UNIVERSITÀ DEGLI STUDI DI PADOVA
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The Besicovitch-Federer Projection Theorem

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To my family, and my dearest loved ones.

Contents

Introduction

One of the most fundamental concepts in Geometric Measure Theory is *Rectifiability*. Given a curve in $\mathbb{R}^2$, for example a C^1 function $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$ with I interval, we say that it is rectifiable if this has finite length. The length is here defined as the supremum of the sum of segments' lengths with endpoints lying on the curve.
We now could try to generalize this concept to a larger family of subsets of $\mathbb{R}^n$. This is done by saying that a set is rectifiable if it can be approximated, in some sense, by rectifiable sets. Let us be more precise: let k be an integer with $0 < k < n$, then a set $E \subset \mathbb{R}^n$ is called *k-rectifiable* if there are at most countably many C^1 submanifolds of $\mathbb{R}^n$ Γ_i with dimension k such that

$$\mathcal{H}^k\left(E \setminus \left(\bigcup_i \Gamma_i\right)\right) = 0,$$

where $\mathcal{H}^k$ is the k-Hausdorff measure.

Hence a k-rectifiable set can be seen, not considering a set of null $\mathcal{H}^k$-measure, as a union of at most countably many C^1 submanifolds with dimension k in $\mathbb{R}^n$. It can be proved that, instead of taking C^1 submanifolds in the definition of rectifiability, one can take images of Lipschitz functions, and the two definitions are equivalent. This can be done because any Lipschitz function can be approximated by C^1 functions: given $f : \mathbb{R}^k \to \mathbb{R}^n$ Lipschitz, we can find for all $\varepsilon > 0$ a C^1 function $g : \mathbb{R}^k \to \mathbb{R}^n$ such that

$$\mathcal{H}^k(\{x \in \mathbb{R}^k \mid f(x) \neq g(x)\}) < \varepsilon.$$

The opposite concept of k-rectifiable sets are *purely k-unrectifiable* sets, which don't contain k-rectifiable sets of positive $\mathcal{H}^k$-measure. It can be proved that any set of finite measure can be decomposed in an "unique" way as the union of a rectifiable set and a purely unrectifiable set, see Theorem 4.3. Rectifiable sets could be more complicated than a simple curve of finite length in $\mathbb{R}^2$. For example, let $\mathbb{Q}^2 = \{q_i\}_{i=1,\dots}$, $[0, 2\pi] \cap \mathbb{Q} = \{\theta_i\}_{i=1,\dots}$ and let $[q_i]$ be the closed segment with midpoint q_i, length 2^{-i} and angle θ_i with respect to the x-axis; the set $E \subset \mathbb{R}^2$ defined as

$$E = \bigcup_i [q_i]$$

is 1-rectifiable, and $\mathcal{H}^1(E) < +\infty$ i.e. it has finite length. We can note that the set E we have defined does not have a tangent at any point, when instead rectifiable curves have it almost everywhere. Therefore we need to change also the notion of tangent in a point x of E rectifiable. Let

$$X(x, L, \theta) = \{y \in \mathbb{R}^2 \mid |P_{L^\perp}(y - x)| < \sin\theta|y - x|\}$$

which represents a two-sided cone in $\mathbb{R}^2$ with vertex x, and direction L. θ is the angle formed by the boundary of the cone with L. If E had a tangent L_x in the usual sense at x, one would see that for all $\theta \in (0, \pi/2)$, $E \cap B(x, r)$ is entirely contained in $X(x, L, \theta)$ for $r > 0$ small enough and for all θ. Therefore we may ask that a line L is "tangent" at $x \in E$ if $E \cap B(x, r)$ is mostly contained in $X(x, L, \theta)$ for r small, i.e.

$$\lim_{r \to 0} r^{-1}\mathcal{H}^1(E \cap B(x, r) \setminus X(x, L, \theta)) = 0. \tag{1}$$

With this new notion of "tangent" line, one could verify that for $\mathcal{H}^1$-almost all $x \in E$ the equation referred by (1) is verified. The opposite holds for purely 1-unrectifiable sets: at almost all of their points there is not a "tangent" line. We will give an example of a set with finite positive $\mathcal{H}^1$-measure that is purely 1-unrectifiable. A first example we can give is the set defined as follows: we take at first $B_0 = B(0, 1)$ and $r_0 = 1$. Then we define B_1 as the union of 4 disjoint balls inside B_0 of radius $r_1 = 1/4r_0$ and disposed as in Figure 1. Proceeding in this way for each ball at each iteration, we can define B_i as union of 4^i balls of radius 4^{-i}. Then

$$A := \bigcap_i B_i$$

is purely 1-unrectifiable.

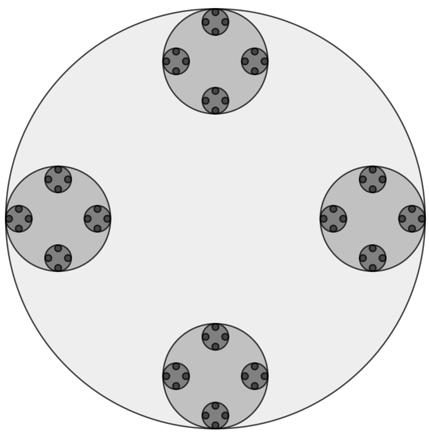

Figure 1: First 4 iterations of A

In this thesis our main objective is to prove the "Besicovitch-Federer projection theorem", which gives a characterization for k-rectifiable sets of $\mathbb{R}^n$ in terms of their orthogonal projections on k-dimensional subspaces. It is possible to define a measure $\gamma_{n,k}$ on the grassmanian of k-planes of $\mathbb{R}^n$ $G(n,k)$. We shall indicate with P_V the orthogonal projection on $V \in G(n,k)$.

Let $E \subset \mathbb{R}^n$ with $\mathcal{H}^k(E) < +\infty$. The Besicovitch-Federer Theorem states that:

1. E is k-rectifiable if and only if $\mathcal{H}^k(P_V B) > 0$ for each $B \subset E$ of positive measure and for $\gamma_{n,k}$-almost all $V \in G(n,k)$.

2. E is purely k-unrectifiable if and only if $\mathcal{H}^k(P_V E) = 0$ for $\gamma_{n,k}$-almost all $V \in G(n,k)$.

The two assertions are equivalent, and we will prove the following:

1. if E is k-rectifiable and with positive $\mathcal{H}^k$-measure then $\mathcal{H}^k(P_V E) > 0$ for $\gamma_{n,k}$-almost all $V \in G(n,k)$

2. if E is purely k-unrectifiable then $\mathcal{H}^k(P_{V^\perp} E) = 0$ for $\gamma_{n,n-k}$-almost all $V \in G(n, n-k)$.

To prove the first assertion we first need to show that E is k-rectifiable if and only if it is k-weakly linearly approximable (see definition 4.3); from this the first assertion will follow. The difficult part is to show that, if E is k-weakly linearly approximable, then E is k-rectifiable. We present here below a sketch of the proof:

1. We select a compact subset $F \subset E$ such that

$$0 < cr^k \leq \mathcal{H}^k(E \cap B(a,r)) \leq Cr^k < +\infty$$

   for $0 < r < r_0$ and $a \in F$, and where the conditions of definition 4.3 hold uniformly. Taking a smaller r, we can consider a ball $B(a,r)$ such that $r^{-k}\mathcal{H}^k((E \setminus F) \cap B(a,r))$ is small and $F \cap B(a,r)$ is close to a $W \in A(n,k)$, i.e. an affine k-plane with $a \in W$.

2. We shall assume, by contradiction, that the projection of $E \cap B(a,r)$ on some $V \in G(n,k)$ is small.

3. We will find many disjoint open cylinders C_i of radii $\rho_i \ll r$ and orthogonal to V such that the same cylinders, but with radii $5\rho_i$, are disjoint, such that $F \cap B(a,r) \cap C_i = \emptyset$ and such that $B(a,r) \cap \partial C_i$ contains a point e_i of F.

4. For some large $N > 0$, $E \cap B(e_i, N\rho_i)$ are approximated by a k-affine plane W_i. Since in $B(a,r) \cap C_i$ there is a little of E, W_i must be almost orthogonal to V. This will give us so many disjoint balls $B(x_{i,j}, \rho_i) \subset B(a,r)$ with $x_{i,j} \in F$ that $r^{-k}\mathcal{H}^k\left(E \cap \bigcup_{i,j} B(x_{i,j}, \rho_i)\right)$ will be much greater than C, and this will lead to a contradiction.

The second statement will be proved in this way:

1. Let $V \in G(n, n-k)$ be fixed; we will define $E_{1,\delta}(V), E_{2,\delta}(V), E_{3,\delta}(V) \subset E$ and show that $\mathcal{H}^k(P_{V^\perp}(E_{i,\delta}(V))) = 0$ for $i = 1, 2, 3$ and for all $V \in G(n, n-k)$.
2. We shall prove that $E = E_0 \cup E_{1,\delta}(V) \cup E_{2,\delta}(V) \cup E_{3,\delta}(V)$ with $\mathcal{H}^k(E_0) = 0$ for $\gamma_{n.n-k}$-almost all $V \in G(n, n-k)$. This will be the most difficult part of the proof.
3. Taking $V \in G(n, n-k)$ such that 2. holds for all $\delta = 1/i$ with $i = 1, 2, \ldots,$ we show that $\mathcal{H}^k(P_{V^\perp}(E)) = 0$.

A sort of generalization can be proved for the Besicovitch-Federer Theorem: as is showed in [H], given $E \subset \mathbb{R}^n$ purely k-unrectifiable then for all $f \in C^k(U, \mathbb{R}^k)$ with Jacobian of constant rank k exists $f_\varepsilon \in C^k(U, \mathbb{R}^k)$ with Jacobian of constant rank k such that

$$\|f - f_\varepsilon\|_{C^1} < \varepsilon$$

and

$$\mathcal{H}^k(f_\varepsilon(E)) = 0.$$

The thesis use [A] as main reference. The basic notions of the Measure Theory are presented in Chapter 1 . In Chapter 2 we discuss about the differentiation of measures, and we define a measure on $G(n, k)$, the set of all the k-dimensional subspaces of $\mathbb{R}^n$. In Chapter 3 we define the Hausdorff measures $\mathcal{H}^s$ and study some properties for Lipschitz functions and some density Theorems. In Chapter 4 we will introduce the concept of Rectifiability and prove some important properties. In Chapter 5 we shall prove the Besicovitch-Federer Theorem and to conclude we will use it to show some examples of 1-rectifiable and purely 1-unrectifiable sets in $\mathbb{R}^2$.

Chapter 1

Basic notions on measure theory

1.1 Measures

In this first Chapter we will introduce the basic notions of Measure Theory, and we will enunciate and prove some covering Theorems of Besicovitch and Vitali.

Definition 1.1. Let $X \neq \emptyset$ be a set. We will call $\mu : \mathcal{P}(X) \longrightarrow [0, +\infty]$ a measure on X if:

1. $\mu(\emptyset) = 0$
2. $\mu(A) \leq \mu(B)$ for all $A, B \in \mathcal{P}(X)$ such that $A \subseteq B$
3. $\mu(\bigcup_{i\in\mathbb{N}} A_i) \leq \sum_{i\in\mathbb{N}} \mu(A_i)$ for all $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{P}(X)$.

We shall call (X, μ) *measure space.*

The easiest examples of measures on a space X are

1. $\mu \equiv \mathbf{0}$ i.e. the null measure, which is for istance $\mathbf{0}(A) = 0 \ \forall A \subseteq X$.
2. The *Dirac* measure: let $x \in X$ be fixed, then $\delta_x : \mathcal{P}(X) \longrightarrow [0, +\infty]$ is defined for all $A \subseteq X$ as
$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$
3. The Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$ which is one of the most famous measure. It is defined as follows: let $\mathcal{R}$ be the family of n-rectangles; a n-rectangle is a set of the form $\prod_{i=1}^n (a_i, b_i)$. We define $V(A) := \prod_{i=1}^n |a_i - b_i|$ and then finally
$$\mathcal{L}^n(A) := \inf \Big\{ \sum_{i=1}^{+\infty} V(E_i) \Big| A \subseteq \bigcup_{i\in\mathbb{N}} E_i \ \text{ and } \ \{E_i\}_{i\in\mathbb{N}} \ \text{ are rectangles} \Big\}.$$

Definition 1.2. Let $\mathcal{A} \subseteq \mathcal{P}(X)$. $\mathcal{A}$ is a σ-algebra if:

1. $\emptyset \in \mathcal{A}$
2. $A \in \mathcal{A} \Longrightarrow A^c = X \setminus A \in \mathcal{A}$
3. $\{A_i\}_{i\in\mathbb{N}} \subset \mathcal{A} \Longrightarrow \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{A}$

Remark 1. If $\mathcal{A}$ is a σ-algebra of X, it can be easily proved that $\mathcal{A}$ is closed also under countably many intersection.

Proposition 1.1. *Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be closed under complement and finite union; a family like $\mathcal{B}$ is called an algebra. If $\mathcal{B}$ is closed also under countably union of pairwise disjoint sets, then $\mathcal{B}$ is a σ-algebra.*

Proof. $F_0 := A_0$ and $F_n := A_n \setminus (\bigcup_{i\leq n-1} A_i)$. Then $F_i \in \mathcal{B}$ for all $i \in \mathbb{N}$ and they are pairwise disjoint. So $\bigcup_{i\in\mathbb{N}} A_i = \bigcup_{i\in\mathbb{N}} F_i \in \mathcal{B}$ $\square$

Definition 1.3. We will say that $A \subseteq X$ is μ-measurable if

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A)$$

for all $E \subseteq X$.

Remark 2. Of course, $E \setminus A = E \cap A^c$. If we want to show that a set A is μ-measurable we just need to show that $\mu(E) \geq \mu(E \cap A) + \mu(E \setminus A)$ for all $E \subseteq X$ because the other inequality always holds.

The concept of "$\mu-$measurable" is important: we can see that in the definition of measure we want to have $\mu(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i)$ for all $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{P}(X)$ such that $A_i \bigcap A_j = \emptyset$ for $i \neq j$. This, with the definition of measure we provided, in general is false. But if the family $\{A_i\}_{i\in\mathbb{N}}$ is composed by μ-measurable disjoint sets, our previous equation holds. More precisely, the following theorem holds.

Theorem 1.2. *Let μ be a measure on X and let $\mathcal{M} \subset \mathcal{P}(X)$ be the set of all the μ-measurable sets. Then*

1. *$\mathcal{M}$ is a σ-algebra*
2. *If $A \subset X$ and $\mu(A) = 0$ then $A \in \mathcal{M}$*
3. *$\mu(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \mu(A_i)$ for all $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{M}$ which are pairwise disjoint*
4. *If $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{M}$ is such that $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{N}$ then*
$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i)$$
5. *If $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{M}$ is such that $A_i \supseteq A_{i+1}$ for all $i \in \mathbb{N}$ then*
$$\mu\left(\bigcap_{i\in\mathbb{N}} A_i\right) = \lim_{i\to\infty} \mu(A_i)$$
provided $\mu(A_j) < +\infty$ for some $j \in \mathbb{N}$.

Proof. 1. Let us prove that $\mathcal{M}$ is an algebra, and that it is closed under union of countably pairwise disjoint sets. Then by proposition 1.1 we conclude. If $A \in \mathcal{M}$ then

$$\mu(E) = \mu(E \cap A) + \mu(E \cap A^c) = \mu(E \cap A^c) + \mu(E \cap A)$$

and so $A^c \in \mathcal{M}$ ($(A^c)^c = A$). Let $A, B \in \mathcal{M}$, let us see that $A \cup B \in \mathcal{M}$: we have for all $E \subseteq X$,

$$\begin{aligned}\mu(E) &= \mu(E \cap A) + \mu(E \cap A^c) = \\ &= \mu(E \cap A \cap B) + \mu(E \cap A \cap B^c) + \mu(E \cap A^c \cap B) + \mu(E \cap A^c \cap B^c) \geq \\ &\geq \mu(E \cap (A \cup B)) + \mu(E \cap (A \cup B)^c)\end{aligned}$$

where the last inequality holds because $A^c \cap B^c = (A \cup B)^c$ and by the equality

$$E \cap (A \cup B) = (E \cap (A \cap B)) \cup (E \cap (A^c \cap B)) \cup (E \cap (A \cap B^c))$$

we get the last inequality by subadditivity (third property of measure, see definition 1.1). Therefore, by induction, we can say that $\mathcal{M}$ is closed under finite unions. Hence $\mathcal{M}$ is an algebra. Let now $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{M}$ be a family of pairwise disjoint sets and define the set $B = \bigcup_{i=0}^{+\infty} A_i$. We have that $A_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ as well as $B_n = \bigcup_{i=0}^{n} A_i$, therefore

$$\begin{aligned}\mu(E \cap B_n) &= \mu(E \cap B_n \cap A_n) + \mu(E \cap B_n \cap A_n^c) = \\ &= \mu(E \cap A_n) + \mu(E \cap B_{n-1}) = \mu(E \cap A_n) + \mu(E \cap A_{n-1}) + \mu(E \cap B_{n-2}) = \cdots\end{aligned}$$

and by induction we get

$$\sum_{i=0}^{n} \mu(E \cap A_i) = \mu(E \cap B_n).$$

Moreover, because $B^c \subseteq B_n^c$, we obtain that

$$\mu(E) = \mu(E \cap B_n) + \mu(E \cap B_n^c) \geq \sum_{i=0}^{n} \mu(E \cap A_i) + \mu(E \cap B^c)$$

and taking the limit as n tends to infinity, we obtain

$$\mu(E) \geq \sum_{i=0}^{+\infty} \mu(E \cap A_i) + \mu(E \cap B^c) \geq \mu(E \cap B) + \mu(E \cap B^c)$$

because $\mu(E \cap B) \leq \sum_{i=0}^{+\infty} \mu(E \cap A_i)$. This proves that $B \in \mathcal{M}$ and that $\mathcal{M}$ is a σ-algebra.
2. Let A such that $\mu(A) = 0$ and $E \subseteq X$; then $\mu(E \cap A) \leq \mu(A) = 0$ and

$$\mu(E) \leq \mu(E \cap A) + \mu(E \cap A^c) = \mu(E \cap A^c) \leq \mu(E).$$

3. By 1. , taking $E = B = \bigcup_{i=0}^{+\infty} A_i$

$$\mu\left(\bigcup_{i=0}^{+\infty} A_i\right) = \sum_{i=0}^{+\infty} \mu(B \cap A_i) + \mu(B \cap B^c) = \sum_{i=0}^{+\infty} \mu(A_i).$$

4. The limit exists because $\{\mu(A_i)\}_{i=0,\dots}$ is a monotone sequence of real numbers and we can suppose that $\mu(A_i) < +\infty$ (otherwise the conclusion holds trivially). Let $E_0 = \emptyset$, $E_i := A_i \setminus A_{i-1}$, then $\bigcup_{i=0}^{+\infty} E_i = \bigcup_{i=0}^{+\infty} A_i$ and $\sum_{i=0}^{n} \mu(E_i) = \sum_{i=1}^{n}(\mu(A_i) - \mu(A_{i-1})) = \mu(A_n)$, and so

$$\mu\left(\bigcup_{i=0}^{+\infty} A_i\right) = \sum_{i=0}^{+\infty} \mu(E_i) = \lim_{n\to+\infty} \sum_{i=0}^{n} \mu(E_i) = \lim_{n\to+\infty} \mu(A_n).$$

5. We can suppose $\mu(A_0) < +\infty$. Let $F_i = A_0 \setminus A_i$ for $i \in \mathbb{N}$. Then $F_i \subseteq F_{i+1}$ for all i; moreover $\mu(F_i) = \mu(A_0) - \mu(A_i)$, therefore

$$\mu\left(\bigcup_{i=0}^{+\infty} F_i\right) = \lim_{n\to+\infty} \sum_{i=0}^{n} \mu(F_i) = \mu(A_0) - \lim_{n\to+\infty} \mu(A_n)$$

and $\mu(\bigcup_{i=0}^{+\infty} F_i) = \mu(A_0 \setminus \bigcap_{i=1}^{+\infty} A_i) = \mu(A_0) - \mu(\bigcap_{i=1}^{+\infty} A_i)$; then

$$\mu\left(\bigcup_{i=0}^{+\infty} F_i\right) = \mu(A_0) - \mu\left(\bigcap_{i=1}^{+\infty} A_i\right) = \mu(A_0) - \lim_{n\to+\infty} \mu(A_n)$$

and we conclude that

$$\mu\left(\bigcap_{i=1}^{+\infty} A_i\right) = \lim_{n\to+\infty} \mu(A_n).$$

□

Remark 3. If A is a μ-measurable set with $\mu(A) < +\infty$ and $A \subset B$, then

$$\mu(B) = \mu(B \setminus A) + \mu(A)$$

and since $\mu(A) < +\infty$ we get $\mu(B \setminus A) = \mu(B) - \mu(A)$. In 4. and 5. of Theorem 1.2 we were allowed to use this formula.

Given $\mathcal{F} \subseteq \mathcal{P}(X)$, one can easily show that exists a σ-algebra that contains $\mathcal{F}$ and it is the smallest σ-algebra containing $\mathcal{F}$. If μ is a measure defined on a topological space, the family of the *Borel* sets is the smallest σ-algebra that contains the open sets of X, or, equivalently, the closed sets of X. This family will be indicated by $\mathcal{B}_X$, and its elements will be called *Borel sets*.

Definition 1.4. Let μ be a measure on a metric space X. We shall say that μ is

1. *locally finite* if $\forall x \in X$, $\exists r > 0$ such that $\mu(B(x,r)) < +\infty$.
2. a *Borel measure* if every Borel set is μ-measurable.
3. *regular* if $\forall A \in \mathcal{P}(X)$ exists B with $A \subseteq B$ μ-measurable such that $\mu(A) = \mu(B)$. If B is also a Borel set, then μ is *Borel regular*.

4. a *Radon measure* if it's a Borel measure and:
   (a) $\mu(K) < +\infty \; \forall K$ compact subset of X
   (b) $\mu(U) = \sup\{\mu(K) \mid K \subseteq U compact\}$ for all open subsets U.
   (c) $\mu(A) = \inf\{\mu(V) \mid A \subseteq V open\}$ for all $A \subseteq X$.

The Lebesgue measure $\mathcal{L}^n$ is a Radon measure on $\mathbb{R}^n$. In Theorem 1.2 in statements (4) and (5), $\{A_i\}_{i\in\mathbb{N}}$ do not need to be μ-measurable if μ is regular/Borel regular; let us briefly prove this. Given $\{A_i\}_{i\in\mathbb{N}}$ as in (4) of Theorem 1.2 (the sets are increasing) but not measurable, exists B_i such that $\mu(A_i) = \mu(B_i)$ for each i. Then we set $C_i := \bigcup_{k\geq i} B_k$; therefore $A_i \subset C_i$ and $\mu(A_i) = \mu(C_i)$. Using $\{C_i\}_{i\in\mathbb{N}}$ the reader can show the thesis. We recall that, given $A, B \subset X$ metric space with d metric, $d(A,B) = \inf\{d(a,b) \mid a \in Ab \in B\}$.

Theorem 1.3 (Carathéodory criterion)**.** *Let μ be a measure on X metric space. Then μ is a Borel measure if and only if*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for all $A, B \subset X$ such that $d(A,B) > 0$.

Proof. Let μ be a Borel measure. Let $A, B \subset X$ with $d(A,B) > 0$; then there is an open set U such that $A \subset U$ and $U \cap B = \emptyset$. Then

$$\mu(A \cup B) = \mu(A \cup B \setminus U) + \mu((A \cup B) \cap U) = \mu(A) + \mu(B).$$

Let us show the converse. Let $A \subseteq X$ and U and open set that contains A; define $A_0 = \emptyset$, $A_n := A \cap \{x \in U | d(x, U^c) \geq 1/n\}$; note that $A_i \subseteq A_{i+1}$ and the union of them gives A. We first show that $\mu(A) = \lim_{n\to+\infty} \mu(A_n)$. We define $D_{n+1} := A_{n+1} \setminus A_n$ so that $d(D_{2i}, D_{2(i-1)}) > 0$. Then we have

$$\mu(A_{2n+1}) \geq \sum_{i=0}^{n} \mu(D_{2i}) \qquad \mu(A_{2n}) \geq \sum_{i=1}^{n} \mu(D_{2i-1});$$

if one of the series is divergent the desired conclusion is trivial, otherwise we can conclude by this relation

$$\mu(A) \leq \mu(A_{2n}) + \sum_{i=n}^{+\infty} \mu(D_{2i}) + \sum_{i=n+1}^{+\infty} \mu(D_{2i-1})$$

and by the fact that $\mu(A_n) \leq \mu(A)$. So $\mu(A) = \lim_{n\to+\infty} \mu(A_n)$. Let us take $E \subseteq X$, and U open subset. Let $A = E \cap U$, so $d(A_n, E \setminus A) > 0$ and we have

$$\mu(E) \geq \mu(A_n \cup (E \setminus A) = \mu(A_n) + \mu(E \setminus A).$$

Taking the limit as n tends to infinity we get the μ-measurability of U open, and μ is a Borel measure. ☐

This criterion will be useful to show in particular that $\mathcal{H}^s$, the s-Hausdorff measure on $\mathbb{R}^n$, is a Borel measure. Given a measure μ on X we can define for any $A \subseteq X$ a new measure restricting μ to A:

Definition 1.5. For μ measure on X and $A \subseteq X$ we define a measure $\mu \llcorner A$ by

$$(\mu \llcorner A)(B) = \mu(A \cap B)$$

for $B \subseteq X$. $\mu \llcorner A$ is called the restriction of μ to A.

μ and $\mu \llcorner A$ are related as we can imagine, as the next proposition tells us:

Proposition 1.4. *Let μ be a measure on X, $A \subset X$. Then:*

1. *every μ-measurable set is $\mu \llcorner A$-measurable.*
2. *if μ is Borel regular and $\mu(A) < +\infty$ with A μ-measurable then $\mu \llcorner A$ is Borel regular.*

Proof. The first statement is easy to show, and we leave it as an exercise. Let μ Borel regular, then exist B Borel set such that contains A and $\mu(A) = \mu(B)$. Let now $C \subset X$ and let D be a Borel set such that $\mu(B \cap C) = \mu(D)$ and $B \cap C \subset D$. Now we set $E = D \cup (X \setminus B)$ which contains C. So

$$(\mu \llcorner A)(E) \leq \mu(B \cap E) = \mu(B \cap D) \leq \mu(D) = \mu(B \cap C) = \mu(A \cap C) = (\mu \llcorner A)(C)$$

and so $(\mu \llcorner A)(C) = (\mu \llcorner A)(E)$ and $\mu \llcorner A$ is Borel regular. $\square$

Let us point out that the measurability of A in the last proof is used in the equation $\mu(B \cap C) = \mu(A \cap C)$. In fact,

$$\mu(B \cap C) = \mu((B \cap C) \setminus A) + \mu(B \cap C \cap A) \leq \mu(B \setminus A) + \mu(A \cap C) = \mu(A \cap C).$$

Next we shall enunciate an approximation theorem which will be useful for us.

Theorem 1.5. *Let μ be a Borel regular measure on X, $A \subset X$ a μ-measurable set and $\epsilon > 0$. Then:*

1. *If $\mu(A) < +\infty$ exists a closed set $C \subset A$ with $\mu(A \setminus C) < \epsilon$.*
2. *If exist $\{U_i\}_{i \in \mathbb{N}}$ open sets such that $A \subseteq \bigcup_{i \in \mathbb{N}} U_i$ and $\mu(U_i) < +\infty$ for all $i \in \mathbb{N}$, then exists U open such that $A \subset U$ and $\mu(U \setminus A) < \epsilon$.*

Remark 4. When $X = \mathbb{R}^n$ and $\mu = \mathcal{L}^n$, in the first statement C can be taken compact, because every closed subset of R^n can be written as countable union of compact sets. In 2. if X is a separable metric space and μ is locally finite, every subset can be covered by countably many open balls of finite measure, therefore the hypotheses are verified.

The proof of the following corollary follows from Theorem 1.5 and by the last remark and is left as exercise.

Corollary 1.6. *A measure on $\mathbb{R}^n$ is a Radon measure if and only if it is locally finite and Borel regular.*

1.2 Integrals

In this section we shall give the basic notions of integral on a set X with a measure μ; the details are explained in many text books. Given a measure μ on X and a suitable $f : X \longrightarrow \mathbb{R}$ we can define the integral of f on X with respect to μ whose notation is

$$\int_X f(x)\, d\mu(x) = \int f(x)\, d\mu(x).$$

In order to define it however, we need to introduce some further notions.

Definition 1.6. Let (X, μ) be measure space. We'll say that $f : X \longrightarrow \mathbb{R}$ is a measurable function if $\overleftarrow{f}(B)$ is μ-measurable for all $B \in \mathcal{B}_X$.

In the definition of measurable function, B can be replaced by $(-\infty, a)$, $(a, +\infty)$, (a, b) and their closure; this is true because $\mathcal{B}_{\mathbb{R}}$ is generated by those sets. So for example, f is measurable if and only if $\overleftarrow{f}((-\infty, a))$ for all $a \in \mathbb{R}$. If $\overleftarrow{f}((-\infty, a))$ is a Borel set for all $a \in \mathbb{R}$ we will say that f is a Borel function.

Proposition 1.7. *If $f : X \longrightarrow \overline{\mathbb{R}}$ is upper semicontinuous, which means*

$$\limsup_{y \to x} f(y) \leq f(x),$$

then f is a Borel function. The same holds for lower semicontinuous functions: $\liminf_{y \to x} f(y) \geq f(x)$.

Proof. We show that $A = \{x \in X | f(x) < t\}$ is open. By hypothesis, $\forall \varepsilon > 0$ exists δ such that if $|x - y| < \delta \Longrightarrow f(y) \leq \limsup_{y \to x} f(y) + \varepsilon \leq f(x) + \varepsilon$. Then choosing $\varepsilon = t - f(x)$ for $x \in A$ we find δ such that $|x - y| < \delta \Longrightarrow f(y) \leq f(x) + \varepsilon = t \Longleftrightarrow B(x, \delta) \subset A$. $\square$

Corollary 1.8. *If $f : X \longrightarrow \overline{\mathbb{R}}$ is a continuous functions then it is a Borel function.*

Proposition 1.9. *Let $f, g : X \longrightarrow \overline{\mathbb{R}}$ be measurable functions. Then $f + g$ is measurable and, provided that $g(x) \neq 0 \forall x \in X$, f/g is measurable.*

Proposition 1.10. *Let $\{f_j\}_{j \in \mathbb{N}}$ be a sequence of measurable functions $f_j : X \longrightarrow \overline{\mathbb{R}}$. Then*

$$F_1(x) = \sup_{j \in \mathbb{N}} f_j(x) \qquad F_3(x) = \limsup_{j \to +\infty} f_j(x)$$
$$F_2(x) = \inf_{j \in \mathbb{N}} f_j(x) \qquad F_4(x) = \liminf_{j \to +\infty} f_j(x)$$

are measurable functions.

Proposition 1.11. *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a measurable function. Then f^+ and f^- are measurable. It follows that $|f|$ is measurable.*

One of the simplest example of function we can give is the characteristic function of a subset A of X, defined as

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Functions like this are measurable if and only if A is measurable. Then a *simple* function is a finite sum of characteristic functions of disjoint measurable sets, for instance

$$\varphi = \sum_{i=1}^{n} c_i \chi_{E_i}$$

where $\{E_i\}_{i\in\mathbb{N}}$ are pairwise disjoint measurable sets and $c_i \geq 0$. We can define for positive simple functions

$$\int_X \varphi(x)\, d\mu(x) := \int_X \sum_{i=1}^{n} c_i \chi_{E_i}(x)\, d\mu(x) = \sum_{i=1}^{n} c_i \mu(E_i).$$

For a general simple function $g : X \longrightarrow \overline{\mathbb{R}}$ we say that is "integrable" if either $\int_X g^+\, d\mu < +\infty$ or $\int_X g^-\, d\mu < +\infty$ and

$$\int_X g\, d\mu(x) := \int_X g^+\, d\mu - \int_X g^-\, d\mu.$$

Definition 1.7. Let $f : X \longrightarrow \overline{\mathbb{R}}$. The *upper integral* of f is

$$\int^* f(x)\, d\mu(x) := \inf \left\{ \int_X g\, d\mu(x) \mid g \geq f \text{ with g simple and integrable} \right\}.$$

The *lower integral*:

$$\int_* f(x)\, d\mu(x) := \sup \left\{ \int_X g\, d\mu(x) \mid g \leq f \text{ with g simple and integrable} \right\}.$$

If upper and lower integral coincide we say that f is integrable and

$$\int_X f(x)\, d\mu(x) = \int^* f(x)\, d\mu(x) = \int_* f(x)\, d\mu(x).$$

If f is a positive measurable function then it is integrable. For more properties and details on the argument the reader is referred to [C], or [B]. We shall make vast use of the following results.

Theorem 1.12 (Fubini)**.** *Let X, Y be separable metric spaces and μ, ν locally finite Borel measures on X, Y respectively. Let $f : X \times Y \longrightarrow \mathbb{R}_{\geq 0}$ be a Borel function. Then*

$$\int\int f(x,y)\, d\mu(x)\, d\nu(y) = \int\int f(x,y)\, d\nu(y)\, d\mu(x).$$

Given A a Borel set of $X \times Y$, $A_y := \{x \in X \mid (x,y) \in A\}$ is μ-measurable, so χ_{A_y} is a measurable function on X. The same holds for $A^x := \{y \in Y \mid (x,y) \in A\}$. Then by Fubini theorem we get

$$\int_Y \int_X \chi_{A_y}(x)\, d\mu(x)\, d\nu(y) = \int_X \int_Y \chi_{A_y}(x)\, d\nu(y)\, d\mu(x) = \\ = \int_X \int_Y \chi_{A^x}(y)\, d\nu(y)\, d\mu(x) = \int_X \int_Y \chi_{A^x}(y)\, d\nu(y)\, d\mu(x)$$

which means that

$$\int_Y \mu(\{x \in X \mid (x,y) \in A\})\, d\nu(y) = \int_X \nu(\{y \in Y \mid (x,y) \in A\})\, d\mu(y).$$

We are now ready to prove a useful formula:

Proposition 1.13. *Let μ be a Borel measure on X separable metric space and $f : X \longrightarrow \mathbb{R}_{\geq 0}$ a Borel function. Then*

$$\int_X f(x)\, d\mu(x) = \int_0^{+\infty} \mu(\{x \in X \mid f(x) \geq t\})\, dt.$$

Proof. The proof is quite simple: let $A_t := \{(x,t) \mid f(x) \geq t\}$, then

$$\int_0^{+\infty} \mu(\{x \in X \mid f(x) \geq t\})\, dt = \int_0^{+\infty} \mu(\{x \in X \mid (x,t) \in A\})\, dt = \\ = \int_X \mathcal{L}^1(\{t \in [0,+\infty) \mid (x,t) \in A\})\, dt = \int_X \mathcal{L}^1([0, f(x)])\, d\mu(x) = \\ = \int_X f(x)\, d\mu(x)$$

$\square$

We shall now introduce the notion of *image measures*:

Definition 1.8. Let μ be a measure on X and $f : X \longrightarrow Y$ a map. Then the image of μ under f, $f_\sharp\mu$, is the measure on Y defined by

$$f_\sharp\mu(A) := \mu(\overleftarrow{f}(A))$$

for all $A \subseteq Y$.

A set $A \subseteq Y$ is $f_\sharp\mu$-measurable if $\overleftarrow{f}(A)$ is μ-measurable: let $F \subseteq Y$, then we get

$$f_\sharp\mu(F) = \mu(\overleftarrow{f}(A \cap F)) + \mu(\overleftarrow{f}(F \setminus A)) = f_\sharp\mu(A \cap F) + f_\sharp\mu(F \setminus A).$$

Then if f is a Borel function and μ a Borel measure, them $f_\sharp\mu$ is a Borel measure on Y.

Definition 1.9. Let μ be a Borel measure on X metric space. We call the *support* of μ the smallest closed set S such that $\mu(X \setminus S) = 0$ and we indicate it with $spt\mu$. For instance

$$spt\mu = X \setminus \{x \in X \mid \exists r > 0 \ \ s.t \ \ \mu(B(x,r)) = 0\}$$

A measure could have the whole space as support as is the case of the Lebesgue measure in $\mathbb{R}^n$; let's give another example: let X be a separable metric space and $F \subset X$ a countable set such that $\overline{F} = X$. Let's say $F = \{f_i\}_{i \geq 1}$. Then

$$\mu := 6/\pi^2 \sum_{i=1}^{+\infty} 1/i^2 \delta_{f_i}$$

is a measure on X (easy exercise) whose support is X. If $spt\mu$ is compact then we'll say that μ has compact support.

Theorem 1.14. *Let X, Y be separable metric spaces, $f : X \longrightarrow Y$ be a continuous map, and μ be a Radon measure on X with compact support. Then $f_\sharp \mu$ is a Radon measure on Y with compact support and $spt(f_\sharp \mu) = f(spt\mu)$.*

Proposition 1.15. *Suppose $f : X \mapsto Y$ is a Borel function, μ a Borel measure and g a non negative Borel function. Then*

$$\int_Y g \, df_\sharp \mu = \int_X g \circ f \, d\mu.$$

1.3 Covering Theorems

In this section we will prove two fundamental covering theorems by Vitali and Besicovitch. Let X be a metric space; we shall denote with $B(x,r)$ the closed ball of centre x and radius r, with $U(x,r)$ the open one.

Recall that $d(B)$ denotes the diameter of B ball, for instance, $d(B) := \sup_{x,y \in B \ x \neq y} d(x,y)$. Let us also notice that, in a generic metric space X, it is not always true that $d(B(x,r)) = 2r$. If $B := B(x,r)$ then $5B := B(x,5r)$. In general we can set

$$5B := \bigcup \{C \mid C \text{ is a closed ball with } C \cap B \neq \emptyset \text{ and } d(C) \leq 2d(B)\}.$$

Theorem 1.16. *Let X be a boundedly compact metric space and $\mathcal{B}$ a family of closed balls in X such that*

$$\sup\{d(B) \mid B \in \mathcal{B}\} < +\infty.$$

Then there is a countable (or finite) sequence $\{B_i\}_{i \in \mathbb{N}} \subset \mathcal{B}$ of disjoint balls such that

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{i \in \mathbb{N}} 5B_i.$$

Proof. Let $C := \{x \in X \mid B(x,r) \in \mathcal{B} \ \exists r\}$ the set of centres. Then, for each $x \in C$ we take $r(x)$ such that

$$r(x) > \frac{14}{15}\sup\{r \mid B(x,r) \in \mathcal{B}\}.$$

Then, from now on, we will work on balls of $\mathcal{B}$ of the form $B(x, r(x))$ for $x \in C$. Let $c \in C$ be fixed and let

$$M = \sup\{r(x) \mid x \in C\}$$

and

$$C_1 := \left\{x \in C \mid \frac{3}{4}M < r(x) \leq M\right\}.$$

M is finite because $r(x) \leq d(B(x,r(x)))$ and $M \leq \sup\{d(B) \mid B \in \mathcal{B}\} < +\infty$. Now choose $x_1 \in C_1$ arbitrarily and set $l(x_1) := 2d(x_1, c)$. Choose $x_2 \in C_1 \cap B(c, l(x_1)) \setminus B(x_1, \frac{8}{3}r(x_1))$, and then by induction

$$x_k \in C_1 \cap B(c, l(x_1)) \setminus \bigcup_{i=1}^{k-1} B\left(x_i, \frac{8}{3}r(x_i)\right).$$

The balls $B(x_i, r(x_i))$ are disjoint: let $x \in B(x_i, r(x_i)) \cap B(x_j, r(x_j))$ with $j > i$; then $d(x_j, x) \leq r(x_j)$ and $d(x_i, x) \leq r(x_i)$ but $d(x_j, x_i) > \frac{8}{3}r(x_i)$. It follows that

$$\frac{8}{3}r(x_i) < d(x_j, x_i) \leq r(x_j) + r(x_i)$$

which implies $\frac{5}{3}r(x_i) < r(x_j) \Longrightarrow \frac{5}{3}M < r(x_j)$ contradiction. The centres of these balls are in $B(c, l(x_1))$ which is compact, then the process must stop for a certain index k_1 because a compact set cannot be filled with infinitely many disjoint balls of radius greater than $\frac{3}{4}M$. So we get

$$C_1 \cap B(c, l(x_1)) \subset \bigcup_{i=1}^{k_1} B\left(x_i, \frac{8}{3}r(x_i)\right).$$

Now, let $C_{1,1} := \{x \in C_1 \mid B(x, r(x)) \cap B(x_i, r(x_i)) = \emptyset \ \forall i = 1, \cdots, k_1\}$ and take $x_{k_1+1} \in C_{1,1} \setminus B(c, l(x_1))$; if such x_{k_1+1} does not exist then $\forall x \in C_1 \setminus B(c, l(x_1))$ exists i such that $B(x, r(x)) \cap B(x_i, r(x_i)) \neq \emptyset$, then, since $r(x) < 5/4M < 5/3r(x_i)$,

$$d(x, x_i) \leq r(x) + r(x_i) < \frac{8}{3}r(x_i)$$

and $x \in \bigcup_{i=1}^{k_1} B\left(x_i, \frac{8}{3}r(x_i)\right)$ and we can cover C_1 with the disjoint balls we found (note that $r(x) \leq 5/4M = (5/3)(3/4)M < 5/3r(x_i)$). So, as we did, we found $k_2 - k_1$ balls such that

$$C_{1,1} \cap (B(c, l(x_{k_1+1})) \setminus B(c, l(x_1))) \subseteq \bigcup_{j=k_1+1}^{k_2} B\left(x_j, \frac{8}{3}r(x_j)\right)$$

and so

$$C_1 \cap B\left(c, l(x_{k_1+1})\right) \subset \bigcup_{i=1}^{k_2} B\left(x_i, \frac{8}{3} r(x_i)\right).$$

Proceeding in this way we find a countable, or finite, collection of disjoint balls $\{B\left(x_i, r(x_i)\right)\}_{i \in \mathcal{I}_1}$ such that

$$C_1 \subset \bigcup_{i \in \mathcal{I}_1} B\left(x_i, \frac{8}{3} r(x_i)\right).$$

Because $r(x) < \frac{3}{2}M < 2r(x_i)$ for all $x \in C_1$ we get

$$\bigcup_{x \in C_1} B\left(x, r(x)\right) \subset \bigcup_{i \in \mathcal{I}_1} B\left(x_i, \frac{14}{3} r(x_i)\right).$$

Now let

$$C_2 = \left\{ x \in C \mid \left(\frac{3}{4}\right)^2 M < r(x) \leq \frac{3}{4} M \right\}$$

and

$$C_2^{'} := \{x \in C_2 \mid B\left(x, r(x)\right) \cap B\left(x_i, r(x_i)\right) = \emptyset \ \ \forall i \in \mathcal{I}_1\}.$$

If $x \in C_2 \setminus C_2^{'}$ then for some $i \in \mathcal{I}_1$ $B\left(x, r(x)\right) \cap B\left(x_i, r(x_i)\right) \neq \emptyset$ and so

$$d(x, x_i) \leq r(x) + r(x_i) < \frac{8}{3} r(x_i)$$

because $r(x) < (5/4)M < (5/3)r(x_i)$; then

$$C_2 \setminus C_2^{'} \subset \bigcup_{i \in \mathcal{I}_1} B\left(x_i, \frac{8}{3} r(x_i)\right). \tag{1.1}$$

Then we can work on $C_2^{'}$ as we did on C_1 and find a countable, or finite, family of disjoint balls such that

$$C_2^{'} \subset \bigcup_{i \in \mathcal{I}_2} B\left(x_i, \frac{8}{3} r(x_i)\right) \tag{1.2}$$

and so, combining 1.1 and 1.2 we find that

$$C_2 \subset \bigcup_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} B\left(x_i, \frac{8}{3} r(x_i)\right).$$

Let be $\mathcal{I} := \bigcup_{h=1}^{+\infty} \mathcal{I}_h$, then

$$\bigcup_{x \in C} B\left(x, r(x)\right) \subset \bigcup_{i \in \mathcal{I}} B\left(x_i, \frac{14}{3} r(x_i)\right). \tag{1.3}$$

Let us call $S(x) := \sup\{r \mid B(x,r) \in \mathcal{B}\}$; then we have that

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{x \in C} B(x, S(x))$$

and by the inclusion 1.3 we get

$$\bigcup_{x \in C} B\left(x, \frac{14}{15}S(x)\right) \subset \bigcup_{x \in C} B(x, r(x)) \subset \bigcup_{i \in \mathcal{I}} B\left(x_i, \frac{14}{3}r(x_i)\right)$$

and scaling by a factor of $\frac{15}{14}$ we get exactly the thesis

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{x \in C} B(x, S(x)) \subset \bigcup_{i \in \mathcal{I}} B(x_i, 5r(x_i))$$

and the proof is complete. □

At this point we can prove Vitali's covering theorem for the Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$.

Definition 1.10. Let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$. $\mathcal{B}$ is a *fine cover* of a set $A \subset \mathbb{R}^n$ if

$$A \subseteq \bigcup_{B \in \mathcal{B}} B$$

and $\forall x$ which is a centre of some $B \in \mathcal{B}$

$$\inf\{d(B) \mid x \in B \text{ and } B \in \mathcal{B}\} = 0.$$

Theorem 1.17. *Let $A \subset \mathbb{R}^n$ and let $\mathcal{B}$ be a fine cover of A. Then there are disjoint balls $B_i \in \mathcal{B}$ such that*

$$\mathcal{L}^n\left(A \setminus \bigcup_i B_i\right) = 0.$$

Moreover, given $\varepsilon > 0$ the balls can be chosen such that $\sum_{i=1}^{+\infty} \mathcal{L}^n(B_i) \leq \mathcal{L}^n(A) + \varepsilon$.

Proof. We can assume A to be bounded. In fact, if we prove the theorem in this case, then noting that $\mathbb{R}^n = \bigcup_i \overline{Q_i}$ where $\{Q_i\}_i$ are disjoint open cubes and that $\mathcal{L}^n(A \setminus \bigcup_i Q_i) = 0$, we can apply the theorem on $A \cap Q_i$ which is bounded and conclude. A is bounded, so $\mathcal{L}^n(A) < +\infty$ and we can choose an open set such that $\mathcal{L}^n(U) \leq (1 + 7^{-n})\mathcal{L}^n(A)$. Now, considering the subfamily of closed balls of $\mathcal{B}$ which are contained in U, we can apply Theorem 1.16 and find countably many disjoint balls $B_i = B(x_i, r_i) \in \mathcal{B}$ such that $B_i \subset U$ and

$$A \subset \bigcup_i B(x_i, 5r_i).$$

Then $5^{-n}\mathcal{L}^n(A) \leq 5^{-n}\sum_i \mathcal{L}^n(B(x_i, 5r_i)) = \sum_i \mathcal{L}^n(B(x_i, r_i))$, and we can take k_1 such that

$$6^{-n}\mathcal{L}^n(A) \leq \sum_{i=1}^{k_1} \mathcal{L}^n(B_i).$$

Let $A_1 := A \setminus \bigcup_{i=1}^{k_1} B_i$; we have that

$$\mathcal{L}^n(A_1) \leq \mathcal{L}^n\left(U \setminus \bigcup_{i=1}^{k_1} B_i\right) = \mathcal{L}^n(U) - \sum_{i=1}^{k_1} \mathcal{L}^n(B_i) \leq (1 + 7^{-n} - 6^{-n})\mathcal{L}^n(A) = \varepsilon \mathcal{L}^n(A)$$

where we have set $\varepsilon = 1 + 7^{-n} - 6^{-n} < 1$. Now we work on A_1 which is contained in the open set $\mathbb{R}^n \setminus \bigcup_{i=1}^{k_1} B_i$. We can find an open set $U_1 \subset \mathbb{R}^n \setminus \bigcup_{i=1}^{k_1} B_i$ such that, like before, $\mathcal{L}^n(U_1) \leq (1 + 7^{-n})\mathcal{L}^n(A_1)$ and again applying theorem 1.16 we get

$$\mathcal{L}^n(A_2) \leq \varepsilon \mathcal{L}^n(A_1) \leq \varepsilon^2 \mathcal{L}^n(A)$$

where $A_2 = A_1 \setminus \bigcup_{i=k_1+1}^{k_2} B_i = A \setminus \bigcup_{i=1}^{k_2} B_i$ and all the balls are disjoint. Now after q steps we get

$$\mathcal{L}^n\left(A \setminus \bigcup_{i=1}^{k_q} B_i\right) \leq \varepsilon^q \mathcal{L}^n(A)$$

and since $\varepsilon < 1$ we can find the required disjoint balls $\{B_i\}_{i \in \mathbb{N}}$ such that

$$\mathcal{L}^n\left(A \setminus \bigcup_i B_i\right) = 0.$$

The last assertion follows from the proof we presented. $\square$

If instead of $\mathcal{L}^n$ we had a general Radon measure on $\mathbb{R}^n$, the theorem we proved could not be valid anymore. Take for example μ a Radon measure on $\mathbb{R}^n$ defined as follow:

$$\mu(A) = \mathcal{L}^1(\{x \in \mathbb{R} \mid (x, 0) \in A\})$$

then the family $\mathcal{B}\{B((x, y), y) \mid x \in \mathbb{R} \text{ and } 0 < y < +\infty\}$ covers $A = \{(x, 0) \mid x \in \mathbb{R}\}$ but

$$\mu\left(A \cap \bigcup_{i=1}^{+\infty} B_i\right) = 0$$

for any countable subcollection of $\mathcal{B}$. But if we assume that each point of A is the centre for a certain closed ball of $\mathcal{B}$ fine cover of A, then the result holds. To prove this we need another covering theorem.

Theorem 1.18 (Besicovitch covering Theorem)**.** *Let $A \subset \mathbb{R}^n$ be bounded, and $\mathcal{B}$ a family of closed balls such that each point of A is the centre of some ball in $\mathcal{B}$. Then exist $P(n)$ and $Q(n)$ constants depending only on n such that:*

1. *there is a countable, or finite, collection of balls $B_i \in \mathcal{B}$ such that they cover A and each $x \in \mathbb{R}^n$ belongs to at most $P(n)$ balls B_i; more precisely,*

$$\chi_A \leq \sum_i \chi_{B_i} \leq P(n)$$

2. *there exist* $\mathcal{B}_1, \cdots, \mathcal{B}_{Q(n)}$ *subfamilies of* $\mathcal{B}$ *such that* $\mathcal{B}_i$ *is formed by disjoint balls and*

$$A \subset \bigcup_{i=1}^{Q(n)} \bigcup_{B \in \mathcal{B}_i} B.$$

In order to prove this we need the following two lemmas.

Lemma 1.19. *Let* $a, b \in \mathbb{R}^2$*,* $0 < |a| < |a - b|$ *and* $0 < |b| < |a - b|$*. Then*

$$\left| \frac{a}{|a|} - \frac{b}{|b|} \right| > 1$$

Proof. We can see that $a \notin B(b, |b|)$ and $b \notin B(a, |a|)$. Then one can see that the angle α is grater than $\pi/3$ (see figure 1.1). Then calling $c' = \left| \frac{a}{|a|} - \frac{b}{|b|} \right|$, $a' = \frac{a}{|a|}$ and $b' = \frac{b}{|b|}$ we get

$$(c')^2 = 2 - 2\cos(\alpha) > 1.$$

If the reader is not happy with this proof, there's another way to prove it; we will now indicate the length of the vectors $a, b, a - b$ with a, b, c respectively; let α be the angle between a, b and let us suppose that $\cos \alpha \geq 1/2$. Then $c^2 = a^2 + b^2 - 2ab \cos \alpha \leq a^2 + b^2 - ab$ and since $a^2 < c^2$ and $b^2 < c^2$ we get that $a(a - b) > 0$ and $b(b - a) > 0$, a contradiction. Therefore $\cos \alpha < 1/2$ and $\alpha > \pi/3$. $\square$

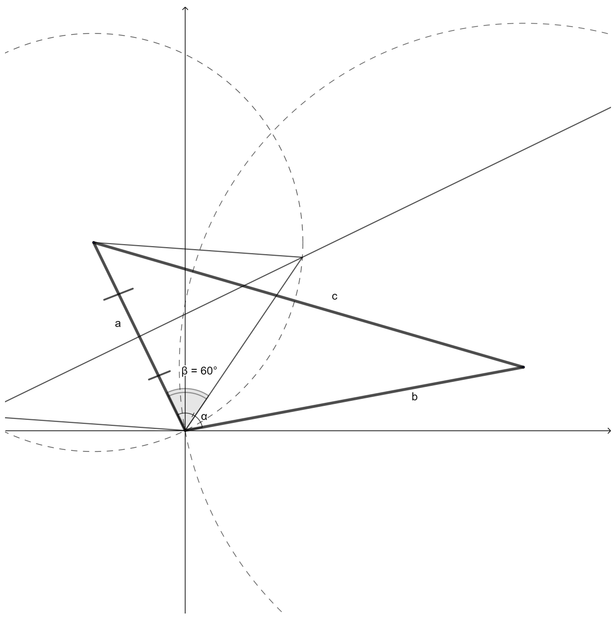

Figure 1.1: Geometric proof of lemma 1.19

Lemma 1.20. *There exist a positive integer $N(n)$ depending only on n with this property: let $\{a_i\}_{i=1,\cdots,k} \subset \mathbb{R}^n$ and $\{r_i\}_{i=1,\cdots,k}$ such that*

$$a_i \notin B(a_j, r_j) \text{ for } j \neq i \quad \text{and} \quad \bigcap_{i=1}^{k} B(a_i, r_i) \neq \emptyset,$$

then $k \leq N(n)$.

Proof. We can assume that

$$0 \in \bigcap_{i=1}^{k} B(a_i, r_i)$$

and so $a_i \neq 0$ for all i. Then

$$|a_i| \leq r_i < |a_i - a_j| \text{ for } i \neq j.$$

Working in the 2-dimensional plane containing $0, a_i, a_j$ we have by Lemma 1.19 that

$$\left| \frac{a_i}{|a_i|} - \frac{a_j}{|a_j|} \right| > 1 \quad \text{for} i \neq j.$$

Since S^{n-1} is compact there is a number $N(n)$ such that if $x_1, \cdots, x_k \in S^{n-1}$ and $|x_i - x_j| > 1$ for $i \neq j$, then $k \leq N(n)$. That $N(n)$ is the integer we were looking for. $\square$

Proof. (of Theorem 1.18) By hypothesis, for each $x \in A$ we pick a ball $B\left(x, r(x)\right] \in \mathcal{B}$. Then, since A is bounded, we can suppose that

$$M_1 = \sup_{x \in A} r(x) < +\infty$$

otherwise we conclude taking $P(n) = Q(n) = 1$. We can then choose $x_1 \in A$ such that $r(x_1) \geq M_1/2$ and inductively

$$x_k \in A \setminus \bigcup_{i=1}^{k-1} B\left(x_i, r(x_i)\right] \text{ with } r(x_k) \geq M_1/2.$$

Since A is bounded, the process terminates after k_1 iterations. Let now

$$M_2 = \sup \left\{ r(x) \mid x \in A \setminus \bigcup_{i=1}^{k_1} B\left(x_i, r(x_i)\right) \right\}$$

and choose

$$x_{k_1+1} \in A \setminus \bigcup_{i=1}^{k_1} B\left(x_i, r(x_i)\right) \text{ with } r(x_{k_1+1}) \geq M_2/2.$$

Again, inductively

$$x_{k_1+l} \in A \setminus \bigcup_{i=1}^{k_1+l-1} B\left(x_i, r(x_i)\right) \text{ with } r(x_{k_1+l}) \geq M_2/2.$$

Following this process, we obtain a sequence of numbers $k_1 < k_2 < \cdots$ and a decreasing sequence of numbers $2M_{i+1} \leq M_i$. In fact, each point taken in $A \setminus \bigcup_{i=1}^{k_1} B\left(x_i, r(x_i)\right)$ has radius less than $M_i/2$. Therefore taking the supremum for all the centres in $A \setminus \bigcup_{i=1}^{k_i} B\left(x_i, r(x_i)\right)$ we get $M_{i+1} \leq M_i/2$. Moreover, we get a sequence of balls $B_i = B\left(x_i, r(x_i)\right) \in \mathcal{B}$ such that the following properties hold:
let $\mathcal{I}_j = \{k_{j-1}+1, \cdots, k_j\}$ with $j \geq 1$ (and $k_0 = 0$) then

$$M_j/2 \leq r(x_j) \leq M_j \text{ for } i \in \mathcal{I}_j$$

$$x_{j+1} \in A \setminus \bigcup_{i=1}^{j} B_i \text{ for } j \geq 1$$

$$x_i \in A \setminus \bigcup_{m \neq k} \bigcup_{j \in \mathcal{I}_m} B_j \text{ for } i \in \mathcal{I}_k.$$

The first two are trivial by construction. For the third: let k be fixed , $m \neq k$, $j \in \mathcal{I}_m$ and $i \in \mathcal{I}_k$. Then, either $m < k$, or $m > k$. In the first case we have $x_i \notin B_j$ by construction (or by 2.), in the second case we have $r(x_j) \leq r(x_i)$, $x_j \notin B_i$ and so $x_i \notin B_j$ $(d(x_i, x_j) > r(x_i) \geq r(x_j))$. Since $\lim_{i \to +\infty} M_i = 0$, it follows that

$$A \subset \bigcup_{i=1}^{+\infty} B_i.$$

Suppose that

$$x \in \bigcap_{i=1}^{p} B_{m_i}$$

then we will show that $p \leq 10^n N(n) =: P(n)$, where $N(n)$ is the number in lemma 1.20. Let us consider $\{j \geq 1 \mid \mathcal{I}_j \cap \{m_1, \cdots, m_p\} \neq \emptyset\}$ then, for each block we can select one index $m_{n_j} \in \mathcal{I}_j$. By the third property, we can apply lemma 1.20 and find out that

$$|\{j \geq 1 \mid \mathcal{I}_j \cap \{m_1, \cdots, m_p\} \neq \emptyset\}| \leq N(n). \tag{1.4}$$

Now fix j and $\mathcal{I}_j \cap \{m_1, \cdots, m_p\} = \{p_1, \cdot, p_q\}$, then the balls $B\left(x_{p_i}, \frac{1}{4}r(x_{p_i})\right)$ for $i = 1, \cdots, q$ are disjoint. In fact if this does not hold for some indices $i < j$ we get

$$M_j/2 \leq r(x_{p_i}) < d(x_{p_i}, x_{p_j}) \leq 1/4(r(x_{p_i}) + r(x_{p_j})) \leq M_j/2$$

a contradiction. Moreover for each index we have $B\left(x_{p_i}, 1/4r(x_{p_i})\right) \subset B\left(x, 5/4M_j\right)$: let $y \in B\left(x_{p_i}, 1/4r(x_{p_i})\right)$, then

$$d(x, y) \leq 5/4r(x_{p_i}) \leq 5/4M_j.$$

Hence,

$$q\alpha(n)(M_j/8)^n \leq \sum_{i=1}^{q} \mathcal{L}^n(B\,(x_{p_i}, 1/4r(x_{p_i}))) \leq \mathcal{L}^n(B\,(x, 5/4M_j)) = \alpha(n)(5/4M_j)^n$$

where $\alpha(n) := \mathcal{L}^n(B\,(0,1))$. This implies $q \leq 10^n$ which means

$$|\mathcal{I}_j \cap \{m_1, \cdots, m_p\}| \leq 10^n. \tag{1.5}$$

Now let N the set of indices j such that $\mathcal{I}_j \cap \{m_1, \cdots, m_p\} \neq \emptyset$. Because we have

$$p = \sum_{j\in N} |\{\mathcal{I}_j \cap \{m_1, \cdots, m_p\}\}|$$

we get by 1.4 and 1.5 that $p \leq 10^n N(n)$ and the first part of the proof is concluded.
By construction of the balls $B_i = B\,(x_i, r_i)$ from the first part, we can assume that the sequence $\{r_i\}_{i=1\dots}$ is decreasing ($r_i = r(x_i)$). Let $B_{1,1} = B_1$; if we have chosen $B_{1,1}, \cdots, B_{1,m}$, then $B_{1,m+1} = B_k$, where k is the smallest integer with

$$B_k \cap \bigcup_{i=1}^{j} B_{1,j} = \emptyset.$$

We continue this process as long as possible, until we get a countable, or finite, disjoint subfamily $\mathcal{B}_1 := \{B_{1,1}, B_{1,2}, \dots\}$ of $\{B_1, \dots\}$. If A is not covered by $\bigcup_{B\in\mathcal{B}_1} B$ we define $B_{2,1} = B_h$ where h is the smallest index with $B_h \notin \mathcal{B}_1$ and again we repeat the process as we did for $\mathcal{B}_1$ now finding $\mathcal{B}_2$. We claim that

$$A \subset \bigcup_{k=1}^{m} \bigcup_{B\in\mathcal{B}_k} B \quad \text{for some } m \leq 4^n P(n) + 1.$$

Suppose m such that $\exists x \in A \setminus \bigcup_{k=1}^{m} \bigcup_{B\in\mathcal{B}_k} B$, then $m \leq 4^n P(n)$. The balls B_i cover A, so $x \in B_i$ for some i. Therefore $B_i \notin \mathcal{B}_k$ for all $1 \leq k \leq m$ which means by construction that $B_i \cap B_{k,i_k} \neq \emptyset$ and $r_i \leq r_{k,i_k}$ (the radii of B_i and B_{k,i_k} respectively). Hence there are $\widetilde{B_k}$ balls of radius $r_i/2$ contained in $(2B_i) \cap B_{k,i_k}$ (see figure 1.2. The details are left to the reader).

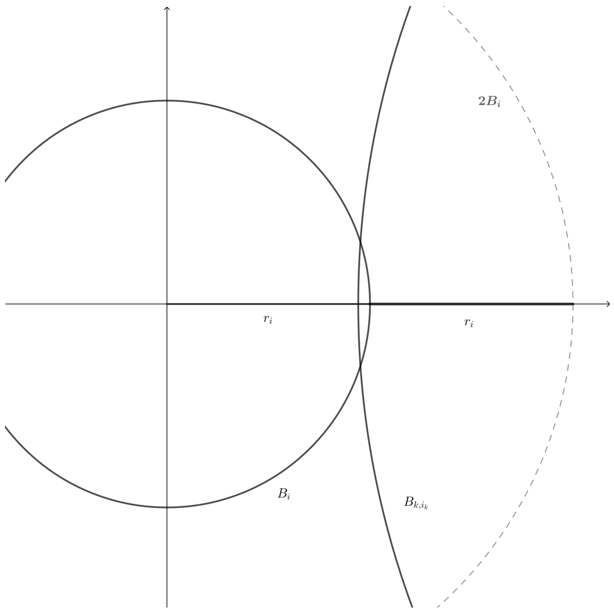

Figure 1.2: A ball of radius $r_i/2$ is contained in $(2B_i) \cap B_{k,i_k}$

Since every point of $\mathbb{R}^n$ is contained in at most $P(n)$ balls B_{k,i_k} this is also true for $\widetilde{B_k}$. Therefore

$$\sum_{k=1}^{m} \chi_{\widetilde{B_k}} \leq P(n)\chi_{\bigcup_{k=1}^{m} \widetilde{B_k}}$$

and we get

$$\begin{aligned} 2^n\alpha(n)r_i^n = \mathcal{L}^n(2B_i) \geq \mathcal{L}^n\left(\bigcup_{k=1}^{m} \widetilde{B_k}\right) = \\ = \int \chi_{\bigcup_{k=1}^{m} \widetilde{B_k}}\, d\mathcal{L}^n \geq P(n)^{-1} \int \sum_{k=1}^{m} \chi_{\widetilde{B_k}}\, d\mathcal{L}^n = P(n)^{-1} \sum_{k=1}^{m} \mathcal{L}^n(\widetilde{B_k}) = \\ = mP(n)^{-1}2^{-n}\alpha(n)r_i^n. \end{aligned}$$

This leads to $m \leq 4^n P(n)$ and thus the proof is complete. □

Before stating Vitali's covering theorem for Radon measures on $\mathbb{R}^n$ let us first state another lemma:

Lemma 1.21. *Let μ be a Radon measure on $\mathbb{R}^n$ and let $\{H_\lambda\}_{\lambda\in\Lambda}$ a family of disjoint Borel sets such that Λ is an uncountable set. Then $|\{\lambda \in \Lambda \mid \mu(H_\lambda) > 0\}|$ is at most countable.*

Proof. Write $\mathbb{R}^n = \bigcup_{i=1}^{+\infty} K_i$ where $\{K_i\}_{i\geq 1}$ is an increasing sequence of compact sets. Let $\Lambda_i := \{\lambda \mid \mu(K_i \cap H_\lambda) > 1/i\}$, then

$$\{\lambda \in \Lambda \mid \mu(H_\lambda) > 0\} = \bigcup_{i=1}^{+\infty} \Lambda_i. \tag{1.6}$$

One inclusion is trivial. Let us show that $\{\lambda \in \Lambda \mid \mu(H_\lambda) > 0\} \subseteq \bigcup_{i=1}^{+\infty} \Lambda_i$. Then $\mu(H_\lambda) > 1/i_0$ for some i_0, but since $\lim_{i\to+\infty} \mu(H_\lambda \cap K_i) = \mu(H_\lambda)$ we have

$$1/i < 1/i_0 - \varepsilon < \mu(H_\lambda) - \varepsilon < \mu(H_\lambda \cap K_i)$$

for ε small and i big enough, and we conclude. By 1.6 we just need to show that $|\Lambda_i|$ is finite for all i. Let $J \subset \Lambda_i$ be finite, this implies that

$$\mu(K_i) \geq \mu\left(K_i \cap (\bigcup_{\lambda\in\Lambda_i} H_\lambda)\right) \geq \mu\left(K_i \cap (\bigcup_{\lambda\in J} H_\lambda)\right) = \sum_{\lambda \in J} \mu(K_i \cap H_\lambda) > |J|/i$$

and then $|J| < \mu(K_i)i < +\infty$ which means that $|\Lambda_i| < \mu(K_i)i < +\infty$. We conclude that only at most countably many H_λ have positive measure. $\square$

Theorem 1.22. *Let μ be a Radon measure on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$ and $\mathcal{B}$ a fine cover of A, such that each point of A is the centre of some ball of $\mathcal{B}$. Then exist $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{B}$ pairwise disjoint such that*

$$\mu\left(A \setminus \bigcup_{i=1}^{+\infty} B_i\right) = 0.$$

Proof. We may assume $\mu(A) > 0$ (otherwise the theorem is trivially true); we can also suppose that A is bounded, otherwise one can proceed as in theorem 1.17 noting that we can take n-rectangles with the union of the boundaries of measure zero by Lemma 1.21. Since μ is a Radon measure we can find U open such that $A \subset U$ and

$$\mu(U) \leq (1 + (4Q(n))^{-1})\mu(A)$$

where $Q(n)$ is the number as in Besicovitch's Covering Theorem 1.18. Therefore we have that

$$A \subset \bigcup_{i=1}^{Q(n)} \bigcup_{B\in\mathcal{B}_i} B \subset U$$

leading to

$$\mu(A) \leq Q(n) \sum_{B\in\mathcal{B}_i} \mu(B) \text{ for some } i \in \{1, \cdots, Q(n)\}.$$

Therefore exists a subfamily $\widetilde{\mathcal{B}_i}$ of $\mathcal{B}_i$ such that $\mu(A) \leq 2Q(n) \sum_{B\in\widetilde{\mathcal{B}_i}} \mu(B)$. We get then

$$\begin{aligned}\mu\Big(A \setminus \bigcup_{B\in\widetilde{\mathcal{B}_i}} B\Big) \leq \mu\Big(U \setminus \bigcup_{B\in\widetilde{\mathcal{B}_i}} B\Big) \leq \mu(U) - \sum_{B\in\widetilde{\mathcal{B}_i}} \mu(B) \leq \\ \leq (1 + 1/4Q(n)^{-1} - 1/2Q(n)^{-1})\mu(A) = (1 - 1/4Q(n)^{-1})\mu(A).\end{aligned}$$

Setting $\varepsilon = 1 - 1/4Q(n)^{-1} < 1$ we conclude as did for Theorem 1.17. $\square$

Chapter 2

Differentiation of measures and the Grassmannian of m-planes

In this chapter we will use covering theorems proved in the last chapter to discuss the differentiation of measures on $\mathbb{R}^n$. Later we will see some properties of $G(n, m)$ the set of all the m-dimensional vectorial subspace of $\mathbb{R}^n$ and we will define a measure on this set.

In general we will say that a property $\mathcal{P}$ holds for μ-almost all $x \in X$ if there exists a set $N \subset X$ with $\mu(N) = 0$ such that $\mathcal{P}$ holds $\forall x \in X \setminus N$.

2.1 Differentiation of measures

Definition 2.1. Let μ an ν be locally finite Borel measures on $\mathbb{R}^n$. The upper derivative of μ with respect to ν at $x \in \mathbb{R}^n$ is defined as

$$\overline{D}(\mu, \nu, x) := \begin{cases} \limsup\limits_{r \to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} & \text{if } \nu(B(x,r) > 0 \text{ for all } r \text{ small enough} \\ +\infty & \text{if } \nu(B(x,r) = 0 \text{ for some } r > 0 \end{cases}.$$

The lower derivative of μ with respect to ν at $x \in \mathbb{R}^n$ is

$$\underline{D}(\mu, \nu, x) := \begin{cases} \liminf\limits_{r \to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} & \text{if } \nu(B(x,r) > 0 \text{ for all } r \text{ small enough} \\ +\infty & \text{if } \nu(B(x,r) = 0 \text{ for some } r > 0 \end{cases}.$$

If for some $x \in \mathbb{R}^n$ we have that $\overline{D}(\mu, \nu, x) = \underline{D}(\mu, \nu, x)$, we define the derivative of μ with respect to ν at x as

$$D(\mu, \nu, x) := \overline{D}(\mu, \nu, x) = \underline{D}(\mu, \nu, x).$$

Lemma 2.1. *Let μ and ν be Radon measures on $\mathbb{R}^n$, $0 < \alpha < +\infty$ and $A \subset \mathbb{R}^n$.*

1. *If $\underline{D}(\mu, \nu, x) \leq \alpha$ for all $x \in A$, then $\mu(A) \leq \alpha\nu(A)$.*

2. *If $\overline{D}(\mu,\nu,x) \geq \alpha$ for all $x \in A$, then $\mu(A) \geq \alpha\nu(A)$.*

Proof. 1. Let $\varepsilon > 0$; because ν is a Radon measure, we find U open with $A \subseteq U$ such that $\nu(U) \leq \nu(A) + \varepsilon$. Then, by Vitali's covering theorem, and using the definition of $\liminf$, we find B_i disjoint balls contained in U such that

$$\mu(B_i) \leq (\alpha+\varepsilon)\nu(B_i) \text{ and } \mu\Big(A \setminus \bigcup_{i\in\mathbb{N}} B_i\Big) = 0.$$

So we get

$$\mu(A) \leq \sum_{i\in\mathbb{N}} \mu(B_i) \leq (\alpha+\varepsilon)\sum_{i\in\mathbb{N}} \nu(B_i) \leq (\alpha+\varepsilon)\nu(U) \leq (\alpha+\varepsilon)(\nu(A)+\varepsilon)$$

and letting $\varepsilon \to 0$ we get $\mu(A) \leq \alpha\nu(A)$.
2. $\limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \geq \alpha$ implies $\liminf_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} \leq 1/\alpha$. □

Definition 2.2. Let μ and ν be measures on $\mathbb{R}^n$. We say that μ is absolutely continuous with respect to ν if $\forall A \subset X$

$$\nu(A) = 0 \Longrightarrow \mu(A) = 0,$$

and we will write $\mu \ll \nu$.

Theorem 2.2. *Let μ and ν be Radon measures on $\mathbb{R}^n$. Then:*

1. *The derivative $D(\mu,\nu,x)$ exists and is finite for ν-almost all $x \in \mathbb{R}^n$.*

2. *$x \mapsto D(\mu,\nu,x)$ is a Borel function.*

3. *For all Borel sets $B \subset \mathbb{R}^n$,*

$$\int_B D(\mu,\nu,x)d\nu(x) \leq \mu(B)$$

and the equality holds if $\mu \ll \nu$.

4. *$\mu \ll \nu$ if and only if $\underline{D}(\mu,\nu,x) < +\infty$ for μ-almost all $x \in \mathbb{R}^n$.*

Proof.

1. Let us consider for $0 < \alpha < \beta < +\infty$ the sets

$$I = \{x \in \mathbb{R}^n \mid D(\mu,\nu,x) = +\infty\} \quad E_{\alpha,\beta} = \{x \in \mathbb{R}^n \mid \underline{D}(\mu,\nu,x) \leq \alpha < \beta \leq \overline{D}(\mu,\nu,x)\}.$$

We have that $I \subset \{x \in \mathbb{R}^n \mid D(\mu,\nu,x) > k\} = I_k$ for all $k \in \mathbb{N}$; then by lemma 2.1

$$\mu(I) \geq k\nu(I)$$

which means $\nu(I) \leq 1/k\mu(I)$ and letting $k \to +\infty$ we get $\nu(I) = 0$. Again, by lemma 2.1 we have

$$\mu(E_{\alpha,\beta}) \leq \alpha\nu(E_{\alpha,\beta}) \qquad \mu(E_{\alpha,\beta}) \geq \beta\nu(E_{\alpha,\beta})$$

which implies that $\nu(E_{\alpha,\beta}) = 0$ for all $0 < \alpha < \beta < +\infty$. Now take α, β as before, but rational. Then $\{x \in \mathbb{R}^n \mid D(\mu,\nu,x)$ does not exist $\} = I \cup \Big(\bigcup_{0<\alpha<\beta<+\infty} E_{\alpha,\beta}\Big)$ which as ν measure null.

2. For a fixed $r > 0$, $x \mapsto \mu(B(x,r))$ is upper semicontinuous, and so is $x \mapsto \nu(B(x,r))$. Let $x_k \to x$, $f_k := \chi_{B(x_k,r)}$ and $f = \chi_{B(x,r)}$. Then $\limsup_{k\to+\infty} f_k \le f$, implying

$$\liminf_{k\to+\infty}(1-f_k) \ge 1-f.$$

Since $(1-f_k)$ are positive measurable functions we can apply Fatou's Lemma (see [C] 2.18) and we obtain

$$\int_{B(x,2r)} (1-f)\,d\mu \le \int_{B(x,2r)} \liminf_{k\to+\infty}(1-f_k)\,d\mu \le \liminf_{k\to+\infty} \int_{B(x,2r)} (1-f_k)\,d\mu$$

which means $\mu(B(x,2r)) - \mu(B(x,r)) \le \mu(B(x,2r)) - \limsup_{k\to+\infty}\mu(B(x_k,r)) \iff \limsup_{k\to+\infty}\mu(B(x_k,r)) \le \mu(B(x,r))$. Then $x \mapsto \mu(B(x,r))$ is Borel regular, as $x \mapsto \nu(B(x,r))$; therefore for all $r > 0$

$$d_r(x) := \begin{cases} \mu(B(x,r))/\nu(B(x,r)) & \text{if } \nu(B(x,r)) > 0 \\ +\infty & \text{if } \nu(B(x,r)) = 0 \end{cases}$$

are Borel functions, but since

$$D(\mu,\nu,x) = \lim_{r\to 0} d_r(x) = \liminf_{k\to+\infty} d_{1/k}(x)$$

we conclude that $D(\mu,\nu,x)$ is a Borel function.

3. Let $1 < a < +\infty$ and $B_p := \{x \in B | a^p \le D(\mu,\nu,x) < a^{p+1}\}$ for $p \in \mathbb{Z}$. Then, except for a set of ν-measure zero, $B = \bigcup_{p\in\mathbb{Z}} B_p$ and we get by lemma 2.1

$$\int_B D(\mu,\nu,x)\,d\nu(x) = \sum_{p=-\infty}^{+\infty} \int_{B_p} D(\mu,\nu,x)\,d\nu(x) \le \sum_{p=-\infty}^{+\infty} a^{p+1}\nu(B_p) \le a \sum_{p=-\infty}^{+\infty} \mu(B_p) \le a\mu(B).$$

Letting $a \to 1$ we obtain $\int_B D(\mu,\nu,x)\,d\nu(x) \le \mu(B)$. If $\mu \ll \nu$ then $D(\mu,\nu,x)$ exists for μ-almost $x \in \mathbb{R}^n$ and then $\sum_{p=-\infty}^{+\infty}\mu(B_p) = \mu(B)$. Therefore,

$$\int_B D(\mu,\nu,x)\,d\nu(x) = \sum_{p=-\infty}^{+\infty} \int_{B_p} D(\mu,\nu,x)\,d\nu(x) \ge \sum_{p=-\infty}^{+\infty} a^{p}\nu(B_p) \ge \frac{1}{a} \sum_{p=-\infty}^{+\infty} \mu(B_p) = \frac{1}{a}\mu(B)$$

and letting $a \to 1$ we obtain the other inequality.

4. Let $\mu \ll \nu$ then $\underline{D}(\mu,\nu,x) < +\infty$ ν-almost everywhere $\Longrightarrow$ $\underline{D}(\mu,\nu,x) < +\infty$ μ-almost everywhere. Let $A \subset \mathbb{R}^n$ such that $\nu(A) = 0$; if $\underline{D}(\mu,\nu,x) < +\infty$ μ-almost everywhere, then

$$\mu(\{x \in A \mid \underline{D}(\mu,\nu,x) \le a\}) \le a\nu(A) = 0$$

for all $a \in \mathbb{N}$ and since $\mu(A) = \mu(\bigcup_{a\in\mathbb{N}}\{x \in A \mid \underline{D}(\mu,\nu,x) \le a\}) = 0$ we conclude. $\square$

Remark 5. In point 2. of Theorem 2.2 we did not say what is the domain of the Borel function; the reader can imagine that if we have $f : X \setminus N \longrightarrow \overline{\mathbb{R}}$ a measurable function with $\mu(N) = 0$ then this can be easily extended to a measurable function $\tilde{f}$ with domain X ($\tilde{f}(x) = 0 \forall x \in N$).

Corollary 2.3 (Lebesgue Density Theorem)**.** *Let ν be a Radon measure on $\mathbb{R}^n$.*

1. *If $A \subset \mathbb{R}^n$ is ν-measurable, then*

$$\lim_{r\to 0} \frac{\nu(A \cap B(x,r))}{\nu(B(x,r))} = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in \mathbb{R}^n \setminus A \end{cases}$$

for ν-almost all $x \in \mathbb{R}^n$.

2. *If $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is locally ν-integrable, then*

$$\lim_{r\to 0} \frac{1}{\nu(B(x,r))} \int_{B(x,r)} f(z)\, d\nu(z) = f(x) \text{ for } \nu\text{-almost all } x \in \mathbb{R}^n.$$

Remark 6. In order to show that $\lim_{r\to 0} \frac{\nu(A\cap B(x,r))}{\nu(B(x,r))} = 1$ when $x \in A$ the measurability of A is not required.

Proof. 1. Follows from 2. with $f = \chi_A$.
2. Since $f = f^+ - f^-$ we can suppose that $f \geq 0$, then $\mu(A) := \int_A f(x)\, d\nu(x)$ is a Radon measure (defined on the σ-algebra of all the ν-measurable sets), and $\mu \ll \nu$. Then by Theorem 2.2 we have

$$\int_B D(\mu,\nu,x)\, d\nu(x) = \mu(B) = \int_B f(x)\, d\nu(x)$$

for all Borel sets B. Then $\lim_{r\to 0} \frac{1}{\nu(B(x,r))} \int_{B(x,r)} f(z)\, d\nu(z) = D(\mu,\nu,x) = f(x)$ for ν-almost all $x \in \mathbb{R}^n$. □

Definition 2.3. Given μ, ν Radon measures on $\mathbb{R}^n$ we say that they are mutually singular if there is a Borel set $A \subset \mathbb{R}^n$ such that $\nu(A) = \mu(\mathbb{R}^n \setminus A) = 0$. In this case we will write $\nu \perp \mu$.

Theorem 2.4. *Let μ and ν be finite Radon measures on $\mathbb{R}^n$. Then there is a Radon measure λ and a Borel function such that $\nu \perp \lambda$ and*

$$\mu(B) = \int_B f(x)\, d\nu(x) + \lambda(B)$$

for all B Borel sets. Moreover, $D(\lambda,\nu,x) = 0$ for ν-almost all x.

Proof. Set $A = \{x \in \mathbb{R}^n | \underline{D}(\mu, \nu, x) < +\infty\}$ and

$$\mu_1 = \mu \llcorner A \qquad \lambda = \mu \llcorner (\mathbb{R}^n \setminus A).$$

We have that μ_1 and λ are Radon measures and $\mu = \mu_1 + \lambda$. Because $\underline{D}(\mu_1, \nu, x) \leq \underline{D}(\mu, \nu, x) < +\infty$ μ_1-almost everywhere we have $\mu_1 \ll \mu$, so $\mu_1(B) = \int_B D(\mu_1, \nu, x)\, d\nu(x)$ and we can take $f = D(\mu_1, \nu,)$ which is a Borel function. Of course, $\lambda(A) = 0$, and because of Theorem 2.2 1. we have that $\nu(\mathbb{R}^n \setminus A) = 0$ and $\nu \perp \lambda$. Then we can see that considering the set

$$C = \{x \in A \mid D(\lambda, \nu, x) \geq n\}$$

we have

$$\nu(C)n \leq \lambda(C) \leq \lambda(A) = 0$$

so that $D(\lambda, \nu, x) = 0$ ν-almost everywhere and then $D(\mu, \nu, x) = D(\mu_1, \nu, x)$ ν-almost everywhere. $\square$

2.2 Haar measure and The orthogonal group

In the next chapters we will need to compare a set with its orthogonal projection on a m-dimensional subspace of $\mathbb{R}^n$ and most statements do not hold for every subspace. We could ask ourself if some kind of property does it hold for almost all m-subspaces. In this section we shall give some definitions and some properties of $\mathcal{O}(n)$, the n-orthogonal group.

Definition 2.4. A topological group G is a group with a structure of topological space such that the group operations

$$\begin{aligned} f\colon G \times G &\to G & \qquad \alpha\colon G &\to G \\ (g, h) &\mapsto gh & g &\mapsto g^{-1} \end{aligned}$$

are continuous.

Definition 2.5. A measure on G is invariant if for all $A \subset G$ and $g \in G$

$$\mu(A) = \mu(gA) = \mu(Ag)$$

where $gA = \{ga \mid a \in A\}$ and $Ag = \{ag \mid a \in A\}$.

Theorem 2.5. *If G is a compact topological group, there is a unique invariant Radon measure μ on G such that $\mu(G) = 1$.*

A measure on a compact topological space like the one described in Theorem 2.5 is called Haar measure. Given μ Haar measure on G, we see that $\nu(A) := \mu(\{a^{-1} \mid a \in A\})$ also defines an Haar measure on G, hence

$$\mu(A) = \mu(\{a^{-1} \mid a \in A\}). \tag{2.1}$$

At this point we will make a small digression on uniformly distributed measures, which, however, will be useful for us.

Definition 2.6. A Borel regular measure μ on a metric space X is *uniformly distributed* if

$$0 < \mu(B(x,r)) = \mu(B(y,r)) < +\infty$$

for all $x, y \in X$ and $r > 0$.

Theorem 2.6. *Let μ, ν be uniformly distributed Borel regular measures on a separable metric space X. Then there exists a constant $c \geq 0$ such that $\mu = c\nu$.*

Proof. Let $g(r) = \mu(B(x,r))$ and $h(r) = \nu(B(x,r))$ functions defined for all $r > 0$ and for $x \in X$. Let $U \neq \emptyset$ be a bounded open subset of X. Then for almost all $x \in U$ the limit $\lim_{r\to 0} \nu(U \cap B(x,r))/\nu(B(x,r))$ exists and it is equal to 1. Hence, by Fatou's Lemma and Fubini's Theorem,

$$\begin{aligned}\mu(U) = \int_U \lim_{r\to 0} \nu(U \cap B(x,r))/\nu(B(x,r))\, d\mu(x) \leq \\ \leq \liminf_{r\to 0} \frac{1}{h(r)} \int_X \nu(U \cap B(x,r))\, d\mu(x) = \liminf_{r\to 0} \frac{1}{h(r)} \int_U \mu(B(x,r))\, d\nu(x) = \\ = \liminf_{r\to 0} \frac{g(r)}{h(r)} \nu(U).\end{aligned}$$

Interchanging μ and ν we get

$$\nu(U) \leq \liminf_{r\to 0} \frac{h(r)}{g(r)} \mu(U).$$

This shows that $\lim_{r\to 0} \frac{g(r)}{h(r)} = c$ exists and so $\mu(U) = c\nu(U)$ for all U open subsets of X (details are left to the reader). Let $E \subset X$, then, because X is separable, we are able to find V_i open bounded sets such that $E \subset \bigcup_i V_i$. Then, since μ, ν are Borel regular, $\mu(E) = \sup\{\mu(U) \mid E \subset U \text{open}\} = c \sup\{\nu(U) \mid E \subset U \text{open}\} = c\nu(E)$, so $\mu = c\nu$. $\square$

Now we shall discuss about the properties of orthogonal group $\mathcal{O}(n)$, which consist of all linear maps $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ preserving the inner product, or, equivalently, the distance:

$$g(x) \cdot g(y) = x \cdot y \Longleftrightarrow |g(x) - g(y)| = |x - y|$$

for all $g \in \mathcal{O}(n)$ and $x, y \in \mathbb{R}^n$. $\mathcal{O}(n)$ is closed and limited in the normed space of the endomorphism of $\mathbb{R}^n$, $\mathcal{L}(\mathbb{R}^n)$, which is of finite dimension. Therefore $\mathcal{O}(n)$ is a compact subspace of $\mathcal{L}(\mathbb{R}^n)$, and the norm is the usual operator norm:

$$\|g\| = \sup_{|x|=1} |g(x)|$$

therefore

$$d(g,h) = \|g - h\| = \sup_{|x|=1} |g(x) - h(x)|.$$

Considering also the composition as operation on $\mathcal{O}(n)$, this becomes also a topological group, and we shall denote the unique Haar measure on $\mathcal{O}(n)$ as θ_n ($\theta_n(\mathcal{O}(n)) = 1$). Since $d(gk, hk) = d(g, h)$ for all $g, h, k \in \mathcal{O}(n)$, we have that

$$\theta_n(B(g, r)) = \theta_n(B(g, r)k) = \theta_n(B(gk, r))$$

and θ_n is uniformly distributed. In order to define a $g \in \mathcal{O}(n)$ we can take two different ordered orthonormal bases of $\mathbb{R}^n$ $\{v_1, \cdots, v_n\}, \{u_1, \cdots, u_n\}$ and define g such that it sends one basis in the other, $g(v_i) = u_i$. It is easy to check that $\mathcal{O}(n)$ acts transitively on S^{n-1}: the action is defined as $g(x)$ for $g \in \mathcal{O}(n)$, $x \in S^{n-1}$ and for all $x, y \in S^{n-1}$ there exists $g \in \mathcal{O}(n)$ such that $g(x) = y$.
With σ^{n-1} we will denote the normalized surface measure on S^{n-1}, for instance, $\sigma^{n-1}(S^{n-1}) = 1$ and it can be defined as follows: recall that $\alpha(n) = \mathcal{L}^n(B(0, 1))$, then

$$\sigma^{n-1}(A) = \alpha(n)^{-1}\mathcal{L}^n(\{tx \mid x \in A, 0 \leq t \leq 1\}) \text{ for } A \subset S^{n-1}.$$

σ^{n-1} can be viewed as $\mathcal{H}^{n-1} \llcorner S^{n-1}$ where $\mathcal{H}^{n-1}$ is the $n-1$-Hausdorff measure, which is Borel regular.

Theorem 2.7. *For any $x \in S^{n-1}$ and $A \subset S^{n-1}$,*

$$\theta_n(\{g \in \mathcal{O}(n) \mid g(x) \in A\}) = \sigma^{n-1}(A).$$

Proof. Let us consider $f_x : \mathcal{O}(n) \longrightarrow S^{n-1}$ defined as $f_x(g) = g(x)$ for a fixed $x \in S^{n-1}$. Since θ_n is a Radon measure with compact support and f_x is a continuous function, $f_{x\sharp}\theta_n$ is a Radon measure with compact support and $f_{x\sharp}\theta_n(S^{n-1}) = 1$. We have that

$$f_{x\sharp}\theta_n(A) = \theta_n(\{g \in \mathcal{O}(n) | g(x) \in A\}).$$

Since both σ^{n-1} and $f_{x\sharp}\theta_n$ are Borel regular measures on S^{n-1} with same value on the whole space, we just need to show that $f_{x\sharp}\theta_n$ is a uniformly distributed measure; then $f_{x\sharp}\theta_n = c\sigma^{n-1}$ with $c = 1$ and we conclude. Given $y, z \in S^{n-1}$ there exists $h \in \mathcal{O}(n)$ such that $y = h(z)$; then

$$\begin{aligned} f_{x\sharp}\theta_n(B(y, r)) &= \theta_n(\{g \in \mathcal{O}(n) \mid g(x) \in B(y, r)\}) = \\ &= \theta_n(\{g \mid |g(x) - h(z)| < r\}) = \theta_n(\{g \mid |h^{-1} \circ g(x) - z| < r\}) = \\ &= \theta_n(\{g \in \mathcal{O}(n) \mid |g(x) - z| < r\}) = f_{x\sharp}\theta_n(B(z, r)) \end{aligned}$$

and we are done. □

Lemma 2.8. *For $x, y \in \mathbb{R}^n$, $x \neq 0$, and $\delta > 0$,*

$$\theta_n(\{g \in \mathcal{O}(n) \mid |x - g(y)| \leq \delta\}) \leq c\frac{\delta^{n-1}}{|x|^{n-1}}$$

where c is a constant depending only on n.

Proof. If $||x|-|y||>\delta$ then $\{g \mid |x-g(y)|\leq \delta\}=\emptyset$, so $\theta_n(\{g \mid |x-g(y)|\leq\delta\})=0$. We can assume that $||x|-|y||\leq\delta$, $x\neq 0$ and $y\neq 0$. Then $|x-g(y)|\leq\delta$ implies

$$|x-g(\frac{|x|y}{|y|})|\leq |x-g(y)|+|(1-|x|/|y|)g(y)|=|x-g(y)|+||y|-|x||\leq 2\delta$$

which means that $|x/|x|-g(y/|y|)|\leq 2\delta/|x|$. Therefore

$$\theta_n(\{g \mid |x-g(y)|\leq\delta\})\leq\theta_n(\{g \mid g(y/|y|)\in B(x/|x|,\delta/|x|)\})=$$
$$=\sigma^{n-1}(\{B(x/|x|,\delta/|x|)\cap S^{n-1}\})\leq c\frac{\delta^{n-1}}{|x|^{n-1}}$$

where c is a constant depending only on n. □

2.3 The Grassmannian of m-planes

Let $0<m<n$. As we anticipated $G(n,m):=\{V\leq\mathbb{R}^n \mid dim_{\mathbb{R}}V=m\}$ and in this section we shall define a measure on $G(n,m)$.
We can identify every element $V\in G(n,m)$ with

$$P_V:\mathbb{R}^n\longrightarrow\mathbb{R}^n$$

the orthogonal projection on $V\in G(n,m)$. Then we can define for $V,W\in G(n,m)$ a distance, using the operator norm:

$$d(V,W)=\|P_V-P_W\|:=\sup_{|x|=1}|P_V(x)-P_W(x)|.$$

Proposition 2.9. *$(G(n,m),d)$ is a compact metric space.*

Proof. Since $G(n,m)=\{P_V:\mathbb{R}^n\to\mathbb{R}^n\}\subset GL(n)$, $\|P_V\|\leq 1$, and $dim_{\mathbb{R}}(GL(n))=n^2$ we just need to show that $G(n,m)$ is closed in $GL(n)$. Let then $\{P_{V_k}\}$ a converging sequence in $GL(n)$. Since S^{n-1} is compact we can choose $v_{i,k}$ orthonormal vectors such that $V_k=\langle\{v_{k,j}\}_{1\leq j\leq m}\rangle$ and such that $v_{k,j}\to v_j$ for $k\to+\infty$. Limits are also orthonormal vectors. Let $V=\langle\{v_j\}_{1\leq j\leq m}\rangle$. Let us show that

$$\|P_{V_k}-P_V\|\to 0$$

as $k\to+\infty$. Since any norm in $GL(n)$ is equivalent, we just need to show that the entries of the matrix P_{V_k} converge to the entries of the matrix of P_V (matrices are written here with respect to the canonical base). Let $\{e_1,\ldots,e_n\}$ the canonical base of $\mathbb{R}^n$. We have that

$$e_i=P_{V_k}(e_i)+P_{V_k^\perp}(e_i)=\alpha_{k,1}v_{k,1}+\ldots+\alpha_{k,m}v_{k,m}+P_{V_k^\perp}(e_i).$$

We can easily see that $\alpha_{k,j}=e_i\cdot v_{k,j}\to e_i\cdot v_j=:\alpha_j$ for $k\to+\infty$. Therefore

$$\lim_{k\to+\infty}P_{V_k}(e_i)=\alpha_1 v_1+\ldots+\alpha_m v_m$$

and

$$\lim_{k\to+\infty} P_{V_k^\perp}(e_i) = e_i - \alpha_1 v_1 + \ldots + \alpha_m v_m$$

for all $i = 1, \ldots, n$. It is trivial that $\alpha_1 v_1 + \ldots + \alpha_m v_m \in V$. Noting that

$$0 \equiv P_{V_k^\perp}(e_i) \cdot v_{k,j} \to \lim_{k\to+\infty} P_{V_k^\perp}(e_i) \cdot v_j$$

we can deduce that $\lim_{k\to+\infty} P_{V_k^\perp}(e_i) \in V^\perp$. We have proved that

$$\lim_{k\to+\infty} P_{V_k}(e_i) = P_V(e_i)$$

for all $i = 1, \ldots, n$. Since each entry of $v_{k,j}$ converges to the corresponding entry of v_j we can conclude that the entries of the i-th column of the matrix of P_{V_k} converge to the entries of the matrix P_V. This holds for all $i = 1, \ldots, n$ we conclude that

$$\|P_{V_k} - P_V\| \to 0$$

and the proof is complete. □

One can also easily see that $G(n, m)$ is also a separable metric space, since it is a subset of a separable metric space. We can see that $\mathcal{O}(n)$ acts on $G(n, m)$ and this action preserves the distance:

$$d(gV, gW) = d(V, W) \text{ for } g \in \mathcal{O}(n), \quad V, W \in G(n, m).$$

Moreover, by standard linear algebra, the action is transitive: $\forall\, V, W \in G(n, m)$ we have $gV = W$. Finally we are ready to define a Radon measure on $G(n, m)$: fix $V \in G(n, m)$

$$\gamma_{n,m}(A) = \theta_n(\{g \mid gV \in A\}).$$

Taking $f_V : \mathcal{O}(n) \longrightarrow G(n, m)$ such that $f_V(g) = gV$ we have that $\gamma_{n,m} = f_{V\sharp}\theta_n$. Since $\{g \mid gV \in hA\} = \{hg \mid gV \in A\}$ and θ_n is invariant then $\gamma_{n,m}$ is $\mathcal{O}(n)$ invariant, that is

$$\gamma_{n,m}(gA) = \gamma_{n,m}(A)$$

for all $g \in \mathcal{O}(n)$ and $A \subset G(n, m)$. Therefore we can notice that the transitivity and the distance preservation of the action of $\mathcal{O}(n)$ imply that $\gamma_{n,m}$ is uniformly distributed. The invariant measure is unique, and in particular $\gamma_{n,m}$ does not depend on the choice of $V \in G(n, m)$. In order to prove equalities with $\gamma_{n,m}$, it will be sufficient to prove that the other side of the equation is $\mathcal{O}(n)$ invariant. For example

$$\gamma_{n,m}(A) = \gamma_{n,n-m}(\{V^\perp \mid V \in A\}) \tag{2.2}$$

or, for $A \subseteq G(n, 1)$,

$$\gamma_{n,1}(A) = \sigma^{n-1}\left(\bigcup_{L\in A} L \cap S^{n-1}\right)$$

and again, for $A \subseteq G(n, n-1)$,

$$\gamma_{n,n-1}(A) = \sigma^{n-1}\left(\bigcup_{V\in A} V^\perp \cap S^{n-1}\right).$$

Lemma 2.10. *For any $x \in \mathbb{R}^n \setminus \{0\}$ and $0 < \delta < +\infty$,*

$$\gamma_{n,m}\left(\{V \mid d(x,V) \leq \delta\}\right) \leq c\frac{\delta^{n-m}}{|x|^{n-m}}$$

with $c = 2^n \alpha(n)^{-1}$. Moreover, from equation 2.2, we have that

$$\gamma_{n,m}\left(\{V \mid |P_V(x)| \leq \delta\}\right) \leq c\frac{\delta^{m}}{|x|^{m}}.$$

Proof. Fix $x \in \mathbb{R}^n \setminus \{0\}$, and $W = \{x \in \mathbb{R}^n \mid x_{m+1} = \ldots = x_n = 0\} \in G(n,m)$. Then $d(x,V) = |x| d(x/|x|, V)$ and by equation 2.1,

$$\gamma_{n,m}\left(\left\{V \mid d\big(x/|x|, V\big) \leq \frac{\delta}{|x|}\right\}\right) = \theta_n\left(\left\{g \mid d\big(x/|x|, gW\big) \leq \frac{\delta}{|x|}\right\}\right) =$$
$$= \theta_n\left(\left\{g \mid d\big(g^{-1}(x/|x|), W\big) \leq \frac{\delta}{|x|}\right\}\right) = \theta_n\left(\left\{g \mid d\big(g(x/|x|), W\big) \leq \frac{\delta}{|x|}\right\}\right) =$$
$$= \sigma^{n-1}(\{y \in S^{n-1} \mid d(y,W) \leq \delta/|x|\}) = \sigma^{n-1}\left(\left\{y \in S^{n-1} \mid \sqrt{\sum_{i=m+1}^{n} y_i^2} \leq \frac{\delta}{|x|}\right\}\right) \leq$$
$$\leq \alpha(n)^{-1}\mathcal{L}^n(\{y \mid |y_i| \leq 1 \text{ for } i \leq \text{m }, |y_i| \leq \delta/|x| \text{ for } i > m \ \}) =$$
$$= \alpha(n)^{-1} 2^n (\delta/|x|)^{n-m}.$$

$\square$

Corollary 2.11. *For $0 < s < m$*

$$\int |P_V(x)|^{-s}\, d\gamma_{n,m} V \leq c|x|^{-s}$$

where c is a constant depending only on m, n and s.

Proof. $V \mapsto |P_V(x)|$ is a measurable function because it is continuous. Another way to see this is the following : fix $W \in G(n,m)$, and recalling the comment after definition 1.8, we have to show that

$$\{V \mid |P_V(x)| > \alpha\}$$

is $\gamma_{n,m}$-measurable for all $x \in \mathbb{R}^n \setminus \{0\}$. This is true since $\{g \mid |P_W(gx)| > \alpha\}$ is θ_n-measurable, since it is open in $\mathcal{O}(n)$. To see it one can consider $P_W \circ f_x : \mathcal{O}(n) \mapsto \mathbb{R}_{\geq 0}$ where $f_x(g) = g(x)$. By proposition 1.13 we get

$$\int |P_V(x)|^{-s}\, d\gamma_{n,m} V = \int_0^{+\infty} \gamma_{n,m}(\{V \mid |P_V(x)| \leq t^{-1/s}\})\, dt =$$
$$= \int_0^{|x|^{-s}} dt + \int_{|x|^{-s}}^{+\infty} \gamma_{n,m}(\{V \mid |P_V(x)| \leq t^{-1/s}\})\, dt \leq$$
$$\leq |x^{-s}| + c|x|^{-m} \int_{|x|^{-s}}^{+\infty} t^{-m/s}\, dt = \left(1 + \frac{cs}{m-s}\right)|x|^{-s}$$

where the third equalities hold since if $t \leq |x|^{-s} \iff |x| \leq t^{-1/s}$ then $|P_V(x)| \leq t^{-1/s}$ for all $V \in G(n,m)$ and so $\gamma_{n,m}(\{V \mid |P_V(x)| \leq t^{-1/s}\}) = 1$ for $t \leq |x|^{-s}$. $\square$

When $m = 0$, then $G(n,0) = \{0\}$ and $\gamma_{n,0} = \delta_0$ on $G(n,0)$.

Lemma 2.12. *Let $V \in G(n,m)$ for $0 < m < n$. Then*

$$\gamma_{n,m}(\{V\}) = 0.$$

Proof. The proof is trivial: fix $\{v_i\}_{i=1,\dots,m}$ an orthonormal base of V. Then

$$\gamma_{n,m}(\{V\}) = \theta_n(\{g \mid gV = V\}) = \\ = \theta_n\left(\bigcap_{i=1}^{m}\{g \mid g(v_i) \in V \cap S^{n-1}\}\right) \leq \sigma^{n-1}(\{V \cap S^{n-1}\}) = 0.$$

$\square$

Lemma 2.13. *Let $k, m \in \mathbb{N}$ such that $1 \leq k \leq n-1$, $0 \leq m \leq n-1$, $k+m \leq n$, and let $W \in G(n,m)$. Then*

$$\gamma_{n,m}(\{V \mid V \cap W \neq \{0\}\}) = 0.$$

Proof. For $n = 2$ the lemma is true. We can proceed by induction on n. We may assume $m \geq 1$ and that the lemma is true for $n-1$. Then

$$\gamma_{n,m}(A) = \int \gamma_{L^\perp,m-1}(\{U \subset L^\perp \mid L + U \in A\})\, d\gamma_{n,1}L \tag{2.3}$$

where the variable is $L \in G(n,1)$, and $\gamma_{L^\perp,m-1}$ is the invariant measure on the subspaces of $L^\perp$ of dimension $m-1 \leq n-2$.
Let us prove 2.3. By Proposition 1.15,

$$\int \gamma_{L^\perp,m-1}(\{U \subset L^\perp \mid U{+}L \in gA\})\, d\gamma_{n,1}gL = \int \gamma_{gL^\perp,m-1}(\{U \subset gL^\perp \mid U{+}gL \in gA\})\, d\gamma_{n,1}L = \\ = \int \gamma_{L^\perp,m-1}(\{g^{-1}U \subset L^\perp \mid g^{-1}U{+}L \in gA\})\, d\gamma_{n,1}L = \int \gamma_{L^\perp,m-1}(\{U \subset L^\perp \mid L{+}U \in A\})\, d\gamma_{n,1}L$$

where with $\gamma_{n,1}gL$ we intend $g_\sharp\gamma_{n,1}$, with g defined from $G(n,1)$ into itself and $g(V) = gV$. Moreover $g_\sharp\gamma_{n,1} = \gamma_{n,1}$. Now take $A = \{V \in G(n,m) \mid V \cap W \neq \{0\}\}$ By hypothesis,

$$\gamma_{n,1}(\{L \in G(n,1) \mid L \subset W\}) = 0,$$

so we can integrate over the lines L such that $L \not\subseteq W$. Then assuming that $L \not\subseteq W$, the conditions $(L+U) \cap W \neq \{0\} \iff (W+L) \cap U \neq \{0\}$ and $U \subset L^\perp$ imply that

$$L^\perp \cap (W+L) \cap U = (W+L) \cap U \neq \{0\}.$$

Noting that $dim_{\mathbb{R}}(L^{\perp} \cap (W + L)) \leq k$ and supposing that $k \leq n - 2$, by induction we have that

$$\gamma_{L^{\perp},m-1}(\{U \subset L^{\perp} \mid (L+U) \cap W \neq \{0\}\}) \leq \gamma_{L^{\perp},m-1}(\{U \subset L^{\perp} \mid L^{\perp} \cap (L+W) \cap U \neq \{0\}\}) = 0.$$

If $k = n - 1$ then $m - 1 = 0$ and the above inequality holds true without using the inductive hypothesis. Finally integrating 2.3 over the lines L such that $L \subsetneqq W$ we conclude. $\square$

Taking $V, W \in G(n, m)$, $V^{\perp} \cap W = \{0\}$ if and only if $P_V|_W : W \longleftrightarrow V$ is injective. Therefore

Corollary 2.14. *Let $W \in G(n, m)$. Then $P_V|_W : W \longrightarrow V$ is injective for $\gamma_{n,m}$-almost all $V \in G(n, m)$.*

Chapter 3

Hausdorff measures and Lipschitz maps on $\mathbb{R}^n$

3.1 Carathéodory's construction

The Caratheodory's construction is useful in order to define measures starting from a family of sets $\mathcal{F} \subset \mathcal{P}(X)$. The construction is very similar to the definition of Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$. The idea is to approximate areas of generic sets by covering them with sets of known area (for example n-rectangles in the case of $\mathcal{L}^n$), subsequently sum them and taking the infimum over all the possible coverings. Let us do this in general and precisely.

Let X be a metric space and $\mathcal{F} \subset \mathcal{P}(X)$ such that:

1. $\forall \delta > 0$ there are $\{E_i\}_{i\geq 1} \subset \mathcal{F}$ such that $X = \bigcup_{i=1}^{+\infty} E_i$ and $d(E_i) \leq \delta$ for all $i = 1, 2, \dots$
2. $\forall \delta > 0$ there exist $E_\delta \in \mathcal{F}$ such that $\zeta(E_\delta) \leq \delta$ and $d(E_\delta) \leq \delta$.

where $\zeta : \mathcal{F} \longrightarrow \mathbb{R}_{\geq 0}$ is a non negative function. Let $E \subset X$. We define for $0 < \delta \leq +\infty$

$$\psi_\delta(E) = \inf \left\{ \sum_{i=1}^{+\infty} \zeta(E_i) \mid E \subset \bigcup_{i=1}^{+\infty} E_i,\ d(E_i) \leq \delta,\ E_i \in \mathcal{F} \right\}. \tag{3.1}$$

Proposition 3.1. *ψ_δ, defined in 3.1, is a measure.*

Proof. $\psi_\delta(\emptyset) = 0$: let $0 < \varepsilon < \delta$. Taking E_ε such that $\zeta(E_\varepsilon) \leq \varepsilon$ and $d(E_\varepsilon) \leq \varepsilon$ we have $\psi_\delta(\emptyset) < \varepsilon$. Let $A \subset B$; a cover of B of elements in $\mathcal{F}$ is also a cover for A, then $\psi_\delta(A) \leq \psi_\delta(B)$. A similar argument holds for the last property of measure, see definition 1.1. □

Let $\varepsilon < \delta \leq +\infty$ and $E \subset X$. Since

$$\left\{ \{E_i\}_{i\geq 1} \subset \mathcal{F} \mid E \subset \bigcup_{i=1}^{+\infty} E_i,\ d(E_i) \leq \varepsilon \right\} \subseteq \left\{ \{E_i\}_{i\geq 1} \subset \mathcal{F} \mid E \subset \bigcup_{i=1}^{+\infty} E_i,\ d(E_i) \leq \delta \right\}$$

we have that $\psi_\delta(E) \leq \psi_\varepsilon(E)$. We can define a measure

$$\psi(E) = \lim_{\delta \to 0} \psi_\delta(E) = \sup_{\delta > 0} \psi_\delta(E)$$

for all $E \subset X$. It is easy to see that ψ is a measure. If we want to specify $\mathcal{F}$ and ζ, we will write $\psi = \psi(\mathcal{F}, \zeta)$.

Theorem 3.2. *Let $\mathcal{F}$ and ζ as above, and $\psi = \psi(\mathcal{F}, \zeta)$. Then*

1. *ψ is a Borel measure.*

2. *If $\mathcal{F}$ is formed by Borel sets, ψ is Borel regular.*

Proof.

1. Recalling Theorem 1.3, let A, B such that $d(A, B) > 0$. We can then choose $0 < \delta < d(A, B)/2$. If we take a cover $\{E_i\}_{i=1,\dots}$ of $A \cup B$ with $d(E_i) < \delta$ for all $i = 1, \dots,$ then there are no sets of this cover that meet both A and B. Therefore

$$\sum_i \zeta(E_i) \geq \sum_{A \cap E_i \neq \emptyset} \zeta(E_i) + \sum_{B \cap E_i \neq \emptyset} \zeta(E_i) \geq \psi_\delta(A) + \psi_\delta(B). \tag{3.2}$$

The inequality 3.2 holds for all the covers of $A \cup B$. Taking the infimum we get $\psi_\delta(A) + \psi_\delta(B) \leq \psi_\delta(A \cup B)$ and since ψ_δ is a measure, $\psi_\delta(A) + \psi_\delta(B) = \psi_\delta(A \cup B)$. Taking the limit as δ tends to zero we conclude.

2. Let $A \subset X$ and choose for $1 \leq i \in \mathbb{N}$ sets $E_{i,j}$ such that $A \subset \bigcup_{j=1}^{+\infty} E_{i,j}$, $d(E_{i,j}) \leq \frac{1}{i}$ and

$$\sum_{j=1}^{+\infty} \zeta(E_{i,j}) \leq \psi_{\frac{1}{i}}(A) + \frac{1}{i}.$$

Let $B = \bigcap_{i=1} \bigcup_j E_{i,j}$. Then B is a Borel set such that $A \subset B$. $\psi(A) \leq \psi(B)$ and

$$\psi_{\frac{1}{i}}(B) \leq \sum_{j=1}^{+\infty} \zeta(E_{i,j}) \leq \psi_{\frac{1}{i}}(A) + \frac{1}{i}.$$

Letting $i \to +\infty$ we get $\psi(A) = \psi(B)$. $\square$

3.2 Hausdorff measures

Let X be a separable metric space, $0 \leq s < +\infty$. Let also $\mathcal{F} = \mathcal{P}(X)$ and $\zeta_s(E) = d(E)^s$. We will interpret $0^0 = 1$ and $d(\emptyset)^s = 0$. Then the measure $\psi(\mathcal{F}, \zeta_s)$ is called the s-dimensional *Hausdorff measure* and it will be denoted by $\mathcal{H}^s$. $\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E)$ where

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{i=1}^{+\infty} d(E_i)^s \mid E \subset \bigcup_{i=1}^{+\infty} E_i,\ d(E_i) \leq \delta \right\}$$

for all $E \subset X$. $\mathcal{H}^0$ is the counting measure, for instance $\mathcal{H}^0(A) = |A|$. $\mathcal{H}^1$ can be viewed as a generalized length measure. In many textbooks the definition of Hausdorff measure is slightly different when $X = \mathbb{R}^n$. For instance see [B, 2.1]. It is defined analogously but with a factor $\alpha(n)/2^s$, so that $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$. The equality is not trivial, for a proof see [B, 2.2]. With our definition we have

$$\mathcal{H}^n = 2^n \alpha(n)^{-1} \mathcal{L}^n$$

and therefore

$$\mathcal{H}^n(B(x,r)) = (2r)^n.$$

Definition 3.1. Given $A \subset \mathbb{R}^n$, we define the *convex hull* of A, $\iota_0 A$ as

$$\iota_0 A = \bigcap_{C \supseteq A \ convex} C.$$

The convex hull of a set can be defined also as the smallest (and unique) convex set containing A.

Proposition 3.3. *Let $A \subset \mathbb{R}^n$. Then $d(A) = d(\iota_0 A)$.*

Proof. Since $A \subset \iota_0 A$ then $d(A) \leq d(\iota_0 A)$. Let $x, y \in \iota_0 A$, then there exist $x_1, \ldots, x_p \in A$ and $y_1, \ldots, y_q \in A$ such that

$$x = \sum_{i=1}^{p} \lambda_i x_i \quad y = \sum_{i=1}^{q} \mu_i y_i$$

with $\lambda_i, \mu_i \in \mathbb{R}_{\geq 0}$ and $\sum_{i=1}^{p} \lambda_i = \sum_{j=1}^{q} \mu_j = 1$.Then

$$d(x,y) \leq \sum_{i=1}^{p} \lambda_i d\left(x_i, \sum_{j=1}^{q} \mu_j y_j\right) \leq$$
$$\leq \sum_{i=1}^{p} \lambda_i \sum_{j=1}^{q} \mu_j d(x_i, y_j) \leq d(A) \sum_{i=1}^{p} \left(\lambda_i \sum_{j=1}^{q} \mu_j\right) = d(A).$$

□

Theorem 3.4. *Let $0 \leq s < +\infty$ and $\zeta_s(E) = d(E)^s$ for $E \subset X$ with X separable metric space. Then, if*

1. *$\mathcal{F} = \{U \subset X \mid U \text{ is open}\}$ or*
2. *$\mathcal{F} = \{C \subset X \mid C \text{ is closed}\}$ or*
3. *if $X = \mathbb{R}^n$, $\mathcal{F} = \{K \subset \mathbb{R}^n \mid K \text{ is convex}\}$*

we have that $\psi(\mathcal{F}, \zeta_s) = \mathcal{H}^s$.

Remark 7. The hypothesis that X is separable is necessary, because otherwise the families of sets we consider in Theorem 3.4 could not be used to define $\psi(\mathcal{F}, \zeta_s)$.

Proof.

1. Let $E \subset X$ and $E_\varepsilon = \{x \in X \mid d(x, E) < \varepsilon\}$. E_ε is open and $d(E_\varepsilon) \leq d(E) + 2\varepsilon$. It is true that $\psi_\delta(E) \geq \mathcal{H}^s_\delta(E)$, therefore the inequality $\psi(E) \geq \mathcal{H}^s(E)$ is verified. Let $E_1, E_2, \ldots$ with $d(E_i) \leq \delta$ such that $E \subset \bigcup_{i=1}^{+\infty} E_i$, and
$$\sum_{i=1}^{+\infty} d(E_i)^s \leq \mathcal{H}^s_\delta(E) + \delta.$$
Therefore
$$E \subset \bigcup_{i=1}^{+\infty} E_{i,\delta\nu_i/2},$$
where $\nu_i > 0$. Then $d(E_{i,\delta\nu_i/2})^s \leq (d(E_i)+\delta\nu_i)^s$. Since $(d(E_i)+\delta\nu_i)^s \to d(E_i)^s$ as $\nu_i \to 0$ we can choose, for each i, ν_i small enough to have $(d(E_i) + \delta\nu_i)^s \leq d(E_i)^s + \delta/2^i$. Let $\nu := 2\sup_i\{\nu_i\}$, we can suppose that $\nu \to 0$ as $\delta \to 0$. Then, since $d(E_{i,\delta\nu_i/2}) \leq \delta + \delta\nu$, we obtain
$$\psi_{\delta+\delta\nu}(E) \leq \sum_{i=1}^{+\infty} d(E_{i,\delta\nu_i/2})^s \leq \sum_{i=1}^{+\infty} d(E_i)^s + \delta/2^i \leq \mathcal{H}^s_\delta(E) + 2\delta$$
which means that $\psi_{\delta(1+\nu)}(E) \leq \mathcal{H}^s_\delta(E) + 2\delta$. Letting $\delta \to 0$ we obtain $\psi(E) \leq \mathcal{H}^s(E)$.
2. True, since $d(E) = d(\overline{E})$ where $\overline{E}$ is the closure of E.
3. Trivial, since $d(E) = d(\iota_0 E)$. □

Remark 8. If $X = \mathbb{R}^n$ for $n \geq 2$, $\mathcal{F} = \{B(x,r) | x \in \mathbb{R}^n, \ r > 0\}$ and $\zeta_s(E) = d(E)^s$ the resulting measure from the Caratheodory's construction is not $\mathcal{H}^s$ for $0 < s < n$. It is called the s-spherical measure and it is indicated by $\mathcal{S}^s$.

Corollary 3.5. *Let $0 \leq s < +\infty$ and $A \subset \mathbb{R}^n$. Then*

1. *$\mathcal{H}^s$ is Borel regular*
2. *$\mathcal{H}^s(aA) = a^s\mathcal{H}^s(A)$ where $aA = \{ax \mid x \in A\}$*
3. *$\mathcal{H}^s(LA) = \mathcal{H}^s(A)$ for all $L : \mathbb{R}^n \to \mathbb{R}^n$ rigidities of $\mathbb{R}^n$ ($|Lx| = |x|$).*

Proof. The first point follows from 3.4 and 3.2 even when X is a generic separable metric space. The last two assertions are easy to show (see also Theorem 3.13). □

The following lemma is very useful to see if a set has $\mathcal{H}^s$ null measure for some s.

Lemma 3.6. *Let $E \subset X$, $0 \leq s < +\infty$ and $0 \leq \delta \leq +\infty$. Then, the following are equivalent:*

1. $\mathcal{H}^s(E) = 0$.

2. $\mathcal{H}^s_\delta(E) = 0$.

3. $\forall \varepsilon > 0\ \exists E_1, E_2, \ldots \subset X$ *such that* $E \subset \bigcup_{i=1}^{+\infty} E_i$ *and* $\sum_{i=1}^{+\infty} d(E_i)^s < \varepsilon$.

Proof. $1 \Longrightarrow 2$ is trivial, as $2 \Longrightarrow 3$. Let us assume 3 and let $\varepsilon > 0$; since $\sum_{i=1}^{+\infty} d(E_i)^s < \varepsilon$ is convergent, let $\nu^s = \sup_{i\geq 1} d(E_i)^s$. We have that $\nu \leq \varepsilon^{1/s}$, then

$$\mathcal{H}^s_\nu(E) \leq \sum_{i=1}^{+\infty} d(E_i)^s < \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain $\mathcal{H}^s(E) = 0$. □

The next theorem will be fundamental to define the Hausdorff dimension of X:

Theorem 3.7. *Let* $0 \leq s < t < +\infty$ *and* $E \subset X$ *then:*

1. $\mathcal{H}^s(E) < +\infty \Longrightarrow \mathcal{H}^t(E) = 0$,

2. $\mathcal{H}^t(E) > 0 \Longrightarrow \mathcal{H}^s(E) = +\infty$.

Proof. Of course $2 \Longleftrightarrow 1$. Let us show that the first assertion is true. Take E_i such that $E \subset \bigcup_{i=1}^{+\infty} E_i$, $d(E_i) \leq \delta$ and $\sum_{i=1}^{+\infty} d(E_i)^s \leq \mathcal{H}^s_\delta(E) + 1$. Therefore

$$\mathcal{H}^t_\delta(E) \leq \sum_{i=1}^{+\infty} d(E_i)^t \leq \delta^{t-s} \sum_{i=1}^{+\infty} d(E_i)^s \leq \delta^{t-s}(\mathcal{H}^s_\delta(E) + 1),$$

which gives, as $\delta \to 0$, $\mathcal{H}^t(E) = 0$. □

Now we can give the definition of Hausdorff dimension.

Definition 3.2. Let $E \subset X$. Then the *Hausdorff dimension* is defined as

$$\begin{aligned} dimE &:= \sup\{s \mid \mathcal{H}^s(E) > 0\} = \sup\{s \mid \mathcal{H}^s(E) = +\infty\} = \\ &= \inf\{s \mid \mathcal{H}^s(E) < +\infty\} = \inf\{s \mid \mathcal{H}^s(E) = 0\}. \end{aligned}$$

The Hausdorff dimension has some natural properties:

Proposition 3.8. *Let* $E \subset X$. *Then*

1. $dimE \leq dimF$ *for* $E \subset F$

2. $dim \bigcup_{i=1}^{+\infty} E_i = \sup_i dimE_i$ *for* $E_i \subset X$.

Proof. The first statement is easy and it is left as exercise. Let $d = \sup_i dimE_i$. The case $d = 0$ is left to the reader. If $d < +\infty$ let c, e such that $c < d < e$. Then there exists E_j such that

$$+\infty = \mathcal{H}^c(E_j) \leq \mathcal{H}^c\left(\bigcup_{i=1}^{+\infty} E_i\right)$$
$$\mathcal{H}^e\left(\bigcup_{i=1}^{+\infty} E_i\right) \leq \sum_{i=1}^{+\infty} \mathcal{H}^e(E_i) = 0.$$

Therefore $d = dim \bigcup_{i=1}^{+\infty} E_i$. If $d = +\infty$ the proof is similar. □

Remark 9. In general, for a set $A \subset X$, if we find an s such that $0 < \mathcal{H}^s(A) < +\infty$ then, $s = dimA$.

If we take $X = \mathbb{R}^n$, since $0 < \mathcal{H}^n(B(0,r)) = (2r)^n < +\infty$, we have that $dimB(0,r) = n$. Then, because $\mathbb{R}^n = \bigcup_{n=1}^{+\infty} B(0,n)$, it follows that $dim\mathbb{R}^n = n$. In $\mathbb{R}^n$ it is interesting to consider Hausdorff measures only when $0 \leq s \leq n$. Moreover one can show that for each $0 \leq s \leq n$ there is $E \subseteq \mathbb{R}^n$ such that $dimE = s$.

Remark 10. $\mathcal{H}^s$ is a Borel regular measure, but in general it is not locally finite and therefore Radon. For example take $s < n$, then $\mathcal{H}^s B(x,r) = +\infty$ for each $x \in \mathbb{R}^n$ and $r > 0$. But, if $E \subset X$ is $\mathcal{H}^s$-measurable and $\mathcal{H}^s(E) < +\infty$ then $\mathcal{H}^s \llcorner E$ is a Radon measure.

3.3 Cantor set in $\mathbb{R}$

In general it is not easy to compute the Hausdorff dimension of a set. In this section we will briefly calculate the dimension of the Cantor set. First of all let us define it: let $0 < \lambda < 1/2$ and $I_{0,1} = [0,1]$. From now on, with the first index of the intervals we will indicate the "iteration" and the second index will serve us to enumerate the intervals. Remove from $I_{0,1}$ an open ball with centre $1/2$ and diameter $(1-2\lambda)$; we obtain $I_{1,1} = [0,\lambda]$ and $I_{1,2} = [1-\lambda, 1]$ each of length λ. Then we can iterate this process for $I_{1,1}, I_{1,2}$ by removing an open interval of length $(1-2\lambda)\lambda$ and midpoint the midpoint of each $I_{1,i}$ for $i = 1, 2$. We obtain $I_{2,1}, I_{2,2}, I_{2,3}, I_{2,4}$ closed intervals of length λ^2. In this way at the k-th iteration we have $\{I_{k,i}\}_{i=1,\ldots,2^k}$ intervals of length λ^k, see figure 3.1. Note that $\bigcup_i I_{k,i} \subset \bigcup_i I_{k-1,i}$. We can now define the λ-Cantor set $C(\lambda)$ as

$$C(\lambda) = \bigcap_{k=0}^{+\infty} \bigcup_{i=1}^{2^k} I_{k,i}.$$

$C(\lambda)$ is non-empty and compact, because in a metric space if $K_1 \supset K_2 \supset K_3 \supset \ldots$ are compact then $K = \cap_i K_i$ is a non-empty compact set. Moreover $C(\lambda)$ has no interior

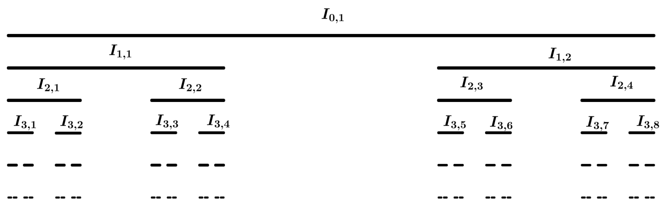

Figure 3.1: Cantor set (5 iterations) with $\lambda = \frac{1}{3}$

points and Lebesgue measure equal to 0: in fact the length of the intervals that we are removing is

$$\sum_{j=0}^{+\infty}(2\lambda)^j(1-2\lambda) = 1.$$

Moreover $C(\lambda)$ is uncountable. Taking $\{I_{k,i}\}_{i=1,\dots,2^k}$ as a cover of $C(\lambda)$ we have that

$$\mathcal{H}^s_{\lambda^k}(C(\lambda)) \leq \sum_{i=1}^{2^k} d(I_{k,i})^s = 2^k(\lambda^k)^s = (2\lambda^s)^k. \tag{3.3}$$

If $s = log(2)/log(1/\lambda)$ then $2\lambda^s = 1$ and therefore, letting $k \to +\infty$, we get that $\mathcal{H}^s(C(\lambda)) \leq 1$ and so $dimC(\lambda) \leq s$. Now we will show that $\mathcal{H}^s(C(\lambda)) \geq 1/4$ and it will follow that

$$dimC(\lambda) = \frac{log(2)}{log(1/\lambda)}.$$

Let $s = log(2)/log(1/\lambda)$. Since $C(\lambda)$ is compact we can cover it with finitely many intervals J_l with $l = 1, \dots, n$. We will show that $\sum_{l=1}^n d(J_l)^s \geq 1/4$. $C(\lambda)$ has no interior points, then we can assume that the endpoints of each interval J_l are outside $C(\lambda)$ (otherwise $C(\lambda)$ would have a non-empty interior). Then each endpoint has distance at least, for instance, $\delta > 0$. Choosing k big enough (so that $\delta > \lambda^k$), we have that each $\{I_{k,i}\}_{i=1,\dots,2^k}$ is contained in some J_l. Now fix a general I open interval which intersect $(0,1)$, then I contains some $I_{w,i}$. Let n be the smallest integer such that $I_{n,i}$ is contained in I for some i. Let $I_{n,i_1}, \dots, I_{n,i_p}$ intervals contained in I. Then $p \leq 4$, otherwise there exists an $I_{n-1,i}$ contained in I. Then

$$4d(I)^s \geq \sum_{m=1}^{p} d(I_{n,i_m})^s = \sum_{m=1}^{p} \sum_{I_{w,i} \subset I_{n,i_m}} d(I_{w,i})^s \geq \sum_{I_{w,i} \subset I} d(I_{w,i})^s$$

where the equality holds since there are 2^{w-n} intervals $I_{w,i}$ in I_{n,i_m} of length λ^w. This leads to

$$2^{w-n}(\lambda^w)^s = \lambda^{sn}(2\lambda^s)^w = \lambda^{sn} = d(I_{n,i_m})^s.$$

Therefore, since each $\{I_{k,i}\}_{i=1,\dots,2^k}$ is contained in some J_l,

$$4\sum_{l=1}^{n} d(J_l)^s \geq \sum_{i=1}^{2^k} d(I_{k,i})^s = 2^k(\lambda^s)^k = 1$$

which gives us

$$\sum_{l=1}^{n} d(J_l)^s \geq 1/4.$$

We can choose the intervals J_l such that

$$1/4 \leq \sum_{l=1}^{n} d(J_l)^s \leq \mathcal{H}^s(C(\lambda)) + \varepsilon$$

and letting $\varepsilon \to 0$ we obtain that $1/4 \leq \mathcal{H}^s(C(\lambda))$. Moreover one can show that $\mathcal{H}^s(C(\lambda)) = 1$.

3.4 Density Theorems for Hausdorff measures

In this section we will present a couple of theorems on Hausdorff density.

Definition 3.3. Let $0 \leq s \leq n$, $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The upper s-density of E at x is defined by

$$\Theta^{*s}(E,x) := \limsup_{r\to 0} \frac{\mathcal{H}^s(E\cap B(x,r))}{(2r)^s}$$

and the lower s-density of E at x is

$$\Theta^s_*(E,x) := \liminf_{r\to 0} \frac{\mathcal{H}^s(E\cap B(x,r))}{(2r)^s}.$$

Naturally, $\Theta^s_*(E,x) \leq \Theta^{*s}(E,x)$ and if they are equal we define the s-dimensional density of E at x as

$$\Theta^s(E,x) = \Theta^s_*(E,x) = \Theta^{*s}(E,x).$$

Remark 11. For $s > n$ one could define upper and lower densities, but they are always equal to 0. For $s = n$ we obtain the usual Lebesgue densities. By Corollary 2.3 we have that $\Theta^n(E,x) = 1$ if $x \in E$ and, if E is Lebesgue measurable, $\Theta^n(E,x) = 0$ if $x \notin E$.

Theorem 3.9. *Let $E \subset \mathbb{R}^n$ $\mathcal{H}^s$-measurable with $\mathcal{H}^s(E) < +\infty$. Then*

1. $2^{-s} \leq \Theta^{*s}(E,x) \leq 1$ *for $\mathcal{H}^s$-almost all $x \in E$.*
2. $\Theta^{*s}(E,x) = 0$ *for $\mathcal{H}^s$-almost all $x \in \mathbb{R}^n \setminus E$.*

Remark 12. The measurability of E is not required to show that $2^{-s} \leq \Theta^{*s}(E,x)$; moreover we can notice that $\mathcal{H}^s \llcorner E$ is a Borel regular measure (and Radon) and so $x \mapsto \Theta^{*s}(E,x), \Theta^s_*(E,x)$ are Borel functions.

Proof.

1. We first prove that $2^{-s} \leq \Theta^{*s}(E,x)$. Let $B = \{x \in E \mid 2^{-s} > \Theta^{*s}(E,x)\}$; using the definition of Θ^{*s} as $\limsup$ the reader can see that, setting

$$B_k := \{x \in E \mid \mathcal{H}^s(E \cap B(x,r)) < (k/k+1)r^s \text{ for } 0 < r < 1/k\}$$

for $k = 1, 2, \dots,$

$$B = \bigcup_{k=1}^{+\infty} B_k.$$

Therefore we only have to show that $\mathcal{H}^s(B_k) = 0$ for all $k \geq 1$. We can cover B_k with E_i such that $d(E_i) < 1/k$ and

$$\sum_{i=1}^{+\infty} d(E_i)^s \leq \mathcal{H}^s(B_k) + \varepsilon.$$

We can suppose that $B_k \cap E_i \neq \emptyset$ for all i, then let $x_i \in B_k \cap E_i$ and $r_i = d(E_i)$. Since $B_k \cap E_i \subset B(x_i, r_i) \cap E$ we have that

$$\mathcal{H}^s(B_k) \leq \sum_{i=1}^{+\infty} \mathcal{H}^s(B_k \cap E_i) \leq \sum_{i=1}^{+\infty} \mathcal{H}^s(B(x_i,r_i) \cap E) \leq$$

$$\leq \sum_{i=1}^{+\infty} \frac{k}{k+1} r_i^s \leq \frac{k}{k+1}(\mathcal{H}^s(B_k) + \varepsilon)$$

By hypothesis $\mathcal{H}^s(B_k) < +\infty$, then letting $\varepsilon \to 0$, and since $0 < k/(k+1) < 1$ we conclude that $\mathcal{H}^s(B_k) = 0$. Now we prove that $\Theta^{*s}(E,x) \leq 1$ for $\mathcal{H}^s$-almost every $x \in E$. Let

$$C := \{x \in E \mid \Theta^{*s}(E,x) > \alpha\}$$

for $\alpha > 1$. Since E is $\mathcal{H}^s$-measurable of finite $\mathcal{H}^s$-measure and $\Theta^{*s}(E,x)$ is a Borel function, we can find an open set U that contains C and such that $\mathcal{H}^s(E \cap U) < \mathcal{H}^s(C) + \varepsilon$ (Theorem 1.5). For every $x \in C$ we have that

$$\alpha < \inf_{\delta > 0} \sup_{r < \delta} \mathcal{H}^s(E \cap B(x,r))/(2r)^s.$$

Then we can find an arbitrarily small r such that $0 < r < \delta/2$, that $B(x,r) \subset U$ and such that

$$\alpha < \mathcal{H}^s(E \cap B(x,r))/(2r)^s.$$

We can then apply Vitali's covering Theorem 1.22 to $\mathcal{H}^s \llcorner U$ and find disjoint balls $\{B_i\}_{i=1,\dots}$ of radius less than $\delta/2$ such that

$$\mathcal{H}^s\left(C \setminus \bigcup_{i=1}^{+\infty} B_i\right) = 0.$$

We can notice that $\mathcal{H}^s_\delta(C) = \mathcal{H}^s_\delta(C \cap \bigcup_{i=1}^{+\infty} B_i)$. Therefore

$$\mathcal{H}^s(C)+\varepsilon > \mathcal{H}^s(E\cap U) \geq \sum_{i=1}^{+\infty} \mathcal{H}^s(E\cap B_i) > \alpha \sum_{i=1}^{+\infty} d(B_i)^s \geq \alpha \mathcal{H}^s_\delta \left(C \cap \bigcup_{i=1}^{+\infty} B_i \right) = \alpha \mathcal{H}^s_\delta(C).$$

Letting $\varepsilon, \delta \to 0$ we conclude that $\mathcal{H}^s(C) \geq \alpha \mathcal{H}^s(C)$ and since $\alpha > 1$ we conclude that $\mathcal{H}^s(C) = 0$.

2. Let $B = \{x \in \mathbb{R}^n \setminus E \mid \Theta^{*s}(E, x) > \alpha\}$ for $\alpha > 0$. Let $\varepsilon > 0$. Since $(\mathcal{H}^s \llcorner E)(B) = 0$ we can find U open such that $B \subset U$ and $\mathcal{H}^s(E \cap U) < \varepsilon$. Again, we can find $0 < r < \delta/2$ small enough such that for each $x \in B$, $B(x, r) \subset U$ and

$$\mathcal{H}^s(E \cap B(x, r)) > \alpha(2r)^s.$$

By Theorem 1.16 we can find $B_i = B(x_i, r_i)$ disjoint with $x_i \in B$, such that $5B_i$ covers B. Therefore

$$\mathcal{H}^s_\infty(B) \leq \sum_{i=1}^{+\infty} d(5B_i)^s = 5^s \sum_{i=1}^{+\infty} d(B_i)^s <$$
$$< \frac{5^s}{\alpha} \sum_{i=1}^{+\infty} \mathcal{H}^s(E \cap B_i) \leq \frac{5^s}{\alpha} \mathcal{H}^s(E \cap U) < \frac{5^s \varepsilon}{\alpha}.$$

Letting $\varepsilon \to 0$ we obtain $\mathcal{H}^s_\infty(B) = 0$ which implies $\mathcal{H}^s(B) = 0$ by Lemma 3.6. If in place of $\mathcal{H}^s_\infty(B)$ we had estimated $\mathcal{H}^s_{5^s\delta}(B)$, we would have reached the same conclusion. □

Corollary 3.10. *Let E, F be $\mathcal{H}^s$-measurable subset of $\mathbb{R}^n$ with $E \subset F$ and $\mathcal{H}^s(F) < +\infty$. Then for $\mathcal{H}^s$-almost all $x \in E$*

$$\Theta^{*s}(E, x) = \Theta^{*s}(F, x)$$

and

$$\Theta^s_*(E, x) = \Theta^s_*(F, x).$$

Proof. For $\mathcal{H}^s$-almost all $x \in E$ $\Theta^{*s}(F \setminus E, x) = 0$. Therefore

$$\Theta^{*s}(E, x) \leq \Theta^{*s}(F, x) \leq \Theta^{*s}(F \setminus E, x) + \Theta^{*s}(E, x) = \Theta^{*s}(E, x).$$

□

The last corollary tells us somehow that densities are preserved if the set is enlarged. Moreover, if A, B are measurable and both are of finite measure, then $A \cup B$ has the same density as $A \cap B$ for almost all $x \in A \cap B$.

3.5 Lipschitz functions

In this section we will discuss about Lipschitz functions and some of their properties.

Definition 3.4. A function $f : D \to \mathbb{R}^n$ for $D \subset \mathbb{R}^m$ is a *Lipschitz map* if there is a constant L such that

$$|f(x) - f(x)| \leq L|x - y|, \tag{3.4}$$

for all $x, y \in D$. The smallest constant such that 3.4 holds will be called the *Lipschitz constant* of f and it will be denoted by $Lip(f)$.

There is another definition of function that extends the property of being Lipschitz:

Definition 3.5. A function $f : D \to \mathbb{R}^n$ for $D \subset \mathbb{R}^m$ is *Hölder continuous* of parameter $0 < \alpha \leq 1$ if there is a constant C such that

$$|f(x) - f(x)| \leq C|x - y|^{\alpha}, \tag{3.5}$$

for all $x, y \in D$. The smallest constant C such that 3.5 holds will be called the *Hölder constant* of f and it will be denoted by $L_{\alpha}(f)$.

If a function is Hölder continuous then it is continuous, and if $\alpha = 1$ we have that f is Lipschitz. Every Lipschitz function $f : D \to \mathbb{R}^n$ defined on a proper subset of $\mathbb{R}^m$ can be extended to $\mathbb{R}^m$:

Theorem 3.11. *Let $f : D \to \mathbb{R}^n$ be a Lipschitz map with Lipschitz constant $Lip(f)$ and $D \subset \mathbb{R}^n$. Then there exists $g : \mathbb{R}^m \to \mathbb{R}^n$ such that $f = g_{|D}$, and $Lip(g) \leq \sqrt{n} Lip(f)$.*

Proof. Let $f = (f_1, \dots, f_n)$, where f_i are the coordinate functions for $i = 1, \dots, n$. We define

$$g_i(x) := \inf_{d \in D} \big(f_i(x) + Lip(f_i)|x - d| \big).$$

of course $g_i(x) = f_i(x)$ for all $x \in D$. Then

$$g_i(x) \leq \inf_{d \in D} \big(f_i(y) + Lip(f_i)|x - y| + Lip(f_i)|y - d| \big) = g_i(y) + Lip(f_i)|x - y|$$

and similarly, $g_i(y) \leq g_i(x) + Lip(f_i)|x - y|$ and g_i is Lipschitz with $Lip(g_i) \leq Lip(f_i) \leq Lip(f)$ for all i. Finally, setting $g = (g_1, \dots, g_n)$,

$$|g(x) - g(y)|^2 = \sum_{i=1}^{n} |g_i(x) - g_i(y)|^2 \leq n Lip(f)^2 |x - y|^2.$$

□

This last theorem can be adapted also for a Hölder continuous map. From the above proof, when $n = 1$, we can conclude that $Lip(g) = Lip(f)$. If $n > 1$ we can just say that $Lip(g) \leq \sqrt{n} Lip(f)$. It is however true that f can be extended in such a way that $Lip(f) = Lip(g)$ but it is not easy to show it (Kirszbraun's theorem); for a proof see [E, 2.10.43].

One of the most important property for Lipschitz maps is that they are differentiable almost everywhere. This is the so-called Rademacher's Theorem. Let us first recall

Definition 3.6. A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *differentiable* at $x \in \mathbb{R}^m$ if there is a (unique) linear map

$$L : \mathbb{R}^m \to \mathbb{R}^n$$

such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - L(y-x)|}{|y-x|} = 0.$$

We call L the derivative of f at x and it is denoted with $Df[x]$.

Theorem 3.12 (Rademacher's Theorem)**.** *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map. Then f is differentiable $\mathcal{L}^m$-almost everywhere.*

Proof. We can assume that $n = 1$, since we could study the coordinate functions. We shall also consider the case $m = 1$ to be known since in one dimension Lipschitz functions are absolutely continuous and so they are differentiable almost everywhere. For $e \in S^{m-1}$ we denote, for $x \in \mathbb{R}^m$, $\partial_e f(x)$ the partial derivative of f in the direction of e. Let B_e the set of points such that $\partial_e f(x)$ does not exist. Since f is a continuous function,

$$\overline{D}_e f(x) := \limsup_{r \to 0} \frac{f(x+te) - f(x)}{t}, \quad \underline{D}_e f(x) := \liminf_{r \to 0} \frac{f(x+te) - f(x)}{t}$$

are Borel function, and

$$B_e = \{x \in \mathbb{R}^m \mid \underline{D}_e f(x) < \overline{D}_e f(x)\}$$

is a Borel set. Let $L_e = \{x + te \mid t \in \mathbb{R}\}$. Applying the one dimensional case to $t \mapsto f(x + te)$, we have

$$\mathcal{H}^1(B_e \cap L_e) = 0.$$

By Fubini's Theorem 1.12 we obtain that $\mathcal{L}^m(B_e) = 0$, therefore $\partial_e f(x)$ exists for $\mathcal{L}^m$-almost all $x \in \mathbb{R}^m$.

Now we will show that $\partial_e f(x) = e \cdot \nabla f(x)$ where $\nabla f(x) = (\partial_1 f(x), \ldots, \partial_m f(x))$ for $\mathcal{L}^m$-almost all $x \in \mathbb{R}^m$. Here $\partial_i f(x) = \partial_{e_i} f(x)$ where e_i is the i-th vector of the canonical base of $\mathbb{R}^m$. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^m)$. Then, for $h \neq 0$,

$$\int h^{-1}(f(x+he) - f(x))\varphi(x)\,dx = -\int h^{-1}(\varphi(x) - \varphi(x-he))f(x)\,dx.$$

Since f is Lipschitz we can apply Lebesgue's dominated convergence Theorem, (see [C, 2.14]) letting $h \to 0$ and applying partial integration we obtain

$$\begin{aligned}\int \partial_e f(x)\varphi(x)\,dx = &-\int f(x)\partial_e\varphi(x)\,dx = \\ &= -\int f(x)(e\cdot\nabla\varphi(x))\,dx = -\sum_{i=1}^m e\cdot e_i\int f(x)\partial_i\varphi(x)\,dx = \\ &= \sum_{i=1}^m e\cdot e_i\int \varphi(x)\partial_i f(x)\,dx = \int \varphi(x)(e\cdot\nabla f(x))\,dx.\end{aligned}$$

Then

$$\int(\partial_e f(x) - e\cdot\nabla f(x))\varphi(x)\,dx = 0$$

and since this holds for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^m)$ we have that $\partial_e f(x) = e\cdot\nabla f(x)$ for $\mathcal{L}^m$-almost all $x\in\mathbb{R}^m$.

Let $\{v_i\}_{i=1,\dots,}$ be a dense subset of S^{m-1}. For each i, let A_i be the set of $x\in\mathbb{R}^m$ for which $\nabla f(x)$ and $\partial_{v_i} f(x)$ exist and $\partial_{v_i} f(x) = v_i\cdot\nabla f(x)$. Let $A = \bigcap_{i=1}^{+\infty} A_i$. Therefore we can say, for what we have proved, that $\mathcal{L}^m(\mathbb{R}^m\setminus A) = 0$.

Now we will finally show that f is differentiable at all $x\in A$: let $x\in A$ and $e\in\mathcal{S}^{m-1}$ and $h>0$. Let

$$Q(x,e,h) = h^{-1}(f(x+he)-f(x)) - e\cdot\nabla f(x).$$

We are going to show that $\lim_{h\to 0} Q(x,e,h) = 0$ **uniformly in e**. Since $|\partial_i f(x)| \le Lip(f)$ then $|\nabla f(x)| \le \sqrt{m}Lip(f)$ and by Cauchy-Schwartz inequality we have, for $e,v\in\mathcal{S}^{m-1}$, that

$$\begin{aligned}|Q(x,e,h) - Q(x,v,h)| &\le \\ &\le |h^{-1}(f(x+he)-f(x+hv))| + |(e-v)\cdot\nabla f(x)| \le \\ &\le (1+\sqrt{m})Lip(f)|e-v|.\end{aligned}$$

Now, since $\mathcal{S}^{m-1}$ is compact, there is an N big enough such that $\forall e\in\mathcal{S}^{m-1}$ there exists $k\in\{1,\dots,N\}$ such that

$$|e-v_k| \le \frac{\varepsilon}{2\,(1+\sqrt{m})\,Lip(f)}.$$

Moreover, we have that

$$\lim_{h\to 0} Q(x,v_k,h) = 0 \text{ for all } k\in\{1,\dots,N\},$$

which means that there exists $\delta>0$, good for all v_k with $k=1,\dots,N$, such that

$$|Q(x,v_k,h)| < \frac{\varepsilon}{2} \text{ for } 0<|h|<\delta \text{ and } k=1,\dots,N.$$

Then for all $e \in \mathcal{S}^{m-1}$ there is v_k with $k \in \{1, \dots, N\}$ such that, for all $0 < |h| < \delta$

$$|Q(x,e,h)| \leq |Q(x,v_k,h)| + |Q(x,e,h) - Q(x,v_k,h)| < \frac{\varepsilon}{2}(1+\sqrt{m})Lip(f)|e - v_k| < \varepsilon$$

and δ does not depend on e, which means that the convergence is uniform in e.
We can now conclude: let $y \in \mathbb{R}^m$ and $e := \frac{y-x}{|y-x|}$, so $y = x + te$ with $t = |y - x|$. It holds that

$$f(y) - f(x) - \nabla f(x) \cdot (y - x) = f(x + te) - f(x) - t\nabla f(x) \cdot e = o(t) = o(|y - x|)$$

which means exactly that f is differentiable at $x \in A$ with $Df[x](y) := \nabla f(x) \cdot y$. $\square$

Proposition 3.13. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Hölder continuous map of parameter $0 < \alpha \leq 1$, $0 \leq s \leq m$ and $A \subset \mathbb{R}^m$, then*

$$\mathcal{H}^s(f(A)) \leq L_\alpha(f)^s \mathcal{H}^{s\alpha}(A),$$

and therefore $dim f(A) \leq \alpha dim A$.

Proof. Let $L = L_\alpha(f)$ and let $\{E_i\}_{i=1,\dots}$ such that $E \subseteq \bigcup_{i=1}^{+\infty} E_i$, $d(E_i) \leq \delta$ and

$$\sum_{i=1}^{+\infty} L^s d(E_i)^{s\alpha} \leq L^s \mathcal{H}^{s\alpha}_\delta(E) + \varepsilon.$$

Then $f(E) \subset \bigcup_{i=1}^{+\infty} f(E_i)$ and $d(f(E_i)) \leq L d(E_i)^\alpha \leq L\delta^\alpha$. Therefore

$$\mathcal{H}^s_{L\delta^\alpha}(f(E)) \leq \sum_{i=1}^{+\infty} d(f(E_i))^s \leq \sum_{i=1}^{+\infty} L^s d(E_i)^{s\alpha} \leq L^s \mathcal{H}^{s\alpha}_\delta(E) + \varepsilon$$

and letting $\varepsilon, \delta \to 0$ we obtain that $\mathcal{H}^s(f(A)) \leq L_\alpha(f)^s \mathcal{H}^{s\alpha}(A)$. $\square$

When $\alpha = 1$ we obtain that for a Lipschitz map f, $\mathcal{H}^s(f(A)) \leq Lip(f)^s \mathcal{H}^s(A)$ and $dim f(A) \leq dim A$.

Theorem 3.14. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map, and let*

$$A = \{x \in \mathbb{R}^m \mid dim(\nabla f(x) \cdot \mathbb{R}^m) < m\}.$$

Then $\mathcal{H}^m(f(A)) = 0$.

Proof. First we define $A_R = A \cap B(0, R)$ for $0 < R < +\infty$. Let $\varepsilon > 0$ and $L = Lip(f)$. Let $x \in A_R$, $W_x := f(x) + \nabla f(x) \cdot \mathbb{R}^m$; then for sufficiently small $r > 0$,

$$fB(x, r) \subset B(f(x), Lr) \cap \{y \in \mathbb{R}^n \mid d(y, W_x) \leq \varepsilon r\}.$$

Let $k = dim W_x$. $fB(x, r)$ is contained in a cylinder C with base an k-ball of radius Lr and $n - k$-height $2\varepsilon r$:

$$C := \{z \in \mathbb{R}^n \mid |P_{W_x} z - f(x)| \leq Lr, |Q_{W_x} - f(x)| \leq \varepsilon r\}.$$

Now we cover C with some balls B_i of radius εr. Let N be the number of balls we use, then $N\mathcal{L}^n(B_i) \geq (Lr)^k(\varepsilon r)^{n-k}$. We get, taking $k = m - 1$, that

$$N \geq \frac{L^{m-1}}{\alpha(n)\varepsilon^{m-1}}$$

and, letting $c = \frac{2^{m+1}}{\alpha(n)}$, we have the following estimate

$$\mathcal{H}^m_\infty(fB(x,r)) \leq 2N(2\varepsilon r)^m \leq cL^{m-1}\varepsilon r^m.$$

By Vitali's covering theorem we can find disjoint balls $B_i = B(x_i, r_i)$ such that

$$\mathcal{L}^m\left(A_R \setminus \bigcup_{i=1}^{+\infty} B_i\right) = 0, \qquad \sum_{i=1}^{+\infty} \mathcal{L}^m(B_i) < \mathcal{L}^m(A_R) + \varepsilon.$$

We have that $fA_R \subseteq \left(\bigcup_{i=1}^{+\infty} fB_i\right) \cup f(A_R \setminus \bigcup_{i=1}^{+\infty} B_i)$ and by Proposition 3.13 $\mathcal{H}^m(f(A_R \setminus \bigcup_{i=1}^{+\infty} B_i)) = 0$. Then

$$\mathcal{H}^m_\infty(fA_R) \leq \sum_{i=1}^{+\infty} \mathcal{H}^m_\infty(fB_i) \leq cL^{m-1}\varepsilon \sum_{i=1}^{+\infty} r_i^m \leq cL^{m-1}\varepsilon\alpha(m)^{-1}(\mathcal{L}^m(A_R) + \varepsilon)$$

and for $\varepsilon \to 0$ we conclude since $\mathcal{L}^m(A_R) < +\infty$ and since $A = \bigcup_{R>0} A_R$. $\square$

Theorem 3.15. *Let $A \subset \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$ be a Lipschitz map. If $m \leq s \leq n$, then*

$$\int^* \mathcal{H}^{s-m}(A \cap \overleftarrow{f}\{y\})\, d\mathcal{L}^m y \leq \alpha(m) Lip(f)^m \mathcal{H}^s(A).$$

Proof. We cover A with closed sets $E_{k,i}$ with $d(E_{k,i}) \leq 1/k$ and

$$\sum_{i=1}^{+\infty} d(E_{k,i})^s \leq \mathcal{H}^s_{1/k}(A) + 1/k.$$

Let $F_{k,i} = \{y \in \mathbb{R}^m \mid E_{k,i} \cap \overleftarrow{f}\{y\} \neq \emptyset\}$. If we take $x, y \in F_{k,i}$ we can find $v, u \in E_{k,i} \cap A$ such that $f(v) = x$ and $f(u) = y$. Then

$$|x - y| \leq Lip(f)|v - u| \leq Lip(f)d(E_{k,i})$$

which means that $d(F_{k,i}) \leq Lip(f)d(E_{k,i})$. Therefore

$$\mathcal{L}^m(F_{k,i}) \leq \alpha(m)(Lip(f)d(E_{k,i}))^m.$$

Using Fatou's Lemma we obtain

$$\int^* \mathcal{H}^{s-m}(A \cap \overleftarrow{f}\{y\})\, d\mathcal{L}^m y =$$

$$= \int^* \lim_{k\to+\infty} \mathcal{H}^{s-m}_{1/k}(A \cap \overleftarrow{f}\{y\})\, d\mathcal{L}^m y \le$$

$$\le \int \liminf_{k\to+\infty} \sum_{i=1}^{+\infty} d(E_{k,i} \cap \overleftarrow{f}\{y\})^{s-m}\, d\mathcal{L}^m y \le$$

$$\le \liminf_{k\to+\infty} \sum_{i=1}^{+\infty} \int_{F_{k,i}} d(E_{k,i} \cap \overleftarrow{f}\{y\})^{s-m}\, d\mathcal{L}^m y \le$$

$$\le \liminf_{k\to+\infty} \sum_{i=1}^{+\infty} d(E_{k,i})^{s-m} \mathcal{L}^m(F_{k,i}) \le$$

$$\le \alpha(m) Lip(f)^m \liminf_{k\to+\infty} \sum_{i=1}^{+\infty} d(E_{k,i})^s \le$$

$$\le \alpha(m) Lip(f)^m \liminf_{k\to+\infty} (\mathcal{H}^s_{1/k}(A) + 1/k) = \alpha(m) Lip(f)^m \mathcal{H}^s(A).$$

□

Lemma 3.16. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map and let $A \subset \mathbb{R}^m$ be $\mathcal{L}^m$-measurable. Then $f(A)$ is a $\mathcal{H}^m$-measurable set.*

Proof. A set $A \subset \mathbb{R}^m$ is $\mathcal{L}^m$-measurable if and only if it can be written as $F \cup N$ where $F \subset A$ is a countable, or finite, union of compact sets and $\mathcal{L}^m(N) = 0$. We can suppose A of finite measure, since $\mathbb{R}^m = \bigcup_{n=1}^{+\infty} B(0,n)$. Therefore we can find compact sets $K_i \subset A$ such that $\mathcal{L}^m(A \setminus K_i) \le 1/i$. Taking $F = \bigcup_{i=1}^{+\infty} K_i$ and $N = A \setminus F$ we are done. The converse is trivially true. This argument holds for generic Borel regular measures, such as $\mathcal{H}^m$ (see Theorem 1.5). Then $f(A) = f(F) \cup f(N)$. $f(N)$ has $\mathcal{H}^m$-measure zero, because $\mathcal{H}^m(f(N)) \le Lip(f)\mathcal{H}^m(N) = 0$. Since f is Lipschitz, then continuous, sends compact sets into compact sets, so $f(F)$ is a countable, or finite, union of compact sets. Then $f(A)$ is $\mathcal{H}^m$-measurable. □

Theorem 3.17. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz map and let $A \subset \mathbb{R}^m$ be $\mathcal{L}^m$-measurable. Then $\Theta^m_*(fA, x) > 0$ for $\mathcal{H}^m$-almost all $x \in fA$.*

Proof. We may assume that $\mathcal{L}^m(A) < +\infty$. Let $E = fA$, $\varepsilon > 0$ and $F = \{x \in E \mid \Theta^m_*(E,x) < \varepsilon\}$. By Lemma 3.16, E is $\mathcal{H}^m$-measurable and of finite measure by Proposition 3.13, therefore F is measurable and of finite measure. Let C be a compact subset of F and U open set such that $A \subset U$ and $\mathcal{L}^m(U) < +\infty$. Let V be an open set of finite measure such that $E \subset V$: $\mathcal{H}^m \llcorner V$ is a Radon measure. We can find for each $x \in C$ a closed ball $B = B(x,r)$ such that $\mathcal{H}^m(E \cap B) < \varepsilon d(B)^m$. With a smaller r, we can also suppose that $D = B(y, r/L) \subset U$ with $y \in A$, $f(y) = x \in C$ and also that

$B \subset V$. Therefore, using Vitali's covering theorem on $\mathcal{H}^m \llcorner V$, we can find $B_i = B(x_i, r_i)$ disjoint closed balls such that

$$\begin{aligned}
&\mathcal{H}^m(E \cap B_i) < \varepsilon d(B_i)^m \\
&D_i = B(y_i, r_i/L) \subset U \\
&\mathcal{H}^m\left(C \setminus \bigcup_{i=1}^{+\infty} B_i\right) = 0.
\end{aligned}$$

D_i are disjoint because $fD_i \subset B_i$. We have then

$$\mathcal{H}^m(C) = \sum_{i=1}^{+\infty} \mathcal{H}^m(C \cap B_i) \leq \varepsilon \sum_{i=1}^{+\infty} d(B_i)^m = \varepsilon c \sum_{i=1}^{+\infty} \mathcal{L}^m(D_i) \leq c\varepsilon \mathcal{L}^m(U)$$

where $c = (2L)^m \alpha(m)^{-1}$ depends only on m and L. Therefore

$$\mathcal{H}^m(C) \leq c\varepsilon \mathcal{L}^m(U)$$

and, since F is $\mathcal{H}^m$-measurable and of finite measure, we have, by Theorem 1.5, that

$$\mathcal{H}^m(\{x \in E \mid \Theta_*^m(E,x) = 0\}) \leq \mathcal{H}^m(F) = \sup\{\mathcal{H}^m(C) \mid C \subset F \text{ compact }\} \leq c\varepsilon \mathcal{L}^m(A)$$

which shows that $\{x \in E \mid \Theta_*^m(E,x) = 0\}$ has $\mathcal{H}^m$-measure zero. $\square$

Chapter 4

Rectifiability

In this chapter we present one of the most fundamental concept in measure theory: Rectifiability.

Definition 4.1. A set $E \subset \mathbb{R}^n$ is m-rectifiable if there exist Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^n$ with $i = 1, \dots$ such that

$$\mathcal{H}^m \left(E \setminus \bigcup_{i=1}^{+\infty} f_i(\mathbb{R}^m) \right) = 0.$$

A set $F \subset \mathbb{R}^n$ is called purely m-rectifiable if $\mathcal{H}^m(E \cap F) = 0$ for every $E \subset \mathbb{R}^m$ m-rectifiable.

Another equivalent definition of m-rectifiable is the following: E is m-rectifiable if there are at most countably many C^1 submanifolds of $\mathbb{R}^n$ Γ_i with dimension m such that

$$\mathcal{H}^m \left(E \setminus \left(\bigcup_i \Gamma_i \right) \right) = 0.$$

As a consequence of the extension theorem 3.11 for Lipschitz maps, we have the following lemma (whose proof is left to the reader).

Lemma 4.1. *$E \subset \mathbb{R}^m$ is m-rectifiable if and only if there exist $\{A_i\}_{i=1,\dots}$ subsets of $\mathbb{R}^m$ and Lipschitz maps $f_i : A_i \to \mathbb{R}^n$ such that $\mathcal{H}^m(E \setminus \bigcup_{i=1}^{+\infty} f_i(A_i)) = 0$.*

Note that $\{A_i\}_{i=1,\dots}$ can be taken $\mathcal{H}^m$-measurable and such that $f_i(A_i) \subset E$. We leave to the reader also the proof of the following fact, which lists some properties of rectifiable sets:

Lemma 4.2. *Let $E \subset \mathbb{R}^n$ m-rectifiable. Then*

1. *E has σ-finite $\mathcal{H}^m$-measure, i.e. E is the union of countably many $\mathcal{H}^m$-measurable sets of finite measure*

2. *Any subset of E is m-rectifiable*

3. *There exists a m-rectifiable Borel set U such that $E \subset U$ and $\mathcal{H}^m(E) = \mathcal{H}^m(B)$*

4. *Union of countably many m-rectifiable sets is m-rectifiable.*

Theorem 4.3. *Let $A \subset \mathbb{R}^n$ with $\mathcal{H}^m(A) < +\infty$. There is a Borel m-rectifiable set E and a purely m-unrectifiable set F such that $A = E \cup F$. This decomposition is unique up to $\mathcal{H}^m$ null sets.*

Proof. Let S the supremum of $\{\mathcal{H}^m(A \cap B) \mid B \text{ is m-rectifiable and Borel}\}$. Then, there is E_i Borel m-rectifiable contained in A such that $\mathcal{H}^m(E_i) \geq S - \frac{1}{i}$. Let us set $E := \bigcup_{i=1}^{+\infty} E_i$ which is m-rectifiable, and $F := A \setminus E$. F is purely m-unrectifiable: this can be seen by contradiction, details are left to the reader. Then

$$A = E \cup F$$

and the decomposition is unique up to $\mathcal{H}^m$ null sets. □

An important property of m-rectifiable sets is the existence $\mathcal{H}^m$-almost everywhere of tangent planes, which approximate the set in some sense. We will indicate with $A(n,m)$ the set of all the affine subspaces of dimension m in $\mathbb{R}^n$. We recall that $W(\varepsilon) := \{x \in \mathbb{R}^n \mid d(x,W) \leq \varepsilon\}$ for every $W \in A(n,m)$.

Definition 4.2 (Linearly approximable)**.** We will say that $E \subset \mathbb{R}^n$ is m-linearly approximable if for $\mathcal{H}^m$-almost all $e \in E$ we have this property: for every $\eta > 0$ there exists $r_0, \lambda > 0$ and $W \in A(n,m)$, with $e \in W$, such that

$$\mathcal{H}^m\big(E \cap B(x, \eta r)\big) \geq \lambda r^m \quad \text{for } x \in W \cap B(e,r) \tag{4.1}$$
$$\mathcal{H}^m\big(E \cap B(e,r) \setminus W(\eta r)\big) < \eta r^m \tag{4.2}$$

for all $0 < r < r_0$.

For a geometric representation of the properties described in 4.2 see figure 4.1. A weaker form of this definition can be given, where $W \in A(n,m)$ depends on $0 < r < r_0$:

Definition 4.3 (Weakly linearly approximable)**.** We will say that $E \subset \mathbb{R}^n$ is m-weakly linearly approximable if, for $\mathcal{H}^m$-almost all $e \in E$, we have this property: for every $\eta > 0$ there exist $r_0, \lambda > 0$ such that, for all $0 < r < r_0$, there is $W_r \in A(n,m)$, with $e \in W_r$, such that 4.1 and 4.2 hold with W_r in place of W.

If these conditions hold, then $\Theta^m_*(E,e) > 0$ for $\mathcal{H}^m$-almost all $e \in E$. Moreover, if $\mathcal{H}^m(E) < +\infty$, then properties 4.1 and 4.2 are preserved $\mathcal{H}^m$-almost everywhere for subsets of E (see Theorem 3.9).

Theorem 4.4. *Let E be an $\mathcal{H}^m$-measurable and m-rectifiable subset of $\mathbb{R}^n$ with $\mathcal{H}^m(E) < +\infty$. Then E is m-linearly approximable.*

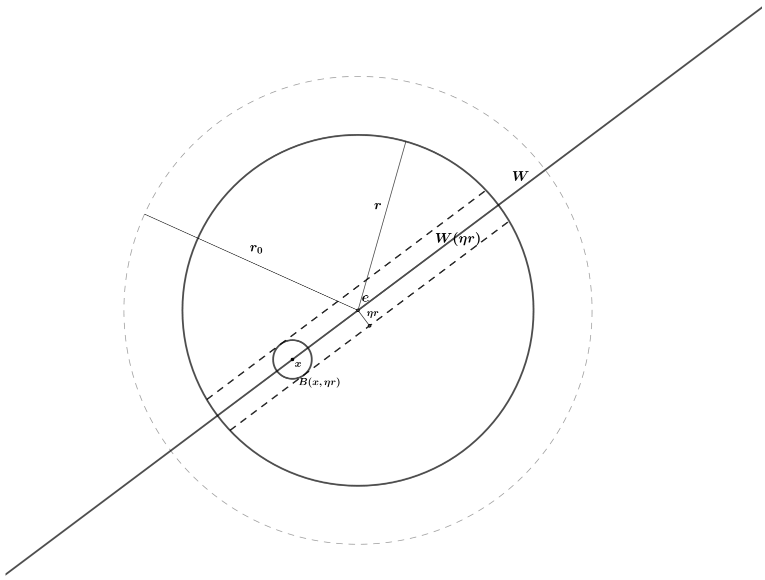

Figure 4.1: Representation of the sets described in definition 4.2. Condition 4.1 implies that $B(x, \eta r)$ contains a good portion of E for all $x \in W \cap B(e, r)$; 4.2 implies that most of $E \cap B(e, r)$ lies in $W(\eta r)$.

Proof. Let $0 < \eta < \frac{1}{2}$. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a Lipschitz function and let $B \subset \mathbb{R}^m$ be a measurable set of finite Lebesgue measure with $fB \subset E$. We have to verify 4.1 and 4.2 for $\mathcal{H}^m$-almost all $a \in fB$. By Theorem 3.17 $\Theta^m_*(fB, a) > 0$ for $\mathcal{H}^m$-almost all $a \in fB$, then we may assume that there is $r_0 > 0$ and $\lambda > 0$ such that

$$\mathcal{H}^m(E \cap B(a, r)) \geq \lambda r^m \tag{4.3}$$

for all $0 < r < r_0$ and all $a \in fB$. In fact B is the union of the sets $(\overleftarrow{f}\{a \in fB \mid \Theta^m_*(fB, a) > 1/i\}) \cap B$ and of a set of measure zero. Therefore we may consider each of them separately. By Theorem 3.12, f is differentiable $\mathcal{H}^m$-almost everywhere in B. Let $L_x = Df[x] - Df[x](x) + f(x)$ and $W_x = L_x\mathbb{R}^m$. By Theorem 3.14, we can suppose that $dim W_x = m$ for $\mathcal{H}^m$-almost all $x \in B$. Moreover for such $x \in B$ there is

$$0 < l(x) := \min_{u \in \mathcal{S}^{m-1}} |Df[x](u)|,$$

which means that $|L_x y - L_x x| \geq l(x)|y - x|$ for all $y \in \mathbb{R}^m$. Let $\varepsilon > 0$. By the Lebesgue density Theorem (see Corollary 2.3) $\mathcal{H}^m$-almost all $x \in B$ have density equal to 1. Let $0 < \delta < 1$ and $x \in B$ of density 1; let us suppose that $\forall r > 0$ there exist $y_r \in B(x, r/\delta)$ such that $d(y_r, B) \geq \delta^2 r$. We have that $B(y_r, \bar{\delta}^2 r) \subset \mathbb{R}^m \setminus B$ for some $\bar{\delta} < \delta$ fixed. Letting $N = \delta^2 + \frac{1}{\delta}$ we obtain

$$\frac{\mathcal{L}^m(B \cap B(x, Nr))}{\mathcal{L}^m(B(x, Nr))} \leq \frac{\mathcal{L}^m(B(x, Nr) \setminus B(y_r, \bar{\delta}^2 r))}{\mathcal{L}^m(B(x, Nr))} = 1 - \left(\frac{\delta\bar{\delta}^2}{(\delta^3 + 1)}\right)^m < 1$$

and x does not have density 1, a contradiction. Therefore we can find a compact subset $C \subset B$ and two numbers $r_0 > 0$ and $\delta < \min\{\eta/4, 1/L\}$ such that $\mathcal{L}^m(B \setminus C) < \varepsilon$, such that for almost all $x \in C$,

$$|f(y) - L_x y| < \delta^2 |y - x| \quad \text{for all } y \in B(x, r_0) \tag{4.4}$$

$$l(x) \geq 2\delta \tag{4.5}$$

$$d(y, B) < \delta^2 r \quad \text{for all } y \in B(x, r/\delta), \;\; 0 < r < r_0. \tag{4.6}$$

We then partition C into finitely many Borel subsets C_i with $d(C_i) < r_0$. Let $x \in C_i$ for some i fixed, $a = f(x)$ such that $\Theta^m(E \setminus fC_i, a) = 0$. We can verify 4.1 and 4.2 in these points of fC_i. Let $0 < r < \delta r_0/2$ and $b \in W_x \cap B(a, r)$ and $b = L_x y$; since $l(x) \geq 2\delta$, $y \in B(x, r/\delta)$. There exists $z \in B$ such that $|y - z| < \delta^2 r$, whence $|x - z| < 2r/\delta < r_0$. Therefore, because $\|Df[x]\| \leq L < 1/\delta$,

$$|f(z) - b| \leq |f(z) - L_x z| + |L_x z - b| < \delta^2 |z - x| + |L_x z - b| \leq \delta^2 |z - x| + L|z - y| < 3\delta r$$

and, since $4\delta < \eta$, it follows by 4.3 that

$$\mathcal{H}^m(E \cap B(b, \eta r)) \geq \mathcal{H}^m(E \cap B(f(z), \delta r)) \geq \lambda \delta^m r^m$$

and 4.1 is verified.
By 4.4 we obtain that

$$f(C_i \cap B(x, r/\delta)) \subset W_x(\delta r) \subset W_x(\eta r).$$

Since $d(C_i) < r_0$ and by 4.5, 4.6, we have for $z \in C_i \setminus B(x, r/\delta)$

$$|a - f(z)| \geq |L_x x - L_x z| - |L_x z - f(z)| \geq 2\delta |x - z| - \delta^2 |x - z| > \delta |x - z| \geq \delta r$$

whence

$$f(C_i \setminus B(x, r/\delta)) \subset \mathbb{R}^n \setminus B(a, r).$$

Therefore $fC_i \cap B(a, r) \subset W_x(\eta r)$ and since $\Theta^m(E \setminus fC_i, a) = 0$ we obtain 4.2. $\square$

Before proving that E has almost everywhere an approximate tangent plane (whose definition still must be given) we need to show some results, and give some notations. Let $V \in G(n, n - m)$ and $Q_V := P_{V^\perp}$. Then we set

$$X(a, V, s) := \{x \in \mathbb{R}^n \mid d(x - a, V) < s|x - a|\} = \{x \in \mathbb{R}^n \mid |Q_V(x - a)| < s|x - a|\}$$

and

$$X(a, r, V, s) := X(a, V, s) \cap B(a, r)$$

for $a \in \mathbb{R}^n$, $0 < s < 1$ and $r > 0$. For a representation of $X(a, V, s)$ see figure 4.2.

Lemma 4.5. *Suppose $E \subset \mathbb{R}^n$, $V \in G(n, n - m)$, $0 < s < 1$, and $r > 0$. If $E \cap X(a, r, V, s) = \emptyset$ for all $a \in E$ then E is m-rectifiable.*

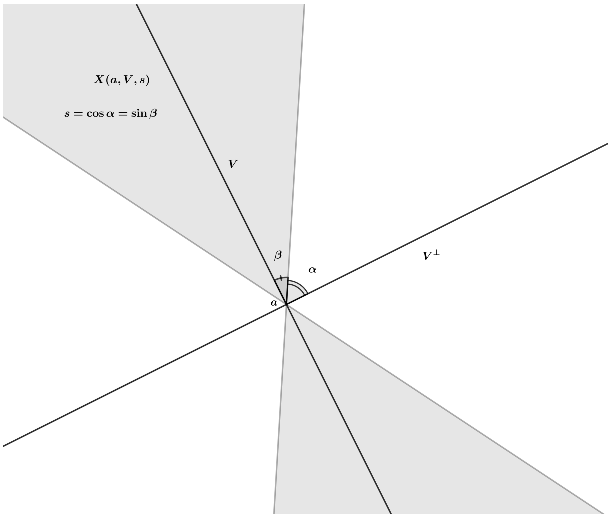

Figure 4.2: Representation of $X(a, V, s)$ in $\mathbb{R}^2$.

Proof. We can suppose that $d(E_i) < r$. Then $|Q_V(b) - Q_V(a)| \geq s|b - a|$ for all $b \in E$. Hence $Q_{V|E}$ is Lipschitz injective and the inverse $f = Q_{V|E}^{-1}$ is Lipschitz. Since $Q_V E$ lies on an m-plane and $E = f(Q_V E)$, E is m-rectifiable. $\square$

Lemma 4.6. *Let $V \in G(n, n-m)$, $0 < s < 1$, $\delta, \lambda > 0$. If A is purely m-unrectifiable and*

$$\mathcal{H}^m(A \cap X(x, r, V, s)) \leq \lambda r^m s^m$$

for all $x \in A$, $0 < r < \delta$, then

$$\mathcal{H}^m(A \cap B(a, \delta/6)) \leq 2\lambda 20^m \delta^m$$

for all $a \in \mathbb{R}^n$.

Proof. We can assume that $A \subset B(a, \delta/6)$ and that

$$A \cap X(x, V, s/4) \neq \emptyset$$

for $x \in A$. The set where this fails, by Lemma 4.6, has $\mathcal{H}^m$-measure zero. Let

$$h(x) = \sup\{|y - x| \mid y \in A \cap X(x, V, s/4)\}$$

for $x \in A$. Then $0 < h(x) \leq \delta/3$. Letting

$$C_x = Q_V^{-1}\left(Q_V B(x, sh(x)/4)\right)$$

and $x' \in A \cap X(x, V, s/4)$ with $|x - x'| \geq 3h(x)/4$, we obtain that

$$A \cap C_x \subset X(x, 2h(x), V, s) \cup X(x', 2h(x), V, s)$$

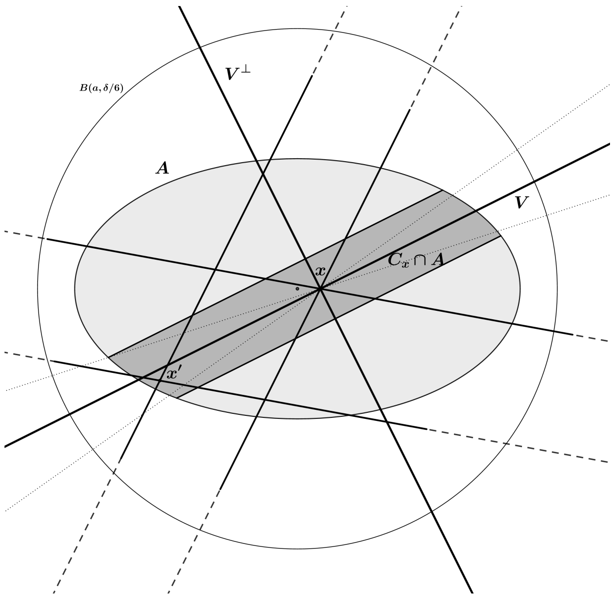

Figure 4.3: Geometric representation, in $\mathbb{R}^2$, of the fact that $A \cap C_x \subset X(x, 2h(x), V, s) \cup X(x', 2h(x), V, s)$.

for $x \in A$ (see figure 4.3). Let us prove it: if $z \in A \cap C_x$, then $|Q_V(x - z)| \leq sh(x)/4$, moreover $|x - z| \leq h(x)$ since if $|x - z| > h(x)$, then $|Q_V(x - z)| < s/4|x - z|$ and $z \in X(x, V, s/4)$, therefore we obtain $h(x) < h(x)$ a contradiction. Hence it is true that $|x' - z| \leq 2h(x)$. Suppose that $z \notin X(x', 2h(x), V, s)$. It follows that

$$
\begin{aligned}
s|x' - z| \leq |Q_V x' - Q_V z| \leq |Q_V(x' - x)| + |Q_V(x - z)| < \\
< s|x - x'|/4 + sh(x)/4 \leq sh(x)/2,
\end{aligned}
$$

and knowing that $|x - x'| \geq 3h(x)/4$, we have that

$$
|x - z| > 3h(x)/4 - h(x)/2 = h(x)/4 \geq |Q_V(x - z)|/s
$$

which means that $z \in X(x, 2h(x), V, s)$. Then by hypothesis we have

$$
\mathcal{H}^m(A \cap C_x) \leq 2\lambda(2h(x))^m s^m.
$$

By Theorem 1.16 there exists $B \subset A$ countable such that the balls $Q_V B(x, sh(x)/20) \subset V^{\perp}$ for $x \in B$ are disjoint and

$$
Q_V A \subset \bigcup_{x \in B} Q_V B(x, sh(x)/4),
$$

which means that $A \subset \bigcup_{x \in B} C_x$. Therefore, recalling that $\mathcal{H}^m(V^{\perp} \cap B(y, r)) = 2^m r^m$

for all $y \in V^{\perp}$, we have that

$$\mathcal{H}^m(A) \leq \sum_{x \in B} \mathcal{H}^m(A \cap C_x) \leq \lambda 2^{m+1} \sum_{x \in B} (sh(x))^m =$$
$$= 2\lambda 20^m \sum_{x \in B} \mathcal{H}^m(V^{\perp} \cap B(Q_V x, sh(x)/20)) \leq$$
$$\leq 2\lambda 20^m \mathcal{H}^m(V^{\perp} \cap B(Q_V a, \delta/2)) = 2\lambda 20^m \delta^m.$$

□

Corollary 4.7. *If $V \in G(n, n-m)$, $\delta > 0$ and $A \subset \mathbb{R}^n$ is purely m-unrectifiable with $\mathcal{H}^m(A) < +\infty$, then*

$$\limsup_{s \to 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m(A \cap X(a, r, V, s)) > 0$$

for $\mathcal{H}^m$-almost all $a \in A$.

Proof. Let Z be the set of points in A such that the statement does not hold. Then let us define

$$Z_i = \left\{ a \in A \mid \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m(A \cap X(a, r, V, s)) < \lambda \text{ for } 0 < s < 1/i \right\}.$$

We have that $Z_1 \subset Z_2 \dots$ and $Z \subset \bigcup_{i=1}^{+\infty} Z_i$. Therefore by the last Lemma we proved, we have that $\mathcal{H}^m(Z_i \cap B(a, \delta/6)) \leq 2\lambda 20^m \delta^m$ for all i, which implies (for $i \to +\infty$) that

$$\mathcal{H}^m(Z \cap B(a, \delta/6)) < 2\lambda 20^m \delta^m.$$

Letting $\lambda \to 0$ we obtain that $\mathcal{H}^m(Z) = 0$ (since Z intersects every ball of radius $\delta/6$ in a set of measure zero). □

Corollary 4.8. *Let $V \in G(n, n-m)$, $0 < s < 1$ and $A \subset \mathbb{R}^n$ a purely m-unrectifiable set with $\mathcal{H}^m(A) < +\infty$. Then*

$$\Theta^{*m}(A \cap X(a, V, s), a) \geq \frac{s^m}{240^{m+1}}$$

for $\mathcal{H}^m$-almost all $a \in A$.

Proof. The set of points were this fails is contained in the union of A_i, where A_i is the set of points $a \in A$ such that

$$\mathcal{H}^m(A \cap X(a, r, V, s)) < \lambda s^m r^m$$

for all $0 < r < 1/i$, with $\lambda = 120^{-m}/3$.

Taking $0 < \delta < 1/i$ we have that $\mathcal{H}^m(A_i \cap B(a, \delta/6)) \leq 2\lambda 20^m \delta^m$ which implies that

$$\Theta^{*m}(A_i, a) \leq 2 \cdot 60^m \lambda = \frac{2^{-m+1}}{3} < 2^{-m}$$

which means, by Theorem 3.9, that $\mathcal{H}^m(A_i) = 0$ and we can conclude. □

Definition 4.4 (Approximate tangent plane)**.** Let $E \subset \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $V \in G(n, m)$. We say that V is an *approximate tangent m-plane* for A in a if $\Theta^{*m}(A, a) > 0$ and

$$\lim_{r \to 0} r^{-m} \mathcal{H}^m \left(A \cap B(a, r) \setminus X(a, V, s) \right) = 0$$

for all $0 < s < 1$.

We will indicate the set of all approximate m-tangent planes for A in a with $apTan^m(A, a)$. If the set is formed by just one element V we will simply write $V = apTan^m(A, a)$. Note that $apTan^m(A, a)$ could be empty in general. The following lemma is a consequence of Theorem 3.9.

Lemma 4.9. *Let $A \subset B \subset \mathbb{R}^n$ be two $\mathcal{H}^m$-measurable sets, with $\mathcal{H}^m(B) < +\infty$. Then $apTan^m(A, a) = apTan^m(B, a)$ for $\mathcal{H}^m$-almost all $a \in A$.*

Now we are ready to state the equivalence of m-rectifiability and existence of an approximate tangent m-plane, which moreover will be unique. The hypothesis that $\mathcal{H}^m(E) < +\infty$ is important. Let $\mathbb{Q}^2 = \{q_i\}_{i=1,\ldots}$ and $I_{i,j} = [q_i, q_j]$ the closed segment with endpoints q_i and $i < j$. Then

$$E = \bigcup_{i<j} I_{i,j}$$

is a 1-rectifiable set of $\mathbb{R}^2$. Since $\mathcal{H}^1(E \cap B(x, r)) = +\infty$ for all $r > 0$ and $x \in E$, E does not have an approximate tangent line at any point $x \in E$, even if it is 1-rectifiable.

Theorem 4.10. *Let $E \subset \mathbb{R}^n$ an $\mathcal{H}^m$-measurable set with $\mathcal{H}^m(E) < +\infty$. Then the following are equivalent:*

1. *E is m-rectifiable.*
2. *E is linearly approximable.*
3. *There is a unique approximate tangent m-plane for E at e for $\mathcal{H}^m$-almost all $e \in E$.*
4. *There is an approximate tangent m-plane for E at e for $\mathcal{H}^m$-almost all $e \in E$.*

Proof. That 1. implies 2. was showed in Theorem 4.4. Let us suppose that 2. holds; by Theorem 3.9 $\Theta^{*m}(E, e) > 0$ for $\mathcal{H}^m$-almost all $e \in E$, let $W \in A(n, m)$ be as in definition 4.2 with $e \in W$. Let $V := W - e = \{w - e \mid w \in W\}$, and $x \in B(e, r) \setminus X(e, V, s)$. If $|x - e| > \varepsilon r$, then $d(x - e, V) \geq s|x - e| > s\varepsilon r$ and so $x \notin W(s\varepsilon r)$. Hence

$$B(e, r) \setminus X(e, V, s) \subseteq \Big(B(e, r) \setminus W(s\varepsilon r) \Big) \cup B(e, \varepsilon r).$$

We can suppose that $\Theta^{*m}(E, e) < +\infty$, therefore there is δ such that

$$\mathcal{H}^m(E \cap B(e, \varepsilon r)) < \Big(\Theta^{*m}(E, e) + 1 \Big) (2r)^m \varepsilon^m$$

for all $0 < r < \delta$. Moreover $\mathcal{H}^m(E \cap B(e,r) \setminus W(s\varepsilon r)) < \varepsilon s r^m$ for all $0 < r < r_0$. Therefore we can estimate $r^{-m}\mathcal{H}^m(E \cap B(e,r) \setminus X(e,V,s))$ (for $0 < r < \min\{\delta, r_0\}$):

$$r^{-m}\mathcal{H}^m(E \cap B(e,r) \setminus X(e,V,s)) \leq r^{-m}\mathcal{H}^m\Big(\big(B(e,r) \cap E \setminus W(s\varepsilon r)\big) \cup B(e,\varepsilon r) \cap E\Big) \leq$$
$$\leq r^{-m}\mathcal{H}^m\left(B(e,r) \cap E \setminus W(s\varepsilon r)\right) + r^{-m}\mathcal{H}^m(E \cap B(e,\varepsilon r)) \leq$$
$$\leq \varepsilon s + \Big(\Theta^{*m}(E,e) + 1\Big)2^m\varepsilon^m$$

which means that V is an approximate tangent m-plane for E at e. Let us suppose that U is another approximate tangent m-plane for E at e, and, by contradiction, that $V \neq U$. Then there are η, s small enough such that for all $r > 0$ there is $z \in W \cap B(e,r)$ such that $B(z,\eta r) \cap X(e,U,s) = \emptyset$. Then $E \cap B(z,\eta r) \subseteq E \cap B(e, r(1+\eta)) \setminus X(e,U,s)$. This leads to

$$0 < \lambda \leq \limsup_{r \to 0} r^{-m}\mathcal{H}^m(E \cap B(z,\eta r)) \leq \lim_{r \to 0} r^{-m}\mathcal{H}^m\Big(E \cap B(e, r(1+\eta)) \setminus X(e,U,s)\Big) = 0,$$

which is a contradiction. Then, $U = V = apTan^m(E,e)$.

That 3. implies 4. it is trivial. To show that 4. implies 1. we can show that if E is purely m-unrectifiable then E does not have an approximate tangent m-plane $\mathcal{H}^m$-almost everywhere; then we can conclude using Lemma 4.9 and Theorem 4.3. Let us assume that F is a purely m-unrectifiable $\mathcal{H}^m$-measurable set of finite $\mathcal{H}^m$-measure. Since $G(n,m)$ is compact, it can be covered with finitely many balls $B(W, 1/3)$. Let us fix $W \in G(n,m)$. It is sufficient to show that the set $D_W = \{a \in F \mid \exists V_a \in apTan^m(F,a) \cap B(W,1/3)\}$ has $\mathcal{H}^m$-measure zero. Let us suppose that $\mathcal{H}^m(D_W) > 0$. Therefore the set of points $a \in C$ of D_W for which, for some $\delta > 0$,

$$\sup_{0<r<\delta} r^{-m}\mathcal{H}^m(D_W \cap B(a,r) \setminus X(a, V_a, 1/3)) < \lambda 3^{-m}$$

has positive $\mathcal{H}^m$-measure. For $r > 0$, if $|P_W(x-a)| < \frac{1}{3}|x-a|$ implies $|P_{V_a}(x-a)| < \frac{2}{3}|x-a|$ which implies $|Q_{V_a}(x-a)| > \frac{1}{3}|x-a|$: this shows that

$$X(a,r,W^\perp, 1/3) \subset B(a,r) \setminus X(a, V_a, 1/3).$$

Therefore for $a \in C$,

$$\mathcal{H}^m(C \cap X(a,r,W^\perp,1/3)) < \lambda 3^{-m} r^m$$

for all $0 < r < \delta$ and choosing $2^{-m}\lambda < 240^{-m-1}$ Corollary 4.8 leads to a contradiction. Then $\mathcal{H}^m(D_W) = 0$. □

Corollary 4.11. *Let $F \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with finite $\mathcal{H}^m$ measure. F is purely m-unrectifiable if and only if $apTan^m(F,a) = \emptyset$ for $\mathcal{H}^m$-almost all $a \in F$.*

Lemma 4.12. *Let V be a vectorial space of finite dimension and $S : V \times V \to \mathbb{R}$ a bilinear symmetric map. Then there exists $\{e_1, \ldots, e_m\}$ orthogonal basis for V such that $S(e_i, e_j) = 0$ for $i \neq j$. Moreover $S(e_j, e_j) \geq S(e_i, e_i)$ for $j > i$.*

Proof. Inductively, we choose e_i in the compact set $C_i := \{x \in V \mid |x| = 1, \;\; x \cdot e_j = 0 \; j < i\}$ such that $S(e_i, e_i) \geq S(x, x)$ for all $x \in C_i$. Then, noting that $|e_i + te_j|^{-1}(e_i + te_j) \in C_i$ for all $t \in \mathbb{R}$ with $i < j$, we can deduce that $S(e_i, e_j) = 0$. $\square$

Now we will state a lemma on projections of m-unrectifiable sets that are weakly m-linearly approximable.

Lemma 4.13. *Let $A \subset \mathbb{R}^n$ be a $\mathcal{H}^m$-measurable with $\mathcal{H}^m(A) < +\infty$. Then if A is both purely m-unrectifiable and weakly m-linearly approximable, $\mathcal{H}^m(P_V A) = 0$ for all $V \in G(n, m)$.*

Proof. Let $0 < \varepsilon < 1/2$ and $V \in G(n, m)$. We can find $C' \subset A$ such that $\mathcal{H}^m(A \setminus C') < \varepsilon/2$ and such that there exist $0 < \delta < 1$, and $r_1 > 0$ for which

$$\mathcal{H}^m(A \cap B(a, r)) \geq \delta r^m \tag{4.7}$$

for all $0 < r < r_1$ and $a \in C'$. This is possible because A is the union of the sets $A_i := \{a \in A \mid \Theta^m_*(A, a) > 1/i\}$ and a set of measure zero $\{a \in A \mid \Theta^m_*(A, a) = 0\}$. This is true since A is weakly m-linearly approximable. We can find a compact set $C \subset C'$ such that the following properties hold: $\mathcal{H}^m(C' \setminus C) < \varepsilon/2$ and there are two positive numbers $\eta < \delta\varepsilon$ and $r_0 < r_1$ such that for all $a \in C$ and $0 < r < r_0$ we have that there is $W \in A(n, m)$ with $a \in W$ for which

$$\mathcal{H}^m(A \cap B(a, 2r) \setminus W(\eta r/2)) < \delta(\eta r/2)^m.$$

Let us assume that there exists $z \in C \cap B(a, r)$ such that $d(z, W) > \eta r$. Then $B(z, \eta r/2) \subset B(a, 2r) \setminus W(\eta r/2)$. Therefore

$$\mathcal{H}^m(B(a, 2r) \setminus W(\eta r/2)) \geq \mathcal{H}^m(B(z, \eta r/2)) \geq \delta(\eta r/2)^m,$$

which is a contradiction.

Then we have found $C \subset A$ compact with $\mathcal{H}^m(A \setminus C) < \varepsilon$ and $\delta > 0$, $\eta > 0$, $r_0 > 0$ such that $\eta < \delta\varepsilon < \varepsilon$ and the following hold:

- the inequality referred by 4.7 holds for all $0 < r < r_0$ and $a \in C$.
- For all $0 < r < r_0$ and $a \in C$ there is $W \in A(n, m)$ with $a \in W$ for which

$$C \cap B(a, r) \setminus W(\eta r) = \emptyset. \tag{4.8}$$

We also have that

$$\mathcal{H}^m\left(P_V(A \setminus C)\right) < \varepsilon \tag{4.9}$$

and since C is purely m-unrectifiable, we have from Lemma 4.5 that

$$\mathcal{H}^m\left(\bigcup_{i=1}^{+\infty}\{a \in C \mid C \cap X(a, 1/i, V^\perp, \eta) = \emptyset\}\right) = 0.$$

Then for $\mathcal{H}^m$-almost all points $a \in C$ there are $b \in C$ arbitrarily close to a such that

$$|P_V(b-a)| < \eta|b-a|;$$

let us take $a, b \in C$ with this property, and $r = |a-b| < r_0$. Let $W \in A(n,m)$ with $a \in W$ such that 4.8 holds and $c = P_W b$. Therefore, using 4.8, we have that

$$|c-b| \leq \eta r;$$

we have also that $|c-a| \leq r$ and $|c-a| \geq |b-a|-|c-b| \geq (1-\eta)r > r/2$ since $\eta < \varepsilon < 1/2$. So $r/2 < |c-a| \leq r$ and one gets easily that $|P_V(c-a)| < 2\eta r$. By Lemma 4.12 we can select an orthonormal basis $\{e_1, \ldots, e_m\}$ for $W - a$ such that $P_V(e_i) \cdot P_V(e_j) = 0$ for $j \neq i$. Then for some i we have that

$$|P_V e_i| \leq 2r^{-1}|P_V(c-a)| < 4\eta$$

because otherwise

$$|P_V(c-a)|^2 = \sum_{i=1}^{m} |(c-a) \cdot e_i|^2 |P_V e_i|^2 > 4r^{-2}|P_V(c-a)|^2|c-a|^2 > |P_V(c-a)|^2.$$

We deduce that $P_V(W \cap B(a,r))$ is contained in an m-rectangle with $m-1$ sides of length $2r$ and one of length $8\eta r$. Therefore $P_V(C \cap B(a,r))$ is contained in a m-rectangle of length $10\eta r, 2r + 2\eta r, \ldots, 2r + 2\eta r$. Since $\eta < 1/2$ we have that

$$\mathcal{H}^m(P_V(C \cap B(a,r))) \leq c\eta r^m \tag{4.10}$$

for a suitable constant depending only by m. Using Vitali's covering theorem we can find disjoint balls $B(a_i, r_i)$ satisfying 4.10 with $a_i \in C$ such that

$$\mathcal{H}^m\left(C \setminus \bigcup_{i=1}^{+\infty} B(a_i, r_i)\right) = 0.$$

Using 4.7 and 4.10 we get

$$\mathcal{H}^m(P_V(C)) \leq \sum_{i=1}^{+\infty} \mathcal{H}^m(P_V(C \cap B(a_i, r_i))) \leq$$
$$\leq c\eta \sum_{i=1}^{+\infty} r_i^m \leq c\eta\delta^{-1} \sum_{i=1}^{+\infty} \mathcal{H}^m(A \cap B(a_i, r_i)) \leq c\varepsilon\mathcal{H}^m(A).$$

Using also 4.9 we have that $\mathcal{H}^m(P_V(A)) < (1 + c\mathcal{H}^m(A))\varepsilon$ and we can conclude letting $\varepsilon \to 0$. $\square$

We now prove a Theorem which tells us basically that m-rectifiable sets are related to m-weakly linearly approximable sets. Moreover, we will see a relation with E and the orthogonal projections of E.

Theorem 4.14. *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with $0 < \mathcal{H}^m(E) < +\infty$. Then E is m-rectifiable if and only if E is m-weakly linearly approximable. Moreover,*

$$\Theta^m(E,x) = 1 \quad \text{for } \mathcal{H}^m\text{-almost all } x \in E \tag{4.11}$$

$$\mathcal{H}^m(P_V(E)) > 0 \quad \text{for } \gamma_{n,m}\text{-almost all } V \in G(n,m)\ . \tag{4.12}$$

Proof. If $E \subset \mathbb{R}^n$ is m-rectifiable then it is also m-linearly approximable, in particular it is weakly m-linearly approximable. Let us suppose that E is weakly m-linearly approximable and let $\varepsilon > 0$. Since E has positive lower density $\mathcal{H}^m$-almost everywhere, we have that

$$E = \bigcup_{n\in\mathbb{N}} E_n$$

where $E_n = \{x \in E \mid \Theta^m_*(E,x) > 1/n\}$ for $n \geq 1$ and $\mathcal{H}^m(E_0) = 0$. Since $\mathcal{H}^m(E) = \lim_{n\to+\infty} \mathcal{H}^m(E_n)$, we can find a compact subset F of E such that $\mathcal{H}^m(E \setminus F) < \varepsilon$ and find δ, r_0 such that

$$\mathcal{H}^m(E \cap B(a,r)) > \delta r^m \tag{4.13}$$

for all $a \in F$ and $0 < r < r_0$. Let $\eta > 0$, $1/2 < u < 1$ and $0 < \gamma \leq 1$ with $\eta < \gamma(1-u)/8$. As we did in Lemma 4.13 we can find $F_1 \subset F$ and $r_1 \leq r_0$ with $\mathcal{H}^m(F \setminus F_1) < \varepsilon$ such that, for any $a \in F_1$ and for all $0 < r < r_1$, there exists $W \in A(n,m)$ with $a \in W$ for which

$$F_1 \cap B(a,r) \setminus W(\eta r) = \emptyset \tag{4.14}$$

$$W \cap B(a,r) \subset F(\eta r). \tag{4.15}$$

Let us explain how to obtain the property 4.15. Note that fixed $\eta > 0$, $\lambda = \lambda(a)$ in Definition 4.3 depends only on a and we can choose $F' \subset F$ large enough such that $\lambda(a) \geq \lambda_0 > 0$ for all $a \in F'$. Let us take $F_1 \subset \{x \in F' \mid \Theta^m(E \setminus F) = 0\}$ compact approximating F'. Let $a \in F_1$. Suppose that, $\forall r > 0$, there exists $z_r \in W \cap B(a,r)$ such that $d(z_r, F) > \eta r$; then $B(z_r, \eta r) \cap F = \emptyset$. Since $z_r \to a$, for r small enough

$$\mathcal{H}^m((E \setminus F) \cap B(a,r)) \geq \mathcal{H}^m((E \setminus F) \cap B(z,\eta r)) \geq \lambda(a) r^m \geq \lambda_0 r^m.$$

This implies that $\Theta^m_*(E \setminus F, a) > 0$ which is a contradiction. Therefore $W \cap B(a,r) \subset F(\eta r)$. We can notice that for $\mathcal{H}^m$-almost all $a \in F_1$, $\Theta^{*m}(F_1, a) \leq 1$ and $\Theta^m(E\setminus F_1, a) = 0$. Therefore, as before, for $\mathcal{H}^m$-almost all $a \in F_1$ there exists a positive number $r_2 \leq r_1$ such that for all $0 < r < r_2$ there is $W \in A(n,m)$ with $a \in W$ for which

$$F_1 \cap B(a,r) \setminus W(\eta r) = \emptyset, \tag{4.16}$$

$$W \cap B(a,r) \subset F_1(\eta r), \tag{4.17}$$

$$\mathcal{H}^m(E \cap B(a,r)) < 3^m r^m, \tag{4.18}$$

$$\mathcal{H}^((E \setminus F_1) \cap B(a,r)) < 400^{-m} t \delta r^m \tag{4.19}$$

with $t = 2^m \gamma^m (u^m - u^{2m})$.

Now, fix such a, r and W and let $V \in G(n,m)$ be such that $P_{V|W-a} : W - a \to V$ is injective and $\gamma \leq \|(P_{V|W-a})^{-1}\|^{-1}$. Then

$$|P_V x - P_V y| \geq \gamma |x - y| \quad \text{for } x, y \in W. \tag{4.20}$$

We will show , given δ, u, γ, and for η small enough, that

$$\mathcal{H}^m(P_V(E \cap B(a,r))) \geq (2\gamma u^2 r)^m. \tag{4.21}$$

With 4.21 we can show everything: if E were not rectifiable, we would apply Lemma 4.13 to an unrectifiable subset of E of positive measure, finding a contradiction. Taking $V \in G(n,m)$ with $V = W + a$, then $\gamma = 1$ and we have that

$$\mathcal{H}^m(E \cap B(a,r)) \geq (2u^2 r)^m$$

for all $0 < r < r_2$; this means that $\Theta^m(E,x) = 1$ since $\Theta^{*m}(E,x) \leq 1$ for $\mathcal{H}^m$-almost all $x \in E$. Then, recalling Corollary 2.14, $P_{V|W-a}$ is injective (as well as $P_{V|W}$) for $\gamma_{n,m}$-almost all $V \in G(n,m)$ and since $\gamma_{n,m}(\{V \mid \|(P_{V|W-a})^{-1}\|^{-1} < \gamma\}) \to 0$ for $\gamma \to 0$ we obtain that $\mathcal{H}^m(P_V(E)) > 0$ for almost all $V \in G(n,m)$.

We now suppose that 4.21 fails $\mathcal{H}^m$-almost everywhere in F_1. Set

$$C = P_V(F_1 \cap B(a,r)) \quad \text{and} \quad D = P_V(W \cap B(a,ur)) \setminus C,$$

C is compact and by hypothesis we have that

$$\mathcal{H}^m(C) < (2\gamma u^2 r)^m.$$

By inequality 4.20 we have that $V \cap B(P_V a, \gamma ur) \subset P_V(W \cap B(a,ur))$: let $z \in V \cap B(P_V a, \gamma ur)$, then there is $b \in W$ such that $P_V b = z$ and (using 4.20) $|a - b| < ur$. We obtain that

$$\begin{aligned}\mathcal{H}^m(D) = \mathcal{H}^m\Big(P_V(W \cap B(a,ur))\Big) - \mathcal{H}^m(C) \geq \\ \geq \mathcal{H}^m\Big(V \cap B(P_V a, \gamma ur)\Big) - 2^m\gamma^m u^{2m} r^m = 2^m \gamma^m (u^m - u^{2m}) r^m = tr^m\end{aligned}$$

which means that

$$\mathcal{H}^m(D) \geq tr^m. \tag{4.22}$$

We now cover D with balls $B(b,\rho)$ with $b \in D$ and $C \cap U(b,\rho) = \emptyset$, $C \cap \partial B(b,\rho) \neq \emptyset$. Then we apply Theorem 1.16 to the balls $B(b, 5\rho)$, and we find a finite collection of disjoint balls $B(b_i, 5\rho_i)$ such that $B(b_i, 25\rho_i)$ covers D. Then, by estimate 4.22, we get

$$\sum_{i=1}^{p} \rho_i^m \geq 50^{-m} tr^m. \tag{4.23}$$

By 4.17 we have that $\rho_i \leq \eta r$ for $i = 1, \ldots, p$. Now we set

$$S_i = P_V^{-1}(B(b_i, \rho_i/2)) \cap W(\gamma(1-u)r/4)$$

and we can suppose that for $i = 1, \dots, q$ S_i does not contain any point of F. Let $c_i \in F \cap S_i$ for $i = q+1, \dots, p$. We have that $b_i = P_V b'_i$ with $b'_i \in W \cap B(a, ur)$, and from the last considerations we obtain that

$$\begin{aligned}|a - c_i| \le |a - b'_i| + |b'_i - P_W c_i| + |P_W c_i - c_i| \le \\ \le ur + |b_i - P_V(P_W c_i))|/\gamma + (1-u)r/4 \le \\ \le ur + |b_i - P_V c_i|/\gamma + |P_V(c_i - P_W c_i)|/\gamma + (1-u)r/4 \le \\ \le ur + \rho_i/(2\gamma) + (1-u)r/2 \le \eta r/\gamma + (1+u)r/2.\end{aligned}$$

Since $\eta < \gamma(1-u)/3$, this gives

$$B(c_i, \rho_i/4) \subset B(a, r).$$

Moreover $P_V(B(c_i, \rho_i/4)) \subset V \cap U(b_i, \rho_i) \subset V \setminus C$ which implies that

$$\bigcup_{i=q+1}^{p} E \cap B(c_i, \rho_i/4) \subset (E \setminus F_1) \cap B(a, r). \tag{4.24}$$

We can then deduce that the balls $B(c_i, \rho_i/4)$ are disjoint, and combining 4.13, 4.24 and 4.19 we obtain that

$$\delta 4^{-m} \sum_{i=q+1}^{p} \rho_i^m < 400^{-m} t \delta r^m$$

and by 4.23 that

$$\sum_{i=1}^{q} \rho_i^m > 100^{-m} t r^m. \tag{4.25}$$

Now we will work for $i = 1, \dots, q$; let $v_i \in \partial B(b_i, \rho_i) \cap C$ then $v_i = P_V e_i$ with $e_i \in F_1 \cap B(a, r) \subset W(\eta r)$ and we obtain that

$$e_i \in P_V^{-1}(\partial B(b_i, \rho_i)) \cap W(\eta r) \cap F_1.$$

We have $\eta^{-1}\rho_i \le r < r_1$ and by 4.15 we obtain

$$A_i = B(e_i, \eta^{-1}\gamma(1-u)\rho_i/16) \cap W_i \subset F((1-u)\rho_i/16).$$

Let us suppose that $b_i \in P_V A_i$, then there is $x \in A_i$ with $P_V x = b_i$ and we can find a point $y \in F$ such that $|x - y| \le (1-u)\rho_i/16$. Then $P_V y \in B(b_i, \rho_i/2)$ and since $\eta < \gamma(1-u)/8$, $\rho_i \le \eta r$. Recalling where we have taken e_i we have

$$d(y, W) \le |y - x| + |x - e_i| + d(e_i, W) < (1-u)\rho_i/16 + \eta^{-1}\gamma(1-u)\rho_i/16 + \eta r < \gamma(1-u)r/4$$

and so $y \in S_i \cap F$, which is impossible. Then $b_i \notin P_V A_i$.
With ∂_V we will indicate the boundary relative to V. Let I_i be the closed segment with end-points b_i and $P_V e_i$, then $I_i \cap \partial_V P_V(A_i) \ne \emptyset$, since $b_i \in I_i \setminus P_V(A_i)$ and $P_V e_i \in I_i \cap P_V(A_i)$. We have that $\partial_V P_V(A_i) = P_V(\partial_{W_i} A_i)$, so we can select $a_i \in \partial_{W_i} A_i$ such

that $P_V(a_i) \in I_i$. Let J_i the closed segment connecting e_i and a_i. Therefore $J_i \subset A_i$ and $P_V J_i \subset I_i$. Since $e_i \in P_V^{-1}(\partial B(b_i, \rho_i)) \cap W(\eta r) \cap F_1$ then

$$|P_V x - b_i| \leq \rho_i \tag{4.26}$$

for all $x \in J_i$. Since $J_i \subset A_i \subset F((1-u)\rho_i/16)$, J_i is contained in the union of the balls $B(x, \rho_i)$ for $x \in F$. The length of J_i is $\eta^{-1}\gamma(1-u)\rho_i/16$ and we can find a finite number of these balls, let us say $B(x_{i,j}, \rho_i)$ for $j = 1, \ldots, k$, such that

$$J_i \cap B(x_{i,j}, \rho_i) \neq \emptyset \tag{4.27}$$

$$B(x_{i,j}, \rho_i) \cap B(x_{i,l}, \rho_i) = \emptyset \text{ for } j \neq l \tag{4.28}$$

$$k > \gamma(1-u)/(160\eta). \tag{4.29}$$

To do so, one can use Theorem 1.16.
We set

$$B_i = \bigcup_{j=1}^{k} B(x_{i,j}, \rho_i)$$

for $i = 1, \ldots, q$. Using 4.26 and 4.27 we have that $P_V B_i \subset B(b_i, 3\rho_i)$ (by standard estimates). Since $B(b_i, 5\rho_i)$ are disjoint then so are the sets $P_V B_i$, as well as B_i. Since $\rho_i \leq \eta r$, by 4.27 and since $\eta < (1-u)/8$,

$$|x_{i,j} - e_i| \leq \mathcal{H}^1(J_i) + \rho_i = \eta^{-1}\gamma(1-u)\rho_i/16 + \rho_i < \gamma(1-u)r/16 + (1-u)r/8 < (1-u)r/4.$$

Taking $b_i' \in W \cap B(a, ur)$ with $P_V b_i' = b_i$, as before

$$\begin{aligned} |e_i - b_i'| &\leq |e_i - P_W e_i| + |P_W e_i - b_i'| \leq \\ &\leq \eta r + |P_V(P_W e_i) - b_i|/\gamma \leq \eta r + |P_V(P_W e_i) - P_V e_i|/\gamma + |P_V e_i - b_i|/\gamma \leq \\ &\leq \eta r + \eta r/\gamma + \rho_i/\gamma \leq 3\eta r/\gamma < (1-u)r/2. \end{aligned}$$

Let $z \in B(x_{i,j}, \rho_i)$, then

$$|z - a| \leq |z - x_{i,j}| + |x_{i,j} - e_i| + |e_i - b_i'| + |b_i' - a| \leq \cdots \leq (7r + ur)/8 < r$$

and we obtain that $B_i \subset B(a, r)$. Now we use 4.28, 4.29 and 4.13 to write

$$\mathcal{H}^m(E \cap B_i) = \sum_{j=1}^{k} \mathcal{H}^m(E \cap B(x_{i,j}, \rho_i)) > k\delta\rho_i^m > 160^{-1}\gamma(1-u)\eta^{-1}\delta\rho_i^m$$

for all $i = 1, \ldots, q$ from which follows, by 4.18 and 4.25, that

$$\begin{aligned} 3^m r^m > \mathcal{H}^m(E \cap B(a, r)) &\geq \sum_{i=1}^{q} \mathcal{H}^m(E \cap B_i) \geq \\ &\geq 160^{-1}\gamma(1-u)\eta^{-1}\delta \sum_{i=1}^{q} \rho_i^m > 100^{-m} 160^{-1}\gamma(1-u)\eta^{-1}\delta t r^m. \end{aligned}$$

We come to a contradiction, since we can choose η as small as much we desire for given δ, u and γ. Then the inequality referred by 4.21 holds and the proof is complete. □

As a consequence of 4.14, a set as in Lemma 4.13 has $\mathcal{H}^m$-measure zero. Now we shall prove a Theorem which will be useful for us in the next chapter. First we prove the following lemma:

Lemma 4.15. *Let μ be a measure on $\mathbb{R}^n$, $F \subset \mathbb{R}^n$ closed and $\delta, M > 0$. If*

$$\mu(B(x,r)) \leq Mr^n$$

for all $0 < r < \delta$ and $B(x,r) \cap F \neq \emptyset$, then

$$\lim_{r \to 0} r^{-n} \mu\left(B(x,r) \setminus F\right) = 0$$

for $\mathcal{L}^n$-almost all $x \in F$.

Proof. Let $x \in F$ and $0 < r < \delta/5$. Let $s_y = d(y,F)/2$ for $y \in B(x,r) \setminus F$. Then $0 < s_y \leq r/2$ and $B(y, s_y) \subset B(x, 2r) \setminus F$. By Theorem 1.16 there is a countable set $S \subset B(x,r) \setminus F$ such that $B(y, s_y)$ with $y \in S$ are disjoint and

$$B(x,r) \setminus F \subset \bigcup_{y \in S} B(y, 5s_y).$$

Therefore,

$$\mu(B(x,r) \setminus F) \leq 5^n M \sum_{y \in S} s_y^n \leq 5^n M \alpha(n)^{-1} \mathcal{L}^n(B(x,2r) \setminus F)$$

and by Lebesgue density Theorem

$$\lim_{r \to 0} r^{-n} \mathcal{L}^n(B(x,2r) \setminus F) = 0$$

for $\mathcal{L}^n$-almost all $x \in F$. This proves the Lemma. □

Theorem 4.16. *Let μ a measure on $\mathbb{R}^n$ and E a $\mathcal{L}^n$-measurable set with $\mu(E) = 0$. Then, for $\mathcal{L}^n$-almost all $x \in E$, we have that*

$$\limsup_{r \to 0} r^{-n} \mu(B(x,r)) = 0$$

or

$$\limsup_{r \to 0} r^{-n} \mu(B(x,r)) = +\infty.$$

Proof. We may assume that E is closed and that the set of points $x \in E$ for which $\limsup_{r \to 0} r^{-n} \mu(B(x,r)) = +\infty$ has $\mathcal{L}^n$-measure zero. Then we set, for $j = 1, 2, \ldots$,

$$F_j = \{x \in E \mid \mu(B(x,r)) \leq jr^n \text{ for } 0 < r < 1/j\}.$$

Each F_j is closed: let $x_k \to x$ a converging sequence of elements in F_j with $x \in E$, and let $0 < r < 1/j$. Let $0 < \varepsilon < 1/j - r$ and $r_k \to r + \varepsilon$ as $k \to +\infty$. Then

$$\mu(B(x,r)) \leq \limsup_{k \to +\infty} \mu(B(x_k, r_k)) \leq j(r + \varepsilon)^m$$

which means that $\mu(B(x,r)) \leq jr^m$ for all $0 < r < 1/j$, i.e. $x \in F_j$. Moreover

$$\{x \in E \mid \limsup_{r \to 0} r^{-n}\mu(B(x,r)) < +\infty\} = \bigcup_{j=1}^{+\infty} F_j.$$

Now we just need to show that the limit is 0 for $\mathcal{L}^n$-almost all $x \in F_j$. Let $x \in F_j$, then $B(x,r) \cap F_j \neq \emptyset$ and $\mu(B(x,r)) \leq jr^n$ for all $0 < r < 1/j$. Then, since $\mu(E) = 0$ we have

$$\begin{aligned}
&\lim_{r \to 0} r^{-n}\mu(B(x,r)) = \\
&\qquad = \lim_{r \to 0} r^{-n}\mu(B(x,r) \setminus F_j) + \lim_{r \to 0} r^{-n}\mu(B(x,r) \cap F_j) = \\
&\qquad\qquad = \lim_{r \to 0} r^{-n}\mu(B(x,r) \setminus F_j) = 0
\end{aligned}$$

□

The proof of this last theorem, and of the lemma we used, works also when $\mathbb{R}^n$ and $\mathcal{L}^n$ are replaced by $\mathcal{S}^{n-1}$ and $\mathcal{H}^{n-1} \llcorner \mathcal{S}^{n-1}$.

Chapter 5

Besicovitch-Federer projection Theorem

In this Chapter we enunciate and prove the Besicovitch-Federer Theorem, which gives a characterization of rectifiable sets in terms of some properties of their orthogonal projection. The Theorem was proved by Besicovitch for $n = 2$ and $m = 1$, while the proof for the general case is credited to Federer. First we state the Theorem:

Theorem 5.1. *Let $A \subset \mathbb{R}^n$ be $\mathcal{H}^m$-measurable with $\mathcal{H}^m(A) < +\infty$. Then*

1. *A is m-rectifiable if and only if $\mathcal{H}^m(P_V B) > 0$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$ for all $B \subset A$ $\mathcal{H}^m$-measurable with $\mathcal{H}^m(B) > 0$.*

2. *A is purely m-unrectifiable if and only if $\mathcal{H}^m(P_V A) = 0$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$.*

These statements are equivalent, and in view of Theorem 4.14, we just need to show that a purely m-unrectifiable set projects in a set of $\mathcal{H}^m$-measure zero for $\gamma_{n,m}$-almost all $V \in G(n,m)$. To do so we will divide the proof in 6 Lemmas.

Lemma 5.2. *Let A be purely m-unrectifiable. Let $\delta > 0$, $V \in G(n, n-m)$ and*

$$A_{1,\delta}(V) = \left\{ a \in A \mid \limsup_{s \to 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m(A \cap X(a,r,V,s)) = 0 \right\}.$$

Then $\mathcal{H}^m(A_{1,\delta}(V)) = 0$.

Proof. It follows immediately from Corollary 4.7. □

Lemma 5.3. *Let $\delta > 0$, $V \in G(n, n-m)$ and*

$$A_{2,\delta}(V) = \left\{ a \in A \mid \limsup_{s \to 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m(A \cap X(a,r,V,s)) = +\infty \right\}.$$

Then $\mathcal{H}^m(Q_V(A_{2,\delta}(V))) = 0$.

Proof. Let $0 < M < +\infty$. For all $a \in A_{2,\delta}(V)$ there are arbitrarily small $s > 0$ and some r, $0 < r < \delta$, such that

$$\mathcal{H}^m(A \cap X(a, r, V, s)) \geq M(rs)^m = M2^{-m}\mathcal{H}^m(Q_V X(a, r, V, s)). \tag{5.1}$$

We can notice that $Q_V X(a, r, V, s) = U(Q_V a, rs) \cap V^\perp$. Considering the cover $Q_V X(a, r_a, V, s_a)$ of $Q_V(A_{2,\delta}(V))$ where r_a, s_a are such that 5.1 holds, we can apply Vitali's covering Theorem and find a countably many disjoint balls such that

$$\mathcal{H}^m\left(Q_V(A_{2,\delta}(A)) \setminus \bigcup_{i=1}^{+\infty} Q_V X(a_i, r_i, V, s_i)\right) = 0.$$

Hence we obtain

$$\begin{aligned}\mathcal{H}^m(Q_V(A_{2,\delta}(V))) \leq \sum_{i=1}^{+\infty} \mathcal{H}^m(Q_V X(a_i, r_i, V, s_i)) \leq \\ \leq \sum_{i=1}^{+\infty} M^{-1}2^m\mathcal{H}^m(A \cap X(a_i, r_i, V, s_i)) \leq M^{-1}2^m\mathcal{H}^m(A).\end{aligned}$$

Letting $M \to +\infty$ we obtain $\mathcal{H}^m(Q_V(A_{2,\delta}(V))) = 0$. □

Lemma 5.4. *Let $V \in G(n, n-m)$ and*

$$A_3(V) = \{a \in A \mid |A \cap (V + a)| = +\infty\}.$$

Then $\mathcal{H}^m(Q_V(A_3(V))) = 0$.

Proof. Recall that $|A \cap (V + a)| = \mathcal{H}^0(A \cap (V + a))$. This Lemma follows from Theorem 3.15:

$$\int_{V^\perp}^* \mathcal{H}^0(A \cap (V + y))\, d\mathcal{H}^m y = \int_{V^\perp}^* \mathcal{H}^0(A \cap \overleftarrow{Q_V}\{y\})\, d\mathcal{H}^m y \leq c\mathcal{H}^m(A) < +\infty,$$

and we have that $\mathcal{H}^0(A \cap (V + y)) < +\infty$ for $\mathcal{H}^m$-almost all $y \in V^\perp$. From this, our assertion follows . □

Now we will introduce some more notations, which will be useful for us in the next two lemmas. Let $m + 1 < n$. We set

$$X^{m+1}(0, L, s) = \{x \in \mathbb{R}^{m+1} \mid d(x, L) < s|x|\}$$

for $L \in G(m + 1, 1)$. For $0 < s < 1$ and $j \in \{m + 1, \ldots, n\}$ we set

$$Z(j, s) = \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^{m} x_i^2 < \frac{s^2}{1 - s^2} x_j^2\right\}.$$

Noting that

$$\sqrt{\sum_{i=1}^{m} x_i^2} < s\sqrt{\sum_{i=1}^{m+1} x_i^2} \Longleftrightarrow \sum_{i=1}^{m} x_i^2 < \frac{s^2}{1-s^2} x_{m+1}^2$$

we obtain that

$$Z(m+1,s) = X^{m+1}(0, L_{m+1}, s) \times \mathbb{R}^{n-m-1},$$

where $L_{m+1} = \langle e_{m+1} \rangle$, the $m+1$-axis. Let $0 < s* < 1$ such that

$$\frac{s^{*2}}{1-s^{*2}} = (n-m)\frac{s^2}{1-s^2}.$$

Using the notations we have just introduced, we can now prove the next Lemma

Lemma 5.5. *Let $V = \{0\} \times \mathbb{R}^{n-m} \in G(n, n-m)$, then we have that*

$$\bigcup_{j=m+1}^{n} Z(j,s) \subset X(0,V,s) \subset \bigcup_{j=m+1}^{n} Z(j,s^*).$$

Proof. We have that

$$X(0,V,s) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{m} x_i^2 < s^2 \sum_{i=1}^{n} x_i^2 \right\} = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{m} x_i^2 < \frac{s^2}{1-s^2} \sum_{i=m+1}^{n} x_i^2 \right\}.$$

The lemma then follows immediately. □

Lemma 5.6. *Let $\delta > 0$. For $\mathcal{H}^m$-almost all $a \in A$ either*

$$\limsup_{s\to 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m(A \cap X(a,r,V,s)) = 0$$

or

$$\limsup_{s\to 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m(A \cap X(a,r,V,s)) = +\infty$$

or

$$(A \setminus \{a\}) \cap (V+a) \cap B(a,\delta) \neq \emptyset,$$

and this holds for $\gamma_{n,n-m}$-almost all $V \in G(n, n-m)$.

Proof. We shall prove the assertion for $m = n-1$ and then for general m. Since A is $\mathcal{H}^m$-measurable of finite measure, by Theorem 1.5 we can assume that A is σ-compact. We can assume, in order to simplify notations, that $a = 0$. For $\theta \in \mathcal{S}^{n-1}$ and $B \subset \mathcal{S}^{n-1}$ let

$$L_\theta = \{t\theta \mid t \in \mathbb{R}\} \quad \text{and} \quad L(B) = \bigcup_{\theta \in B} L_\theta.$$

We can now define a measure μ on $\mathcal{S}^{n-1}$:

$$\mu(B) := \sup_{0<r<\delta} r^{-(n-1)} \mathcal{H}^{n-1}(A \cap B(r) \cap L(B))$$

for all $B \subset \mathcal{S}^{n-1}$. We then set

$$C = \{\theta \in \mathcal{S}^{n-1} \mid (A \setminus \{0\}) \cap B(\delta) \cap L_\theta \neq \emptyset\}$$

which is σ-compact since so is A. Therefore, letting $E = \mathcal{S}^{n-1} \setminus C$, $\mu(E) = 0$. Then, by theorem 4.16, we obtain that for almost all $\theta \in \mathcal{S}^{n-1}$ either

$$\limsup_{t \to 0} t^{-(n-1)} \mu(\mathcal{S}^{n-1} \cap B(\theta, t)) = 0$$

or

$$\limsup_{t \to 0} t^{-(n-1)} \mu(\mathcal{S}^{n-1} \cap B(\theta, t)) = +\infty$$

or

$$\theta \in C.$$

We have that for any $x, \theta \in \mathcal{S}^{n-1}$ with $x \cdot \theta \geq 0$

$$d(x, L_\theta) \leq |x - \theta| \leq 2d(x, L_\theta).$$

Therefore we obtain

$$X(0, r, L_\theta, s) \subset B(r) \cap L(\mathcal{S}^{n-1} \cap B(\theta, 2s)) \setminus \{0\} \subset X(0, r, L_\theta, 3s)$$

which tells us that the three conditions we found are equivalent to the three condition of the lemma for $m = n-1$. Let now $m < n-1$. We can say more from what we have proved: if A is σ-compact and $\mathcal{H}^m(A) < +\infty$ for $\gamma_{m+1,1}$-almost all $L \in G(m+1, 1)$ either

$$\limsup_{s \to 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m\Big(A \cap B(r) \cap (X^{m+1}(0, L, s) \times \mathbb{R}^{n-m-1})\Big) = 0$$

or

$$\limsup_{s \to 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m\Big(A \cap B(r) \cap (X^{m+1}(0, L, s) \times \mathbb{R}^{n-m-1})\Big) = +\infty$$

or

$$(A \setminus \{0\}) \cap B(\delta) \cap (L \times \mathbb{R}^{n-m-1}) \neq \emptyset.$$

To get this, we can apply Theorem 4.16, defining

$$\begin{aligned}
&L(B) = \bigcup_{\theta \in B} (L_\theta \times \mathbb{R}^{n-m-1}) \subset \mathbb{R}^n \\
&C = \{\theta \in \mathcal{S}^m \mid (A \setminus \{0\}) \cap B(\delta) \cap (L_\theta \times \mathbb{R}^{n-m-1}) \neq \emptyset\} \\
&\mu(B) := \sup_{0<r<\delta} r^{-m} \mathcal{H}^m(A \cap B(r) \cap L(B))
\end{aligned}$$

for $B \subset \mathcal{S}^m$ and proceeding as we did for $m = n-1$. Using lemma 5.5 we just need to show that θ_n-almost all $g \in \mathcal{O}(n)$ either

$$\begin{aligned}
&\limsup_{s \to 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m\big(A \cap B(r) \cap gZ(j, s)\big) = 0 \text{ or} \\
&\limsup_{s \to 0} \sup_{0<r<\delta} (rs)^{-m} \mathcal{H}^m\big(A \cap B(r) \cap gZ(j, s)\big) = +\infty \text{ or} \\
&(A \setminus \{0\}) \cap gV \cap B(\delta) \neq \emptyset.
\end{aligned}$$

We will prove this for $j = m+1$. Let χ be the characteristic function of those $g \in \mathcal{O}(n)$ for which none of the three alternatives hold. Since A is σ-compact, χ is a Borel function. We set

$$\mathcal{O}(m+1) = \{g \in \mathcal{O}(n) \mid g|\{0\} \times \mathbb{R}^{n-m-1} \text{ is the identity } \}$$

and since $gZ(m+1,s) = X^{m+1}(0, gL_{m+1}, s) \times \mathbb{R}^{n-m-1}$ for $g \in \mathcal{O}(m+1)$ we obtain, from the first part of the proof, that

$$\int_{\mathcal{O}(m+1)} \chi \, d\theta_{m+1} = 0.$$

For any $h \in \mathcal{O}(n)$, $h^{-1}(A)$ is σ-compact and since the characteristic function corresponding to $h^{-1}(A)$ is $g \mapsto \chi(h \circ g)$ we have

$$\int_{\mathcal{O}(m+1)} \chi(h \circ g) \, d\theta_{m+1} g = 0.$$

Then, since θ_{m+1} is invariant, for any $g \in \mathcal{O}(m+1)$ we have

$$\int_{\mathcal{O}(n)} \chi(h) \, d\theta_n h = \int_{\mathcal{O}(n)} \chi(h \circ g) \, d\theta_n h.$$

Therefore, using Fubini's Theorem, we obtain

$$\begin{aligned}
\int_{\mathcal{O}(n)} \chi(h) \, d\theta_n h &= \int_{\mathcal{O}(m+1)} \int_{\mathcal{O}(n)} \chi(h) \, d\theta_n h \, d\theta_{m+1} = \\
&= \int_{\mathcal{O}(m+1)} \int_{\mathcal{O}(n)} \chi(h \circ g) \, d\theta_n h \, d\theta_{m+1} g = \\
&= \int_{\mathcal{O}(n)} \int_{\mathcal{O}(m+1)} \chi(h \circ g) \, d\theta_{m+1} g \, d\theta_n h = 0
\end{aligned}$$

and we can conclude. $\square$

We are now ready to prove Theorem 5.1:

Proof of Besicovith-Federer projection Theorem. For all the notations recall Lemmas 5.2, 5.3, and 5.4. Let $V \in G(n, n-m)$ and $\delta > 0$. Let

$$A_{3,\delta}(V) = \{a \in A \mid (A \setminus \{a\}) \cap (V + a) \cap B(a, \delta) \neq \emptyset\}.$$

By lemma 5.6 we have for $\gamma_{n,n-m}$-almost all $V \in G(n, n-m)$ that

$$\mathcal{H}^m\big(A \setminus (A_{1,\delta}(V) \cup A_{2,\delta}(V) \cup A_{3,\delta}(V))\big) = 0. \tag{5.2}$$

Now we shall show that if $V \in G(n, n-m)$ is such that 5.2 holds for all $\delta_i = 1/i$, with $i = 1, 2, \ldots$, then $\mathcal{H}^m(Q_V A) = 0$ and the Theorem will be proved. Since $\mathcal{H}^m(A_{1,\delta}(V)) = 0$,

$$\mathcal{H}^m\big(A \setminus (A_{2,\delta}(V) \cup A_{3,\delta}(V))\big) = 0.$$

Moreover

$$\bigcap_{\delta_i} A_{3,\delta_i}(V) \subset A_3(V)$$

and

$$A \setminus \left(\bigcup_{\delta_i} A_{2,\delta_i}(V) \cup A_3(V) \right) \subset A \setminus \left(\bigcup_{\delta_i} A_{2,\delta_i}(V) \cup \bigcap_{\delta_i} A_{3,\delta_i}(V) \right) \subset$$
$$\subset \bigcup_{\delta i} \Big(A \setminus (A_{2,\delta_i}(V) \cup A_{3,\delta_i}(V)) \Big).$$

Then

$$\mathcal{H}^m \left(Q_V A \setminus \left(Q_V(\bigcup_{\delta_i} A_{2,\delta_i}(V)) \cup Q_V(A_3(V)) \right) \right) \leq$$
$$\leq \mathcal{H}^m \left(Q_V \Big(A \setminus (\bigcup_{\delta_i} A_{2,\delta_i}(V) \cup A_3(V)) \Big) \right) \leq \sum_{\delta_i} \mathcal{H}^m \Big(A \setminus (A_{2,\delta_i}(V) \cup A_{3,\delta_i}(V)) \Big) = 0$$

but, by lemmas 5.3, 5.4 we have that $\mathcal{H}^m(Q_V \left(\bigcup_{\delta_i} A_{2,\delta_i}(V)\right)) = 0$ and $\mathcal{H}^m(Q_V(A_3(V))) = 0$. It follows that $\mathcal{H}^m(Q_V A) = 0$. We have then proved the Besicovitch-Federer projection Theorem. □

5.1 Cantor set in $\mathbb{R}^2$

We now focus our attention on a particular subset of $\mathbb{R}^2$ which has positive and finite $\mathcal{H}^1$-measure and it is purely 1-unrectifiable. We prove first a version of the Besicovitch-Federer Projection theorem in $\mathbb{R}^2$ for 1-rectifiable sets. We can notice that when $n = 2$ and $m = 1$, every $L \in G(2,1)$ forms an angle $\theta \in [0, \pi)$ with the x-axis. We can then identify $G(2,1)$ with $[0, \pi]$ where π is identified with 0 and then $\gamma_{2,1} = \mathcal{L}^1 \llcorner [0, \pi)$. With L_θ we will indicate the element of $G(2,1)$ forming an angle θ with the x-axis.

Theorem 5.7. *Let $E \subset \mathbb{R}^2$ be 1-rectifiable and $0 < \mathcal{H}^1(E) < +\infty$. Then there exists at most one direction $\theta \in [0, \pi)$ such that $\mathcal{H}^1(P_{L_\theta}(E)) = 0$.*

Proof. Since E is 1-rectifiable it is the union of 1-rectifiable curves Γ_i and a null set, then $E \subset \bigcup_i \Gamma_i$. Therefore we can suppose that $E \subset \Gamma$, where Γ is a 1-rectifiable curve. Moreover, $\Theta^m(E, x) = 1$ for $\mathcal{H}^1$-almost all $x \in E$ by Theorem 4.14. Let $x \in E$ such that $\Theta^m(\Gamma, x) = 1$. Then $\forall \varepsilon > 0$ we can find an $r > 0$ small enough such that

$$\mathcal{H}^1(E \cap B(x,r)) > (1 - \varepsilon^2)2r$$

and

$$\mathcal{H}^1(\Gamma \cap B(x,r)) < (1 + \varepsilon)2r.$$

This implies that

$$\mathcal{H}^1(E \cap B(x,r)) > (1-\varepsilon)\mathcal{H}^1(\Gamma \cap B(x,r))$$

whence

$$\mathcal{H}^1((\Gamma \setminus E) \cap B(x,r)) < \varepsilon \mathcal{H}^1(\Gamma \cap B(x,r)).$$

$\Gamma \cap B(x,r)$ is the union of at most countably many disjoint arcs and we can choose an arc $\Gamma_0 \subset \Gamma \cap B(x,r)$ such that

$$\mathcal{H}^1(\Gamma_0 \setminus E) < \varepsilon \mathcal{H}^1(\Gamma_0) < 2\varepsilon|y-z|$$

where x, y are the endpoints of Γ_0. We take then an $L_\theta \in G(2,1)$ such that it forms an angle φ with the segment $[x,y]$ for which $|\cos(\varphi)| > 2\varepsilon$. Then

$$\begin{aligned}\mathcal{H}^1(P_{L_\theta}(E)) > |cos\varphi||y-z| - \mathcal{H}^1(P_{L_\theta}(\Gamma_0 \setminus E)) \geq \\ \geq |cos\varphi||y-z| - \mathcal{H}^1(\Gamma_0 \setminus E) > |cos\varphi||y-z| - \varepsilon\mathcal{H}^1(\Gamma_0) > \\ > (|\cos(\varphi)| - 2\varepsilon)|y-z| > 0.\end{aligned}$$

Thus $\mathcal{H}^1(P_{L_\theta}(E)) > 0$ for all $\theta \in [0,\pi)$ except for a set of directions of length $2\cos^{-1}(2\varepsilon)$ for all $\varepsilon > 0$, which means that $\mathcal{H}^1(P_{L_\theta}(E)) = 0$ for at most one direction $\theta \in [0,\pi)$. □

From this theorem we can deduce that if $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < +\infty$ such that it projects in a set of $\mathcal{H}^1$-measure zero for two distinct directions we can conclude that E is purely 1-unrectifiable. Therefore by Theorem 5.1 we can conclude that $\mathcal{H}^1(P_L(E)) = 0$ for $\gamma_{2,1}$-almost all $L \in G(2,1)$. We now present a family of Cantor-type subsets of $\mathbb{R}^2$: $Q_\lambda = C_\lambda \times C_\lambda$ where C_λ is the λ-Cantor set presented in Chapter 3, with $0 < \lambda < 1/2$. In order to visualize it we can take, as first step, a square Q_0 with sides of length 1 and vertices $(0,0), (1,0), (0,1), (1,1)$. Then (see figure 5.1 for $\lambda = 1/4$) we proceed to take 4 squares $Q_{1,j}$ with sides of length λ, then of length λ^2 and so on.

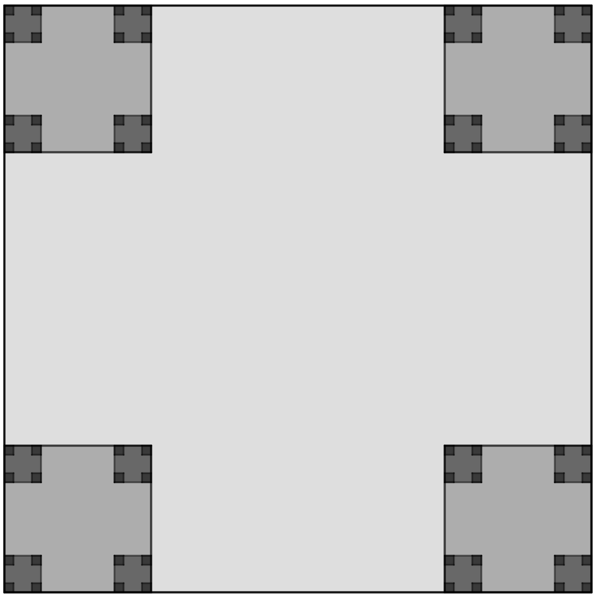

Figure 5.1: First 4 iterations of $C_{1/4} \times C_{1/4}$

We wish first to estimate the dimension of $C_\lambda \times C_\lambda$. Let us call $\{Q_{k,j}\}_{j=1,\ldots,4^k}$ the squares of the k-th iteration; we have that $d(Q_{k,j}) = \sqrt{2}\lambda^k$. Then we can estimate

$$\mathcal{H}^s_{\sqrt{2}\lambda^k}(Q_\lambda) \leq \sum_{j=1}^{4^k} d(Q_{k,j})^s \leq \sqrt{2}^s (4\lambda^s)^k$$

and choosing $s = \frac{\log 4}{\log(1/\lambda)}$ we have that

$$\mathcal{H}^s(Q_\lambda) \leq \sqrt{2}^s$$

which tells us that $dimQ_\lambda \leq \frac{\log 4}{\log(1/\lambda)}$. By a similar argument to the one presented in Chapter 3 one can see also that $0 < \mathcal{H}^s(Q_\lambda)$, which tells us that

$$dimQ_\lambda = \frac{\log 4}{\log(1/\lambda)}.$$

Let us consider the Cantor set with $\lambda = 1/4$. Then, $dimQ_{1/4} = 1$ and its measure is positive and finite. It is easy to check that the projections of $Q_{1/4}$ on the x and y axis have $\mathcal{H}^1$-measure zero. Then by Theorem 5.1 we can conclude that $Q_{1/4}$ is purely 1-unrectifiable. It is also possible to find some lines, for instance 4, where the projection of Q_λ has positive $\mathcal{H}^1$-measure (see figure 5.2); of course the set of those lines have $\gamma_{2,1}$-measure zero, in particular the set formed by these 4 distinct lines has measure zero.

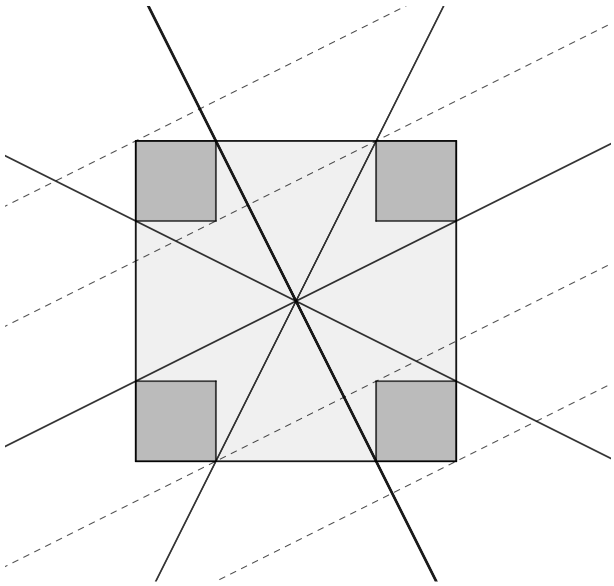

Figure 5.2: Representation of 4 directions where the projection of $Q_{1/4}$ has positive $\mathcal{H}^1$-measure.

To show that $Q_{1/4}$ is purely 1-unrectifiable we can also show that

$$apTan^1(Q_{1/4}, x) = \emptyset$$

for all $x \in Q_{1/4}$, which means that for all x we have to find an $0 < s < 1$ such that

$$\limsup_{r \to 0} r^{-1} \mathcal{H}^1(Q_{1/4} \cap B(x,r) \setminus X(x,L,s)) > 0.$$

Let us fix $x \in Q_{1/4}$, and take $L \in G(2,1)$. If $x \in Q_{1/4}$ we can find a sequence of squares forming $Q_{1/4}$ $\{Q_{k,j_k}\}_{k=1,\dots,}$ such that $x \in Q_{k,j_k}$ for all $k \in \mathbb{N}$. Then, setting $r_k = \sqrt{2}4^{-k}$, we have that $Q_{1/4} \cap Q_{k,j_k} \subset Q_{1/4} \cap B(x, r_k)$. We can then choose $0 < s < 1$ small enough such that there is an $Q_{k+1,i_{k+1}}$ entirely contained in $Q_{1/4} \cap Q_{k,j_k} \setminus X(x,L,s)$. For instance, we could take $0 < s < 1/\sqrt{5}$. Therefore

$$\begin{aligned} r_k^{-1}\mathcal{H}^1(Q_{1/4} \cap B(x,r_k) \setminus X(x,L,s)) &\geq \\ \geq r_k^{-1}\mathcal{H}^1(Q_{1/4} \cap Q_{k,j_k} \setminus X(x,L,s)) &\geq r_k^{-1}\mathcal{H}^1(Q_{1/4} \cap Q_{k+1,i_{k+1}}) = \\ &= \frac{\mathcal{H}^1(Q_{1/4})4^k}{4^{k+1}\sqrt{2}} = \frac{\mathcal{H}^1(Q_{1/4})}{4\sqrt{2}} > 0. \end{aligned}$$

Letting $k \to +\infty$ we can conclude. Another easier way to see that Q_λ is purely 1-unrectifiable, is to notice that given $E_1, E_2 \subset \mathbb{R}$ with $\mathcal{H}^1(E_1) = \mathcal{H}^1(E_2) = 0$ and

$$\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$$

a C^1 function we have (locally) that $\gamma(t) = (t, y(t))$ and so $(E_1 \times E_2) \cap im\gamma \subset \gamma(E_1)$. Therefore

$$\mathcal{H}^1((E_1 \times E_2) \cap im\gamma) \leq \mathcal{H}^1(\gamma(E_1)) \leq (Lip\gamma)\mathcal{H}^1(E_1) = 0$$

and $E_1 \times E_2$ is purely 1-unrectifiable. Now we give an example of a set in $\mathbb{R}^2$ which is 1-rectifiable. We will verify this in three different ways. Let $\{q_1, \ldots, q_k, \ldots\}$ the set of points with rational coordinates contained in $B(0,1)$. Let $S_i = \partial B(q_i, 2^{-i})$, and define

$$E = \bigcup_{i=1}^{+\infty} S_i.$$

It is pretty easy to verify that $\mathcal{H}^1(E) = 2\pi$, and it is still easy to check that the orthogonal projection of this set on every line $L \in G(2,1)$ has positive measure, which means that E is 1-rectifiable. One can also verify that E has an approximate tangent line in $\mathcal{H}^1$-almost all $x \in E$. Let us fix $i \in \mathbb{N}$; by Theorem 3.9,

$$\lim_{r \to 0} r^{-1}\mathcal{H}^1\left(\left(\bigcup_j S_j \setminus S_i\right) \cap B(x,r)\right) = 0$$

for $\mathcal{H}^1$-almost all $x \in E$, and for such $x \in E$, the tangent line through $x \in S_i$ is our desired approximate tangent line of E. It is also easy to see that E is the countably union of Lipschitz curves. For instance, $f_i(t) = q_i + 2^{-i}(\cos t, \sin t)$ with $t \in [0, 2\pi)$.

5.2 Conclusions

The main objective of this thesis was to enunciate and prove the Besicovitch-Federer Theorem, which is what we did in Chapters 4 and 5. Most of the arguments in Chapters 1-3 were presented for the reader that is approaching for the first time the study of Geometric measure Theory.

We conclude now presenting some generalizations to what we did, such as the definition of Rectifiability. Let μ be a measure on $\mathbb{R}^n$. In [E], $E \subset \mathbb{R}^n$ is called *countably (μ, m)-rectifiable* if there are Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\mu\left(E \setminus \bigcup_{i=1}^{+\infty} f_i(\mathbb{R}^m)\right) = 0.$$

$E \subset \mathbb{R}^n$ will be called *(μ, m)-rectifiable* if $\mu(E) < +\infty$. $F \subset \mathbb{R}^n$ is *purely (μ, m)-unrectifiable* if $\mu(F \cap E) = 0$ for all $E \subset \mathbb{R}^n$ (μ, m)-rectifiable sets. Let us suppose that μ is a Borel regular measure on $\mathbb{R}^n$ and A is a Borel set with $\mu(A) < +\infty$ such that

1. $|A \cap P_V^{-1}\{y\}| < +\infty$ for $\mathcal{H}^m$-almost all $y \in V$ and
2. $\mathcal{H}^m(P_V(B)) = 0$ for all $B \subset A$ and $\mu(B) = 0$.

From the proof of Theorem 5.1 we can deduce that if A is purely (μ, m)-unrectifiable, then $\mathcal{H}^m(P_V(A)) = 0$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$. This is exactly what Federer proved. We finish presenting an interesting relation for 1-rectifiable sets in $\mathbb{R}^2$ and the *Menger curvature.* Let $x, y, z \in \mathbb{R}^2$ and let $R(x, y, z)$ be the radius of the circle passing through these three points. If x, y, z are aligned then $R(x, y, z) = +\infty$. The Menger curvature of the triple (x, y, z) is

$$c(x, y, z) = \frac{1}{R(x, y, z)}.$$

We can notice that x, y, z are aligned if and only if $c(x, y, z) = 0$. An explicit formula for $c(x, y, z)$ is

$$c(x, y, z) = \frac{4A(x, y, z)}{|x-y||x-z||y-z|}$$

where $A(x, y, z)$ is the area of the triangle with vertexes x, y, z. Let us set

$$c^2(A) = \int_A \int_A \int_A c(x, y, z)^2 \, d\mathcal{H}^1 x \, d\mathcal{H}^1 y \, d\mathcal{H}^1 z$$

for any $\mathcal{H}^1$-measurable set $A \subset \mathbb{R}^2$.

Let $E \subset \mathbb{R}^2$ be a $\mathcal{H}^1$-measurable set with $\mathcal{H}^1(E) < +\infty$. It can be proved that if $c^2(E) < +\infty$, then E is 1-rectifiable. This is a Theorem of David and Léger, whose proof can be found in [I].

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