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TESI DI LAUREA MAGISTRALE

TRACE THEOREM FOR THE MARTINET DISTRIBUTION

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[...] nel silenzio sordo del mondo
la bellezza non tacerà.

Gianni Rodari

Contents

Introduction

Tracing functions in the Euclidean case; a very brief "historical" overview. Given $\Omega \subseteq \mathbb{R}^n$ open (bounded) of class $\mathscr{C}^1$, consider the Dirichlet problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $g : \partial\Omega \to \mathbb{R}$ is a given function, together with its weak formulation:

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = 0 \qquad \forall\, \varphi \in \mathscr{C}_c^\infty(\Omega).$$

We look for a solution in the Sobolev space $W^{1,2}(\Omega)$. Some questions arise: what does it mean that $u = g$ on $\partial\Omega$ for functions $u \in W^{1,2}(\Omega)$? Which are the admissible functions g?

The *main problem* could then be reformulated in these terms: given $\Omega \subseteq \mathbb{R}^n$ open of class $\mathscr{C}^l$, we want to define a linear continuous operator $\mathrm{Tr} : W^{l,p}(\Omega) \to L^p(\partial\Omega)$ such that $\mathrm{Tr} f = f_{|\partial\Omega}$ for all $f \in W^{l,p}(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$; moreover we want to describe explicitely the space $\mathrm{Tr}(W^{l,p}(\Omega))$. The problem with $l = 1$ and $p > 1$ was solved by Emilio Gagliardo in [8]; before [8] only the case $p = 2$ had been treated (by Nachman Aronszajn in [1], L. N. Slobodetskij and and Mikhail Vasil'evich Babich in [2]). In his article, Gagliardo used the definition of trace of a function given by Charles B. Morrey in [13].

Before stating the main result, which is due to many successive refinements, we recall that $\|\cdot\|_{W^{l,p}(\mathbb{R}^n)}$ is the *Sobolev norm* while the *Besov-Nikol'skiĭ norm* $\|\cdot\|_{B^l_p(\mathbb{R}^n)}$ is defined by

$$\|f\|_{B^l_p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^N} \frac{\|\Delta_h^\sigma f(x)\|^p_{L^p_x(\mathbb{R}^n)}}{|h|^{pl+n}} \, dh \right)^{1/p}$$

for $p \in [1, \infty[$, while for $p = \infty$ we set

$$\|f\|_{B^l_\infty(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)} + \sup_{h \neq 0} \frac{\|\Delta_h^\sigma f(x)\|_{L^\infty_x(\mathbb{R}^n)}}{|h|^l}$$

with $l \in]0, \infty[$, $l < \sigma \in \mathbb{N}$, $\Delta_h f(x) = f(x+h) - f(x)$ and recursively

$$\Delta_h^\sigma f(x) = \underbrace{\Delta_h(\Delta_h(\ldots(\Delta_h}_{\sigma \text{ times}} f)\ldots))(x) = \sum_{k=0}^{\sigma} (-1)^\sigma \binom{\sigma}{k} f(x+kh).$$

The space $B^l_p(\mathbb{R}^n)$ is the space of those functions $\in L^p(\mathbb{R}^n)$ such that $\|f\|_{B^l_p(\mathbb{R}^n)} < \infty$. It is called *Besov-Nikol'skiĭ space*.

We also remind that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$ with $m < n$, then g is said to be a *trace of the function* f if there exists a function h equivalent to f on $\mathbb{R}^n$ such that

$$h(\cdot, v) \to g(\cdot) \ \text{ in } \ L^1_{\mathrm{loc}}(\mathbb{R}^m) \ \text{ as } \ v \to 0.$$

The aforementioned main result is the following:

Theorem 1 (Trace theorem for linear subspaces of $\mathbb{R}^n$)**.** *Let $l, m \in \mathbb{N}$, $m < n$, $1 \le p \le \infty$, $l > (n-m)/p$. Then for all $f \in W^{l,p}(\mathbb{R}^n)$ we have $\mathrm{Tr} f \in B_p^{l-\frac{n-m}{p}}(\mathbb{R}^m)$ and there exists $c > 0$ such that*

$$\|\mathrm{Tr}_{\mathbb{R}^m} f\|_{B_p^{l-\frac{n-m}{p}}(\mathbb{R}^m)} \le c\|f\|_{W^{l,p}(\mathbb{R}^n)}.$$

Moreover $\mathrm{Tr}_{\mathbb{R}^m} : W^{l,p}(\mathbb{R}^n) \to B_p^{l-\frac{n-m}{p}}(\mathbb{R}^m)$ *is surjective.*

Definitions of traces of functions, Besov-Nikol'skiĭ spaces and so on may be given also if we replace $\mathbb{R}^N$ with a smooth surface, as we required at the beginning. It requires some work; the complete theory is brilliantly developed for example in [4].

Trace theorem in the Grushin plane. The idea of this thesis comes from another work by my supervisor Professor Roberto Monti and his colleague Professor Daniele Morbidelli (namely [12]); a part of that work has been devoted to the study of the trace theorem in the Grushin plane.

Let us first consider the Carnot-Carathéodory metric d (see Section 1.2) induced on $\mathbb{R}^2$ by the vector fields

$$X_1 = \partial_x, \quad X_2 = |x|^\alpha \partial_y, \ \alpha > 0.$$

If $(x, y) \in \mathbb{R}^2$ and $r \ge 0$, let $B((x, y), r) = \{(\xi, \eta) \in \mathbb{R}^2 \,:\, d((x, y), (\xi, \eta)) < r\}$. The boxes

$$\mathrm{Box}((x, y), r) = [x - r, x + r] \times [y - r(|x| + r)^\alpha, y + r(|x| + r)^\alpha]$$

are equivalent to C-C balls and the metric d can be evaluated rather explicitely. Indeed it can be proved that there exist constants $0 < c_1 < c_2$ such that for all $(x, y) \in \mathbb{R}^2$ and $r \ge 0$

$$\mathrm{Box}((x, y), c_1 r) \subset B((x, y), r) \subset \mathrm{Box}((x, y), c_2 r).$$

Moreover for $\lambda > 0$ and for all $(x, y), (\xi, \eta) \in \mathbb{R}^2$ with $|x| \ge |\xi|$

$$d((x, y), (\xi, \eta)) \simeq \begin{cases} |x - \xi| + \frac{|y-\eta|}{|x|^\alpha} & \text{if } |x|^{\alpha+1} \ge \lambda |y - \eta| \\ |x - \xi| + |y - \eta|^{1/(\alpha+1)} & \text{if } |x|^{\alpha+1} < \lambda |y - \eta| \end{cases}$$

where the equivalence constants depend of λ.

Definition 2. Let $\Omega \subset \mathbb{R}^2$ be an open set with $\partial\Omega$ of class $\mathscr{C}^1$. A point $(0, y_0) \in \partial\Omega$ is said to be *α-admissible*, $\alpha > 0$, if one of the following two conditions holds:

- (Non characteristic case). There exist $\delta > 0$ and $\psi \in \mathscr{C}^1(y_0 - \delta, y_0 + \delta)$ such that $\psi(y_0) = 0$ and

$$\partial\Omega \cap (-\delta, \delta) \times (y_0 - \delta, y_0 + \delta) = \{(\psi(y), y) \,:\, |y - y_0|, |\psi(y)| < \delta\}.$$

- (Characteristic case). There exist $\delta > 0$ and $c > 0$ such that

$$\partial\Omega \cap (-\delta, \delta) \times (y_0 - \delta, y_0 + \delta) = \{(x, \varphi(x)) \in \mathbb{R}^2 \,:\, |x| < \delta\},$$

where $\varphi \in \mathscr{C}^1(-\delta, \delta)$ and $|\varphi'(x)| \leq c|x|^\alpha$ for all $x \in (-\delta, \delta)$.

Finally, Ω is said to be α-*admissible* if all the points of $\partial\Omega \cap \{x = 0\}$ are $\alpha-$admissible.

Finally we introduce on $\partial\Omega$ the measure $\mu := |X\nu|\mathcal{H}^1 \llcorner \partial\Omega$ where $\nu(x, y)$ is the unit normal to $\partial\Omega$ at $(x, y) \in \partial\Omega$ and

$$\begin{aligned}|X\nu(x, y)| &= \left(\langle X_1(x, y), \nu(x, y)\rangle^2 + \langle X_2(x, y), \nu(x, y)\rangle^2\right)^{1/2} \\ &= \left(\nu_1(x, y)^2 + |x|^{2\alpha}\nu_2(x, y)^2\right)^{1/2}.\end{aligned}$$

This measure can be estimated rather explicitely in terms of the Lebesgue measure:

Lemma 3. *Let $\Omega \subset \mathbb{R}^2$ be bounded open set with $\partial\Omega$ of class $\mathscr{C}^1$ and suppose it is α-admissible. Then there esist $0 < m_1 < m_2$ and $r_0 > 0$ such that*

$$m_1 \frac{\mathscr{L}^2(B((x, y), r))}{r} \leq \mu(B(x, y), r) \leq m_2 \frac{\mathscr{L}^2(B((x, y), r))}{r}$$

for all $(x, y) \in \partial\Omega$ and for all $0 < r < r_0$.

The starting point of our thesis is the following theorem:

Theorem 4. *Let $X_1 = \partial_x$ and $X_2 = |x|^\alpha\partial_y$, $\alpha > 0$. Let $1 < p < \infty$ and $s = 1 - 1/p$. If $\Omega \subset \mathbb{R}^2$ is a bounded open set of class $\mathscr{C}^1$ which is α-admissible, then there exist $C > 0$ and $\delta_0 > 0$ such that*

$$\int_{\partial\Omega\times\partial\Omega\cap\{d(z,\xi)<\delta_0\}} \frac{|u(z) - u(\xi)|^p}{d(z, \xi)^{ps}\mu(B(z, d(z, \xi)))}\, d\mu(z)\, d\mu(\xi) \leq C \int_\Omega |Xu(x, y)|^p\, dx\, dy$$

for all $u \in \mathscr{C}^1(\Omega) \cap \mathscr{C}(\overline{\Omega})$.

In the second part of [12] they also prove that the hypothesis of α-admissiblity for the domain Ω in the previous theorem is necessary; more precisely there exist domains of class $\mathscr{C}^1$ that are not α-admissible for which the estimate of the previous theorem fails.

Content of the thesis. The first chapter of this work is dedicated to some theoretical prolegomena, tools that we will use in the second chapter in order to prove the main result.

In the second chapter we prove the trace theorem for the *Martinet distribution* defined in $\mathbb{R}^3$ by the vector fields

$$X = (X_1, X_2) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + \frac{x^2}{2}\frac{\partial}{\partial z}\right).$$

$\mathbb{R}^3$ will be endowed with the sub-Riemannian distance d induced by X: according to the definition given in Section 1.2 if

$$\ell(c) = \int_0^\tau \sqrt{\inf\{u_1^2(t) + u_2^2(t) \,:\, u_1(t)X_1(c(t)) + u_2(t)X_2(c(t)) = \dot{c}(t)\}}\, dt$$

then by definition, if $p, q \in \mathbb{R}^3$, we have

$$d(p,q) = \inf\{\ell(c) \,:\, c : [0,\tau] \to \mathbb{R}^3 \text{ absolutely continuous with } c(0) = p \text{ and } c(\tau) = q\}.$$

This distance can be computed rather explicitely: we prove an estimate for d in terms of the Euclidean metric of $\mathbb{R}^3$ in Lemma 2.3.1. $(\mathbb{R}^3, d)$ is a *Carnot-Carathéodory space.*

A complete and interesting result would be a trace theorem for a suitable $\Omega \subseteq \mathbb{R}^3$ regular enough, but in order to rough-hew the problem we chose a "simpler" case: $\Omega = \{z > 0\}$.

Let us consider then the Euclidean unit normal to $\partial\Omega$ $\nu = (0,0,1)$; we have that

$$|X\nu| = \sqrt{\langle X_1, \nu\rangle^2 + \langle X_2, \nu\rangle^2} = x^2.$$

A point $x \in \partial\Omega$ is said to be *characteristic* if $|X\nu| = 0$ (see also Section 1.2). At these points the vector fields X_1 and X_2 become tangent to the boundary and most of the tools from classical analysis fail to work. For example, near the characteristic set standard surface measure does not scale correctly and fails to satisfy estimates of the kind of Proposition 2.2.1 with respect to sub-Riemannian balls (cfr. introduction of [5]). For these reasons the right surface measure in the LHS cannot be the Hausdorff one. The natural surface measure that takes into account characteristic points in the boundary is

$$\mu := |X\nu| \mathscr{L}^2 \llcorner \partial\Omega.$$

In Proposition 2.2.1 we prove an estimation for the μ-measure of the sub-Riemannian boxes.

The more interesting part of our work is in fact the proof of the trace theorem *even* in the nearby of characteristic points (there are indeed more general trace theorems for spaces without characteristic points - see the first part of [12]).

Let us now state the main result of this thesis:

Theorem 5. *For every $1 < p < \infty$ there exists a constant $C(p) > 0$ such that for every function $u \in \mathscr{C}^1_c(\mathbb{R}^2 \times [0,\infty))$ the following integral estimate holds*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|u(w,0) - u(q,0)|^p}{d(w,q)^{ps}\mu(B(w,d(w,q)))}\, d\mu(w)\, d\mu(q) \le C(p) \int_{\mathbb{R}^2\times[0,\infty)} |Xu(x,y,z)|^p \, dx\, dy\, dz.$$

The proof of Theorem 5 relies on some recurrent ideas; we firstly split the integration domain of LHS, which without losing of generality may be assumed to be for instance $[-1,1]^2 \times [-1,1]^2$ (see Remark 2.4.2), into four subdomains:

$$\begin{aligned}
A_1 &= \{((x,y),(x',y')) \in [-1,1]^2 \times [-1,1]^2 \,:\, |x| \le d(w,q),\ |x'| \le d(w,q)\} \\
A_2 &= \{((x,y),(x',y')) \in [-1,1]^2 \times [-1,1]^2 \,:\, |x| > d(w,q),\ |x'| \le d(w,q)\} \\
A_3 &= \{((x,y),(x',y')) \in [-1,1]^2 \times [-1,1]^2 \,:\, |x| \le d(w,q),\ |x'| > d(w,q)\} \\
A_4 &= \{((x,y),(x',y')) \in [-1,1]^2 \times [-1,1]^2 \,:\, |x| > d(w,q),\ |x'| > d(w,q)\};
\end{aligned}$$

in each A_i we constructed paths linking points $(x,y,0)$ and $(x',y',0)$ using vector fields X_1, X_2, their integral curves and suitable approximations of $[X_1, X_2]$. To fix the ideas, over the set A_1 we firstly estimate

$$|u(x,y,0) - u(x,y',x^2(y'-y)/2)| = \left| \int_0^{y'-y} X_2 u(\underbrace{x, y+t, x^2t/2}_{=\Psi_1(x,y,t)})\, dt \right|,$$

and then

$$|u(x, y', x^2(y'-y)/2) - u(x', y', x^2(y'-y)/2)| = \left| \int_0^{x'-x} X_1 u(x+t, y', x^2(y'-y)/2)\, dt \right|$$

where these equalities are justified basically by fundamental theorem of calculus; the "descent" from $u(x', y', x^2(y'-y)/2)$ to $u(x', y', 0)$ is made following the vector field $[[X_1, X_2], X_1] = -\frac{\partial}{\partial z}$ for a suitable time, approximating the Lie brackets via their components, in order of being able of "reconstructing" the sub-elliptic gradient $|Xu|$. Here Hardy and Minkowski inequalities are also used.

The case A_4 requires some more refined ideas; even though the main technique of "linking" points (subdividing the path into some sub-paths) is used again, because of the constraints on the coordinates (given by the definition of A_4 itself) the integrals to be estimated present a much more entangled situation (in terms of variables dependence); we needed to use Minkowski integral inequality in a specific point of the proof, immediatly followed by the use of spherical coordinates together with the coarea formula.

As far as we know Theorem 5 is a new result.

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Chapter 1

Preliminaries

In this chapter we recall some results which will be used in the second chapter.

1.1 Various results

Commutator of vector fields

Definition 1.1.1 (Lie brackets)**.** Given two vector fields $f, g \in \mathscr{C}^\infty(\mathbb{R}^n)$, their *Lie bracket* is defined as

$$[f, g](x) = D_x g(x) f(x) - D_x f(x) g(x).$$

In the following, we shall use the exponential notation $\theta \mapsto e^{\theta f} x$ to denote the solution of the Cauchy problem

$$\begin{cases} \frac{dw}{d\theta} = f(w) \\ w(0) = x. \end{cases}$$

Lemma 1.1.2 (Characterizations of Lie brackets)**.** *The Lie bracket can be equivalently characterized as*

$$[f, g] = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon^2}(e^{-\epsilon g} e^{-\epsilon f} e^{\epsilon g} e^{\epsilon f} - x).$$

Proof. They are (truncated) Taylor expansions: for $\epsilon \to 0^+$

$$\begin{aligned}
e^{\epsilon f} x &= x + \epsilon f(x) + \frac{\epsilon^2}{2} Df(x) f(x) + o(\epsilon^2); \\
e^{\epsilon f} e^{\epsilon g} x &= e^{\epsilon f} x + \epsilon g(e^{\epsilon f} x) + \frac{\epsilon^2}{2} Dg(e^{\epsilon f} x) g(e^{\epsilon g} x) + o(\epsilon^2) \\
&= x + \epsilon f(x) + \frac{\epsilon^2}{2} Df(x) f(x) + \epsilon (g(x + \epsilon f(x) + o(\epsilon)) + o(\epsilon)) + \epsilon^2 Dg(x) g(x) + o(\epsilon^2) \\
&= x + \epsilon f(x) + \frac{\epsilon^2}{2} Df(x) f(x) + \epsilon (g(x) + \epsilon Dg(x) f(x)) + \frac{\epsilon^2}{2} Dg(x) g(x) + o(\epsilon^2) \\
&= x + \epsilon (f(x) + g(x)) + \epsilon^2 \left(\frac{1}{2} Df(x) f(x) + \frac{1}{2} Dg(x) g(x) + Dg(x) f(x) \right) + o(\epsilon^2); \\
e^{-\epsilon f} e^{\epsilon g} e^{\epsilon f} &= e^{\epsilon g} e^{\epsilon f} x - \epsilon f(e^{\epsilon g} e^{\epsilon f} x) + \frac{\epsilon^2}{2} Df(e^{\epsilon g} e^{\epsilon f} x) f(e^{\epsilon g} e^{\epsilon f} x) + o(\epsilon^2)
\end{aligned}$$

$$
\begin{aligned}
&= x + \epsilon(f(x)+g(x)) + \epsilon^2\left(\frac{1}{2}Df(x)f(x) + \frac{1}{2}Dg(x)g(x) + Dg(x)f(x)\right)\\
&\quad - \epsilon f(x+\epsilon(f(x)+g(x))) + \frac{\epsilon^2}{2}Df(x)f(x) + o(\epsilon^2)\\
&= x + \epsilon(f(x)+g(x)) + \epsilon^2\left(\frac{1}{2}Df(x)f(x) + \frac{1}{2}Dg(x)g(x) + Dg(x)f(x)\right)\\
&\quad - \epsilon f(x) - \epsilon^2 Df(x)(f(x)+g(x)) + \frac{\epsilon^2}{2}Df(x)f(x) + o(\epsilon^2)\\
&= x + \epsilon g(x) - \epsilon^2 Df(x)g(x) + \frac{\epsilon^2}{2}Dg(x)g(x) + \epsilon^2 Dg(x)f(x) + o(\epsilon^2);
\end{aligned}
$$

$$
\begin{aligned}
e^{-\epsilon g}e^{-\epsilon f}e^{\epsilon g}e^{\epsilon f} &= e^{-\epsilon f}e^{\epsilon g}e^{\epsilon f}x - \epsilon g(e^{-\epsilon f}e^{\epsilon g}e^{\epsilon f}x) + \frac{\epsilon^2}{2}Dg(e^{-\epsilon f}e^{\epsilon g}e^{\epsilon f}x)g(x) + o(\epsilon^2)\\
&= x + \epsilon g(x) + \epsilon^2[f,g](x) + \frac{\epsilon^2}{2}Dg(x)g(x) - \epsilon g(x+\epsilon g(x)) + \frac{\epsilon^2}{2}Dg(x)g(x) + o(\epsilon^2)\\
&= x + \epsilon g(x) + \epsilon^2[f,g](x) - \epsilon g(x) - \epsilon^2 Dg(x)g(x) + \epsilon^2 Dg(x)g(x) + o(\epsilon^2)\\
&= x + \epsilon^2[f,g](x) + o(\epsilon^2)
\end{aligned}
$$

that concludes the proof. □

Coarea formula

Theorem 1.1.3 (Coarea formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \geq m$. Then for each $\mathcal{L}^n$-measurable set $A \subseteq \mathbb{R}^n$,*

$$\int_A Jf\,dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))\,dy.$$

Proof. The proof is long and quite difficult, thus it is omitted; see e.g. [6], page 112. □

Functional inequalities

We recall a couple of inequalities of fundamental importance; we will further use them.

Theorem 1.1.4. *Let $(X, \mathcal{M}, \mu)$ be a measurable space. Let p and q be conjugate exponents. Suppose that g is a measurable function on X such that $fg \in L^1$ for all f in the space Σ of simple functions that vanish outside a set of finite measure, and the quantity*

$$M_q(f) = \sup\left\{\left|\int fg\right| : f \in \Sigma \text{ and } \|f\|_{L^p(\mu)} = 1\right\}$$

is finite. Also, suppose either that $S_g = \{x : g(x) \neq 0\}$ is σ-finite or that μ is semifinite. Then $g \in L^q(\mu)$ and $M_q(g) = \|g\|_{L^q(\mu)}$.

Proof. The proof falls outside our aims; it can be found in [7], page 189. □

Theorem 1.1.5 (Minkowski inequality for integrals)**.** *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, and let f be an $(\mathcal{M} \otimes \mathcal{N})$-measurable function on $X \times Y$.*

- *If $f \geq 0$ and $1 \leq p < \infty$, then*

$$\left[\int \left(\int f(x,y)\,d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int \left[\int f(x,y)^p\,d\mu(x)\right]^{1/p} d\mu(y).$$

- *If $1 \leq p \leq \infty$, $f(\cdot,y) \in L^p(\mu)$ for almost every y, and the function $y \mapsto \|f(\cdot,y)\|_{L^p(\mu)}$ is in $L^1(\nu)$, then $f(x,\cdot) \in L^1(\nu)$ for almost every x, the function $x \mapsto \int f(x,y)\,d\nu(y)$ is in $L^p(\mu)$, and*

$$\left\|\int f(\cdot,y)\,d\nu(y)\right\|_{L^p(\mu)} \leq \int \|f(\cdot,y)\|_{L^p(\mu)}\,d\nu(y).$$

Proof. If $p = 1$, the first statement is merely Tonelli theorem. If $1 < p < \infty$, let q be the conjugate exponent to p and suppose $g \in L^q(\mu)$. Then by Tonelli theorem and Hölder inequality,

$$\begin{aligned}\int \left[\int f(x,y)\,d\nu(y)\right] |g(x)|\,d\mu(x) &= \int\int f(x,y)|g(x)|\,d\mu(x)\,d\nu(y)\\ &\leq \|g\|_{L^q(\mu)} \int \left[\int f(x,y)^p\,d\mu(x)\right]^{1/p} d\nu(y).\end{aligned}$$

The first assertion follows then from Theorem 1.1.4. When $p < \infty$, the second assertion follows from the first (with f replaced by $|f|$) and Fubini theorem; when $p = \infty$, it is a simple consequence of the monotonicity of the integral. □

Theorem 1.1.6 (Hardy inequality)**.** *If $p > 1$, $f(x) \geq 0$ and $F(x) = \int_0^x f(t)\,dt$, then*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p\,dx,$$

unless $f \equiv 0$. The constant is the best possible.

Proof. Let

$$G(x) = \frac{1}{x}\int_0^x f(t)\,dt = \int_0^1 f(tx)\,dt$$

and set $f_t(x) = f(tx)$. By Minkowski inequality for integrals

$$\|G(x)\|_{L^p((0,\infty))} \leq \int_0^1 \|f_t(x)\|_{L^p((0,\infty))}\,dt = \int_0^1 \left(\int_0^\infty |f_t(x)|^p\,dx\right)^{1/p} dt.$$

By the change of variables $s = tx$ and using Fubini-Tonelli theorem we get

$$\|G(x)\|_{L^p((0,\infty))} \leq \int_0^1 \left(\int_0^\infty |f(s)|^p \frac{ds}{t}\right)^{1/p} dt = \int_0^1 t^{-1/p}\,dt \left(\int_0^1 |f(s)|^p\,ds\right)^{1/p}.$$

Hence

$$\|G(x)\|_{L^p((0,\infty))} \leq \frac{p}{p-1}\|f(s)\|_{L^p((0,\infty))}$$

which directly leads to

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,dx.$$

□

1.2 Sub-Riemannian Geometry: prolegomena

Distributions. Let M be a $\mathscr{C}^\infty$ manifold of dimension n, and let $m \leq n$. Suppose that for each $x \in M$, we assign an m-dimensional subspace $\Delta_x \subset T_x(M)$ of the tangent space in such a way that for a neighborhood $N_x \subset M$ of x there exist m linearly independent smooth vector fields $X_1, \ldots, X_m$ such that for any point $y \in N_x$, $\text{span}\{X_1(y), \ldots, X_m(y)\} = \Delta_y$. We let Δ refer to the collection of all Δ_x for all $x \in M$ and we then call Δ a *distribution* of dimension m on M. The set of smooth vector fields $\{X_1, \ldots, X_m\}$ is called a *local basis* of Δ.

Sub-Riemannian distance. Let M be a real analytic n-dimensional manifold (a manifold with analytic transition maps) and $X_1, \ldots, X_m$ analytic vector fields on M. We define a *sub-Riemannian metric g on M* by setting, for each $q \in M$ and $v \in T_qM$,

$$g_q(v) = \inf\{u_1^2 + \cdots + u_m^2 \; : \; u_1X_1(q) + \cdots + u_mX_m(q) = v\}.$$

The *length* of an absolutely continuous path $c(t)$ $(0 \leq t \leq \tau)$ is defined as

$$\text{length}(c) = \int_0^\tau \sqrt{g_{c(t)}(\dot{c}(t))}\, dt.$$

Finally the *sub-Riemannian distance* (or *Carnot-Carathéodory distance*) is $d(p, q) = \inf \text{length}(c)$, where the infimum is taken on all the absolutely continuous paths joining p and q.

The manifold M endowed with the distance d, denoted (M, d), is called the *sub-Riemannian manifold* attached to $X_1, \ldots, X_n$.

Singular points. Let $\mathcal{L}^1 = \mathcal{L}^1(X_1, \ldots, X_m)$ be the set of linear combinations, with real coefficients, of the vector fields $X_1, \ldots, X_m$. We define recursively $\mathcal{L}^s = \mathcal{L}^s(X_1, \ldots, X_m)$ by setting, for $s > 1$,

$$\mathcal{L}^s = \mathcal{L}^{s-1} + [\mathcal{L}^{s-1}, \mathcal{L}^1].$$

Due to Jacobi identity $\mathcal{L}^s$ is the set of linear combinations of all commutators of $X_1, \ldots, X_m$ with length $\leq s$. The union $\mathcal{L}$ of all $\mathcal{L}^s$ is a Lie sub-algebra of the Lie algebra of vector fields on M. It is generated by the commutators $[[X_{i_1}, X_{i_2}], \ldots, X_{i_k}]$. Such a commutator is denoted $[X_I]$, where I is the multi-index $I = (i_1, \ldots, i_k)$ and its length is $|I| = k$.

For $p \in M$, let $L^s(p)$ be the subspace of T_pM which consists of the values $X(p)$ taken, at the point p, by the vector fields X belonging to $\mathcal{L}^s$. By Chow's Condition[1], at each point $p \in M$ there is a smallest integer $r = r(p)$ such that $L^{r(p)}(p) = T_pM$ (and so $\dim L^{r(p)}(p) = n$). This integer is called the *degree of nonholonomy* at p. We say that p is a *regular point* if the sequence

$$1 \leq \dim L^1(p) \leq \cdots \leq \dim L^s(p) \leq \cdots \leq \dim L^{r(p)}(p) = n$$

remains constant in a neighborhood of p. Otherwise we say that p is a *singular point*.

Minimal basis. Let (M, d) be the sub-Riemannian manifold attached to a system $X_1, \ldots, X_n$ of vector fields. Let $p \in M$. We call *minimal basis at p* a family of commutators $([X_{I_1}], \ldots, [X_{I_n}])$ which values at p form a basis of T_pM and such that the total length $\sum_{i=1}^n |I_i|$ equals $\sum_{i=1}^n w_i$

[1]**Chow's Condition.** The vector fields $X_1, \ldots, X_m$ and their iterated brackets $[X_i, X_j]$, $[[X_i, X_j], X_k]$ etc. span the tangent space T_pM at every point p of M.

(where the sequence $w_1 \leq \cdots \leq w_n$ is defined by setting $w_j = s$ if $n_{s-1} < j \leq n_s$ with $n_s = \dim L^s(p)$ for $s = 1, \ldots, r$). It implies that, up to a permutation of indices, each $|I_i|$ equals w_i. To a family $\underline{I} = ([X_{I_1}], \ldots, [X_{I_n}])$ we associate the application

$$\phi^p_{\underline{I}} : (u_1, \ldots, u_n) \mapsto p \exp(u_n[X_{I_n}]) \ldots \exp(u_1[X_{I_1}]).$$

When $\underline{I}$ is a minimal basis at p, $\phi^p_{\underline{I}}$ is a diffeomorphism from a neighborhood of $0 \in \mathbb{R}^n$ into a neighborhood of p in M. It defines local coordinates, called *canonical coordinates of the second kind*, which are privileged at p.

Associated basis. Let now $\Omega \subset M$ be compact. We denote by r the maximum of the degree of nonholonomy on Ω.

Let $p \in \Omega$ and $\epsilon > 0$. We consider the families of vector fields $([X_{I_1}], \ldots, [X_{I_n}])$ such that each bracket $[X_{I_j}]$ is of length $|I_j| \leq r$. On the (finite) set of these families, we have a function

$$\left|\det\left([X_{I_1}]\epsilon^{|I_1|}, \ldots, [X_{I_n}]\epsilon^{|I_n|}\right)(p)\right|.$$

We say that the family $\underline{I} = ([X_{I_1}], \ldots, [X_{I_n}])$ is *associated with* (p, ϵ) on Ω if it achieves the maximum of this function. In particular the value at p of a family associated with (p, ϵ) forms a basis of T_pM.

1.3 Uniform estimate for sub-Riemannian balls

Now: for $p \in M$ and $\epsilon > 0$, $B(p, \epsilon)$ denotes the open ball centered at p of radius ϵ for the sub-Riemannian distance d. Moreover for a family $\underline{I} = ([X_{I_1}], \ldots, [X_{I_n}])$ of vector fields we set

$$B_{\underline{I}}(p, \epsilon) = \{p \exp(u_n[X_{I_n}]) \ldots \exp(u_1[X_{I_1}]),\ |u_i| < \epsilon^{|I_i|},\ 1 \leq i \leq n\}.$$

Theorem 1.3.1. *Let $\Omega \subset M$ be a compact set. There exist a constant $\delta_0 > 0$ and functions $k(\delta)$, $K(\delta)$, $0 < k(\delta), K(\delta)$, with $\lim_{\delta\to 0} K(\delta) = 0$, such that: for every $p \in \Omega$, $\epsilon < 1$, $\delta < \delta_0$ and evermy family $\underline{I}$ associated with $p, \epsilon)$ on Ω,*

$$B_{\underline{I}}(p, k(\delta)\epsilon) \subset B(p, \delta\epsilon) \subset B_{\underline{I}}(p, K(\delta)\epsilon).$$

The following corollary will be used in the next chapter:

Corollary 1.3.2. *Let $\Omega \subset M$ be a compact set. There exist constants c, C and $\delta_0 > 0$ such that, for every $p \in \Omega$, $\epsilon < \delta_0$ and every family $\underline{I}$ associated with $(p, \epsilon/\delta_0)$ on Ω,*

$$B_{\underline{I}}(p, c\epsilon) \subset B(p, \epsilon) \subset B_{\underline{I}}(p, C\epsilon).$$

Proofs of these results can be found in [11]. They generalize the Ball-Box theorem due to Bellaïche and Gromov.

Chapter 2

The trace theorem

2.1 Shape of the sub-Riemannian boxes

Our goal is to estimate the sub-Riemannian distance induced on $\mathbb{R}^3$ by Martinet distribution (here we indicate with the same name the distribution and the basis which spans it - see Section 1.2) in terms of the Euclidean one, defined by vector fields[1]

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x^2}{2}\frac{\partial}{\partial z}. \tag{2.1}$$

The only non zero commutators are

$$X_{12} = [X_1, X_2] = x\frac{\partial}{\partial z}, \quad X_{121} = [[X_1, X_2], X_1] = -\frac{\partial}{\partial z};$$

every point in the plane $\{x = 0\}$ is singular. Two families of vector fields have a non identically zero determinant, $\underline{I} = (X_1, X_2, X_{12})$ and $\underline{J} = (X_1, X_2, X_{121})$. We have

$$|\det(\epsilon X_1, \epsilon X_2, \epsilon^2 X_{12})(x, y, z)| = |x|\epsilon^4$$

and

$$|\det(\epsilon X_1, \epsilon X_2, \epsilon^3 X_{121})(x, y, z)| = \epsilon^5.$$

Thus the families associated with $((x, y, z), \epsilon)$ are

$$\underline{J} \text{ if } |x| < \epsilon, \quad \underline{J} \text{ and } \underline{I} \text{ if } |x| = \epsilon, \quad \underline{I} \text{ if } |x| > \epsilon.$$

Therefore at a singular point $p_0 = (0, y, z)$ the minimal basis is $\underline{J}$ while at a regular point $p = (x, y, z)$ (with $x \neq 0$) the minimal basis is $\underline{I}$.

Using Corollary 1.3.2 we compute the sub-Riemannian boxes in order to get an uniform estimate of sub-Riemannian balls associated to the metric we are looking for; we split the computation into two cases: when $|x| < \epsilon$ and when $|x| \geq \epsilon$.

In the case $|x| < \epsilon$ the box of center $p = (x, y, z)$ is the following set of points of $\mathbb{R}^3$:

$$B_{\underline{J}}(p, \epsilon) = \left\{(x, y, z)\exp(u_3 X_{121})\exp(u_2 X_2)\exp(u_1 X_1) \,:\, |u_1| < \epsilon,\, |u_2| < \epsilon,\, |u_3| < \epsilon^3\right\};$$

[1] At the beginning I personally found a bit confusing this way to write vector fields. A brilliant explanation of this notation can be find in Chapter 3 of *Introduction to smooth manifolds* by John M. Lee, version 3.0.

more explicitely we have that

$$\begin{aligned}(x,y,z)\exp(u_3X_{121})\exp(u_2X_2)\exp(u_1X_1) &= (x,y,z-u_3)\exp(u_2X_2)\exp(u_1X_1)\\ &= (x,y+u_2,z-u_3+u_2x^2/2)\exp(u_1X_1)\\ &= (x+u_1,y+u_2,z-u_3+u_2x^2/2)\end{aligned}$$

and this leads to a more explicit writing of the boxes:

$$B_{\underline{J}}(p,\epsilon) = [x-\epsilon,x+\epsilon]\times[y-\epsilon,y+\epsilon]\times\underbrace{\{z-u_3+u_2x^2/2\,:\,|u_2|\le\epsilon,\ |u_3|\le\epsilon^3\}}_{=A}.$$

Notice that

$$\max A = z+\epsilon\frac{x^2}{2}+\epsilon^3 \quad \text{and} \quad \min A = z-\epsilon\frac{x^2}{2}-\epsilon^3;$$

and since we are in the case $|x|<\epsilon$, we have that

$$\boxed{B_{\underline{J}}(p,\epsilon)\simeq[x-\epsilon,x+\epsilon]\times[y-\epsilon,y+\epsilon]\times[z-\epsilon^3,z+\epsilon^3]}.$$

Even though the computations are the same, the second case ($|x|>\epsilon$) presents a little bit more tricky geometric "issue" (which was negligible in the first case):

$$B_{\underline{I}}(p,\epsilon) = \{(x,y,z)\exp(u_3X_{12})\exp(u_2X_2)\exp(u_1X_1)\,:\,|u_1|<\epsilon,\ |u_2|<\epsilon,\ |u_3|<\epsilon^3\};$$

more explicitely we have

$$\begin{aligned}(x,y,z)\exp(u_3X_{121})\exp(u_2X_2)\exp(u_1X_1) &= (x,y,z+u_3x)\exp(u_2X_2)\exp(u_1X_1)\\ &= (x,y+u_2,z+u_3x+u_2x^2/2)\exp(u_1X_1)\\ &= (x+u_1,y+u_2,z+u_3x+u_2x^2/2)\end{aligned}$$

which leads to

$$B_{\underline{I}}(p,\epsilon) = [x-\epsilon,x+\epsilon]\times[y-\epsilon,y+\epsilon]\times\{z+xu_3+u_2x^2/2\,:\,|u_2|\le\epsilon,\ |u_3|\le\epsilon^2\}.$$

However we cannot here just brutally take the supremum and the infimum, since we would get a too rough estimate. To clarify the situation, let's compute the x-section of $B_{\underline{I}}((x,0,0),\epsilon)$: it is the set

$$\left\{(y,z)\in\mathbb{R}^2\,:\,|y|<\epsilon,\ \left|z-\frac{x^2y}{2}\right|<\frac{\epsilon^2x}{2}\right\}.$$

In conclusion

$$\boxed{B_{\underline{I}}(p,\epsilon)\simeq\left\{(x',y',z')\in\mathbb{R}^3\,:\,|x-x'|<\epsilon,\ |y-y'|<\epsilon,\ \left|z'-z-\frac{x^2}{2}(y'-y)\right|<\frac{\epsilon^2x}{2}\right\}}.$$

By Fubini-Tonelli theorem we have that 3-dim Lebesgue measures $\mathscr{L}^3$ of the boxes computed are

$$\mathscr{L}^3(B_{\underline{J}}(p,\epsilon))\simeq\epsilon^5,\quad \mathscr{L}^3(B_{\underline{I}}(p,\epsilon))\simeq\epsilon^4x.$$

Finally for a center $p=(x,y,z)$ we write

$$\operatorname{Box}(p,r) := \begin{cases} B_{\underline{J}}(p,r) & \text{if } |x|\le r\\ B_{\underline{I}}(p,r) & \text{if } |x|>r.\end{cases} \tag{2.2}$$

2.2 Measure of the sub-Riemannian boxes

In order to be usable in computations, the μ-measure of the sub-Riemannian boxes must be estimated in terms of their Lebesgue measures:

Proposition 2.2.1. *Let $\Omega = \{z > 0\}$ and $\mu := |X\nu|\mathscr{L}^2 \llcorner \partial\Omega$, where X is 2.1. Then*

$$\mu(\mathrm{Box}(p, r)) \simeq \frac{\mathscr{L}^3(\mathrm{Box}(p, r))}{r} \tag{2.3}$$

for centers $p = (\bar{x}, \bar{y}, 0)$.

Proof. In our case $|X\nu| = x^2$: indeed if X is 2.1 we have that

$$|X\nu| := \sqrt{\langle X_1, \nu\rangle^2 + \langle X_2, \nu\rangle^2}$$

where ν is the (constant) normal unit vector to Ω, which is $\nu = (0, 0, 1)$. So we have

$$\langle X_1, \nu\rangle = \langle (1, 0, 0), \nu\rangle = 0$$

and

$$\langle X_2, \nu\rangle = \langle (0, 1, x^2), \nu\rangle = x^2.$$

Therefore

$$|X\nu| = x^2. \tag{2.4}$$

Hence we have that

$$\mu(B_{\underline{I}}(p, r)) \simeq \int_{\bar{y}-r}^{\bar{y}+r} \int_{\bar{x}-r}^{\bar{x}+r} x^2 \, dx \, dy = \frac{4}{3} r^4 + 2\bar{x}^2 r^2 \simeq r^4$$

and this proves the proposition in the case $|\bar{x}| \leq r$. The second case ($|\bar{x}| > r$) requires more work; indeed the condition of the boxes, taking into account the fact that $z = \bar{z} = 0$, becomes $|\bar{x}||y - \bar{y}| < r^2 \longrightarrow |y - \bar{y}| < r^2/|\bar{x}|$; now, it's clear that if $2r^2/|\bar{x}| > r$, which holds if and only if $2r > |\bar{x}|$, then the intersection between the box $B_{\underline{I}}(p, r)$ and the plane $\{z = 0\}$ gives a square of side length $= 2r$, which is exactly the previous case. In particular

$$\mu(B_{\underline{I}}(p, r)) \simeq \int_{\bar{y}-r}^{\bar{y}+r} \int_{\bar{x}-r}^{\bar{x}+r} x^2 \, dx \, dy \simeq r^4 \simeq \frac{|\bar{x}| r^4}{r}$$

since $r < |\bar{x}| < 2r$. When conversely $|\bar{x}| \geq 2r$, we have that

$$\begin{aligned} \mu(B_{\underline{I}}(p, r)) &= \int_{|x-\bar{x}|<r} x^2 \int_{|y-\bar{y}|<2r^2/|\bar{x}|} dy \, dx \\ &= \frac{4r^2}{|\bar{x}|} \int_{\bar{x}-r}^{\bar{x}+r} x^2 \, dx = \frac{4r^2}{3|\hat{x}|}((\bar{x}+r)^3 - (\bar{x}-r)^3) \\ &= \frac{4r^2}{3|\bar{x}|}(6\bar{x}^2 r + r^3) \simeq \frac{r^3}{|\bar{x}|}\bar{x}^2 = r^3|\bar{x}| = \frac{\mathscr{L}^3(B_{\underline{I}}(p, r))}{r}. \end{aligned}$$

This concludes the proof. $\square$

2.3 Estimates for the sub-Riemannian distance

It is basically impossible to directly deal with the sub-Riemannian distance; however the computations made in Section 2.1 allow us to estimate it in terms of the Euclidean distance. We sum up such result in the following lemma:

Lemma 2.3.1. *Let $r > 0$. For all $(x, y, z), (x', y', z') \in \mathbb{R}^3$ the following estimates for the sub-Riemannian distance d induced by the Martinet distribution hold:*

$$d((x,y,z),(x',y',z')) \simeq \begin{cases} |x-x'| + |y-y'| + |z-z'|^{1/3} & \text{if } |x| \le r \\ |x-x'| + |y-y'| + \sqrt{|z-z'|/|x|} & \text{if } |x| > r \text{ and } |z-z'| \ge x^2|y-y'| \\ |x-x'| + |y-y'| + \sqrt{|y-y'||x|} & \text{if } |x| > r \text{ and } |z-z'| \le x^2|y-y'|. \end{cases}$$

Proof. The first case is quite simple: let's take $p = (x, y, z) \in \mathbb{R}^3$ with $|x| \le r$; let $B_{\underline{J}}(p, r)$ be the sub-Riemannian box defined above. The point $q = (x', y', z') \in \mathbb{R}^3$ is distant r from q if and only if it belongs to the boundary of $B_{\underline{J}}(p, r)$, i.e. if and only if $|x - x'| \le r$, $|y - y'| \le r$ and $|z - z'| \le r^3$ and at least one of the inequalities is actually an equality. This leads to $d(p, q) \simeq |x - x'| + |y - y'| + |z - z'|^{1/3}$.

Now let's assume that $|z-z'| \ge x^2|y-y'|$; as before $d(p, q) = r$ if and only if $|x-x'| \le r$, $|y - y'| \le r$ and $|z - z' - x^2(y - y')/2| < r^2|x|/2$ and at least one of the inequalities is an equality. We have

$$\begin{aligned} |z-z'| &\le \left| z - z' - \frac{x^2}{2}(y'-y) \right| + \left| \frac{x^2}{2}(y'-y) \right| \\ &\le \left| z - z' - \frac{x^2}{2}(y'-y) \right| + \frac{1}{2}|z-z'| \end{aligned}$$

which leads to

$$\frac{1}{2}|z-z'| \le \frac{r^2|x|}{2}.$$

Moreover

$$\begin{aligned} |z-z'| &\ge \left| z - z' - \frac{x^2}{2}(y'-y) \right| - \left| \frac{x^2}{2}(y'-y) \right| \\ &\ge \left| z - z' - \frac{x^2}{2}(y'-y) \right| - \frac{1}{2}|z-z'| \end{aligned}$$

and therefore

$$\frac{3}{2}|z-z'| \ge \left| z - z' - \frac{x^2}{2}(y'-y) \right|.$$

The argument proves that

$$\left| z - z' - \frac{x^2}{2}(y'-y) \right| \le \frac{r^2|x|}{2} \iff |z-z'| \lesssim \frac{r^2|x|}{2}$$

and this is enough for the second case.

Let's finally assume $|z-z'| \le x^2|y-y'|$; *mutatis mutandis* in the previous computations we get

$$\left|z - z' - \frac{x^2}{2}(y'-y)\right| \le \frac{r^2|x|}{2} \iff \frac{x^2}{2}|y-y'| \le \frac{r^2|x|}{2}$$
$$\iff \sqrt{|x||y'-y|} \le r.$$

The proof is concluded. $\square$

2.4 Trace theorem for the Martinet Distribution

We have now all the tools to proceed with the estimate; our goal is to estimate the following integral (using the Martinet distribution 2.1)

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|u(w,0)-u(q,0)|^p}{d(w,q)^{ps}\mu(B(w,d(w,q)))}\, d\mu(w)\, d\mu(q) \tag{2.5}$$

where $w=(x,y)$, $q=(x',y')$, d is the sub-Riemannian distance induced on $\mathbb{R}^3$ by Martinet distribution 2.1 (see Section 1.2), $s = 1-1/p$ and μ is the measure defined in Proposition 2.2.1.

Theorem 2.4.1. *For every $1<p<\infty$ there exists a constant $C(p)>0$ such that for every function $u \in \mathscr{C}^1_c(\mathbb{R}^2\times[0,\infty))$ the following integral estimate holds*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|u(w,0)-u(q,0)|^p}{d(w,q)^{ps}\mu(B(w,d(w,q)))}\, d\mu(w)\, d\mu(q) \le C(p)\int_{\mathbb{R}^2\times[0,+\infty)} |Xu(x,y,z)|^p\, dx\, dy\, dz.$$

Remark 2.4.2. We are going without losing of generality to prove the theorem for functions $u \in \mathscr{C}^1_c([-1,1]^2\times[0,1])$; however the result can be easily extended considering the transformation $\delta_\lambda : (x,y,z) \mapsto (\lambda x, \lambda y, \lambda^3 z)$. Indeed for $\lambda>1$ let us take $u \in \mathscr{C}^1_c([-\lambda,\lambda]^2\times[0,\lambda^3])$ and $\tilde{u}(x,y,z) = u(\delta_{1/\lambda}(x,y,z)) \in \mathscr{C}^1_c([-1,1]^2\times[0,1])$. With simple computations it turns out that

$$|X\tilde{u}(x,y,z)| = \frac{1}{\lambda}|Xu(\delta_{1/\lambda}(x,y,z))|$$

and then

$$|Xu(x,y,z)| = \lambda|X\tilde{u}(\delta_\lambda(x,y,z))|.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^2\times[0,+\infty)} |Xu(x,y,z)|^p\, dx\, dy\, dz &= \lambda^p \int_{\mathbb{R}^2\times[0,+\infty)} |X\tilde{u}(\delta_\lambda(x,y,z))|\, dx\, dy\, dz\\ &= \lambda^{p-5}\int_{\mathbb{R}^2\times[0,+\infty)} |X\tilde{u}(x,y,z)|\, dx\, dy\, dz.\end{aligned}$$

On the LHS we have that, if $\bar{w} = \delta_\lambda w$ and $\bar{q} = \delta_\lambda q$,

$$d(w,q)^{ps} = d(\delta_{1/\lambda}\bar{w}, \delta_{1/\lambda}\bar{q})^{ps} = \left(\frac{1}{\lambda}\right)^{ps} d(\bar{w},\bar{q})^{ps}$$

and

$$\mu(B(\delta_{1/\lambda}\bar{w}, d(\bar{w},\bar{q})/\lambda)) = \mu(B(\bar{w}, d(\bar{w},\bar{q})))/\lambda^4.$$

In conclusion $d\mu(w) = d\mu(\bar{w})/\lambda^4$ and then

$$\begin{aligned}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|u(w,0)-u(q,0)|^p}{d(w,q)^{ps}\mu(B(w,d(w,q)))}\,d\mu(w)\,d\mu(q) &= \lambda^{ps}\lambda^4\frac{1}{\lambda^4}\frac{1}{\lambda^4}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\tilde{u}(\bar{w},0)-\tilde{u}(\bar{q},0)|^p}{d(\bar{w},\bar{q})^{ps}\mu(B(\bar{w},d(\bar{w},\bar{q})))}\,d\mu(\bar{w})\,d\mu(\bar{q})\\ &= \lambda^{p-5}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{|\tilde{u}(\bar{w},0)-\tilde{u}(\bar{q},0)|^p}{d(\bar{w},\bar{q})^{ps}\mu(B(\bar{w},d(\bar{w},\bar{q})))}\,d\mu(\bar{w})\,d\mu(\bar{q}).\end{aligned}$$

This means that dilations of the support does not affect the trace inequality.

Proof. The integral (2.5) has to be split in a sum of integrals; in particular $[-1,1]^2\times[-1,1]^2 = Q_1\times Q_1 = \bigcup_{i=1}^4 A_i$ where

$$\begin{aligned} A_1 &= \{((x,y),(x',y'))\in Q_1\times Q_1 \,:\, |x|\le d(w,q),\ |x'|\le d(w,q)\}\\ A_2 &= \{((x,y),(x',y'))\in Q_1\times Q_1 \,:\, |x|> d(w,q),\ |x'|\le d(w,q)\}\\ A_3 &= \{((x,y),(x',y'))\in Q_1\times Q_1 \,:\, |x|\le d(w,q),\ |x'|> d(w,q)\}\\ A_4 &= \{((x,y),(x',y'))\in Q_1\times Q_1 \,:\, |x|> d(w,q),\ |x'|> d(w,q)\}.\end{aligned}$$

We want to deal now with the case A_1. The idea is to connect the points w and q with the integral curves of the vector fields X_1, X_2 and their commutators, subdividing the path in some subpaths. Let's do the first step, moving in the y and z directions using the integral curve of X_2; we can assume $y' > y$ (on the contrary we could just switch the two points w and q):

$$|u(x,y,0)-u(x,y',x^2(y'-y)/2)| = \left|\int_0^{y'-y} X_2u(\underbrace{x,y+t,x^2t/2}_{=\Psi_1(x,y,t)})\,dt\right| \tag{2.6}$$

where $\Psi_1\in\mathscr{C}(\mathbb{R}^3;\mathbb{R}^3)$ is defined by $\Psi_1(x,y,t) = (x,y+t,x^2t/2)$. We want to prove that

$$I_1 := \int_{A_1}\frac{|u(w)-u(q)|^p}{d(w,q)^{ps}\mu(B(w,d(w,q)))}\,d\mu(w)\,d\mu(q) \le \sum_{i=1}^{12} I_{1i} \tag{2.7}$$

where the I_{1i}s are the integrals of the subpaths. Using 2.4 (i.e. $d\mu(w) = x^2dx\,dy$ and $d\mu(q) = x'^2dx'\,dy'$), Lemma 2.3.1 (i.e. $d(w,q)\simeq|x-x'|+|y-y'|$) and Proposition 2.2.1 (i.e. $\mu(B(w,d(w,q))\simeq d(w,q)^4)$ we have that

$$I_{11}\simeq\int_{A_1}\frac{|u(x,y,0)-u(x,y',x^2(y'-y)/2)|^p}{(|x-x'|+|y-y'|)^{p+3}}|x|^2|x'|^2\,dx\,dy\,dx'\,dy'.$$

Using $|x'|\le d = d(w,q)$ and 2.6 we find

$$I_{11}\le C\int_{A_1}\frac{1}{d^{p+1}}\left(\int_0^{y'-y}|X_2u(\Psi_1(x,y,t))|\,dt\right)^p|x|^2\,dx\,dy\,dx'\,dy'.$$

We claim now that there exists a constant $C(p)>0$ depending only on p such that

$$\int_{\{|x'|\le d\}}\frac{dx'}{d^{p+1}}\le\frac{C(p)}{h^p} \tag{2.8}$$

where $h = |y' - y|$. Indeed, assuming moreover wlog $x' \geq 0$,

$$\int_{\{x' \leq |x-x'|+h\}} \frac{dx'}{(|x-x'|+h)^{p+1}} = \underbrace{\int_{\{x' \leq |x-x'|+h,\, 0<x'\leq x\}} \frac{dx'}{(|x-x'|+h)^{p+1}}}_{=J_1} + \underbrace{\int_{\{x' \leq |x-x'|+h,\, x\leq x'\}} \frac{dx'}{(|x-x'|+h)^{p+1}}}_{=J_2};$$

since $0 < x' \leq x - x' + h$ iff $x' \leq (x+h)/2$ we get

$$\begin{aligned} J_1 = \int_0^{(x+h)/2} \frac{dx'}{(x-x'+h)^{p+1}} &= \left[\frac{1}{p}(x-x'+h)^{-p}\right]_{x'=0}^{x'=(x+h)/2} \\ &= \frac{1}{p}\left[\left(\frac{x+h}{2}\right)^{-p} - (x+h)^{-p}\right] \\ &= \frac{C(p)}{(x+h)^p} \leq \frac{C(p)}{h^p}. \end{aligned}$$

For J_2 we get that $x' \leq x' - x + h$ holds iff $x \leq h$; moreover $x' \leq 1$. Thus

$$\begin{aligned} J_2 = \int_x^1 \frac{dx'}{(x'-x+h)^{p+1}} &= \left[-\frac{(x'-x+h)^{-p}}{p}\right]_x^1 \\ &= -\frac{1}{p}[(1-x+h)^{-p} - h^{-p}] \\ &\leq \frac{1}{ph^p}. \end{aligned}$$

This prove the claim 2.8. Thus using 2.8 we get

$$\begin{aligned} &\int_{A_1} \frac{1}{d^{p+1}} \left(\int_0^{y'-y} |X_2 u(\Psi_1(x,y,t))|\, dt\right)^p |x|^2\, dx\, dy\, dx'\, dy' \\ &\leq C(p) \int_0^1 \frac{1}{h^p} \int_{\{|x|\leq 1, |y|\leq 1\}} \left(\int_0^h |X_2 u(\Psi_1(x,y,t))|\, dt\right)^p |x|^2\, dx\, dy\, dh. \end{aligned}$$

By Minkowski inequality for integrals we get now that the last line above is

$$\begin{aligned} &\leq C(p) \int_0^1 \frac{1}{h^p} \left[\int_0^h \underbrace{\left(\int_{\{|x|\leq 1, |y|\leq 1\}} |X_2 u(\Psi_1(x,y,t))|^p |x|^2\, dx\, dy\right)^{1/p}}_{=g(t)} dt\right]^p dh \\ &= C(p) \int_0^1 \left(\frac{1}{h}\int_0^h g(t)\, dt\right)^p dh \leq C_1(p) \int_0^1 g(t)^p\, dt \\ &= C_1(p) \int_0^1 \int_{\{|x|\leq 1, |y|\leq 1\}} |X_2 u(\Psi_1(x,y,t))|^p |x|^2\, dx\, dy\, dt \end{aligned}$$

where we used Hardy inequality in the second line. Here we are using $p > 1$ (if $p = 1$ the constant generated by Hardy inequality blows up). Now we do the change of variables $(\xi, \eta, \tau) = \Psi_1(x, y, t)$; the jacobian matrix of this change of variables is

$$\frac{\partial \Psi_1}{\partial x \partial y \partial t} = \begin{pmatrix} 1 & 0 & tx \\ 0 & 1 & 0 \\ 0 & 1 & x^2/2 \end{pmatrix}$$

whose determinant is $x^2/2$. Therefore

$$\int_{\{|x|\le 1,\, |y|\le 1,\, 0\le t\le 2\}} |X_2 u(\Psi_1(x,y,t))|^p |x|^2 \, dx\, dy\, dt \le C(p) \int_{\{|\xi|\le 1,\, |\eta|\le 2,\, |\tau|\le 1\}} |X_2 u(\xi,\eta,\tau)|^p \, d\eta\, d\xi\, d\tau.$$

The first step has been done. Now we have to move along the x direction, from the point $(x, y', x^2(y'-y)/2)$ to the point $(x', y', x^2(y'-y)/2)$ using the integral curve of the vector field X_1. We have that

$$|u(x, y', x^2(y'-y)/2) - u(x', y', x^2(y'-y)/2)| = \left| \int_0^{x'-x} X_1 u(x+t, y', x^2(y'-y)/2)\, dt \right|.$$

Thus

$$I_{12} \simeq \int_{A_1} \frac{|u(x, y', x^2(y'-y)/2) - u(x', y', x^2(y'-y)/2)|^p}{(|x-x'| + |y-y'|)^{p+3}} |x|^2 |x'|^2 \, dx\, dy\, dx'\, dy'.$$

With the change of variables $x + t = \tau \longrightarrow dt = d\tau$ and $x^2(y'-y)/2 = \eta \longrightarrow x^2/2\, dy = d\eta$ we get a new domain $\widehat{A}_1 = \{|x| \le \widehat{d},\, |x'| \le \widehat{d}\}$ where $\widehat{d} = |x-x'| + 2|\eta|/x^2$ (if $x \ne 0$). Therefore

$$\begin{aligned} I_{12} &\le \int_{A_1} \frac{1}{d^{p+3}} \left| \int_0^{x'-x} |X_1 u(x+t, y', x^2(y'-y)/2)|\, dt \right|^p |x|^2 |x'|^2 \, dx\, dx'\, dy\, dy' \\ &\le 2 \int_{\widehat{A}_1} \frac{1}{\widehat{d}^{p+3}} \left| \int_x^{x'} |X_1 u(\tau, y', \eta)|\, d\tau \right|^p |x'|^2 \, dx\, dx'\, dy'\, d\eta \\ &\le 2 \int_{\widehat{A}_1} \frac{1}{\widehat{d}^{p+1}} \left| \int_x^{x'} |X_1 u(\tau, y', \eta)|\, d\tau \right|^p dx\, dx'\, dy'\, d\eta. \end{aligned}$$

We distinguish four cases:

$$\begin{aligned} \mathbf{I}: &\quad x' \ge x \ge 0; \\ \mathbf{II}: &\quad x' \ge 0 \ge x; \\ \mathbf{III}: &\quad x \ge x' \ge 0; \\ \mathbf{IV}: &\quad x \ge 0 \ge x'. \end{aligned}$$

Case **I**. We firstly split the domain $\widehat{A}_1$ as follow:

$$\begin{aligned} \widehat{A}_1 &= (\widehat{A}_1 \cap \{2|\eta|/x^2 > |x'-x|\}) \cup (\widehat{A}_1 \cap \{2|\eta|/x^2 \le |x'-x|\}) \\ &= \widehat{A}_1^+ \cup \widehat{A}_1^-. \end{aligned}$$

On $\widehat{A}_1^+$ the following holds: $2|\eta|/x^2 \le \widehat{d} \le 4|\eta|/x^2$; moreover we know that

$$\begin{cases} x \le \widehat{d} = |x - x'| + 2|\eta|/x^2 \\ x' \le \widehat{d} = |x - x'| + 2|\eta|/x^2 \end{cases}$$

and those together lead to

$$\widehat{d} = |x' - x| + 2|\eta|/x^2 = x' - x + 2|\eta|/x^2 \ge x'$$

and

$$x \le 2|\eta|/x^2.$$

Thus

$$\begin{aligned} I_{12}^+ &= 2 \int_{\widehat{A}_1^+} \frac{1}{\widehat{d}^{p+1}} \left| \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta \\ &\le 2 \int_{\widehat{A}_1^+} \frac{1}{\widehat{d}} \left| \frac{1}{x'} \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta. \end{aligned}$$

We have that $0 \le x \le \widehat{d} \le 4|\eta|/x^2$, which implies $0 \le x \le \sqrt[3]{4|\eta|}$. Hence the last integral above is

$$\le \int_0^{\sqrt[3]{4|\eta|}} \frac{x^2}{2|\eta|}\, dx \int_D \left| \frac{1}{x'} \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx'\, dy'\, d\eta$$

where $D \subseteq Q_R \times [0, 1]$ is a certain domain. Finally

$$\begin{aligned} I_{12}^+ &\le \frac{4}{3} \int_D \left| \frac{1}{x'} \int_0^{x'} X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx'\, dy'\, d\eta \\ &\le \frac{4}{3} C(p) \int_{\tilde{D}} |X_1 u(\tau, y', \eta)|^p \, d\tau\, dy'\, d\eta \end{aligned}$$

for a suitable $\tilde{D} \subseteq Q_R \times [0, 1]$, where we used again Hardy inequality and Minkowski inequality; we integrated over $\mathbb{R}^2 \times \mathbb{R}^+$ but u has been chosen with compact support.

Similarly we define now

$$I_{12}^- = 2 \int_{\widehat{A}_1^-} \frac{1}{\widehat{d}^{p+1}} \left| \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta.$$

The points of $\widehat{A}_1^-$ fulfill the conditions

$$x' - x = |x - x'| \le \widehat{d} \le 2|x' - x| = 2(x' - x) \le 2x'$$

and then $x' \le \widehat{d} \le 2(x' - x)$ leads to $\widehat{d} \ge x' - x \ge x'/2$. Hence

$$\begin{aligned} I_{12}^- &\le 2 \int_{\widehat{A}_1^-} \frac{1}{x'^{p+1}} \left| \int_0^{x'} X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta \\ &\le 2 \int_0^{2x'} \frac{dx}{x'} \left| \frac{1}{x'} \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta \end{aligned}$$

since $0 \le x \le \widehat{d} \le 2(x'-x) \le 2x'$. Thus we get as in the previous case

$$\begin{aligned} I_{12}^+ &\le 4 \int_{D_1} \left| \frac{1}{x'} \int_0^{x'} X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx'\, dy'\, d\eta \\ &\le 4C(p) \int_{D_1} |X_1 u(\tau, y', \eta)|^p \, d\tau\, dy'\, d\eta \end{aligned}$$

for a certain domain $D_1 \subset [-1,1]^3$. This concludes the case **I**.

Case **II**. We have here

$$\begin{aligned} I_{12} \le & \underbrace{2 \int_{\widehat{A}_1} \frac{1}{\widehat{d}^{p+1}} \left| \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta}_{=L_1} \\ & + \underbrace{2 \int_{\widehat{A}_1} \frac{1}{\widehat{d}^{p+1}} \left| \int_x^{0} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta}_{=L_2} \end{aligned}$$

and the conditions

$$\widehat{d} = |x'-x| + 2|\eta|/x^2 \ge |x'-x| \ge \begin{cases} x' \\ |x| \end{cases}$$

hold. As before

$$L_1 \le C \int_{\widehat{A}_1} \frac{1}{\widehat{d}} \left| \frac{1}{x'} \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta$$

and if we split the domain of integration $\widehat{A}_1$ into $\widehat{A}_1^+ \cup \widehat{A}_1^-$ as we did in the previous case we have that in $\widehat{A}_1^+$ the conditions $2|\eta|/x^2 \le \widehat{d} \le 4|\eta|/x^2$ hold; thus $|x| \le \widehat{d} \longrightarrow -\widehat{d} \le x \le \widehat{d}$ and then $\sqrt[3]{-4|\eta|} \le x \le \sqrt[3]{4|\eta|}$. Hence

$$\begin{aligned} & C \int_{\widehat{A}_1^+} \frac{1}{\widehat{d}} \left| \frac{1}{x'} \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta \\ \le\, & C \int_{\sqrt[3]{-4|\eta|}}^{\sqrt[3]{4|\eta|}} \frac{x^2}{2|\eta|} \, dx \int_D \left| \frac{1}{x'} \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx'\, dy'\, d\eta \end{aligned}$$

where $D \subset [-1,1]^3$ is a certain domain. The final computations are identical to the previous case. On $\widehat{A}_1^-$ we have to be a little bit more careful: $x'-x = |x'-x| \le \widehat{d} \le 2(x'-x)$ still holds, but we cannot conclude that $2(x'-x) \le 2x'$. However if $x' \ge |x|$ we have

$$\begin{aligned} & C \int_{\widehat{A}_1^-} \frac{1}{\widehat{d}} \left| \frac{1}{x'} \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx\, dx'\, dy'\, d\eta \\ \le\, & C \int_{-x'}^{x'} \frac{dx}{x'} \int_D \left| \frac{1}{x'} \int_0^{x'} |X_1 u(\tau, y', \eta)| \, d\tau \right|^p dx'\, dy'\, d\eta \end{aligned}$$

and the conclusive computations are still the same. If on the contrary $|x| \geq x'$ it is obviously

$$\int_0^{x'} |X_1 u(\tau, y', \eta)|\, d\tau \leq \int_0^{|x|} |X_1 u(\tau, y', \eta)|\, d\tau;$$

then

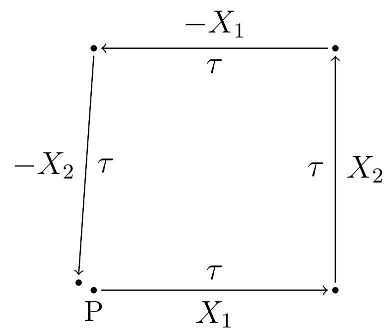
$$\begin{aligned}\int_{\widehat{A}_1^-} \frac{1}{\widehat{d}^{p+1}} \left| \int_0^{x'} |X_1 u(\tau, y', \eta)|\, d\tau \right|^p dx\, dx'\, dy'\, d\eta &\leq \int_{\widehat{A}_1^-} \frac{1}{\widehat{d}^{p+1}} \left| \int_0^{|x|} |X_1 u(\tau, y', \eta)|\, d\tau \right|^p dx\, dx'\, dy'\, d\eta \\ &\leq \int_0^{|x|} \frac{dx'}{|x|} \int_D \left| \frac{1}{|x|} \int_0^{|x|} |X_1 u(\tau, y', \eta)|\, d\tau \right|^p dx\, dy'\, d\eta.\end{aligned}$$

This concludes the estimate for L_1. Noticing that

$$\left| \int_x^0 |X_1 u(\tau, y', \eta)|\, d\tau \right| = \left| \int_0^{|x|} |X_1 u(\tau, y', \eta)|\, d\tau \right|$$

the computations for L_2 are the same. This concludes case **II**. Computations for cases **III** and **IV** are basically the same of **I** and **II**, and thus are left.

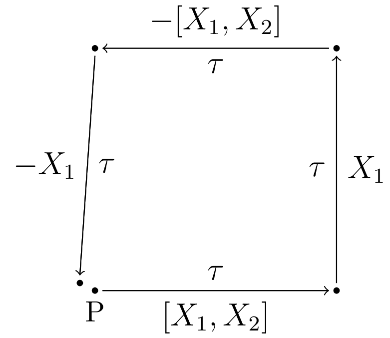
What we need to do at this point is to descend towards $-z$ direction in order to reach the point $(x', y', 0)$. Since we cannot directly use the vector field $[[X_1, X_2], X_1] = -\partial/\partial z$ we need to estimate it in terms of X_1 and X_2 using 1.1.2. Since X_1 and X_2 do not commute, if we apply in order X_1, X_2, $-X_1$ and $-X_2$ for a time τ to a point $P = (\overline{x}, \overline{y}, \overline{z})$ we do not come back to P, but we can estimate the error. The situation is the following:



More explicitely

$$\begin{aligned} P = (\overline{x}, \overline{y}, \overline{z}) &\xrightarrow{X_1} (\overline{x} + \tau, \overline{y}, \overline{z}) \xrightarrow{X_2} (\overline{x} + \tau, \overline{y} + \tau, \overline{z} + \tau(\overline{x} + \tau)^2/2) \\ &\xrightarrow{-X_1} (\overline{x}, \overline{y} + \tau, \overline{z} + \tau(\overline{x} + \tau)^2/2) \\ &\xrightarrow{-X_2} (\overline{x}, \overline{y}, \overline{z} + \tau(\overline{x} + \tau)^2/2 - \tau\overline{x}^2/2) \\ &= (\overline{x}, \overline{y}, \overline{z} + \tau^3/2 + \overline{x}\tau^2). \end{aligned}$$

This allows to estimate $[X_1, X_2]$; we do the same with $[[X_1, X_2], X_1] = -\partial/\partial z$ (remember that $-[X_1, X_2] = [X_1, -X_2]$ because Lie brackets are bilinear):



and explicitley we get

$$P = (\overline{x}, \overline{y}, \overline{z}) \xrightarrow{[X_1, X_2]} (\overline{x}, \overline{y}, \overline{z} + \tau^3/2 + \overline{x}\tau^2) \xrightarrow{X_1} (\overline{x} + \tau, \overline{y}, \overline{z} + \tau^3/2 + \overline{x}\tau^2)$$
$$\xrightarrow{-[X_1, X_2]} (\overline{x} + \tau, \overline{y}, \overline{z} - \tau^3) \xrightarrow{-X_1} (\overline{x}, \overline{y}, \overline{z} - \tau^3).$$

Back to our computation, we have reached the point $(x', y', x^2(y'-y)/2)$; the exact time $\overline{\tau}$ for whom we have to follow $[[X_1, X_2], X_1]$ in order to reach the plane $z = 0$ is therefore $x^2(y'-y)/2 - \tau^3 = 0 \longrightarrow \overline{\tau} = \sqrt[3]{x^2(y'-y)/2}$.

We proceed now with our integral estimates; following the vector field X_1 for a time $\overline{\tau} = \sqrt[3]{x^2(y'-y)/2}$ we reach the point $(x' + \overline{\tau}, y', x^2(y'-y)/2)$. Thus

$$\left|u(x' + \overline{\tau}, y', x^2(y'-y)/2) - u(x', y', x^2(y'-y)/2)\right| = \left|\int_0^{\overline{\tau}} X_1 u(x' + t, y', x^2(y'-y)/2)\, dt\right|.$$

As we did before,

$$I_{13} \simeq \int_{A_1} \frac{|u(x' + \sqrt[3]{x^2(y'-y)/2}, y', x^2(y'-y)/2) - u(x', y', x^2(y'-y)/2)|}{(|x - x'| + |y - y'|)^{p+3}} |x|^2 |x'|^2\, dx\, dy\, dx'\, dy'$$
$$\leq \int_{A_1} \frac{1}{d^{p+1}} \left|\int_0^{\overline{\tau}} |X_1 u(x' + t, y', x^2(y'-y)/2)|\, dt\right|^p |x|^2\, dx\, dy\, dx'\, dy'.$$

Now, if $x' \geq \sqrt[3]{x^2(y'-y)/2} = \overline{\tau}$ we can easily conclude using the change of variables we have done few steps above, i.e. $x' + t = \tau$ and $\eta = x^2(y'-y)/2$. The computations are the same, and they are left (x' will be the increment in the Hardy inequality). If on the contrary $x' < \sqrt[3]{x^2(y'-y)/2}$ we should split into two cases again: if indeed $x \geq y' - y \geq 0$ then $x' < \sqrt[3]{x^2(y'-y)/2} \leq x/\sqrt[3]{2}$ we get

$$\int_{A_1} \frac{1}{d^{p+1}} \left|\int_0^{\overline{\tau}} |X_1 u(x' + t, y', x^2(y'-y)/2)|\, dt\right|^p |x|^2\, dx\, dy\, dx'\, dy'$$
$$\leq \int_{A_1} \frac{1}{d^{p+1}} \left|\int_0^{Cx} |X_1 u(x' + t, y', x^2(y'-y)/2)|\, dt\right|^p |x|^2\, dx\, dy\, dx'\, dy'$$

and with the usual change of variables $x' + t = \tau$ and $\eta = x^2(y'-y)/2$ the last line becomes

$$\leq \int_{\widehat{A}_1} \frac{1}{\widehat{d}^{p+1}} \left|\int_0^{C'x} |X_1 u(\tau, y', \eta)|\, d\tau\right|^p dx\, d\eta\, dx'\, dy'$$

and we conclude again in the same way.

The case $x \le y' - y$ is slightly more difficult:

$$\int_{A_1} \frac{1}{d^{p+1}} \left| \int_0^{\overline{\tau}} |X_1 u(x'+t, y', x^2(y'-y)/2)|\, dt \right|^p |x|^2 \,dx\,dy\,dx'\,dy'$$

$$\le \int_{A_1} \frac{1}{d^{p+1}} \left| \int_0^{(y'-y)/\sqrt[3]{2}} |X_1 u(x'+t, y', x^2(y'-y)/2)|\, dt \right|^p |x|^2 \,dx\,dy\,dx'\,dy'.$$

Now we do a first change of variables in x: $\xi = x^2(y'-y)/2$ from which we get $x = \sqrt{2\xi/(y'-y)} \longrightarrow dx = d\xi/\sqrt{2\xi(y'-y)}$; thus the last line displayed above is

$$= \int_{\tilde{A}_1} \frac{1}{d^{p+1}} \left| \int_0^{y'-y} |X_1 u(x'+t, y', \xi)|\, dt \right|^p \frac{\sqrt{2\xi}}{(y'-y)^{3/2}} \,d\xi\,dx'\,dy\,dy'$$

$$\simeq \int_{\tilde{A}_1} \frac{1}{d^{p+1}} \left| \int_0^{y'-y} |X_1 u(x'+t, y', \xi)|\, dt \right|^p \,d\xi\,dx'\,dy\,dy'$$

$$= \int_{\tilde{A}_1} \frac{1}{\tilde{d}^{p+1}} \left| \int_0^{y'-y} |X_1 u(x'+t, y', \xi)|\, dt \right|^p \,d\xi\,dx'\,dy\,dy'$$

with $\tilde{d} = |\sqrt{\xi/(y'-y)} - x'| + |y'-y|$. With the change of variables $\tau = x' + t$ in t the last line displayed above becomes

$$= \int_{\tilde{A}_1} \frac{1}{\tilde{d}^{p+1}} \left| \int_{y'-y}^{x'+y'-y} |X_1(\tau, y', \xi)|\, d\tau \right|^p \,d\xi\,dx'\,dy\,dy'$$

$$\le \int_{\tilde{A}_1} \frac{1}{\tilde{d}^{p+1}} \left| \int_0^{\sqrt[3]{4}|y'-y|} |X_1(\tau, y', \xi)|\, d\tau \right|^p \,d\xi\,dx'\,dy\,dy'.$$

As we did some pages above, we prove that there exists a real number $C(p)$ depending on p such that

$$\int_{\tilde{A}_1} \frac{dx'}{\tilde{d}^{p+1}} \le \int_{\{|x'| \le \tilde{d}\}} \frac{dx'}{\tilde{d}^{p+1}} \le \frac{C(p)}{h^p}$$

where $h = |y'-y|$. Assuming wlog $x' \ge 0$,

$$\int_{\{|x'|\le\tilde{d}\}} \frac{dx'}{\tilde{d}^{p+1}} = \int_{\{x'\le\tilde{d},\, x'\le\sqrt{\xi/(y'-y)}\}} \frac{dx'}{\tilde{d}^{p+1}} + \int_{\{x'\le\tilde{d},\, \sqrt{\xi/(y'-y)}<x'\}} \frac{dx'}{\tilde{d}^{p+1}};$$

since $x' \le \sqrt{\xi/(y'-y)} - x' + h \longrightarrow x' \le (\sqrt{\xi/(y'-y)} + h)/2$ we have that

$$\int_{\{x'\le\tilde{d},\, x'\le\sqrt{\xi/(y'-y)}\}} \frac{dx'}{\tilde{d}^{p+1}} = \int_0^{(\sqrt{\xi/(y'-y)}+h)/2} \frac{dx'}{(\sqrt{\xi/(y'-y)} - x' + h)^{p+1}}$$

$$= -\frac{1}{p}\left[(\sqrt{\xi/(y'-y)} - x' + h)^{-p} \right]_0^{(\sqrt{\xi/(y'-y)}+h)/2}$$

$$= -\frac{1}{p}\left[\left(\frac{\sqrt{\xi/(y'-y)} + h}{2} \right)^{-p} - (\sqrt{\xi/(y'-y)} + h)^{-p} \right]$$

$$\le \frac{C(p)}{(\sqrt{\xi/(y'-y)} + h)^p} \le \frac{C(p)}{h^p}.$$

Moreover

$$\int_{\{x'\le\tilde d,\ \sqrt{\xi/(y'-y)}<x'\}} \frac{dx'}{\tilde d^{p+1}} \le \int_{\sqrt{\xi/(y'-y)}}^{1} \frac{dx'}{(x'-\sqrt{\xi/(y'-y)}+h)^p}$$
$$= \left[-\frac{(x'-\sqrt{\xi/(y'-y)}+h)^{-p}}{p}\right]_{\sqrt{\xi/(y'-y)}}^{1}$$
$$= -\frac{1}{p}\left[(1-\sqrt{\xi/(y'-y)}-h)^{-p}-h^{-p}\right] \le \frac{1}{ph^p}$$

which leads to the estimate that we claimed.

Let's proceed with our integral estimate. Following the vector field X_2 for a time $\bar\tau = \sqrt[3]{x^2(y'-y)/2}$ we reach the point $(x'+\bar\tau, y'+\bar\tau, x^2(y'-y)/2)+\bar\tau(x'+\bar\tau)^2/2)$. Therefore we have to estimate the integral

$$I_{14} \simeq \int_{A_1} \frac{1}{d^{p+3}} \left|\int_0^{\bar\tau} X_2u(x'+\bar\tau, y'+t, x^2(y'-y)/2+t(x'+\bar\tau)^2/2)\,dt\right|^p |x'|^2|x|^2\,dx\,dy\,dx'\,dy' \tag{2.9}$$

As usual we need to understand which will be the suitable increment for the Hardy inequality; it turns out that $\bar\tau$ is the suitable increment. We better rewrite 2.9 with the change of variables $\bar\tau = \sqrt[3]{x^2/(y'-y)/2}$ in y:

$$I_{14} \simeq 6\int_{\widehat A_1} \frac{1}{d^{p+3}} \left|\int_0^{\bar\tau} X_2u(x'+\bar\tau, y'+t, \bar\tau^3+t(x'+\bar\tau)^2/2)\,dt\right|^p |x'|^2\bar\tau^2\,dx\,d\bar\tau\,dx'\,dy'$$
$$\lesssim \int_{\widehat A_1} \frac{1}{d^{p+1}} \left|\int_0^{\bar\tau} X_2u(x'+\bar\tau, y'+t, \bar\tau^3+t(x'+\bar\tau)^2/2)\,dt\right|^p \bar\tau^2\,dx\,d\bar\tau\,dx'\,dy'$$
$$= \int_{\widehat A_1} \frac{1}{d^{p+1}} \left|\int_0^{\bar\tau} X_2u(\Psi_2(\bar\tau, y', t))\,dt\right|^p \bar\tau^2\,dx\,d\bar\tau\,dx'\,dy'$$

with $\Psi_2(\bar\tau, y', t) = (x'+\bar\tau, y'+t, \bar\tau^3+t(x'+\bar\tau)^2/2)$. We immediatly notice that

$$\frac{\partial\Psi_2}{\partial\bar\tau\partial y'\partial t} = \begin{pmatrix} 1 & 0 & 2\bar\tau^2+(x'+\bar\tau) \\ 0 & 1 & 0 \\ 0 & 1 & (x'+\bar\tau)^2/2 \end{pmatrix}$$

and that $\det\frac{\partial\Psi_2}{\partial\bar\tau\partial y'\partial t} = (x'+\bar\tau)^2/2$. We claim now that there exists $C(p)>0$ such that

$$\int_{\widehat A_1} \frac{dx}{\widehat d^{p+1}} \le \frac{C(p)}{\bar\tau^p}$$

with $\widehat d = |x-x'|+2\bar\tau^3/x^2$. We split computations in various cases.

Case $\boxed{x \ge x' \ge 0}$. Here the condition $|x| \le \widehat d$ becomes $x \le \sqrt{2\bar\tau^3/x'}$; moreover we also know that $|x'| \leq \widehat{d}$ which leads to $x' \leq x \leq \sqrt{2\bar\tau^3/x'}$. Therefore

$$\begin{aligned}\int_{\widehat{A}_1} \frac{dx}{\widehat{d}^{p+1}} &\leq \int_{x'}^{\sqrt{2\bar\tau^3/x'}} \frac{dx}{(x-x'+2\bar\tau^3/x^2)^{p+1}} \leq \int_{x'}^{\sqrt{2\bar\tau^3/x'}} \frac{dx}{x^{p+1}} \\ &= C(p)\left[\left(\sqrt{\frac{\bar\tau^3}{x'}}\right)^{-p} - (x')^{-p}\right] \\ &\leq C(p)\frac{(x')^{p/2}}{\bar\tau^{3p/2}} = \frac{C(p)}{\bar\tau^p}\frac{(x')^{p/2}}{\bar\tau^{p/2}} \leq \frac{C(p)}{\bar\tau^p}\end{aligned}$$

which conclude the estimate in this first case.

Case $\boxed{x' \geq x \geq 0}$. Here the condition $|x'| \leq x'-x+2\bar\tau^3/x^2$ leads to $0 \leq x \leq \sqrt[3]{2}\bar\tau$. Hence

$$\int_{\widehat{A}_1} \frac{dx}{\widehat{d}^{p+1}} \leq \int_0^{\sqrt[3]{2}\bar\tau} \frac{dx}{(\underbrace{x'-x}_{\geq 0}+\bar\tau^3/x^2)^{p+1}} \leq \int_0^{\sqrt[3]{2}\bar\tau} \frac{dx}{\left(\frac{\bar\tau^3}{\sqrt[3]{4}\bar\tau^2}\right)^{p+1}} = \frac{C(p)}{\bar\tau^p}.$$

Case $\boxed{x' \geq 0 > x}$. Both conditions $|x| \leq \widehat{d}$ and $|x'| \leq \widehat{d}$ are empty; thus x moves between 0 and -1:

$$\begin{aligned}\int_{\widehat{A}_1} \frac{dx}{\widehat{d}^{p+1}} &\leq \int_{-1}^0 \frac{dx}{(x'-x+\bar\tau^3/x^2)^{p+1}} \leq \int_{-1}^0 \frac{dx}{(-x+\bar\tau^3/x^2)^{p+1}} \\ &= \int_{-1}^0 \frac{dx}{\left(\frac{1}{x^2}\right)^{p+1}(-x^3+\bar\tau^3)^{p+1}} = \int_{-1}^0 \frac{x^{2(p+1)}}{(-x^3+\bar\tau^3)^{p+1}}\,dx \\ &= \bar\tau\int_0^{1/\bar\tau} \frac{(\bar\tau\xi)^{2(p+1)}}{((\bar\tau\xi)^3+\bar\tau^3)^{p+1}}\,d\xi \leq \frac{1}{\bar\tau^p}\int_0^\infty \frac{\xi^{2(p+1)}}{(1+\xi^3)^{p+1}}\,d\xi \leq \frac{C(p)}{\bar\tau^p}\end{aligned}$$

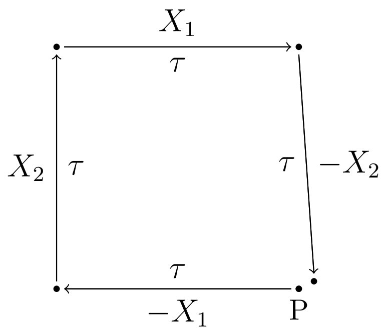
where we changed variable putting $-x = \xi\bar\tau$.

In each case we got the estimate needed. Therefore if $x' \geq 0$ we conclude:

$$\int_{\widehat{A}_1} \frac{1}{d^{p+1}}\left|\int_0^{\bar\tau} X_2u(\Psi_2(\bar\tau,y',t))\,dt\right|^p \bar\tau^2\,dx\,d\bar\tau\,dx'\,dy' \leq \int_{Q_R\times[0,1]} |X_2u(\xi,\eta,\tau)|^p\,d\xi\,d\eta\,d\tau$$

where we used Hardy inequality, Minkowski inequality and the fact that determinant of the jacobian of Ψ_2^{-1} is $= 1/(x'+\bar\tau)^2$, and thus $\bar\tau^2/(x'+\bar\tau)^2 \leq 1$.

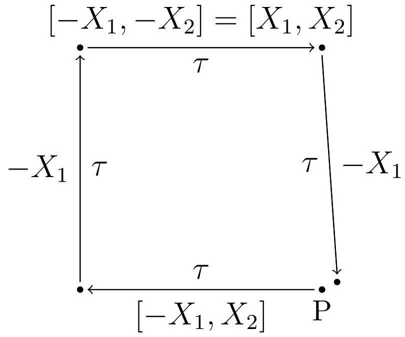
However if $x' \leq 0$ the computations we have done so far do not work anymore, because nothing we know about the quantity $\bar\tau^2/(x'+\bar\tau)^2$; we need a geometric idea to solve this case. Indeed using bilinearity of Lie brackets we have that $[[-X_1,X_2],-X_1] = [[X_1,X_2],X_1] = -\partial/\partial z$. To estimate $[-X_1,X_2]$ we may follow the following scheme:



and explicitely

$$\begin{aligned} P = (\overline{x}, \overline{y}, \overline{z}) &\xrightarrow{-X_1} (\overline{x} - \tau, \overline{y}, \overline{z}) \xrightarrow{X_2} (\overline{x} - \tau, \overline{y} + \tau, \overline{z} + \tau(\overline{x} - \tau)^2/2) \\ &\xrightarrow{X_1} (\overline{x}, \overline{y} + \tau, \overline{z} + \tau(\overline{x} - \tau)^2/2) \\ &\xrightarrow{-X_2} (\overline{x}, \overline{y}, \overline{z} + \tau(\overline{x} - \tau)^2/2 - \tau\overline{x}^2/2) \\ &= (\overline{x}, \overline{y}, \overline{z} + \tau^3/2 - \overline{x}\tau^2). \end{aligned}$$

To conclude we do the same with the vector fields $[-X_1, X_2]$ and $-X_1$:

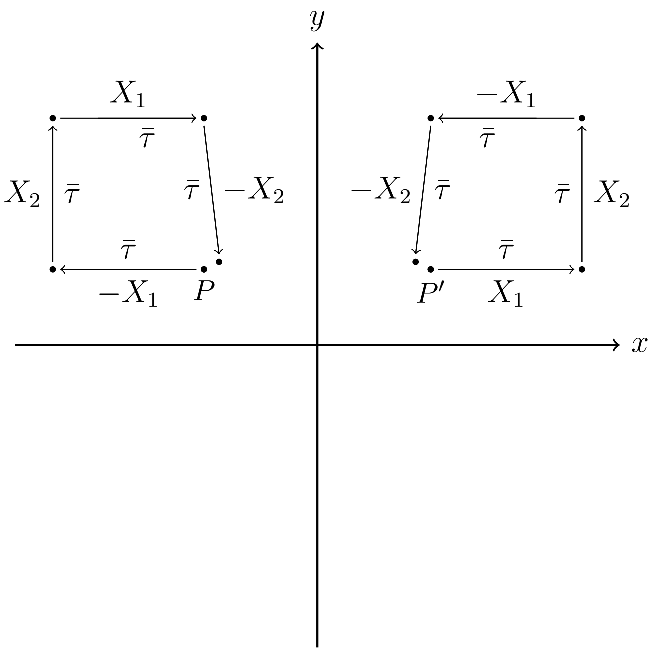


and explicitely

$$\begin{aligned} P = (\overline{x}, \overline{y}, \overline{z}) &\xrightarrow{[-X_1, X_2]} (\overline{x}, \overline{y}, \overline{z} + \tau^3/2 - \overline{x}\tau^2) \xrightarrow{-X_1} (\overline{x} - \tau, \overline{y}, \overline{z} + \tau^3/2 - \overline{x}\tau^2) \\ &\xrightarrow{[-X_1, -X_2]} (\overline{x} - \tau, \overline{y} + \tau, \overline{z} + \tau^3/2 - \overline{x}\tau^2 - \tau^3/2 + (x - \tau)\tau^2) \\ &\xrightarrow{X_1} (\overline{x}, \overline{y}, \overline{z} - \tau^3) \end{aligned}$$

which leads again to $\bar{\tau} = \sqrt[3]{x^2(y' - y)/2}$ as we needed.

Why did we choose a different path to reach the "same" point? Because if $x' < 0$ we want to go away from the singularity. The following picture better depicts the geometric idea behind our choice:



The point P has negative abscissa and if we had used the vector field X_1, we would have moved too much close to the singularity; the same with the point P (and the vector field $-X_1$). Note that we carried out the computations for a point of "type" P' (positive abscissa), but they would be basically the same in the case of a point of "type" P: the first step would be moving from $(x', y', x^2(y'-y)/2)$ to $(x'-\bar{\tau}, y', x^2(y'-y)/2)$ and so on.

This does not conclude the estimate for the case A_1, but the main ideas on how to carry out the further estimates have already been clarified. The other steps can be obtained with similar computations; we may then consider proved the theorem in the case of A_1.

We want to deal now with the case A_4. Here again the idea is to connect the points w and q with the integral curves of some vector fields, but we found out that the "technologies" used so far were not anymore appropriate. Let us consider then the vector field

$$Z = X_1 + X_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{x^2}{2}\frac{\partial}{\partial z};$$

the integral curves of Z can be found solving

$$\begin{cases} \dot{\gamma}(t) = Z(\gamma(t)) \\ \gamma(0) = (x, y, 0) \end{cases}$$

which leads to $\gamma(t) = (x+t, y+t, (x+t)^3/6 - x^3/6)$.

The integral we want to deal with is again

$$\int_{\{x \geq d,\, x' \geq d\}} \frac{|u(x,y,0) - u(x',y',0)|^p}{d^{ps}\mu(B(x,y,0),d)} x^2 x'^2 \, dx\, dy\, dx'\, dy'$$

but notice that in this case, according to Proposition 2.2.1 and Lemma 2.3.1,

$$d^{ps}\mu(B(x,y,0),d) \simeq d^{p+2}|x|$$

and

$$d = |x-x'| + |y-y'| + \sqrt{|y-y'||x|}$$

but since here $x \geq d$, we may assume that the term $|y-y'|$ is negligible; thus

$$d \simeq |x-x'| + \sqrt{|y-y'||x|} \simeq \max\{|x-x'|, \sqrt{|y-y'||x|}\}.$$

We can also suppose $x' \geq x$ without loss of generality. Following the vector field Z for a time $x'-x$ we move from $(x,y,0)$ to $(x', y+x'-x, (x'^3 - x^3)/6)$; hence we have to estimate the integral

$$\int_{\{x \geq d,\, x' \geq d\}} \frac{1}{d^{p+2}|x|} \left(\int_0^{x'-x} |Zu(x+t, y+t, [(x+t)^3 - x^3]/6)|\, dt \right)^p x^2 x'^2 \, dx\, dy\, dx'\, dy'.$$

Using $x'-x \leq d$ we have that the previous integral is obviously

$$\leq \int_{\{x \geq d,\, x' \geq d\}} \frac{1}{d^{p+2}|x|} \left(\int_0^{d} |Zu(x+t, y+t, [(x+t)^3 - x^3]/6)|\, dt \right)^p x^2 x'^2 \, dx\, dy\, dx'\, dy';$$

we notice that the inner integral does not depend on x' and y', thus we use "spherical" coordinates in d. In particular

$$f(x,y) = \frac{1}{d^{p+2}|x|}\left(\int_0^d |Zu(x+t, y+t, [(x+t)^3 - x^3]/6)|\,dt\right)^p$$

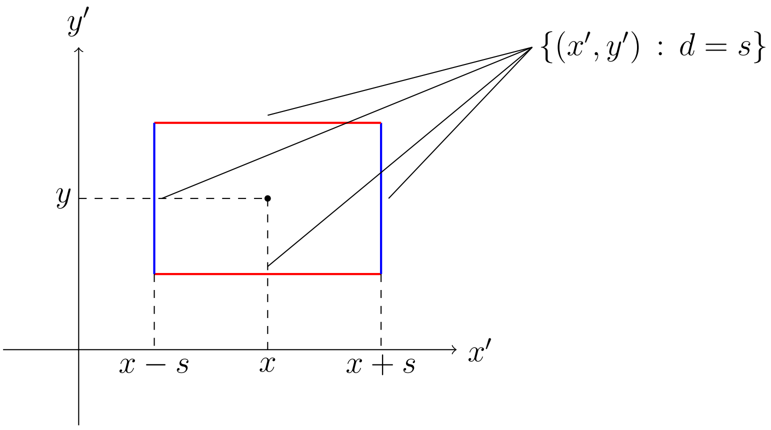
is "radial" in d (in the coordinates x' and y'); using then the Coarea formula (Theorem 1.1.3) we get

$$\int_{\{d<r\}} f\,dx'\,dy' = \int_0^r \left(\int_{\{d=s\}} \frac{f}{|\nabla d|}\,d\mathcal{H}^1(x',y')\right) ds.$$

Therefore we have to compute

$$\int_{\{d=s\}} \frac{1}{|\nabla d|}\,d\mathcal{H}^1(x',y');$$

a picture helps to understand the situation:



Indeed along the red edges it is $d = \sqrt{|y-y'||x|}$ because $|x-x'| < s$ and then

$$|\nabla_{y'} d| = \frac{1}{2}\frac{\sqrt{x}}{\sqrt{|y-y'|}} = \frac{1}{2}\frac{\sqrt{x}}{\sqrt{s^2/x}} = \frac{x}{2s};$$

along the blue edges $d = |x-x'|$ and then $|\nabla d| = 1$. In conclusion

$$\int_{\{d=s\}} \frac{1}{|\nabla d|}\,d\mathcal{H}^1(x',y') \simeq \frac{s}{x}\cdot 2s \simeq \frac{s^2}{x}$$

along red edges and again

$$\int_{\{d=s\}} \frac{1}{|\nabla d|}\,d\mathcal{H}^1(x',y') \simeq \frac{s^2}{x}$$

along blue edges. These computations are enough to conclude the estimate: indeed along

$d=s$ we may suppose $x \simeq x'$ with $s < x/2$ and therefore

$$\int_{\{x\geq d\}}\int_{\{x'\geq d\}}\frac{1}{d^{p+2}|x|}\left(\int_0^d |Zu(x+t,y+t,[(x+t)^3-x^3]/6)|\,dt\right)^p x'^2\,dx'\,dy'\,x^2\,dx\,dy$$
$$\leq \int_{\{x\geq d\}}\int_0^1\int_{\{d=s\}}\frac{1}{d^{p+2}|x|}\left(\int_0^d |Zu(x+t,y+t,[(x+t)^3-x^3]/6)|\,dt\right)^p x'^2\frac{d\mathcal{H}^1(x',y')}{|\nabla d|}\,ds\,dx\,dy$$
$$\leq \int_{\{x\geq d\}}\int_0^1\frac{1}{s^{p+2}|x|}\left(\int_0^s |Zu(x+t,y+t,[(x+t)^3-x^3]/6)|\,dt\right)^p x^2\cdot\frac{s^2}{x}\,ds\,x^2\,dx\,dy$$
$$\leq \int_{\{|x|\leq 1,\,|y|\leq 1\}}\int_0^1\left(\frac{1}{s}\int_0^s |Zu(x+t,y+t,[(x+t)^3-x^3]/6)|\,dt\right)^p x^2\,dx\,dy.$$

Using Hardy inequality the last line above is

$$\leq \int_{\{|x|\leq 1,\,|y|\leq 1\}}\int_0^1 |Zu(x+t,y+t,[(x+t)^3-x^3]/6)|^p\,dt\,x^2\,dx\,dy$$
$$\leq \int_{\{|x|\leq 1,\,|y|\leq 2,\,x\leq\sigma\}}\int_{-1}^1 |Zu(\sigma,y,(\sigma^3-x^2)/6|^p\,d\sigma\,x^2\,dx\,dy$$
$$\leq \int_{\{|\sigma|\leq 1,\,|y|\leq 2,\,0\leq z\leq 1\}} |Zu(\sigma,y,z)|^p\,d\sigma\,dy\,dz$$

where we put $x+t=\sigma$ in s and $(\sigma^3-x^3)/6=z$ in x.

This concludes the estimate along the first path.

In the second step we have to estimate

$$\int_{\{x\geq d,\,x'\geq d\}}\frac{1}{d^{p+2}|x|}|\Delta|^p x^2 x'^2\,dx\,dy\,dx'\,dy'$$

where

$$\Delta = u(x',y+x'-x,(x'^3-x^3)/6)-u(x',y',(x'^3-x^3)/6+(y'-y-x'+x)x'^2/2)$$
$$=\int_0^{y'-y-x'+x} X_2u(x',y+x'-x+t,(x'^3-x^3)/6+tx'^2/2)\,dt.$$

Being u with compact support, we may integrate over $\mathbb{R}^2\times\mathbb{R}^2$ instead of $\{x\geq d,\,x'\geq d\}$. We get then

$$\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{d^{p+2}|x|}\left|\int_0^{h_2-h_1} X_2u\left(x+h_1,y+h_1+t,\frac{(x+h_1)^3-x^3}{6}+t\frac{(x+h_1)^2}{2}\right)dt\right|^p x^2(x+h_1)^2\,dx\,dy\,dh_1\,d$$

after the change of variables $x'=x+h_1$ and $y'=y+h_2$; notice that $|h_2-h_1|\leq|h_2|+|h_1|\leq |x'-x|+\sqrt{|y'-y|}\sqrt{|y'-y|}\leq d$. With the change of variable $h_2x=k_2$ we get that the previous integral is

$$\leq \int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{d^{p+2}}\left|\int_0^d\left|X_2u\left(x+h_1,y+h_1+t,\frac{(x+h_1)^3-x^3}{6}+t\frac{(x+h_1)^2}{2}\right)\right|\,dt\right|^p (x+h_1)^2\,dx\,dy\,dh_1\,dk_2;$$

here $d = |h_1| + \sqrt{|k_2|}$. With the change of variable $k_2 = h_3^2 \longrightarrow dk_2 = 2h_3dh_3$ we finally get

$$\leq \int_{\mathbb{R}^2} dh \int_{\mathbb{R}^2} \frac{1}{|h|^{p+1}} \left| \int_0^{|h|} \left| X_2 u \left(x+h_1, y+h_1+t, \frac{(x+h_1)^3 - x^3}{6} + t\frac{(x+h_1)^2}{2} \right) \right| dt \right|^p (x+h_1)^2 \, dx \, dy$$

where we used $|h_3| \leq d$, $h = (h_1, h_3)$ and then $d \simeq |h|$. We proceed with the chain of inequalities; in order to get rid of the h_1 in the inner integral we put $x + h_1 = \bar{x}$ in x and $y + h_1 + t = \bar{y}$ in y from which we get

$$\leq \int_{\mathbb{R}^2} dh \int_{\mathbb{R}^2} \frac{1}{|h|^{p+1}} \left| \int_0^{|h|} \left| X_2 u \left(\bar{x}, \bar{y}, \frac{\bar{x}^3 - (\bar{x}-h_1)^3}{6} + t\frac{\bar{x}^2}{2} \right) \right| dt \right|^p \bar{x}^2 \, d\bar{x} \, d\bar{y} =: L;$$

then with the last linear change $(\bar{x}^3 - (x-h_1)^3)/6 + t\bar{x}^2/2 = \tau\bar{x}^2/2$ in t; notice that

$$\left| \frac{\bar{x}^3 - (\bar{x}-h_1)^3}{6} \right| = \left| \frac{\bar{x}^3 - \bar{x}^3 - h_1^3 + 3\bar{x}^2 h_1 + 3\bar{x}h_1^2}{6} \right| \leq C\bar{x}^2|h_1| \leq C\bar{x}^2|h|$$

and then $0 \leq \tau \leq C'|h|$ for a suitable constant $C' > 0$. Then

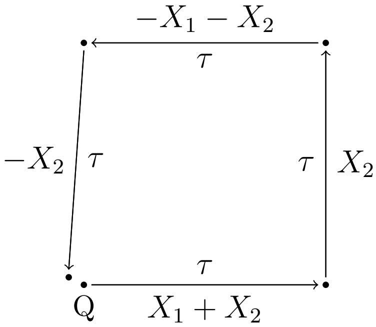
$$\begin{aligned} L &\leq \int_{\mathbb{R}^2} dh \int_{\mathbb{R}^2} \frac{1}{|h|^{p+1}} \left| \int_0^{|h|} \left| X_2 u \left(\bar{x}, \bar{y}, \tau\frac{\bar{x}^2}{2} \right) \right| d\tau \right|^p \bar{x}^2 \, d\bar{x} \, d\bar{y} \\ &\leq \int_{\mathbb{R}^2} \frac{dh}{|h|^{p+1}} \left[\int_0^{C'|h|} \left(\int_{\mathbb{R}^2} \left| X_2 u \left(\bar{x}, \bar{y}, \frac{\bar{x}^2}{2}\tau \right) \right|^p \bar{x}^2 \, d\bar{x} \, d\bar{y} \right)^{1/p} d\tau \right]^p \end{aligned}$$

where we used Minkowski integral inequality in the last step in order to exchange integration in $d\tau$ with integration in $d\bar{x}\,d\bar{y}$. We can now proceed with "spherical" coordinates in $|h|$:

$$\begin{aligned} &\leq \int_0^\infty \left[\frac{1}{s} \int_0^s \left(\int_{\mathbb{R}^2} |X_2 u(\bar{x}, \bar{y}, \bar{x}^2\tau/2)|^p \, \bar{x}^2 \, d\bar{x} \, d\bar{y} \right)^{1/p} d\tau \right]^p ds \\ &\leq \int_0^\infty \int_{\mathbb{R}^2} |X_2 u(\bar{x}, \bar{y}, \bar{x}^2 s/2)|^p \bar{x}^2 \, d\bar{x} \, d\bar{y} \, ds \\ &\leq \int_0^\infty \int_{\mathbb{R}^2} |X_2 u(x, y, z)|^p \, dx \, dy \, dz. \end{aligned}$$

where we used Hardy inequality in the second line and a last change of variable $\bar{x}^2 s/2 = z$ in s. Also the estimate along the second path has been done.

We have now to deal with the descent. As we have done in the case of A_1, we use Lie brackets, noticing that $[X_1 + X_2, X_2] = [X_1, X_2] = x\frac{\partial}{\partial z}$; this time one application of commutators will be enough; using Lemma 1.1.2 we can estimate the time required to reach the plane $z = 0$ which should be comparable with the distance d between the two points w and q. The scheme is the same used few pages above:



where $Q = \left(x', y', \frac{x'^3}{6} - \frac{x^3}{6} + (y' - y - x' + x)\frac{x'^2}{2}\right)$; we have to estimate the time of the descent. More explicitely:

$$\begin{aligned}
Q &= \left(x', y', \frac{x'^3}{6} - \frac{x^3}{6} + (y' - y - x' + x)\frac{x'^2}{2}\right) \\
&\xrightarrow{X_1+X_2} \left(x' + \tau, y' + \tau, \frac{(x'+\tau)^3}{6} - \frac{x^3}{6} + \frac{x'^2}{2}(y' - y + x - x')\right) \\
&\xrightarrow{X_2} \left(x' + \tau, y' + 2\tau, -\frac{x^3}{6} + (y' - y - x' + x)\frac{x'^2}{2} + \frac{(x'+\tau)^3}{6} + \tau\frac{(x'+\tau)^2}{2}\right) \\
&\xrightarrow{-X_1-X_2} \left(x', y' + \tau, \frac{x'^3}{6} - \frac{x^3}{6} + (y' - y - x' + x)\frac{x'^2}{2} + \tau\frac{(x'+\tau)^2}{2}\right) \\
&\xrightarrow{-X_2} \left(x', y', \frac{\tau^3}{2} + \tau^2 x' - \frac{x^3}{6} + \frac{x'^3}{6} + (y' - y - x' + x)\frac{x'^2}{2}\right)
\end{aligned}$$

and then the time $\overline{\tau}$ should solve

$$\frac{\overline{\tau}^3}{2} + \overline{\tau}^2 x' = \overline{\tau}^2\left(x' + \frac{\overline{\tau}}{2}\right) = \underbrace{\frac{x^3}{6} - \frac{x'^3}{6} - (y' - y - x' + x)\frac{x'^2}{2}}_{=z}.$$

We have $\overline{\tau} \le x'$; on the contrary, if $x' \le \overline{\tau}$ hold, it would mean $x'^3 \le \overline{\tau}^3 \le z$ id est $x' \le \sqrt[3]{z} < \sqrt[3]{dx'^2}$, contradiction because we are in the case $x' \ge d$. Thus $\overline{\tau} \le x'$ leads to $\overline{\tau}^2 x' \le z$ and then $\overline{\tau} \le \sqrt{z/x'}$. By Lemma 2.3.1

$$\overline{\tau} \le \sqrt{\frac{z}{x'}} = d((x', y', z), (x', y', 0)) \le c_0 d((x, y, 0), (x', y', 0)) = c_0 d$$

where we also used triangle inequality and the fact that

$$d((x', y', z), (x, y, 0)) \simeq d$$

because we followed the vector fields for a time comparable to d.

Now we estimate

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x^2 x'^2}{d^{p+2}|x|}\,dx\,dy\,dx'\,dy' \left(\int_0^{\overline{\tau}} \left| Zu\left(x' + \tau, y' + \tau, \frac{(x'+\tau)^3}{6} - \frac{x^3}{6} + \frac{x'^2}{2}(y' - y + x - x')\right)\right| d\tau\right)^p.$$

With the change of variables $x' - x = h_1 > 0$ in x' and $y' - y = h_2$ in y' the previous is

$$\leq \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x^2(x+h_1)^2}{d^{p+2}|x|}\, dx\, dy\, dh_1\, dh_2$$
$$\left(\int_0^d \left| Zu\left(x+h_1+\tau, y+h_2+\tau, \frac{(x+h_1+\tau)^3}{6} - \frac{x^3}{6} + \frac{(x+h_1)^2}{2}(h_2-h_1)\right)\right| d\tau\right)^p$$

where now $d = |x'-x| + \sqrt{|y'-y||x|} = |h_1| + \sqrt{|h_2||x|}$. A second change of variables $h_2 x = \widetilde{h}_2$ says that the chain of inequalities continues with

$$\leq \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{(x+h_1)^2}{d^{p+2}}\, dx\, dy\, dh_1\, d\widetilde{h}_2 \left(\int_0^d \left| Zu\left(x+h_1+\tau, y+\frac{\widetilde{h}_2}{x}+\tau, \dots\right)\right| d\tau\right)^p.$$

The third change of variables $\widetilde{h}_2 = \overline{h}_2^2$ leads to

$$\lesssim \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{(x+h_1)^2}{d^{p+1}}\, dx\, dy\, dh_1\, d\overline{h}_2 \left(\int_0^d \left| Zu\left(x+h_1+\tau, y+\frac{\overline{h}_2^2}{x}+\tau, \dots\right)\right| d\tau\right)^p$$

since $\overline{h}_2 = \sqrt{\widetilde{h}_2} = \sqrt{h_2|x|} = \sqrt{|y'-y||x|} < d$. The most tricky part of this computation starts here with the fourth change of variables in τ:

$$(x+h_1+\tau)^2\sigma = \frac{(x+h_1+\tau)^3}{6} - \frac{x^3}{6} + \frac{(x+h_1)^2}{2}(h_2-h_1) \tag{2.10}$$

the differential turns out to be

$$d\sigma = \underbrace{\frac{(x+h_1+\tau)^4/2 - 2(x+h_1+\tau)[(x+h_1+\tau)^3/6 - x^3/6 + (x+h_1)^2(h_2-h_1)/2]}{(x+h_1+\tau)^4}}_{=1/\Phi} d\tau;$$

after some elementary computations (which should be split into two cases: case $-(x+h_1)^2(x+h_1+\tau)(h_2-h_1) > 0$ and case $-(x+h_1)^2(x+h_1+\tau)(h_2-h_1) \leq 0$) we get that $|\Phi| \leq C$ for a suitable constant $C \in \mathbb{R}$, provided that $x \leq d/8$. With similar computations we have that $|\sigma| \leq kd = k|h|$ (remember that $\tau \in [0, |h|]$). Thus our integral inequalities chain continues with

$$\lesssim \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{(x+h_1)^2}{d^{p+1}}\, dx\, dy\, dh_1\, d\overline{h}_2 \left(\int_0^d \left| Zu\left(x+h_1+\widehat{\tau}, y+\frac{\overline{h}_2}{x}+\widehat{\tau}, (x+h_1+\widehat{\tau})^2\sigma\right)\right| d\sigma\right)^p$$

with $\widehat{\tau} = \widehat{\tau}(h_1, h_2, \sigma, x)$. As we did before, we use now Minkowski inequality for integrals in order to exchange integration in $dh = dh_1\, d\overline{h}_2$ with integration in $dx\, dy$:

$$\lesssim C(p) \int_{\mathbb{R}^2} \frac{dh_1\, d\overline{h}_2}{|h|^{p+1}} \left[\int_0^{|h|} \left(\int_{\mathbb{R}^2} \left| Zu\left(x+h_1+\widehat{\tau}, y+\frac{\overline{h}_2}{x}+\widehat{\tau}, (x+h_1+\widehat{\tau})^2\sigma\right)\right|^p dx\, dy\right)^{1/p} d\sigma\right]^p.$$

A fifth change of variables pushes us towards the final result: $\overline{x} = x + h_1 + \widehat{\tau}$ in x and $\overline{y} = y + \overline{h}_2^2/x + \widehat{\tau}$ in y:

$$\lesssim C(p) \int_{\mathbb{R}^2} \frac{dh_1\, d\overline{h}_2}{|h|^{p+1}} \left[\int_0^{|h|} \left(\int_{\mathbb{R}^2} |Zu(\overline{x}, \overline{y}, \overline{x}^2\sigma)|^p \frac{(\overline{x}-\widehat{\tau})^2}{(1+\widehat{\tau}')}\, d\overline{x}\, d\overline{y}\right)^{1/p} d\sigma\right]^p$$
$$\lesssim C(p) \int_{\mathbb{R}^2} \frac{dh_1\, d\overline{h}_2}{|h|^{p+1}} \left[\int_0^{C|h|} \left(\int_{\mathbb{R}^2} |Zu(\overline{x}, \overline{y}, \overline{x}^2\sigma)|^p \frac{\overline{x}^2}{(1+\widehat{\tau}')}\, d\overline{x}\, d\overline{y}\right)^{1/p} d\sigma\right]^p;$$

differentiating the identity 2.10 in x it is possible to show that $1+\tau'$ is bounded from below provided that $d \leq x/16$ (it is enough to use $(x+h_1+\tau) \sim x$ and other trivial inequalities involving x and the distance). "Spherical" coordinates in $|h|$, Hardy inequality and the change of variable $\overline{z} = \overline{x}^2\sigma$ in σ allow us to conclude the estimate. The further steps of the path may be done with similar ideas, and thus are omitted.

This concludes the proof for A_4; the proofs for the cases A_2 and A_3 may be obtained combining techniques from the proof of the case A_1 and techniques from the proof of the case A_4.

□

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