Department of Pure and Applied Mathematics
Bachelor's Degree in Mathematics

Costruction of a fundamental solution for the Kolmogorov equation.

Supervisor:
Prof. Monti Roberto

Candidate:
Accadia Davide

Academic year 2020/2021

Quote:
Sometimes it is the people no one imagines anything of, who do the things that no one can imagine.
(Alan Turing)

Contents

Introduction

The main objective of this thesis is to present fundamental solutions.
There will be a little of definition work to do, we will not give a complete characterisation to distributions, just the necessary tools to define the Fourier transform for distributions and to derive the solution to Kolmogorov equation.
In the first chapter we shall see the definition of the distributions and some necessary properties. In the second chapter we understand what are differential operators and what does it means to have a fundamental solution. The third and last chapter is dedicated to the Kolmogorov equation, introducing the Fourier transform as is needed to derive one fundamental solution.

Chapter 1

Test function and Distribution Space in $\mathbb{R}^n$

First of all, the definitions of our objects

Definition 1.1. (Space of test functions)
A test function on $\mathbb{R}^n$ is a infinitely differentiable function with compact support. The space of test function is denoted as $C_0^\infty(\mathbb{R}^n)$ or $\mathscr{D}(\mathbb{R}^n)$.

Definition 1.2. (Distribution)
A distribution is a linear form on $\mathscr{D}(\mathbb{R}^n)$ that has the following continuity property: for any K compact subset of $\mathbb{R}^n$ there are $k \in \mathbb{N}$ and $C \in \mathbb{R}$ such that

$$\forall \phi \in \mathscr{D}(K); \qquad |u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\partial^\alpha \phi(x)|;$$

where $\alpha \in \mathbb{N}^n$ is a multi-index and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is the weight of α. The space of distribution is denoted as $\mathscr{D}'(\mathbb{R}^n)$.

Remark. Any locally integrable function f induces a distribution u as

$$\phi \in \mathscr{D}(\mathbb{R}^n); \quad u \in \mathscr{D}'(\mathbb{R}^n); \quad u(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx.$$

Example 1.3. If f and g are test functions, the integration by parts formula gives

$$\int_{\mathbb{R}^n} f(x)\partial^\alpha g(x)dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial^\alpha f(x)g(x)dx$$

where $\alpha \in \mathbb{N}^n$ is a multi-index and $\partial^\alpha f$ means $\frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}} f$.

That suggests us the next definition:

Definition 1.4. (Differentiating a distribution)
Let u be a distribution and $\alpha \in \mathbb{Z}^n$ then we shall define a distribution $\partial^\alpha u$ as
$$\partial^\alpha u(\phi) = (-1)^{|\alpha|} u(\partial^\alpha \phi).$$

Theorem 1.5. Let $\phi(x, y)$ in $C^\infty(X \times Y)$ where Y is an open set in $\mathbb{R}^n$, such that there is a compact set $K \subset X$ so that the support of ϕ is in K $\times Y$, and u be a distribution. Then

$$y \to u(\phi(., y))$$

as a function in y, is a C^∞ function and

$$\partial_y^\alpha u(\phi(., y)) = u(\partial_y^\alpha \phi(., y)).$$

proof. The Taylor formula for ϕ in respect of y

$$\phi(x, y + h) = \phi(x, y) + \sum_j h_j \frac{\partial \phi(x, y)}{\partial y_j} + \psi(x, y, h);$$
$$\sup_x |\partial_x^\alpha \psi * x, y, h)| = O(|h|^2)$$

$$u(\phi(., y + h)) = u(\phi(., y)) + \sum_j h_j u(\frac{\partial \phi(x, y)}{\partial y_j}) + O(|h|^2)$$

So $y \to u(\phi(., y))$ is differentiable.

$$\frac{\partial}{\partial y_j} u(\phi(., y)) = \frac{\partial \phi(x, y)}{\partial y_j}.$$

Theorem 1.6. if $u \in \mathscr{D}'(I)$ where I is an open interval in $\mathbb{R}$. and $u' = 0$ then u is a constant.

proof. $u' = 0$ means

$$\forall \phi \in C_0^\infty(A); \quad u(\phi') = 0.$$

Obviously if ϕ is a test function, ϕ' is a test function, and the fundamental theorem of integration holds in this domain.

$$\phi(x) = \int_{-\infty}^{x} \phi'(t) dt$$

The idea is, if $\int \psi = 0$ then we can define the primitive, and that is a test function, so $u(\psi) = 0$ then for any ψ and for $\int \gamma = 1$

$$0 = u(\psi - \gamma \int \psi) = u(\psi) - \int \psi u(\gamma)$$

This proves that u is a constant.

Theorem 1.7. Let u be a distribution in $Y \times I$, where Y is an open set in $\mathbb{R}^{n-1}$ and I is an open interval in $\mathbb{R}$. Then if $\partial_n u = 0$

$$\phi \in \mathscr{D}(Y \times I); u(\phi) = \int_{\mathbb{R}} u_0(\phi(.,x_n))dx_n.$$

Where u_0 is a distribution in $\mathbb{R}^{n-1}$, if fact this means that u is constant respectively to x_n.

proof. The proof is analogue to the previous.

Theorem 1.8. If u and f are continuous functions, and $\partial_j u = f$ as distributions, then $\partial_j u(x)$ exists and is equal to $f(x)$

proof. Assume $j = n$ and that $X = Y\times$ I since the property is local. Let $\tau \in I$ and define v(x) as

$$x = (x', x_n); \quad v(x) = \int_{\tau}^{x_n} f(x', t)dt.$$

$\partial_n(u - v) = 0$ so by the theorem before $u(x) = v(x) + w(x')$ where w is a constant function, the right hand function is differentiable in respect to x^n, with derivative f.

Definition 1.9. (C^1 boundary) If $B \subset A$ open sets in $\mathbb{R}^n$ then we say B has C^1 boundary in A if for every boundary point of Y in X we can find a C^1 function

$$\rho(x_0) = 0; \quad dp(x_0) \neq 0; \quad A_0 \cap B = \{x \in A_0 | \rho(x) < 0\}$$

Remark. Any Borel Measure induces a distribution.

$$\mu(\phi) = \int_{\mathbb{R}} \phi \, d\mu.$$

The space of distribution contains many of already known objects and defines for each derivatives of all orders.

Example 1.10. (Dirac's measure)

Consider δ_0, it is a measure on $\mathbb{R}^n$.

$$\delta_0 : \mathscr{P}(\mathbb{R}^n) \to \mathbb{R}; \qquad \delta_0(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$$

Example 1.11. (Dirac's distribution)
Let's consider the distribution h induced by the Heavyside function $H(x)$. It is clear that H is not differentiable in 0 as a function.

$$H : \begin{cases} H(x) = 1 \text{ if } x \geq 0 \\ H(x) = 0 \text{ if } x < 0 \end{cases} \quad h(\phi) = \int_{\mathbb{R}} H(x)\phi(x)dx$$

If we try to differentiate h as the definition we gave suggests we obtain h'

$$h'(\phi) = -\int_0^\infty \phi' = \phi(0)$$

h' is the distibution induced by the measure δ_0, we call it the Delta distribution centered in 0 and denote it simply as δ_0.

Chapter 2

Fundamental Solutions

Definition 2.1. (LDO with constant coefficient and related equation)

A Linear differential operator is a function : $C^\infty(\mathbb{R}^n;\mathbb{C}) \to C^\infty(\mathbb{R}^n;\mathbb{C})$ such that is linear in ϕ and its derivatives.

$$P = \sum_{|\alpha|<k} c_\alpha \partial^\alpha; \quad c_\alpha \in \mathbb{C}, \alpha \in \mathbb{N}^n.$$

Definition 2.2. (Fundamental solution for a LDO)

Let P be a LDO then we call the distribution E a fundamental solution of P if

$$PE = \delta_0.$$

Remark. (Translation)

We have to note that, when the LDO has constant coefficients, it commutes with translations of $\mathbb{R}^n$.

$$PE(\phi \circ \tau) = \delta_0(\phi \circ \tau) = \delta_{\tau(0)}$$

Example 2.3. (Examples of differential operator)

Differentiation in $\mathbb{R}$: we have already seen that the Heavyside function is a fundamental solution for the simple differentiation in $\mathbb{R}$.

$$P = \frac{d}{dx}$$

Differentiation by $\bar{z}$ in $\mathbb{C}$: We will see that the fundamental solution can be derived from the solution of the Laplace operator.

$$P = \frac{\partial}{\partial \bar{z}}$$

Laplace operator in $\mathbb{R}^n$: its fundamental solution has a very different structure depending on the dimension. The case $n = 1$ is different because the domain is not connected, a wide range of solution is accepted, for the other case we will have to distinguish for $n = 2$ and $n > 2$.

$$P = \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

The heat operator in $\mathbb{R}^{n+1}$: it represents how heat diffuses through time, the fundamental solution is the case where all of the heat is concentrated in a starting point.

$$P = \Delta_x - \partial_t$$

The wave operator $\mathbb{R}^{n+1}$; it represents how waves evolve.

$$P = \Delta_x - \partial_t^2$$

Remark. (Considering $\mathbb{R}^n \setminus \{0\}$)

In the following examples, we will work with LDOs that have the following property, when A is an open subset of $\mathbb{R}^n$ and u is a distribution.

$$u \in \mathscr{D}'(\mathbb{R}^n) \quad Pu \in C^\infty(A) \implies u \in C^\infty(A)$$

Since δ_0 restricted to $\mathbb{R}^n \setminus \{0\}$ is in $C^\infty(\mathbb{R}^n \setminus \{0\})$ then E, the fundamental solution must also be in $C^\infty(\mathbb{R}^n \setminus \{0\})$ when restricted to $\mathbb{R}^n \setminus \{0\}$ must be in $C^\infty(\mathbb{R}^n \setminus \{0\})$.

Definition 2.4. (Homogeneous distributions)

A distribution u in $(\mathbb{R}^n \setminus \{0\})$ is called homogeneous of degree a if for any ϕ in $\mathscr{D}(\mathbb{R} \setminus \{0\})$,

$$\langle u, \phi(x) \rangle = t^\alpha \langle u, t^n \phi(tx) \rangle$$

Theorem 2.5. Given u an homogeneous distribution in $(\mathbb{R}^n \setminus \{0\})$ of degree $a \leq -n$, there is an unique extension of u that is an homogeneous distribution of degree a in $\mathbb{R}^n$

Theorem 2.6. Let $u_1, \cdots, u_n$ be homogeneous distributions of degree $1-n$ in $\mathbb{R}^n \setminus \{0\}$ and assume that $\sum_{j=1}^n \partial_j u_j = 0$

$$\sum_{j=1}^{n} \partial_j \dot{u}_j = \delta_0$$

Lemma 2.7. Differentiation by $\bar{z}$

The second fundamental solution that we see is for the

$$\frac{\partial E_\zeta}{\partial \bar{z}} = \delta_\zeta \text{ where } E_\zeta(z) = \frac{1}{\pi(z-\zeta)}$$

It also derives from the Cauchy's integral formula, but we will solve it in a more elegant way.

$$\phi(\zeta) = -\frac{1}{\pi} \int_Y \partial \frac{\phi(x,y)}{\partial \tilde{z}} \frac{1}{z-\zeta} dxdy + \frac{1}{2\pi i} \int_{\partial Y} \phi(x,y) \frac{1}{z-\zeta}$$

Lemma 2.8. (Gauss-Green)

Let A be a open and limited set in $\mathbb{R}^n$ and such that ∂A is C^1
$F \in C^1(A;\mathbb{R}^n) \cap C(\bar{A};\mathbb{R}^n)$ then

$$\int_A divF dx = -\int_{\partial A} \langle F, N_{\partial A} \rangle d\sigma$$

We will use it like this

$$\int_A \phi \Delta f dx = \int_A \phi div(\nabla f) dx = \int_A (div(\phi \nabla f) - \langle \nabla \phi, \nabla f \rangle) dx$$
$$= \int_{\partial A} \phi \nabla f d\sigma - \int_A \langle \nabla \phi, \nabla f \rangle) dx$$

Theorem 2.9. (Fundamental solution for the Laplace equation) The Laplace operator has a fundamental solution in $\mathbb{R}^n$ for $n \geq 2$.

$$\begin{cases} E(x) = \frac{log|x|}{2\pi} \text{ if } x \in \mathbb{R}^2 \setminus \{0\} \\ E(x) = -\frac{|x|^{2-n}}{(n-2)c_n} \text{ if } x \in \mathbb{R}^n \setminus \{0\} \text{ and } n > 2 \end{cases}$$

where c_n is the area of the unit sphere

proof. If $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\langle \partial_j E, \phi \rangle = -\langle E, \partial_j \phi \rangle = -\lim_{\epsilon \to 0} \int_{|x|>\epsilon} E(x) \partial_j \phi(x) dx =$$
$$\int \partial_j E(x) \phi(x) dx + \lim_{\epsilon \to 0} \int_{|x|=\epsilon} E(x) \phi(x) \frac{x_j}{|x|} d\sigma$$

Using the gauss green theorem for the last equality. The limit is $O(\epsilon)$ or $O(\log(\frac{1}{\epsilon})$ depending on the cases. This tells us that the distribution's derivatives are defined by locally integrable functions. For $x \neq 0$

$$\Delta E = \frac{n|x|^{-n} - \sum n x_j^2 |x|^{-n-2}}{c_j} = 0$$

This tells us that $c\delta_0$ is an extension of ΔE for some c

$$\langle \Delta E, \phi \rangle = \langle E, \Delta \phi \rangle = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} (E \Delta \phi - \phi \Delta E) dx =$$
$$\lim_{\epsilon \to 0} \int_{|x|>\epsilon} div(E \nabla \phi - \phi \nabla E) dx =$$
$$\lim_{\epsilon \to 0} \int_{|x|=\epsilon} \langle \phi \nabla E - E \nabla \phi, \frac{x}{|x|} \rangle d\sigma = \phi(0)$$

Remark. The case n=2 implies the solution of the first operator we discussed, the differentiation by $\bar{z}$.

$$\Delta = 4 \frac{\partial^2}{\partial_z \partial_{\bar{z}}}; \qquad \frac{\partial}{\partial z} E = \frac{1}{4\pi z};$$

Lemma 2.10. (Integrals)

To prove that the fundamental solution for the Heat equation defines a distribution we need that it is locally integrable.

$$(\int_{\mathbb{R}} e^{-t^2} dt)^n = c_n \frac{\Gamma(n/2)}{2}$$

$$\int_{\mathbb{R}} e^{-t^2} dt = \pi^{\frac{1}{2}}$$

$$\int_{\mathbb{R}} e^{-at^2} dt = (\frac{\pi}{a})^{\frac{1}{2}}, \quad a > 0$$

$$\int_{\mathbb{R}^n} exp(-\langle Ax, x\rangle)dx = \pi^{\frac{n}{2}}(det(A))^{-\frac{1}{2}}$$

Theorem 2.11. (Fundamental solution for the Heat equation) Considering the Heat operator introducing before we will prove that the function E is is the fundamental solution of the Heat operator P.

$$E(x,t) = \begin{cases} (4t\pi)^{-\frac{n}{2}} e^{(-\frac{|x|^2}{4t})}, t > 0; \\ 0, t \leq 0 \end{cases}$$

$$(\frac{\partial}{\partial t} - \Delta_x)E = \delta_0$$

proof Calculating the two parts of PE we note that they cancel each other.

$$\frac{\partial E}{\partial_j} = -x_j \frac{E}{2t}; \quad \Delta_x E = -n\frac{E}{2t} + |x|^2 \frac{E}{4t^2} = \frac{\partial}{\partial t}$$

So we know that PE = 0 on $(\mathbb{R}^n \setminus \{0\})$ As before we know that PE has $\{0\}$ as support.

$$\begin{aligned} \langle PE, \phi\rangle &= \langle E, \frac{\partial \phi}{\partial t} + \Delta_x \phi\rangle \\ &= \lim_{\epsilon \to 0} \int_{t > \epsilon} -E(x,t)(\frac{\partial \phi}{\partial t} + \Delta_x \phi) dx dt \\ &= \lim_{\epsilon \to 0} \int E(x,\epsilon)\phi(x,\epsilon)dx \\ &= \lim_{\epsilon \to 0} \int E(x,1)\phi(\sqrt{\epsilon}x, \epsilon)dx = \phi(0) \end{aligned}$$

We used the Gauss green theorem, a change of variables and the bounded convergence theorem to bring the limit inside the integral.

Definition 2.12. (Support of a distribution)

We say that a point x is inside of the support of u if x has no neighborhood in which the restriction of u is 0. The space of distribution with compact support is denoted with $\mathscr{E}'(\mathbb{R}^n)$

Definition 2.13. (Convolution by smooth function)

If u is a distribution and ϕ a test function then the convolution $u * \phi$ is defined as the function of x

$$(u * \phi)(x) = u(\phi(x - y));$$

where ϕ on the righthand is considered a function of y.

Theorem 2.14. (Associativity of the convolution operation)

If u is a distribution and ϕ and ψ are test functions then

$$(u * \phi) * \psi = u * (\phi * \psi).$$

Definition 2.15. (Convolution of distribution)

Let u_1 and u_2 be two distributions one of which has compact support then the convolution of the two is defined as the only distribution u such that

$$\phi \in C_0^\infty(\mathbb{R}^n); \quad u_1 * u_2 * \phi = u * \phi.$$

Remark. (Reason behind fundamental solutions)

Through the fundamental solution of an operator one can construct any solution of that operator, thank to the following convolution and δ_0 properties

$$(\partial^\alpha u) * \phi = u * (\partial^\alpha \phi) = u * (\delta_0 * (\partial^\alpha \phi)) = u * (\partial^\alpha \delta_0) * \phi.$$

The first and last equalities are derived from the definitions; the second is due to the fact that δ_0 is the identity for convolution with test functions. This proves $\partial^\alpha u = (\partial^\alpha \delta_0) * u$ if we remember the definition of linear differential operator we gave before we can just substitute ∂^α with P

This has the important consequence, where E is the fundamental solution

$$E * (Pu) = u, \quad u \in \mathscr{E}'(\mathbb{R}^n)$$
$$P(E * f) = f, \quad f \in \mathscr{E}'(\mathbb{R}^n)$$

The convolution with E is the right and left inverse of P. The second equation show us very clearly the importance behind finding a fundamental solution.

Chapter 3

Kolmogorov equation

In this last chapter we will consider the following equation, relative to a LDO with non constant coefficients.

$$\mathscr{K}u = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t} = 0; \qquad t > 0.$$
$$u(t,.) \to \delta_(x_0, y_0) \quad \text{as} \quad t \to 0.$$

Definition 3.1. (Schwartz space and its dual)
We call $\mathscr{S}(\mathbb{R}^n)$ the space of temperate functions, the set of all $\phi \in C^\infty(\mathbb{R}^n)$ functions such that

$$\sup_x |x^\beta \partial^\alpha \phi(x)| < \infty$$

Definition 3.2. (Fourier transform)
If $u \in \mathscr{S}'(\mathbb{R}^n)$ it is defined the Fourier transform of u $\hat{u}$.

$$\hat{u}(\phi) = u(\hat{\phi})$$

This $\hat{u}$ is still in $\mathscr{S}'(\mathbb{R}^n)$ and the inversion rule holds.

Theorem 3.3. (Transform of a Gaussian function)
If A is a symmetric and non singular matrix in $M(\mathbb{C})$ the function $u(x) = exp(-\langle Ax, x\rangle 2)$ is in $\mathscr{S}'(\mathbb{R}^n)$ if and only if $\langle ReAx, x\rangle \geq 0$ and if $ReA \geq 0$ its Fourier transform is $\hat{u}(x) = (2\pi)^{\frac{n}{2}}(det(A^{-1}))^{\frac{1}{2}}exp(-\frac{\langle A^{-1}x,x\rangle}{2})$

Theorem 3.4. (Construction of the solution to the Kolmogorov equation)
This is the solution of the Kolmogorov equation
proof Let us assume that u is a solution and $\hat{u}$ his Fourier transform respect (x,y) exists and behaves well when $t \to 0$.
Transforming each term of the equation.

$$-\xi^2\hat{u} - \eta\frac{\partial \hat{u}}{\partial \xi} - \frac{\partial \hat{u}}{\partial t} = 0; \qquad t > 0.$$

We have that $d\hat{u} = -\xi^2 \hat{u} dt$ if $d\eta = 0$ and $d\xi = \eta dt$

$$\hat{u}(t, \xi + \eta t, \eta) = \hat{u}(0, \xi, \eta)\, exp \int_0^t -(\xi + \eta s)^2 ds$$
$$= \hat{u}(0, \xi, \eta)\, exp(-\xi^2 t - \xi \eta t^2 - \eta^2 \frac{t^3}{3})$$

The exponential on the right is relative to the matrix A^{-1}

$$A^{-1} = \begin{bmatrix} 2t & t^2 \\ t^2 & \frac{2}{3}t^3 \end{bmatrix} \qquad A = \begin{bmatrix} \frac{2}{t} & -\frac{3}{t^2} \\ -\frac{3}{t^2} & \frac{6}{t^3} \end{bmatrix}$$

If we assume $\hat{u}(0, \xi, \eta) = exp(-ix_0\xi - iy_0\eta)$ and invert the Fourier transform we get

$$u(t, x, y - tx) = 3^{\frac{1}{2}} \frac{1}{2\pi t^2} exp\big(-\frac{1}{t}(x - x_0)^2 + \frac{3}{t^2}(x - x_0)(y - y_0) - \frac{3}{t^3}(y - y_0)^2\big)$$

This function depends on x_0 and y_0, we should rewrite it in a more general form, showing all of the dependencies.

$$E(x, y, t; x_0, y_0) = 3^{\frac{1}{2}} \frac{1}{2\pi t^2} exp\big(-\frac{1}{t}(x - x_0)^2 + \frac{3}{t^2}(x - x_0)(y + tx - y_0) - \frac{3}{t^3}(y + tx - y_0)^2\big)$$

Remark. Conclusions.

Even if the operator does not have constant coefficients, it does commute with translations and we can say that E is a two sided fundamental solution of the Kolmogorov equation.

$$\mathscr{K} u = f;$$
$$u = \int_{s<t} E(t - s, x, y; x_0, y_0) f(s, x_0, y_0) dx_0 dy_0 ds.$$

The function $f \to u$ is a right and left inverse of the operator.

Bibliography

[1] Hörmander, Lars (1989),
The Analysis of linear partial differential operators I, Springer-Verlag, Lund