of Q_r.

Let $E \subset \mathbb{H}^n$ be a set with $0 \in \partial^* E$ and $\nu_E(0) = X_1$. Let $J : H \to H$ be the complex structure and consider $Y_1 = J(X_1)$. We define the lateral closure of Q_r relative to the positive direction Y_1 as:

$$\bar{Q}_r^{Y_1,+} = \big\{(z,t) \in \mathbb{H}^n : -r < y_1 \leq r, |x_1|, |x_i|, |y_i| < r, |t| < r^2,\ i = 2,\dots,n\big\}.$$

We are adding to Q_r the open face of the boundary where $y_1 = r$.

Definition 1.2. We say that a set $E \subset \mathbb{H}^n$ with $0 \in \partial^* E$ and $\nu_E(0) = X_1$ is strongly H-perimeter minimizing in Q_r if for any $0 < s \leq r$ we have

$$P_H(E; Q_s) \leq P_H(F, Q_s)$$

for any set $F \subset \mathbb{H}^n$ such that $E\Delta F \cap \bar{Q}_s$ is a compact subset of $\bar{Q}_s^{Y_1,+}$.

Here and hereafter, $E\Delta F = E \setminus F \cup F \setminus E$ denotes the symmetric difference.

In the second claim of Theorem 3.2, we show that if E is strongly H-perimeter minimizing then the function $\varphi : D \to \mathbb{R}$, where now D is a subset of $\mathbb{H}^{n-1}$, is H-harmonic, i.e., it solves the partial differential equation

$$\Delta_H \varphi = 0, \quad \text{in } D \subset \mathbb{H}^{n-1},$$

where Δ_H is the Kohn Laplacian in the lower dimensional Heisenberg group $\mathbb{H}^{n-1}$

$$\Delta_H = \sum_{j=2}^{n} X_j^2 + Y_j^2. \tag{1.6}$$

It is not clear whether the strong H-perimeter minimality can be relaxed to the natural local minimality. The problem is related to the construction of suitable contact vector fields in $\mathbb{H}^n$ with compact support. This problem is explained in Section 3, along the proof of Theorem 3.2.

The ideas presented in this paper are part of a joint research project with D. Vittone.

2. Blow-up at the reduced boundary of minimizers

In this section, we show that in the blow-up of an H-perimeter minimizing set at a point of the reduced boundary there is a function belonging to a suitable Sobolev space.

We use the box-norm $\|p\|_\infty = \max\{|z|, |t|^{1/2}\}$ for $p = (z,t) \in \mathbb{H}^n$, and the homogeneous balls

$$B_r = \{p \in \mathbb{H}^n : \|p\|_\infty < r\} \quad \text{and} \quad B_r(p) = p * B_r, \quad r > 0.$$

The balls B_r are equivalent to the cubes Q_r.

2.1. **Small excess at the reduced boundary.** Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter in $\mathbb{H}^n$. We say that $0 \in \mathbb{H}^n$ is a point of the H-reduced boundary of E, $0 \in \partial^* E$, if the following three conditions hold: $\mu_E(B_r) > 0$ for all $r > 0$, we have

$$\lim_{r\to 0} \frac{1}{\mu_E(B_r)} \int_{B_r} \nu_E \, d\mu_E = \nu_E(0),$$

and $|\nu_E(0)| = 1$. This definition is introduced and studied in [9]. The horizontal excess of E in B_r, $r > 0$, is

$$\mathrm{Exc}(E, B_r) = \min_{\nu \in \mathbb{S}^{2n}} \frac{1}{r^{Q-1}} \int_{B_r} |\nu_E(p) - \nu|^2 d\mu_E.$$

We refer the reader to [13] for an account on excess in the Euclidean setting. Notice that $r^{1-Q} C_1 \le \mu_E(B_r) \le r^{1-Q} C_2$ for constants $0 < C_1 < C_2 < \infty$ and $Q = 2n+2$. For minimizers, the constants are independent of the point in the reduced boundary. Thus, if $0 \in \partial^* E$ is a point in the H-reduced boundary of E then there exists a sequence $r_h \to 0^+$ such that

$$\frac{1}{\mu_E(B_{r_h})} \int_{B_{r_h}} |\nu_E - \nu_E(0)|^2 d\mu_E < \frac{1}{h},$$

and so we have $\mathrm{Exc}(E, B_{r_h}) < 1/h$.

We consider the anisotropic dilations $(z,t) \mapsto (\lambda z, \lambda^2 t) = \delta_\lambda(z,t)$, $\lambda > 0$. The rescaled sets $E_h = \delta_{1/r_h} E$, $h \in \mathbb{N}$, satisfy $\sup_{h\in\mathbb{N}} P_H(E_h; B_1) < \infty$. Moreover, we have:

i) If E is H-perimeter minimizing near $0 \in \partial E^*$, then each set E_h is H-perimeter minimizing in B_1;
ii) Since excess is scale invariant, there holds $\mathrm{Exc}(E_h, B_1) < 1/h$;
iii) $0 \in \partial^* E_h$.

Rotating each set E_h by an isometry fixing the t-axis, we may assume that

$$\mathrm{Exc}(E_h, B_1) = \int_{B_1} |\nu_{E_h} - \nu|^2 d\mu_{E_h} < \frac{1}{h}, \tag{2.7}$$

where $\nu \in \mathbb{S}^{2n}$ is a vector independent of h. In fact, we may assume that $\nu = \nu_E(0)$. Possibly taking a subsequence, by the compactness theorem for sets with finite H-perimeter, there exists a set $F \subset \mathbb{H}^n$ such that

$$\lim_{h\to\infty} \chi_{E_h} = \chi_F, \quad \text{in } L^1(B_1).$$

Moreover, by the lower semicontinuity of excess we have $\mathrm{Exc}(F, B_1) = 0$. Since $0 \in \partial F$, when $n \geq 2$ this implies that

$$F \cap B_1 = \{(z,t) \in B_1 : \langle z, \nu\rangle \geq 0\},$$

see [9]. When $n = 1$, this fact does no longer hold, i.e., ∂F needs not be flat in any neighborhood of 0, see [14].

2.2. **Lipschitz approximation and intrinsic gradient.** We identify the vertical hyperplane $W = \mathbb{R} \times \mathbb{H}^{n-1} = \big\{(z,t) \in \mathbb{H}^n : x_1 = 0\big\}$ with $\mathbb{R}^{2n}$ via the coordinates $w = (x_2, \dots, x_n, y_1, \dots, y_n, t)$. The line flow of the vector field X_1 starting from the point $(z,t) \in W$ is

$$\exp(sX_1)(z,t) = (z + s\mathrm{e}_1, t + 2y_1 s), \quad s \in \mathbb{R},$$

where $\mathrm{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{2n}$ and $z = (x,y) \in \mathbb{C}^n = \mathbb{R}^{2n}$, with $x = (x_1, \dots, x_n)$, $x_1 = 0$, and $y = (y_1, \dots, y_n)$.

Let $D \subset W$ be a set and let $\varphi : D \to \mathbb{R}$ be a function. The set

$$E_\varphi = \big\{ \exp(sX_1)(w) \in \mathbb{H}^n : s > \varphi(w),\, w \in D\big\} \tag{2.8}$$

is called *intrinsic epigraph* of φ along X_1. The set

$$\mathrm{gr}(\varphi) = \big\{ \exp(\varphi(w)X_1)(w) \in \mathbb{H}^n : w \in D\big\} \tag{2.9}$$

is called *intrinsic graph* of φ along X_1.

We identify $\nu = (1, 0, \dots, 0) \in \mathbb{R}^{2n}$ with $(\nu, 0) \in \mathbb{H}^n$. For any $p \in \mathbb{H}^n$, we let $\nu(p) = \langle p, \nu\rangle\nu \in \mathbb{H}^n$ and we define $\nu^\perp(p) \in W \subset \mathbb{H}^n$ as the unique point such that

$$p = \nu^\perp(p) * \nu(p). \tag{2.10}$$

The (open) cone with vertex $0 \in \mathbb{H}^n$, axis $\nu \in \mathbb{R}^{2n}$, $|\nu| = 1$, and aperture $\alpha \in (0, \infty]$ is the set

$$C(0, \nu, \alpha) = \big\{p \in \mathbb{H}^n : \|\nu^\perp(p)\|_\infty < \alpha \|\nu(p)\|_\infty\big\}. \tag{2.11}$$

The cone with vertex $p \in \mathbb{H}^n$, axis $\nu \in \mathbb{R}^{2n}$, and aperture $\alpha \in (0, \infty]$ is the set $C(p, \nu, \alpha) = p * C(0, \nu, \alpha)$.

Definition 2.1 (Intrinsic Lipschitz graphs)**.** Let $D \subset W$ be a set and let $\varphi : D \to \mathbb{R}$ be a function. The function φ is *L-intrinsic Lipschitz* with $0 < L < \infty$ if for any $p \in \mathrm{gr}(\varphi)$ there holds

$$\mathrm{gr}(\varphi) \cap C(p, \nu, 1/L) = \emptyset. \tag{2.12}$$

The starting point of our argument is the following result of [14] on the Lipschitz approximation of H-minimal boundaries. We denote by $\mathscr{S}^{Q-1}$ the $(2n+1)$-dimensional spherical Hausdorff measure constructed using any homogeneous left invariant metric on $\mathbb{H}^n$. We shall use freely the identity

$$\mathscr{S}^{Q-1} \llcorner \partial^* E = \mu_E. \tag{2.13}$$

Recall that for an H-perimeter minimizing set E, the reduced boundary $\partial^* E$ coincides with the essential boundary, that is denoted by ∂E.

Theorem 2.2. *Let $n \geq 2$. For any $L > 0$ there are constants $k > 1$ and $c(L, n) > 0$ such that for any H-perimeter minimizing set E in B_{kr}, with $0 \in \partial E$ and $\nu_E(0) = \nu = X_1$, there exists an L-intrinsic Lipschitz function $\varphi : W \to \mathbb{R}$ such that*

$$\mathscr{S}^{Q-1}\big((\mathrm{gr}(\varphi) \Delta \partial E) \cap B_r\big) \leq c(L, n)(kr)^{Q-1} \mathrm{Exc}(E, B_{kr}), \quad r > 0. \tag{2.14}$$

Theorem 2.2 holds also for $n = 1$. In this case, the Lipschitz constant L has to be suitably large.

We introduce a nonlinear gradient for functions $\varphi : D \to \mathbb{R}$ with $D \subset W$ open set. The Burgers' operator $\mathscr{B} : \mathrm{Lip}_{\mathrm{loc}}(D) \to L^\infty_{\mathrm{loc}}(D)$ is

$$\mathscr{B}\varphi = \frac{\partial \varphi}{\partial y_1} - 4\varphi \frac{\partial \varphi}{\partial t}. \tag{2.15}$$

When $\varphi \in C(D)$ is only continuous, we say that $\mathscr{B}\varphi$ exists in the sense of distributions and is represented by a locally bounded function, if there exists a function $\vartheta \in L^\infty_{\mathrm{loc}}(D)$ such that for all $\psi \in C^1_c(D)$ there holds

$$\int_D \vartheta \psi \, dw = -\int_D \Big\{ \varphi \frac{\partial \psi}{\partial y_1} - 2\varphi^2 \frac{\partial \psi}{\partial t} \Big\} dw. \tag{2.16}$$

In this case, we let $\mathscr{B}\varphi = \vartheta$.

Next, notice that the vector fields $X_2, \ldots, X_n, Y_2, \ldots, Y_n$ can be naturally restricted to W and that they are self-adjoint.

Definition 2.3 (Intrinsic gradient)**.** Let $D \subset W = \mathbb{R}^{2n}$ be an open set and let $\varphi \in C(D)$ be a continuous function. We say that the intrinsic gradient $\nabla^\varphi \varphi \in L^\infty_{\mathrm{loc}}(D; \mathbb{R}^{2n-1})$ exists in the sense of distributions if the distributional derivatives

$X_i\varphi, \mathscr{B}\varphi, Y_i\varphi$, $i = 2, \dots, n$, are represented by locally bounded functions in D. In this case, we let

$$\nabla^\varphi \varphi = \big(X_2\varphi, \dots, X_n\varphi, \mathscr{B}\varphi, Y_2\varphi, \dots, Y_n\varphi\big), \tag{2.17}$$

and we call $\nabla^\varphi\varphi$ the intrinsic gradient of φ.

When $n = 1$, the intrinsic gradient reduces to $\nabla^\varphi\varphi = \mathscr{B}\varphi$.

Theorem 2.4. *Let $D \subset W$ be an open set and $\varphi : D \to \mathbb{R}$ be a continuous function. The following statements are equivalent:*

A) *We have $\nabla^\varphi\varphi \in L^\infty_{\mathrm{loc}}(D; \mathbb{R}^{2n-1})$.*

B) *For any $D' \subset\subset D$, the function $\varphi : D' \to \mathbb{R}$ is intrinsic Lipschitz.*

Moreover, if A) or B) holds then the intrinsic epigraph $E_\varphi \subset \mathbb{H}^n$ has locally finite H-perimeter in the cylinder $D * \mathbb{R} = \{w * (s\mathrm{e}_1) \in \mathbb{H}^n : w \in D,\ s \in \mathbb{R}\}$, for $\mathscr{L}^{2n}$-a.e. $w \in D$ the inner horizontal normal to ∂E_φ is

$$\nu_{E_\varphi}(w * \varphi(w)) = \Big(\frac{1}{\sqrt{1 + |\nabla^\varphi\varphi(w)|^2}}, \frac{-\nabla^\varphi\varphi(w)}{\sqrt{1 + |\nabla^\varphi\varphi(w)|^2}}\Big), \tag{2.18}$$

and, for any $D' \subset D$, we have the area formula

$$P_H(E_\varphi; D' * \mathbb{R}) = \int_{D'} \sqrt{1 + |\nabla^\varphi\varphi|^2} dw. \tag{2.19}$$

The equivalence between A) and B) is a deep result that is proved in [3], Theorem 1.1. Formula (2.18) for the normal and the area formula (2.20) are proved in [8] Corollary 4.2 and Corollary 4.3, respectively. Part of these results is the fact that $\|\nabla^\varphi\varphi\|_\infty$ is equivalent to the Lipschitz constant. The area formula (2.19) can be improved in the following way

$$\int_{\partial E_\varphi \cap (D' * \mathbb{R})} g(p)\, d\mu_{E_\varphi} = \int_{D'} g(w * \varphi(w)) \sqrt{1 + |\nabla^\varphi\varphi(w)|^2} dw, \tag{2.20}$$

where $g : \partial E_\varphi \to \mathbb{R}$ is a Borel function.

A result related to Theorem 2.4 can be found in [16], where it is proved that if $E \subset \mathbb{H}^n$ is a set with finite H-perimeter having controlled normal ν_E, say $\langle \nu_E, \mathrm{e}_1\rangle \geq k > 0$ μ_E-a.e., then the reduced boundary $\partial^* E$ is an intrinsic Lipschitz graph along X_1.

2.3. **Blow-up of H-minimal boundaries.** Let $E \subset \mathbb{H}^n$ be an H-perimeter minimizing set in a neighborhood of $0 \in \mathbb{H}^n$, with $0 \in \partial E$ and $\nu_E(0) = X_1$. Let E_h be the rescaled sets of E introduced before equation (2.7). The square root of excess

$$\eta_h = \sqrt{\mathrm{Exc}(E_h, B_1)} \tag{2.21}$$

is infinitesimal, and we may assume that $\eta_h > 0$.

Let $\sigma > 0$ be a small number, e.g., $0 < \sigma \leq 1/k$ where $k > 1$ is the geometric constant given by Theorem 2.2, and let $0 < L \leq 1$ be a Lipschitz constant. Since each set E_h is H-perimeter minimizing in the ball B_1, by Theorem 2.2 there exist L-intrinsic Lipschitz functions $\varphi_h : W \to \mathbb{R}$ such that

$$\mathscr{S}^{Q-1}\big((\mathrm{gr}(\varphi_h)\Delta\partial E_h)\cap B_\sigma\big) \le c(L,n,\sigma)\mathrm{Exc}(E_h,B_1) = c_0\eta_h^2, \tag{2.22}$$

where $c_0 = c(L,n,\sigma)$.

In this section we prove the following theorem. Recall that the Sobolev space $W^{1,2}_H(D)$ is the set of all $\varphi \in L^2(D)$ such that the distributional derivatives

$$X_2\varphi,\dots,X_n\varphi,\frac{\partial\varphi}{\partial y_1},Y_2\varphi,\dots,Y_n\varphi \in L^2(D)$$

are squared integrable. In this case, we let

$$\nabla_H\varphi = \Big(X_2\varphi,\dots,X_n\varphi,\frac{\partial\varphi}{\partial y_1},Y_2\varphi,\dots,Y_n\varphi\Big).$$

Theorem 2.5. *Let $n \ge 2$. Under the assumptions made at the beginning of this section, there exist an open neighborhood $D \subset W$ of $0 \in W$, constants $\bar\varphi_h \in \mathbb{R}$, a function $\varphi \in W^{1,2}_H(D)$, and a selection of indices $k \mapsto h_k$ such that, for $k \to \infty$ we have*

$$\begin{aligned} &\frac{\varphi_{h_k}-\bar\varphi_{h_k}}{\eta_{h_k}} \rightharpoonup \varphi \qquad \text{weakly in } L^2(D),\\ &\frac{\nabla^{\varphi_{h_k}}\varphi_{h_k}}{\eta_{h_k}} \rightharpoonup \nabla_H\varphi \quad \text{weakly in } L^2(D;\mathbb{R}^{2n-1}). \end{aligned}$$

In the proof of Theorem 2.5, we use the Poincaré inequality of [8]. As in Section 2.1 of [8] (but with our normalization (1.2) of the vector fields), for $w = (z,t) \in W$ and $\varphi : W \to \mathbb{R}$ we let

$$d_\varphi(w,0) = \frac{1}{2}\max\big\{|z|,|t+4\varphi(w)y_1|^{1/2}\big\} + \frac{1}{2}\max\big\{|z|,|t+4\varphi(0)y_1|^{1/2}\big\}, \tag{2.23}$$

and, for $r>0$,

$$U_\varphi(r) = \big\{w\in W : d_\varphi(w,0) < r\big\}. \tag{2.24}$$

Theorem 2.6. *Let $n \ge 2$ and let $\varphi : W \to \mathbb{R}$ be an L-intrinsic Lipschitz function. There exist constants $C_1, C_2 > 0$ depending on L and n such that*

$$\int_{U_\varphi(r)} |\varphi(w)-\varphi_{U_\varphi(r)}|^2 dw \le C_1 r^2 \int_{U_\varphi(C_2 r)} |\nabla^\varphi\varphi(w)|^2 dw, \quad r>0, \tag{2.25}$$

where

$$\varphi_{U_\varphi(r)} = \frac{1}{\mathscr{L}^{2n}(U_\varphi(r))}\int_{U_\varphi(r)} \varphi(w)dw. \tag{2.26}$$

See Corollary 4.5 in [8].

Proof of Theorem 2.5. By the lower density estimate $P_H(E_h; B_{\sigma/2}) \geq C\sigma^{Q-1}$ with a constant $C > 0$ independent of h and, from (2.22), we deduce that $\mathrm{gr}(\varphi_h) \cap B_{\sigma/2} \neq \emptyset$ for all $h \in \mathbb{N}$ large enough. It follows that (details are omitted) there exists $\varepsilon_1 > 0$ such that

$$\mathrm{gr}(\varphi_h) \cap \{w \in W : |w| < \varepsilon_1\} * \mathbb{R} \subset B_\sigma. \tag{2.27}$$

Without loss of generality we can assume that $\|\varphi_h\|_\infty \leq 1$ for all $h \in \mathbb{N}$. Thus, from (2.23) and (2.24), it follows that there exist $\varepsilon_0 > 0$ and $r > 0$ such that

$$D := \big\{w \in W : |w| < \varepsilon_0\big\} \subset U_{\varphi_h}(r) \subset U_{\varphi_h}(C_2 r) \subset \big\{w \in W : |w| < \varepsilon_1\big\} =: D'. \tag{2.28}$$

Then, by (2.22), we deduce the estimate

$$\mathscr{S}^{Q-1}\big((\mathrm{gr}(\varphi_h) \setminus \partial E_h) \cap D' * \mathbb{R}\big) \leq c_0 \eta_h^2. \tag{2.29}$$

Let $D_h \subset D'$ be the set of the points $w \in D'$ such that

$$\nu_{E_{\varphi_h}}(w * \varphi_h(w)) = \Big(\frac{1}{\sqrt{1 + |\nabla^{\varphi_h}\varphi_h(w)|^2}}, \frac{-\nabla^{\varphi_h}\varphi_h(w)}{\sqrt{1 + |\nabla^{\varphi_h}\varphi_h(w)|^2}}\Big), \tag{2.30}$$

and

$$\nu_{E_{\varphi_h}}(w * \varphi_h(w)) = \nu_{E_h}(w * \varphi_h(w)). \tag{2.31}$$

By Theorem 2.4, see formula (2.18), identity (2.30) holds for $\mathscr{L}^{2n}$-a.e. $w \in D'$. By the locality of H-perimeter (see Corollary 2.5 in [1]) and by the area formula (2.19), identity (2.31) holds for $\mathscr{L}^{2n}$-a.e. $w \in \pi(\mathrm{gr}(\varphi_h) \cap \partial E_h)$, where $\pi : \mathbb{H}^n \to W$ is the projection along X_1.

Since each function φ_h is L-intrinsic Lipschitz with $0 < L \leq 1$, we can assume $\|\nabla^{\varphi_h}\varphi_h\|_\infty \leq 1$. Then for any point $w \in D_h$ we have:

$$|\nu_{E_h}(w * \varphi_h(w)) - \nu|^2 = |\nu_{E_{\varphi_h}}(w) - \nu|^2 \geq \frac{1}{2}|\nabla^{\varphi_h}\varphi_h(w)|^2,$$

where $\nu = (1, 0, \dots, 0) \in \mathbb{S}^{2n}$. By the area formula (2.20) for intrinsic Lipschitz functions and by (2.7), we obtain the estimate

$$\int_{D_h} |\nabla^{\varphi_h}\varphi_h(w)|^2 dw \leq 2\int_{B_1} |\nu_{E_h} - \nu|^2 d\mu_{E_h} \leq 2\eta_h^2. \tag{2.32}$$

Again by $\|\nabla^{\varphi_h}\varphi_h\|_\infty \leq 1$, by the area formula, and by (2.22), we obtain

$$\begin{aligned}\int_{D'\setminus D_h} |\nabla^{\varphi_h}\varphi_h(w)|^2 dw &\leq \mathscr{L}^{2n}(D' \setminus D_h) \\ &\leq \mathscr{S}^{Q-1}((\mathrm{gr}(\varphi_h) \setminus \partial E_h) \cap B_\sigma) \leq c_0\eta_h^2.\end{aligned} \tag{2.33}$$

It follows that the sequence of functions $|\nabla^{\varphi_h}\varphi_h|/\eta_h$, $h \in \mathbb{N}$, is uniformly bounded in $L^2(D')$. Then there exists a function $\Phi \in L^2(D'; \mathbb{R}^{2n-1})$ such that, possibly taking a subsequence, we have as $h \to \infty$

$$\frac{\nabla^{\varphi_h}\varphi_h}{\eta_h} \rightharpoonup \Phi \quad \text{weakly in } L^2(D'; \mathbb{R}^{2n-1}). \tag{2.34}$$

After a relabeling, we assume here and hereafter that the full sequence is converging.

We denote by $\bar{\varphi}_h$ the mean of φ_h defined in (2.26), namely,

$$\bar{\varphi}_h = \frac{1}{\mathscr{L}^{2n}(U_{\varphi_h}(r))} \int_{U_{\varphi_h}(r)} \varphi_h(w)dw, \tag{2.35}$$

where $r > 0$ is such that the inclusions in (2.28) hold. By the Poincaré inequality (2.25), by the inclusions in (2.28), (2.32), and (2.33) we have

$$\begin{aligned}
\int_D |\varphi_h(w) - \bar{\varphi}_h|^2 dw &\leq \int_{U_{\varphi_h}(r)} |\varphi_h(w) - \bar{\varphi}_h|^2 dw \\
&\leq C_1 r^2 \int_{U_{\varphi_h}(C_2 r)} |\nabla^{\varphi_h}\varphi_h(w)|^2 dw \\
&\leq C_1 r^2 \int_{D'} |\nabla^{\varphi_h}\varphi_h(w)|^2 dw \\
&\leq C_1 r^2 (2 + c_0)\eta_h^2.
\end{aligned}$$

Then, the sequence $(\varphi_h - \bar{\varphi}_h)/\eta_h$ is uniformly bounded in $L^2(D)$. It follows that we have $\varphi_h - \bar{\varphi}_h \to 0$ in $L^2(D)$. As the sequence of sets $(E_h)_{h\in\mathbb{N}}$ is converging to a half-plane inside the ball B_1, we deduce that $\bar{\varphi}_h \to 0$ as $h \to \infty$. Finally, by weak compactness there exists a function $\varphi \in L^2(D)$ such that, possibly taking a further subsequence, we have

$$\frac{\varphi_h - \bar{\varphi}_h}{\eta_h} \rightharpoonup \varphi \quad \text{weakly in } L^2(D). \tag{2.36}$$

We claim that $\varphi \in W^{1,2}_H(D)$ and that

$$\Phi = \nabla_H \varphi = \Big(X_2\varphi, \dots, X_n\varphi, \frac{\partial \varphi}{\partial y_1}, Y_2\varphi, \dots, Y_n\varphi \Big), \tag{2.37}$$

in the sense of weak derivatives in $L^2(D)$. Notice that the nonlinear derivative $\mathscr{B}\varphi_h/\eta_h$ is converging to the linear derivative $\partial_{y_1}\varphi$.

By (2.36), for any test function $\psi \in C^1_c(D)$ we have

$$\lim_{h\to\infty} \int_D \frac{\varphi_h - \bar{\varphi}_h}{\eta_h} \psi dw = \int_D \varphi\, \psi\, dw. \tag{2.38}$$

On the other hand, by the distributional definition (2.16) of the derivative $\mathscr{B}\varphi_h$ we have

$$\begin{aligned}\frac{1}{\eta_h}\int_D \psi\, \mathscr{B}\varphi_h\, dw &= -\frac{1}{\eta_h}\int_D \big\{\varphi_h\psi_{y_1} - 2\varphi_h^2\psi_t\big\}dw\\ &= -\frac{1}{\eta_h}\int_D \big\{(\varphi_h-\bar\varphi_h)\psi_{y_1} - 2(\varphi_h^2-\bar\varphi_h^2)\psi_t\big\}dw\\ &= -\int_D \Big\{\frac{\varphi_h-\bar\varphi_h}{\eta_h}\psi_{y_1} - 2\frac{\varphi_h-\bar\varphi_h}{\eta_h}(\varphi_h+\bar\varphi_h)\psi_t\Big\}dw.\end{aligned}$$

Since $\varphi_h+\bar\varphi_h$ is converging to zero strongly in $L^2(D)$ and $(\varphi_h-\bar\varphi_h)/\eta_h$ is uniformly bounded in $L^2(D)$, we obtain

$$\lim_{h\to\infty}\frac{1}{\eta_h}\int_D \psi\, \mathscr{B}\varphi_h\, dw = -\int_D \varphi\, \psi_{y_1} dw.$$

A similar argument shows that for any $Z\in\{X,Y\}$ and $j=2,\dots,n$ we have

$$\lim_{h\to\infty}\frac{1}{\eta_h}\int_D \psi\, Z_j\varphi_h\, dw = -\int_D \varphi\, Z_j\psi dw.$$

This finishes the proof of (2.37). □

3. H-harmoncity of the limit function

In this section, we prove that the limit function φ given by Theorem 2.5 is independent of the variable y_1 dual in the complex sense to the graph direction x_1. If the set E is a *strong* minimizer in the sense of Definition 1.2, we show that the function φ is H-harmonic in $\mathbb{H}^{n-1}$, the lower dimensional Heisenberg group.

3.1. **First variation formula.** We recall the first variation formula for H-perimeter of sets in $\mathbb{H}^n$ that are deformed along a contact flow. A diffeomorphism $\Psi:\Omega\to\Psi(\Omega)$, with $\Omega\subset\mathbb{H}^n$ open set, is a contact map if for any $p\in\Omega$ the differential Ψ_* maps the horizontal space H_p into $H_{\Psi(p)}$. A one-parameter flow $(\Psi_s)_{s\in\mathbb{R}}$ of diffeomorphisms in $\mathbb{H}^n$ is a contact flow if each Ψ_s is a contact map. Contact flows are generated by contact vector fields (see [12]). A contact vector field in $\mathbb{H}^n$ is a vector field of the form

$$V_\psi = \sum_{j=1}^n (Y_j\psi)X_j - (X_j\psi)Y_j - 4\psi T, \tag{3.39}$$

where $\psi\in C^\infty(\mathbb{H}^n)$ is the generating function. For any compact set $K\subset\mathbb{H}^n$ we have the flow $\Psi:[-\delta,\delta]\times K\to\mathbb{H}^n$ that is defined by $\dot\Psi(s,p)=V_\psi(\Psi(s,p))$ and $\Psi(0,p)=p$ for any $s\in[-\delta,\delta]$ and $p\in K$, for some $\delta=\delta(\psi,K)>0$. We call Ψ the flow generated by ψ. We also let $\Psi_s=\Psi(s,\cdot)$.

Related to the generating function ψ, we have, at any point $p \in \mathbb{H}^n$, the real quadratic form $\mathscr{Q}_\psi : H_p \to \mathbb{R}$

$$\mathscr{Q}_\psi\Big(\sum_{j=1}^n x_j X_j + y_j Y_j\Big) = \sum_{i,j=1}^n x_i x_j\, X_j Y_i \psi + x_j y_i\,(Y_i Y_j \psi - X_j X_i \psi) - y_i y_j\, Y_j X_i \psi, \tag{3.40}$$

where $x_j, y_j \in \mathbb{R}$, and ψ with its derivatives are evaluated at p. The quadratic form $\mathscr{Q}_\psi$ appears in the first variation of H-perimeter along the flow generated by ψ. In the following, we identify H_p with $\mathbb{R}^{2n}$ by declaring $X_1, \dots, X_n, Y_n, \dots, Y_n$ an orthonormal basis.

Theorem 3.1. *Let $\Omega \subset \mathbb{H}^n$ be a bounded open set and let $\Psi : [-\delta, \delta] \times \Omega \to \mathbb{H}^n$ be the flow generated by $\psi \in C^\infty(\mathbb{H}^n)$. Then there exists $C = C(\psi, \Omega) > 0$ such that for any set $E \subset \mathbb{H}^n$ with finite H-perimeter in Ω we have*

$$\Big| P_H(\Psi_s(E), \Psi_s(\Omega)) - P_H(E, \Omega) + s \int_\Omega \big\{4(n+1)T\psi + \mathscr{Q}_\psi(\nu_E)\big\} d\mu_E \Big| \leq C\, P_H(E, \Omega)\, s^2 \tag{3.41}$$

for any $s \in [-\delta, \delta]$.

The proof of Theorem 3.1 when $\partial E \cap \Omega$ is a C^∞-smooth hypersurface can be found in [15]. The proof for a set with finite H-perimeter will appear elsewhere.

3.2. ***H*-harmonicity of φ.** Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter in $\mathbb{H}^n$. Assume that $0 \in \mathbb{H}^n$ is a point of the H-reduced boundary of E, $0 \in \partial^* E$, with $\nu_E(0) = (1, 0, \dots, 0) \in \mathbb{R}^{2n}$, and that E is H-perimeter minimizing in a neighborhood of 0, in the sense of Definition 1.1.

Let $(E_h)_{h\in\mathbb{N}}$ be the sequence of rescaled sets introduced in Section 2.1. We can assume that each set E_h is H-perimeter minimizing in the cube

$$Q_R = \{(z, t) \in \mathbb{H}^n : |x_i|, |y_i|, |t|^2 < R,\ i = 1, \dots, n\},$$

for some large $R > 0$. Let $(\varphi_h)_{h\in\mathbb{N}}$ be the sequence of L-intrinsic Lipschitz functions satisfying (2.22), with $0 < L \leq 1$. We can assume that each φ_h is defined on $D_1 = \{(z, t) \in Q_1 : x_1 = 0\}$. Finally, let $\varphi \in W^{1,2}_H(D_1)$ be the limit function of a subsequence of $(\varphi_h)_{h\in\mathbb{N}}$, as in Theorem 2.5. Without loss of generality, we can assume that φ is defined on the whole D_1. Let $D_{1/4} = \{(z, t) \in Q_{1/4} : x_1 = 0\}$.

Theorem 3.2. *Let $n \geq 2$ and let E be a set with locally finite H-perimeter, as above. Then:*

i) *If E is H-perimeter minimizing in a neighborhood of $0 \in \mathbb{H}^n$, then the function $\varphi : D_{1/4} \subset \mathbb{R} \times \mathbb{H}^{n-1} \to \mathbb{R}$ is independent of the variable y_1 of the factor $\mathbb{R}$.*

ii) *If E is strongly H-perimeter minimizing in a neighborhood of $0 \in \mathbb{H}^n$, then the function φ is H-harmonic, i.e., it is of class C^∞ and it solves the partial differential equation*

$$\Delta_H \varphi = 0 \quad in \quad D_{1/4} \cap \{y_1 = 0\}, \tag{3.42}$$

where Δ_H is the Kohn Laplacian (1.6) *in* $\mathbb{H}^{n-1}$.

Proof. Let $\psi \in C^\infty(\mathbb{H}^n)$ be the generating function of a contact vector field V_ψ. We assume that ψ has the following structure. First we assume that we have

$$\psi = \alpha + x_1\beta + \frac{1}{2}x_1^2\gamma,$$

where $\alpha, \beta, \gamma \in C^\infty(\mathbb{H}^n)$ are smooth functions such that

$$X_1\alpha = X_1\beta = X_1\gamma = 0 \text{ in the stripe } \{(z,t) \in \mathbb{H}^n : |x_1| < 1/4\}. \tag{3.43}$$

After a Taylor development in the variable x_1 along the flow of X_1, the function ψ has this structure plus a remainder. The functions β, γ are always assumed to satisfy

$$\beta, \gamma \in C_c^\infty(Q_{1/2}). \tag{3.44}$$

As far as the function α is concerned, we distinguish two cases, according to the claims i) and ii):

i) In this case, we assume also that

$$\alpha \in C_c^\infty(Q_{1/2}). \tag{3.45}$$

ii) In this case, we let

$$\alpha(x_1, y_1, z_2, \dots, z_n, t) = \int_0^{y_1} \vartheta(x_1, s, z_2, \dots, z_n, t)ds, \quad x_1 \in \mathbb{R}, \tag{3.46}$$

where $\vartheta \in C_c^\infty(Q_{1/2})$ is an arbitrary smooth compactly supported function such that $X_1\vartheta = 0$ in $\{|x_1| < 1/4\}$.

We consider the sets $E'_h = \Phi_{s_h}(E_h)$, where $s_h > 0$ are small numbers that will be fixed later. We can assume that $\partial E_h \subset \{|x_1| < 1/4\}$ for all $h \in \mathbb{N}$. In the stripe (3.43), the vector field V_ψ has the form

$$V_\psi = (Y_1\psi)X_1 - (\beta + x_1\gamma)Y_1 + \sum_{j=2}^n (Y_j\psi)X_j - (X_j\psi)Y_j - 4\psi T. \tag{3.47}$$

It follows that $P_H(\Psi_{s_h}(E_h), \Psi(Q_1)) = P_H(E'_h, Q_1)$ for all large $h \in \mathbb{N}$.

In case i), each E_h is H-perimeter minimizing in the cube Q_1; in fact we have $E'_h \Delta E_h \subset\subset Q_1$. In case ii), each E_h is strongly H-perimeter minimizing in the cube

Q_1; in fact, we have $E'_h \Delta E_h \cap \bar{Q}_1 \subset \bar{Q}_1^{Y_1,+}$. In both cases, by Theorem 3.1 the minimality condition $P_H(E_h, Q_1) \le P_H(E'_h, Q_1)$ gives

$$0 \le P_H(E'_h, Q_1) - P_H(E_h, Q_1) = -s_h \int_{Q_1} \Big\{ 4(n+1)T\psi + \mathscr{Q}_\psi(\nu_{E_h}) \Big\} d\mu_{E_h} + O(s_h^2),$$

where $O(s_h^2)/s_h^2$ is bounded by a constant independent of h. We fix $s_h > 0$ such that $s_h = o(\eta_h)$ as $h \to \infty$, where $\eta_h > 0$ is the excess (2.21), and we obtain

$$0 \le -\frac{1}{\eta_h} \int_Q \Big\{ 4(n+1)T\psi + \mathscr{Q}_\psi(\nu_{E_h}) \Big\} d\mu_{E_h} + o(1),$$

where $o(1)$ is infinitesimal as $h \to \infty$. Replacing ψ with $-\psi$ and using the identity $\mathscr{Q}_{-\psi}(\nu_{E_h}) = -\mathscr{Q}_\psi(\nu_{E_h})$, we also have the opposite inequality. We therefore deduce that

$$\lim_{h\to\infty} \frac{1}{\eta_h} \int_{Q_1} \Big\{ 4(n+1)T\psi + \mathscr{Q}_\psi(\nu_{E_h}) \Big\} d\mu_{E_h} = 0. \tag{3.48}$$

Notice that the excess η_h in (2.21) can be equivalently defined using homogeneous cubes in place of balls.

From now on, we let $D = D_1$. Let $E_{\varphi_h} \subset \mathbb{H}^n$ be the intrinsic epigraph of φ_h, as in (2.8). Let $\mathrm{gr}(\varphi_h)$ be the intrinsic graph of φ_h over D, as in (2.9). With a slightly abuse of notation, for any $h \in \mathbb{N}$ let $D_h \subset D$ be the set of points $w \in D$ such that (2.30) and (2.31) hold. By (2.22), (2.31), and (2.20) we have

$$\begin{aligned}
\int_{Q_1} \Big\{ 4(n+1)T\psi + \mathscr{Q}_\psi(\nu_{E_h}) \Big\} d\mu_{E_h} &= \int_{Q_1 \cap \mathrm{gr}(\varphi_h)} \Big\{ 4(n+1)T\psi + \mathscr{Q}_\psi(\nu_{E_h}) \Big\} d\mu_{E_h} + O(\eta_h^2) \\
&= \int_{Q_1 \cap \mathrm{gr}(\varphi_h)} \Big\{ 4(n+1)T\psi + \mathscr{Q}_\psi(\nu_{E_{\varphi_h}}) \Big\} d\mu_{E_{\varphi_h}} + O(\eta_h^2) \\
&= \int_{D_h} \Big\{ 4(n+1)T\psi + \mathscr{Q}_\psi(\nu_{E_{\varphi_h}}) \Big\} \sqrt{1 + |\nabla^{\varphi_h}\varphi_h(w)|^2}\, dw + O(\eta_h^2),
\end{aligned}$$

where $\nu_{E_{\varphi_h}}$ is the vector in (2.30) and the bracket $\{\dots\}$ in the last line is evaluated at $w * \varphi_h(w)$. By (2.22), we have $\mathscr{L}^{2n}(D \setminus D_h) = O(\eta_h^2)$, and so we deduce that

$$\lim_{h\to\infty} \frac{1}{\eta_h} \int_D \Big\{ 4(n+1)T\psi(w * \varphi_h(w)) + \mathscr{Q}_\psi(\nu_{E_{\varphi_h}}(w * \varphi_h(w))) \Big\} dw = 0. \tag{3.49}$$

We compute the limit in (3.49). We start from the integral of $T\psi(w * \varphi_h(w))$. The sequence $(\varphi_h)_{h\in\mathbb{N}}$ is converging to 0 uniformly. We omit details of the proof of this fact. Then we can assume that $\|\varphi_h\|_\infty < 1/4$ and thus, by (3.43), we have $X_1 T\alpha = T X_1 \alpha = 0$. This implies that $T\alpha(w * \varphi_h(w)) = T\alpha(w) = \alpha_t(w)$, where we are using the notation $\alpha_t = \partial\alpha/\partial t$. The same holds for β and γ. Thus we have, for any $w \in D$,

$$T\psi(w * \varphi_h(w)) = \alpha_t + \varphi_h \beta_t + \frac{1}{2}\varphi_h^2 \gamma_t,$$

where the right hand-side is evaluated at w. With abuse of notation, here and in the following we denote by $\psi, \alpha, \beta, \gamma, \vartheta$ also the restriction of the functions to $\{x_1 = 0\}$.

Since we have

$$\operatorname{supp}(\alpha), \operatorname{supp}(\beta), \operatorname{supp}(\gamma) \subset \big\{(z,t) \in \mathbb{H}^n : |t|^2 < 1/2\big\}, \tag{3.50}$$

then there holds

$$\int_D \alpha_t \, dw = \int_D \beta_t \, dw = \int_D \gamma_t \, dw = 0.$$

Let $\bar\varphi_h \in \mathbb{R}$ be the numbers given by Theorem 2.5. By (2.38), we have

$$\lim_{h\to\infty} \frac{1}{\eta_h} \int_D \varphi_h \, \beta_t \, dw = \lim_{h\to\infty} \int_D \frac{\varphi_h - \bar\varphi_h}{\eta_h} \beta_t \, dw = \int_D \varphi \, \beta_t dw, \tag{3.51}$$

and

$$\lim_{h\to\infty} \frac{1}{\eta_h} \int_D \varphi_h^2 \gamma_t \, dw = \lim_{h\to\infty} \int_D \frac{\varphi_h - \bar\varphi_h}{\eta_h} (\varphi_h + \bar\varphi_h) \gamma_t \, dw = 0, \tag{3.52}$$

because $\varphi_h + \bar\varphi_h$ is converging to 0 strongly in L^2. From (3.51) and (3.52), we deduce that

$$\lim_{h\to\infty} \frac{1}{\eta_h} \int_D 4(n+1) T\psi(w * \varphi_h(w)) dw = 4(n+1) \int_D \varphi \, \beta_t \, dw. \tag{3.53}$$

We compute the limit of the integral of $\mathscr{Q}_\psi(\nu_{E_{\varphi_h}})$ in (3.49). Letting

$$\nu_{E_{\varphi_h}} = (\nu_{X_1}, \dots, \nu_{X_n}, \nu_{Y_1}, \dots, \nu_{Y_n}) \in \mathbb{S}^{2n},$$

we isolate in (3.40) the terms containing ν_{X_1}. Namely, we have

$$\begin{aligned} \mathscr{Q}_\psi(\nu_{E_{\varphi_h}}) = (X_1 Y_1 \psi) \nu_{X_1}^2 &+ \sum_{j=2}^n (X_j Y_1 \psi + X_1 Y_j \psi) \nu_{X_1} \nu_{X_j} \\ &+ \sum_{j=1}^n (Y_j Y_1 \psi - X_1 X_j \psi) \nu_{X_1} \nu_{Y_j} \\ &+ \mathscr{E}_\psi(\nu_{E_{\varphi_h}}), \end{aligned} \tag{3.54}$$

where $\mathscr{E}_\psi(\nu_{E_{\varphi_h}})$ is a quadratic form that does not contain ν_{X_1}.

Inserting into formula (3.54) the derivatives

$$\begin{aligned} X_1 Y_1 \psi &= Y_1 X_1 \psi - 4T\psi \\ &= Y_1 \beta + x_1 Y_1 \gamma - 4\Big(\alpha_t + x_1 \beta_t + \frac{1}{2} x_1^2 \gamma_t\Big), \\ X_j Y_1 \psi &= Y_1 X_j \alpha + x_1 Y_1 X_j \beta + \frac{1}{2} x_1^2 Y_1 X_j \gamma, && j \ge 2, \\ X_1 Y_j \psi &= Y_j \beta + x_1 Y_j \gamma, && j \ge 2, \\ Y_j Y_1 \psi &= Y_j Y_1 \alpha + x_1 Y_j Y_1 \beta + \frac{1}{2} x_1^2 Y_j Y_1 \gamma, && j \ge 1, \end{aligned} \tag{3.55}$$

we obtain

$$\begin{aligned}
\mathscr{Q}_\psi(\nu_{E_{\varphi_h}}) &= \Big\{ Y_1\beta + x_1 Y_1\gamma - 4\Big(\alpha_t + x_1\beta_t + \frac{1}{2}x_1^2\gamma_t\Big)\Big\}\nu_{X_1}^2 \\
&+ \sum_{j=2}^n \Big\{ Y_1X_j\alpha + x_1Y_1X_j\beta + \frac{1}{2}x_1^2 Y_1X_j\gamma + Y_j\beta + x_1Y_j\gamma \Big\}\nu_{X_1}\nu_{X_j} \\
&+ \sum_{j=1}^n \Big\{ Y_jY_1\alpha + x_1Y_jY_1\beta + \frac{1}{2}x_1^2 Y_jY_1\gamma - X_j\beta - x_1X_j\gamma \Big\}\nu_{X_1}\nu_{Y_j} \\
&+ \mathscr{E}_\psi(\nu_{E_{\varphi_h}}),
\end{aligned} \tag{3.56}$$

where, by (2.18) and (2.17), we have

$$\begin{aligned}
\nu_{X_1} &= \frac{1}{\sqrt{1+|\nabla^{\varphi_h}\varphi_h|^2}}, \quad \nu_{Y_1} = -\frac{\mathscr{B}\varphi_h}{\sqrt{1+|\nabla^{\varphi_h}\varphi_h|^2}}, \\
\nu_{Z_j} &= -\frac{Z_j\varphi_h}{\sqrt{1+|\nabla^{\varphi_h}\varphi_h|^2}}, \quad Z \in \{X, Y\}, \quad j \geq 2.
\end{aligned} \tag{3.57}$$

Above, $\mathscr{B}\varphi_h$ is the Burgers' operator. In particular, since each φ_h is intrinsic L-Lipschitz with $0 < L \leq 1$ we can assume that $\sup_{h\in\mathbb{N}} \|\nabla^{\varphi_h}\varphi_h\|_\infty < \infty$ and thus there exists an absolute constant $C > 0$ such that

$$|\mathscr{E}_\psi(\nu_{E_{\varphi_h}})| \leq C|\nabla^{\varphi_h}\varphi_h|^2. \tag{3.58}$$

So, from (2.32) and (2.33) we have

$$\lim_{h\to\infty} \frac{1}{\eta_h}\int_D |\mathscr{E}_\psi(\nu_{E_{\varphi_h}}(w * \varphi_h(w))|dw = 0.$$

In other words, the limit (3.49) of the integral of $\mathscr{E}_\psi$ in (3.49) vanishes.

We compute the limit of the integral of the first three lines in (3.56), separately. By (3.43), we have $X_1Y_1\beta = Y_1X_1\beta - 4T\beta = -4T\beta$ and thus

$$Y_1\beta(w * \varphi_h(w)) = \beta_{y_1}(w) - 4\varphi_h(w)\beta_t(w).$$

Similarly, there holds

$$Y_1\gamma(w * \varphi_h(w)) = \gamma_{y_1}(w) - 4\varphi_h(w)\gamma_t(w).$$

The limit of the integral of terms in the first line of (3.56) containing x_1^2 vanishes, by a computation analogous to (3.52). Moreover, by (2.32), (2.33), and (3.57) the function $\nu_{X_1}^2$ may be replaced by 1. Thus, the limit of the integral of the first line in

(3.56) is

$$
\begin{aligned}
\lim_{h\to\infty}\frac{1}{\eta_h}\int_B\Big\{\beta_{y_1}-4\varphi_h\beta_t+\varphi_h(\gamma_{y_1}-4\varphi_h\gamma_t)-4\Big(\alpha_t+\varphi_h\beta_t+\frac12\varphi_h^2\gamma_t\Big)\Big\}\nu_{X_1}^2\,dw=\\
=\lim_{h\to\infty}\int_D(\gamma_{y_1}-8\beta_t)\frac{\varphi_h-\bar\varphi_h}{\eta_h}dw\\
=\int_D(\gamma_{y_1}-8\beta_t)\varphi\,dw.
\end{aligned}
\tag{3.59}
$$

We used Theorem 2.5.

We compute the limit of the integral of the second line in (3.56). In this case, the limit of the integral of terms containing x_1 or x_1^2 vanishes. So we have:

$$
\begin{aligned}
\lim_{h\to\infty}\frac{1}{\eta_h}\int_D\sum_{j=2}^n\Big\{Y_1X_j\alpha+\varphi_hY_1X_j\beta+\frac12\varphi_h^2Y_1X_j\gamma+Y_j\beta+\varphi_hY_j\gamma\Big\}\nu_{X_1}\nu_{X_j}\,dw=\\
=-\lim_{h\to\infty}\int_D\sum_{j=2}^n(Y_1X_j\alpha+Y_j\beta)\frac{X_j\varphi_h}{\eta_h}dw\\
=-\int_D\sum_{j=2}^n\Big(\frac{\partial}{\partial y_1}X_j\alpha+Y_j\beta\Big)X_j\varphi\,dw.
\end{aligned}
\tag{3.60}
$$

We used Theorem 2.5.

Finally, we compute the limit of the integral of the third line in (3.56):

$$
\begin{aligned}
\lim_{h\to\infty}\frac{1}{\eta_h}\int_D\sum_{j=1}^n\Big\{Y_jY_1\alpha+\varphi_hY_jY_1\beta+\frac12\varphi_h^2Y_jY_1\gamma-X_j\beta-\varphi_hX_j\gamma\Big\}\nu_{X_1}\nu_{Y_j}\,dw=\\
=-\lim_{h\to\infty}\int_D\Big\{\frac{\mathscr{B}\varphi_h}{\eta_h}Y_1^2\alpha+\sum_{j=2}^n\Big\{Y_jY_1\alpha-X_j\beta\Big\}\frac{Y_j\varphi_h}{\eta_h}\Big\}dw\\
=-\int_D\Big\{\partial_{y_1}\varphi\,Y_1^2\alpha+\sum_{j=2}^n\Big(Y_jY_1\alpha-X_j\beta\Big)Y_j\varphi\Big\}dw.
\end{aligned}
\tag{3.61}
$$

We used Theorem 2.5.

Putting together (3.53), (3.59), (3.60), and (3.61), we obtain:

$$
\begin{aligned}
\int_D\Big\{\big(4(n+1)\beta_t+\gamma_{y_1}-8\beta_t\big)\varphi-\partial_{y_1}\varphi Y_1^2\alpha-\\
-\sum_{j=2}^n\Big(\frac{\partial}{\partial y_1}X_j\alpha+Y_j\beta\Big)X_j\varphi-\Big(Y_jY_1\alpha-X_j\beta\Big)Y_j\varphi\Big\}dw=0.
\end{aligned}
\tag{3.62}
$$

When $\alpha=\beta=0$, this equation reads

$$
0=\int_D\gamma_{y_1}\varphi\,dw=-\int_D\gamma\varphi_{y_1}dw,
$$

for any test function $\gamma \in C_c^\infty(D_{1/2})$. This implies that φ is independent of y_1. This proves claim i) of the theorem.

When $\alpha = \gamma = 0$, equation (3.62) reads

$$\begin{aligned}0 &= \int_D \Big\{4(n-1)\beta_t\varphi + \sum_{j=2}^n X_j\beta Y_j\varphi - Y_j\beta X_j\varphi\Big\} dw \\ &= \int_D \Big\{4(n-1)\beta_t - \sum_{j=2}^n Y_jX_j\beta - X_jY_j\beta\Big\}\varphi\, dw,\end{aligned}$$

for any $\beta \in C_c^\infty(D_{1/2})$. This information is empty. In fact, the equation is satisfied for any test function because $Y_jX_j - X_jY_j = [Y_j, X_j] = 4T$.

When $\beta = \gamma = 0$, by $Y_1\varphi = 0$ and (3.46) equation (3.62) reads

$$\begin{aligned}0 &= \int_D \Big\{Y_1^2\alpha Y_1\varphi + \sum_{j=2}^n Y_1X_j\alpha X_j\varphi + Y_jY_1\alpha Y_j\varphi\Big\} dw \\ &= -\int_D \frac{\partial\alpha}{\partial y_1} \sum_{j=2}^n (X_j^2\varphi + Y_j^2\varphi) dw \\ &= -\int_D \vartheta\, \Delta_H\varphi\, dw,\end{aligned}$$

for any test function $\vartheta \in C_c^\infty(D_{1/2})$. Then the function $\varphi \in W_H^{1,2}(D)$ solves the partial differential equation $\Delta_H\varphi = 0$ in the weak sense in $D_{1/4} \cap \{y_1 = 0\}$. It follows that φ is smooth, by hypoellipticity, and φ is a classical solution. This proves claim ii). □

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