Theory of functions 2 Sheet 3

Various exercises 12 April 2012

Exercise 1

1) Let μ be a finite Borel measure on $\mathbb{R}$ supported in the interval $[0,1]$. Construct a function $f \in BV_{\text{loc}}(\mathbb{R})$ such that $f' = \mu$ in distributional sense.

2) Find a finite Borel measure μ on $\mathbb{R}^2$ such that there exists no function $f \in BV_{\text{loc}}(\mathbb{R}^2)$ such that $\mu = \|Df\|$.

Exercise 2 Let $f : [0,1] \to [0,1]$ be the Cantor-Vitali function introduced in the lecture. Study the measure $\mu = \|f'\|$. Is it true that $\mu = \beta\mathcal{H}^s \llcorner K$ for some $\beta > 0$, with $s = \log 2/\log 3$ and K is the 1/3-Cantor set?

Exercise 3 (1) Let $\mathcal{K} = \{K \subset \mathbb{R}^n : K \neq \emptyset,\ K\text{compact}\}$. For any $\delta > 0$ let $K_\delta = \{x \in \mathbb{R}^n : \text{dist}(x,K) < \delta\}$. Prove that the function $d : \mathcal{K} \times \mathcal{K} \to [0,\infty)$

$$d(K,H) = \inf\big\{\delta > 0 : K \subset H_\delta \text{ and } H \subset K_\delta\big\}$$

is a distance on $\mathcal{K}$, known as Hausdorff distance. Discuss the validity of the (sequential) lower semicontinuity

$$\mathcal{H}^1(K) \leq \liminf_{m\to\infty} \mathcal{H}^1(K_m),$$

for $K, K_m \in \mathcal{K}$ and $K_m \to K$ in the Hausdorff distance as $m \to \infty$.

(2) Let (X,d) be a metric space and let $\gamma, \gamma_n : [0,1] \to X$ be Lipschitz curves, $n \in \mathbb{N}$. Determine a notion of convergence $\gamma_n \to \gamma$ as $n \to \infty$ ensuring the lower semicontinuity of the length:

$$\text{Var}(\gamma) \leq \liminf_{n\to\infty} \text{Var}(\gamma_n).$$

(3) Let $1 \leq p < \infty$ and let $f_n \in W^{1,p}(\mathbb{R})$, $n \in \mathbb{N}$, be a sequence such that $f_n \to f \in L^p(\mathbb{R})$ as $n \to \infty$ in the $L^p(\mathbb{R})$ norm. Prove or disprove the following statement: $f \in W^{1,p}(\mathbb{R})$ and

$$\int_{\mathbb{R}} |f'(x)|^p dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |f_n'(x)|^p dx.$$

(4) Let $f_k \in BV(\mathbb{R}^n)$, $k \in \mathbb{N}$, be a sequence such that $f_k \to f \in L^1(\mathbb{R}^n)$ as $k \to \infty$ in the $L^1(\mathbb{R}^n)$ norm. Prove or disprove the following statement: $f \in BV(\mathbb{R}^n)$ and

$$\|Df\|(\mathbb{R}^n) \leq \liminf_{k\to\infty} \|Df_k\|(\mathbb{R}^n).$$

Exercise 4 Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$, be a Lipschitz function with Lipschitz constant $0 \leq L < \infty$. Prove that for any $A \subset \mathbb{R}^n$ and for all $0 \leq s \leq n$ we have

$$\mathcal{H}^s(f(A)) \leq L^s \mathcal{H}^s(A).$$

Exercise 5 Nel Postulato IV dell'opera *Sulla sfera e il cilindro* Archimede afferma:

"Tra le superfici aventi le stesse linee terminali giacenti su un piano, sono disuguali quelle tali che, essendo ambedue concave dalla stessa parte, o una è tutta compresa dall'altra e dalla superficie piana avente gli stessi termini, o ha una parte compresa dall'altra, una parte comune con essa. Ed è minore la superficie che è compresa dall'altra."

Riformulare l'affermazione di Archimede nel linguaggio della matematica contemporanea e quindi provarla. Usare l'Esercizio 4.