Theory of functions 2

Sheet 1

Riesz theorem, measure theory

Within 22th March 1012

Exercise 1 Let $f : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, be the function defined by $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ if $|x| \geq 1$, with $x \in \mathbb{R}^n$. Prove that $f \in BV(\mathbb{R}^n)$, compute the measure μ and the vector σ given by the structure theorem of BV-functions. Show that $f \notin W^{1,1}(\mathbb{R}^n)$.

Exercise 2 Let $T : C_c(\mathbb{R}) \to \mathbb{R}$ be the functional

$$T(f) = \int_{-\infty}^{+\infty} f(x)dx, \quad f \in C_c(\mathbb{R}),$$

where the integral is the Riemann-integral. Show that T il linear and bounded (for the sup-norm, when the support of the functions is contained in a fixed compact set). Compute the measure μ given by Riesz theorem, (i.e., prove that μ must be the Lebesgue measure).

Exercise 3 Let $\mathcal{A}$ be a σ-algebra on X, let $\mathcal{B}$ be a σ-algebra on Y and let $f : X \to Y$ be measurable (that is, $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$). Let $\mu : \mathcal{A} \to [0, \infty]$ be a measure. Show that $f_\sharp \mu = \nu$ defined by

$$\nu(B) = \mu(f^{-1}(B)), \quad B \in \mathcal{B},$$

is a measure, called the *push-forward measure of* μ. Prove the following change-of-variable formula

$$\int_Y g(y)d\nu(y) = \int_X g(f(x))d\mu(x),$$

for any $g \in L^1(Y; \nu)$.

Exercise 4 Let μ be the Lebesgue measure on $[0, 1]$. Write $[0, 1] = A \cup B$ where $\mu(A) = 0$ and

$$B = \bigcup_{n=1}^{\infty} K_n,$$

with $K_n \subset [0, 1]$ compact sets containing no open intervals.