Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"
Corso di Laurea Magistrale in Matematica

Reflection Techniques and Applications to the Singular Yamabe Equation

Relatore:
Prof. Roberto Monti

Candidato:
Giacomo Vianello
Matricola 1211137

16 ottobre 2020
Anno Accademico 2019/2020

Contents

Introduction

The aim of this work is to introduce two reflection techniques, the second being a nonstandard, boosted version of the first one, and then to analyze the role played by these techniques in the study of symmetry properties for some elliptic equations, in particular for the singular Yamabe Equation. The first chapter is devoted to list some preliminary instruments needed to prove the results stated in the following parts: in particular, we prove some different versions of the Maximum Principle and the Hopf Lemma, and we furnish the statement (and a sketch of the proof) of a Maximum Principle for weakly subharmonic maps, that in its most general formulation requires just the upper semicontinuity of the function. The second chapter is devoted to the description of a first, standard reflection technique holding for smooth, bounded, connected open subsets. This technique is developed by Gidas, Ni and Niremberg in [2] and is inspired by another reflection technique theorized by Alexandrov in Differential Geometry. We now try to give a description of this reflection technique. Given a unit vector $\gamma \in \mathbb{S}^{n-1}$ and an open, bounded, connected subset $\Omega \subset \mathbb{R}^n$ of class $\mathcal{C}^2$, choosing λ small enough, the hyperplane $T_\lambda : x \cdot \gamma = \lambda$ intersects Ω and so we may consider the open set $\Sigma(\lambda) := \Omega \cap \{x \cdot \gamma > \lambda\}$ and its reflection $\Sigma'(\lambda)$ in the hyperplane T_λ. Denoted by λ_0 the supremum of the values λ for which $\Sigma(\lambda) \neq \varnothing$, we have $\Sigma'(\lambda) \subset \Omega$ for $\lambda_0 - \epsilon < \lambda < \lambda_0$ (ϵ small). Then the necessary (but not sufficient) condition in order that the reflection of $\Sigma(\lambda)$ is no longer contained in Ω turns out to be that $\Sigma'(\lambda)$ becomes internally tangent to the boundary of Ω at some point not belonging to T_λ, or that T_λ reaches a position orthogonal to $\partial\Omega$ at some point. Let $\Sigma_\gamma := \Sigma(\lambda_1)$, where λ_1 is the smallest critical value λ such that one of the two just described conditions holds. This reflection technique allows to prove the following symmetry property

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected subset of class $\mathcal{C}^2$, $\gamma \in \mathbb{S}^{n-1}$ be a unit vector and u be a smooth solution to*

$$\Delta u + b(x)\partial_\gamma u + f(u) = 0, \ \textit{in } \Omega$$

where $b \in \mathcal{C}^0(\overline{\Omega})$, $b \geq 0$ in $\Sigma_\gamma \cup \Sigma'_\gamma$ and $f \in \mathcal{C}^1(\mathbb{R})$. Assume that $u > 0$, $u \in \mathcal{C}^2(\overline{\Omega} \cap \{x \cdot \gamma > \lambda\})$, and $u = 0$ on $\partial\Omega \cap \{x \cdot \gamma > \lambda\}$. Then, for any $\lambda_1 < \lambda < \lambda_0$, one has

$$\partial_\gamma u(x) < 0 \ \textit{and } u(x) < u(x_\lambda), \ \forall x \in \Sigma(\lambda).$$

Therefore $\partial_\gamma u < 0$ in Σ_γ and in addition, if $\partial_\gamma u$ vanishes at some point contained in $T_{\lambda_1} \cap \Omega$, then u is necessarily symmetric with respect to T_{λ_1}, $\Omega = \Sigma_\gamma \cup \Sigma'_\gamma \cup (T_{\lambda_1} \cap \Omega)$ and $b \equiv 0$.

This so technical result becomes more intuitive if one considers a ball centered at the origin: as a matter of fact, in this case, the Theorem above guarantees the radiality of the solutions to

$$\begin{aligned} -\Delta u = f(u) &\quad \text{in } B(0,R[\\ u = 0 &\quad \text{su } \partial B(0,R[. \end{aligned}$$

To verify it, it's enough to apply the Theorem just stated first to a generic unit vector γ and subsequently to its opposite, deducing then that $\partial_\gamma u$ needs to vanish on the hyperplane $x \cdot \gamma = 0$.

The third chapter is then dedicated to the study of work [1] by Caffarelli, Gidas and Spruck: in that paper, the authors introduce a second, more sophisticated, reflection technique and its application to the classification of the singular solutions to the *Yamabe Equation*

$$-\Delta u = u^{\frac{n+2}{n-2}}$$

in the punctured, unit ball. The classification result proved is the following.

Theorem. *Let $u > 0$ be a $\mathcal{C}^2$ solution to*

$$-\Delta u = u^{\frac{n+2}{n-2}}, \text{ in } B(0,1[\setminus\{0\},$$

with a nonremovable, isolated singularity at the origin. Then there is a radial, singular solution $\phi \equiv \phi(|x|)$ to the same equation such that

$$u(x) = (1 + o(1))\,\phi(|x|),$$

as $x \to 0$.

We now aim to try to understand how the authors argue in order to show the validity of the classification Theorem above. To do so, we find convenient to divide the description into some steps, the most delicate of which consists precisely of finding a suitable, smart reflection technique. So let's consider the Yamabe Equation

$$-\Delta u = u^{\frac{n+2}{n-2}},$$

and let u be a solution to this equation in the punctured ball $B(0,1[\setminus\{0\}$.

(1) First we need to apply to u the Kelvin Transform in order to get a function v defined in a neighbourhood of ∞: this map v turns out to solve the Yamabe Equation again. The really smart idea here is to perform such a Kelvin Transform with respect to a point close to the origin, but different from it: in such a way, the singularity of u at the origin is transformed into a singularity for v at a point z distant from the origin. The assets brought by such a choice will be clear in step (3).

(2) The second passage consists in proving that, since u solves the Yamabe Equation in the punctured ball (possibly with an isolated singularity at the origin), it follows that u is a weak solution to the Yamabe Equation in the entire ball, and that v is a weak solution to the same equation in a neighbourhood of ∞ containing the singularity z. This step is very important becuse justifies the idea of the authors of developing a reflection technique valid for weak solutions: in other words, we bypass the problem represented by the presence of the singularity for u and v exploiting a weak notion of solution.

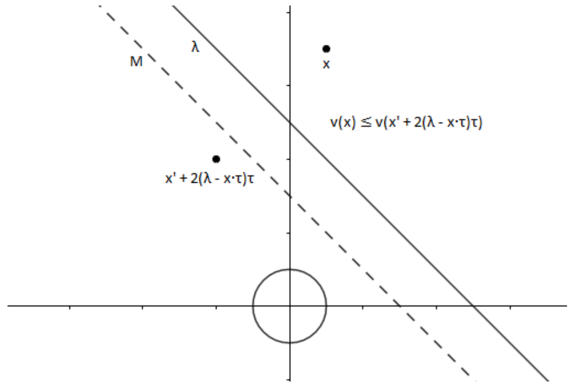
(3) The third step is to prove some decay estimates valid for the Kelvin Transform v of the solution u. In particular we prove that

$$\begin{aligned} v(x) &= \frac{1}{|x|^{n-2}}\Big(a_0 + \sum_{k=1}^{n} a_k \frac{x_k}{|x|^2}\Big) + \mathcal{O}(|x|^{-n}) \\ \frac{\partial v}{\partial x_i}(x) &= -(n-2)a_0 \frac{x_i}{|x|^n} + \mathcal{O}(|x|^{-n}) \\ \frac{\partial^2 v}{\partial x_i \partial x_j}(x) &= \mathcal{O}(|x|^{-n}), \end{aligned}$$

for $|x| \to \infty$. We observe that the validity of these asymptotic expansions is a consequence of the fact that, at the first step, the Kelvin Transform is performed with respect to a point that is different from the origin: in fact, such a choice produces a singularity of v at a point distant from the origin, but doesn't influence the behaviour of v at ∞.

(4) We now get to the most important, delicate passage: to prove a reflection technique for v from which a symmetry property for v and u will follow. We show that, for a "big measure" set of unit vectors τ, for M big enough, one has

$$v(x) \leq v(x' + 2(\lambda - x \cdot \tau)), \text{ for } x \cdot \tau > \lambda \geq M.$$



A part of the difficulty of this step is represented by the estimate of the measure of the collection of unit vectors for which the reflection property above holds true, whose proof requires advanced notions of measure theory and some elements of Harmonic Analysis.

(5) Then, we give the first, general application of the theory developed in the previous parts: the solution u is *asymptotically radial* around the origin, namely

$$u(x) = (1 + \mathcal{O}(|x|)) \fint_{\partial B(0,r[} u(w)\, \mathrm{d}\sigma(w), \text{ for } x \to 0.$$

This property is weaker than the condition stated in the classification Theorem met before: in fact, roughly speaking, it states just a "radiality as $x \to 0$", and not a "proximity to a radial *solution* as $x \to 0$".

(6) Indeed, the final step is precisely devoted to understand how the asymptotic symmetry may be boosted in order to get the classification result stated above. Such a strengthening requires an exhaustive, classificatory survey of the radial solutions to the Yamabe Equation: in fact, analyzing the radial solutions to the Yamabe Equation, one finds a very natural notion of energy that can be generalized to the case of a generic solution and that furnishes a way to measure the "asymptotic distance" between two solutions. Recalling the expression of the Laplace Operator in spherical coordinates, one finds that the radial solutions $u(x) = \phi(|x|)$ are precisely the functions $\phi(r) = r^{\frac{2-n}{2}}\psi(-\ln r)$, where ψ is a solution to the ordinary differential equation

$$\psi'' - \left(\frac{n-2}{2}\right)^2 \psi + \psi^{\frac{n+2}{n-2}} = 0. \tag{1}$$

For the ODE above, we can easily formulate a definition of energy

$$E(\psi, \psi') \equiv \frac{1}{2}(\psi')^2 - \frac{1}{2}\left(\frac{n-2}{2}\right)^2 \psi^2 + \frac{n-2}{2n}\psi^{\frac{2n}{n-2}}:$$

this energy E is constant along the solutions to the ODE above, and then, up to a choice of suitable value of $D = 2E$, one discovers that any solution to the equation above solves

$$(\psi')^2 = \left(\frac{n-2}{2}\right)^2 \psi^2 - \frac{n-2}{n}\psi^{\frac{2n}{n-2}} + D.$$

Now, the idea is to generalize the definition of E to the case of a generic solution u replacing ψ with

$$\beta(t) \equiv r^{\frac{n-2}{2}} \fint_{\partial B(0,r[} u(w)\, \mathrm{d}\sigma(w),$$

where $t = -\ln r$, that is the spherical average of u (up to a change of variable). In this more general case, E is not constant along β, but one can prove the validity of the following energy estimate (as $t \to \infty$)

$$(\beta')^2 = \left(\frac{n-2}{2}\right)^2 \beta^2 - \frac{n-2}{n}\beta^{\frac{2n}{n-2}} + D_\infty + (\beta^2 + (\beta')^2)\mathcal{O}(e^{-t}),$$

where D_∞ is a suitable asymptotic constant. It's then clear that u is close to the radial solution $\phi \equiv \phi(|x|)$ where $\phi(r) = r^{\frac{2-n}{2}}\psi(-\ln r)$ and ψ has energy $E = D_\infty/2$. Combining this observation and the asymptotic symmetry of step (5), arguing in a suitable way, one deduces the statement of the classification Theorem given at the beginning.

Chapter 1

Some preliminary results

In this chapter we present some useful tools that will be used along the work. We start furnishing some classical Maximum Principles and Hopf Lemmas freely following [4]. Next we will expose some properties of the weak solutions, in particular we will prove a Maximum Principle for weakly subharmonic functions, following [5]. Then we will prove an estimate for the (weak) solution to a homogeneous Dirichlet boundary value problem associated to the Poisson Equation.

1.1 Elliptic Differential Operators

Given an open subset $\Omega \subset \mathbb{R}^n$, we consider second order linear differential operators in Ω, namely operators like

$$\mathcal{L} \equiv \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.1}$$

where $\Omega \ni x \to A(x) \equiv (a_{i,j}(x))_{i,j}$ is a symmetric matrix valued map. We also consider a more general notion of second order differential operator, in which we admit the presence of first order terms

$$L \equiv \mathcal{L} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i},$$

or of first and zero order terms

$$L + h \equiv \mathcal{L} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + h, \tag{1.2}$$

where $\mathcal{L}$ is a second order linear differential operator like (1.1), $\Omega \ni x \to b_i(x)$, $1 \le i \le n$, and h are real valued maps. We say that $\mathcal{L}$ is the principal part of the (generalized) second order differential operator. We assume that operators like (1.1) or (1.2) act on $\mathcal{C}^2(\Omega)$ functions, and so the hypothesis about the symmetry of $A(x)$ does not represent a loss of generality.

Definition 1.1.1. A second order linear differential operator like (1.1) is said to be:

(i) *elliptic* at a point $x \in \Omega$ provided that there exists a positive number $\mu \equiv \mu(x) > 0$ such that

$$\sum_{i,j} a_{i,j}(x)\, \xi_i \xi_j \ge \mu(x) \sum_i \xi_i^2,$$

namely if $A(x)$ is positive definite;

(ii) *elliptic* in Ω provided that it's elliptic at any point of Ω;

(iii) *uniformly elliptic* in Ω provided that it's elliptic in Ω and there exists a scalar $\mu_0 > 0$ such that $\mu(x) \ge \mu_0$, for every $x \in \Omega$.

A second order differential operator like (1.2) is said to be *elliptic* at a point x (*elliptic*, *uniformly elliptic* in Ω) if its principal part is so.

The prototypical example of (uniformly) elliptic operator is the well known Laplace operator

$$\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}.$$

Let's consider a generic orthogonal change of coordinates

$$y = Cx, \tag{1.3}$$

for some orthogonal $n \times n$ matrix C.

Lemma 1.1.1. Let $\mathcal{L} \equiv \sum_{i,j} a_{i,j} \dfrac{\partial^2}{\partial x_i \partial x_j}$ be a second order linear differential operator like (1.1) in a open Ω. Then, under the orthogonal transformation (1.3), the operator $\mathcal{L}$ assumes the form

$$\widetilde{\mathcal{L}} \equiv \sum_{k,l} b_{k,l} \frac{\partial^2}{\partial y_k \partial y_l},$$

where $b_{k,l} = \sum_{i,j} a_{i,j} \, c_{k,i} \, c_{l,j} = (C\,A\,C^T)_{k,l}$, namely, for any $u \in \mathcal{C}^2(\Omega)$, one has $\widetilde{\mathcal{L}}u = \mathcal{L}u$ in Ω. Moreover, if $\mathcal{L}$ is elliptic at a point x, then $\widetilde{\mathcal{L}}$ is elliptic at x with the same ellipticity constant $\mu(x)$.

The proof is a trivial computation. Exploiting the Spectral Theorem for orthogonal matrices, one can also infer the validity of the following

Lemma 1.1.2. If $\mathcal{L}$ is a second order linear differential operator like (1.1) elliptic at a point $x \in \Omega$, then there exists an orthogonal change of coordinates like (1.3) such that

$$\widetilde{\mathcal{L}} = \sum_i d_i \frac{\partial^2}{\partial y_i^2},$$

where $d_i \geq \mu(x)$, for every i. In particular, applying another change of coordinates $z_k = \frac{1}{\sqrt{d_k}} y_k$, $\widetilde{\mathcal{L}}$ coincides with the Laplace Operator (at x).

1.1.1 Maximum Principles

Let Ω be an open subset of $\mathbb{R}^n$ and consider a second order differential operator

$$L \equiv \sum_{i,j} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$$

elliptic in Ω. Let $u \in \mathcal{C}^2(\Omega)$ be such that $Lu > 0$ and assume that u has a local maximum at a point $x_0 \in \Omega$. Then we know that

$$\begin{aligned} &\nabla u(x_0) = 0 \\ &Hess[u](x_0) \text{ is negative semidefinite.} \end{aligned}$$

Let $z = Cx$ be the change of coordinates which transforms L into the Laplace Operator at x_0. Observing that

$$Hess_z[u](x_0) = C^T Hess_x[u](x_0)\, C,$$

we obtain that $\dfrac{\partial^2 u}{\partial z_k^2}(x_0) \le 0$, for any k. Therefore one has

$$\begin{aligned} Lu(x_0) &= \mathcal{L}u(x_0) + \sum_i b_i(x_0) \frac{\partial u}{\partial x_i}(x_0) \\ &= \mathcal{L}u(x_0) = \Delta_z u(x_0) \le 0, \end{aligned}$$

a contradiction. Thus we infer that u cannot attain a local maximum at a point of Ω.

Requiring uniform ellipticity, this property may be extended to the case of large differential inequalities. From now on we suppose that the dimension $n \ge 2$.

Theorem 1.1.1 (**Maximum Principle, first version**). *Let Ω be an open connected subset of $\mathbb{R}^n$ and $u \in \mathcal{C}^2(\Omega)$ satisfy the differential inequality*

$$Lu = \sum_{i,j} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial u}{\partial x_i} \ge 0 \tag{1.4}$$

in Ω, where L is a uniformly elliptic differential operator in Ω with uniformly bounded coefficients $a_{i,j}$, b_i. Then, if there exist $M \in \mathbb{R}$, $P \in \Omega$ such that $u \le M$ in Ω and $u(P) = M$, one has

$$u \equiv M \text{ in } \Omega.$$

Proof. By contradiction, suppose that $P, Q \in \Omega$ satisfy the property $u(P) = M > u(Q)$. Then we can find a continuous path $\gamma : [0,1] \to \Omega$ such that

$$\gamma(0) = Q \qquad\qquad \gamma(1) = P.$$

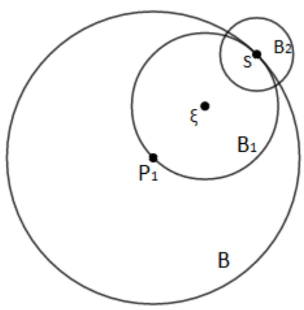
Denoted by $R = \gamma(t_R)$ the first point in which $u(R) = M$, it's clear that

$$u(\gamma(t)) < M, \text{ for any } 0 \le t < t_R.$$

Let $d := \operatorname{dist}(\gamma, \partial\Omega)$ and pick $P_1 = \gamma(t)$, for some $0 < t < t_R$, such that $|P_1 - R| < d/2$: we can consider the biggest open ball B centered at P_1 in which $u < M$. Such a ball needs to have a radius strictly smaller than $d/2$, and then to be contained in Ω. Let $S \in \partial B$ be a point such that $u(S) = M$, and denote by B_1 the only ball tangent to ∂B at S (it's the only ball B_1 with the properties $S \in \partial B_1$, $\overline{B_1} \setminus \{S\} \subset B$): we note that

$$u < M \text{ in } \overline{B_1} \setminus \{S\}.$$

Denoted by r_1 the radius of B_1, let B_2 be the ball of center S and radius $r_2 = r_1/2$.



Consider

$$C_2' := \partial B_2 \cap \overline{B_1} \qquad\qquad C_2'' := \partial B_2 \setminus C_2'.$$

Being $u < M$ on C_2', by compactness of C_2', it needs to exist a value $\zeta > 0$ such that

$$u \le M - \zeta, \text{ on } C_2'.$$

Moreover, $u \le M$ on C_2''. Let $\xi = (\xi_1, ..., \xi_n)$ be the center of B_1 and consider the following function

$$z(x) \equiv e^{-\alpha \sum_{i=1}^n (x_i - \xi_i)^2} - e^{-\alpha r_1^2} \tag{1.5}$$

where $\alpha > 0$ has to be suitably defined. We note that

$$z > 0 \text{ in } B_1$$
$$z = 0 \text{ on } \partial B_1$$
$$z < 0 \text{ elsewhere.}$$

Now

$$\begin{aligned}
Lz &= e^{-\alpha \sum_{k=1}^n (x_k-\xi_k)^2}\Big[4\alpha^2 \sum_{i,j=1}^n a_{i,j}(x_i - \xi_i)(x_j - \xi_j) + \\
&- 2\alpha(a_{i,i} + b_i(x_i - \xi_i))\Big] \ge (\text{by uniform ellipticity and } \sum_{k=1}^n (x_k - \xi_k)^2 \ge r_1^2/4) \\
&\ge \alpha e^{-\alpha \sum_{k=1}^n (x_k-\xi_k)^2}\Big[\alpha\mu_0 r_1^2 - 2\sum_{i=1}^n (a_{i,i} + b_i(x_i - \xi_i))\Big] \\
&> 0 \text{ in } B_2,
\end{aligned}$$

up to a choice of a enough big α. Define $w \equiv u + \epsilon z$, $0 < \epsilon < \frac{\zeta}{1-e^{-\alpha r_1^2}}$. We have:

(1) $w < M$ on C_2' (because $0 \le z \le 1 - e^{-\alpha r_1^2}$ and so $\epsilon z < \zeta$, $w = u + \epsilon z < u + \zeta \le M$);

(2) $w < M$ on C_2'' (because $z < 0$ on C_2'', and $u \le 0$ everywhere);

(3) $w = M$ at S (being $z(S) = 0$).

These three observations imply that w has a maximum in B_2, and we also know that

$$Lw = Lu + \epsilon Lz > 0 \text{ in } B_2.$$

This is a contradiction, thanks to the computation performed before. **QED**

The result just shown can be extended as follows

Theorem 1.1.2 (Maximum Principle, second version). *Let Ω be an open connected subset of $\mathbb{R}^n$ and $u \in \mathcal{C}^2(\Omega)$ satisfy the differential inequality*

$$(L + h)u = \sum_{i,j} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial u}{\partial x_i} + hu \ge 0 \tag{1.6}$$

in Ω, where $L + h$ is a uniformly elliptic differential operator in Ω with uniformly bounded coefficients $a_{i,j}$, b_i, h and $h \le 0$. Then, if there exist $M \ge 0$, $P \in \Omega$ such that $u \le M$ in Ω and $u(P) = M$, one has

$$u \equiv M \text{ in } \Omega.$$

The proof is the same, up to the following straightforward remark: if $(L + h)u > 0$, u cannot attain a nonnegative local maximum in Ω.

Attention! The thesis turns out to be false if $h \nleq 0$: for example $u(x) \equiv e^{-|x|^2}$ solves

$$\Delta u + (2 - 4|x|^2)u = 0 \text{ in } \mathbb{R}^n$$

but has a maximum at the origin.

1.1.2 Hopf Lemmas

Let Ω be an open subset of $\mathbb{R}^n$ and assume that $u : D \to \mathbb{R}$, $D \supset \Omega$, satisfies the differential inequality (1.4). Let $P \in \partial\Omega \cap D$ and suppose that u is continuous at P, $u(P) = \sup_\Omega u = \max_{\Omega\cup\{P\}} u$. It's intuitively clear that any directional derivative of u with respect to a direction pointing to the outside of Ω has to be nonnegative. Actually something stronger holds.

Definition 1.1.2. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $u : \Omega \to \mathbb{R}$ be a function which admits first partial derivatives in Ω. Given $P \in \partial\Omega$ and a vector $\nu \in \mathbb{R}^n$, we say that u is *derivable in the direction ν at P* provided that the limit

$$\lim_{\substack{x\to P\\ x\in\Omega}} \nu \cdot \nabla u(x) := \frac{\partial u}{\partial \nu}(P)$$

exists in $\mathbb{R}$.

If ν points to the outside of Ω and u is derivable in the direction ν at P, $\dfrac{\partial u}{\partial \nu}(P)$ is also called *outer derivative of u in the direction ν at P*. The following result aims to formalize the heuristic observation done at the beginning of the subsection.

Lemma 1.1.3. Let Ω be an open subset in $\mathbb{R}^n$ and, given $P \in \partial\Omega$, suppose that there exists a coordinate cylinder [1] $C \equiv C(P, R, r, \delta)$ for Ω around P such that the map

$$\gamma : B_{\mathbb{R}^{n-1}}(0, r] \to]-\delta, \delta[$$

representing $\partial\Omega$ in C is differentiable at 0. Let $\nu \in \mathbb{R}^n$ satisfy $\nu \cdot n > 0$ where

$$n \equiv R^T\left(-\nabla\gamma(0), 1\right)^T \frac{1}{\sqrt{1 + |\nabla\gamma(0)|^2}}$$

is the outer normal to $\partial\Omega$ at P. Then ν points to the outside of Ω and, if $u : D \to \mathbb{R}$, $D \supset \Omega$, admits first order partial derivatives in Ω, $P \in \partial\Omega \cap D$, u is continuous at P and

$$u(P) = \sup_\Omega u = \max_{\Omega\cup\{P\}} u,$$

then $\dfrac{\partial u}{\partial \nu}(P) \geq 0$.

Proof. For $\epsilon > 0$ enough small, we have that $P - t\nu \in \Omega$, for any $0 < t < \epsilon$, because $-\nu \cdot n < 0$ and thus $-\nu$ points to the interior of Ω. Consider the function

$$]0, \epsilon[\ni t \to v(t) \equiv u(P - t\nu).$$

By contradiction, let $\dfrac{\partial u}{\partial \nu}(P) < 0$. We have $v(0) \geq v(t)$, for any $0 < t < \epsilon$, v is derivable and

$$v'(t) = -\nabla(P - t\nu) \cdot \nu$$

and $\lim_{t\to 0^+} -\nabla(P - t\nu) \cdot \nu = -\dfrac{\partial u}{\partial \nu}(P) > 0$. So, up to a restriction of ϵ, we can assume that $v' > 0$ on $0 < t < \epsilon$. Therefore

$$u(P) = v(0) = \lim_{t\to 0^+} v(t) \leq v(\epsilon/4) < v(\epsilon/2) = u(P - \frac{\epsilon}{2}\nu),$$

and this is a contradiction. **QED**

[1] Given an orthogonal matrix R and $r, \delta > 0$, we say that $C(P, R, r, \delta) := P + R^T(B_{\mathbb{R}^{n-1}}(0, r[\times] - \delta, \delta[)$ is a *coordinate cylinder* for an open Ω around a point $P \in \partial\Omega$ provided that there exists a map $\gamma : B_{\mathbb{R}^{n-1}}(0, r[\to] - \delta, \delta[$ such that $\gamma(0) = 0$, $\gamma < \delta/2$ and

$$R(\Omega - P) \cap (B_{\mathbb{R}^{n-1}}(0, r[\times] - \delta, \delta[) = \{(\eta, y) \in B_{\mathbb{R}^{n-1}}(0, r[\times] - \delta, \delta[\text{ s.t } y < \gamma(\eta)\}.$$

We say that γ *represents* $\partial\Omega$ in the coordinate cylinder.

We now want to state and prove a boosted version of this result, valid for solutions to (1.4). To do so, we need to impose a well known regularity condition.

Definition 1.1.3. Let Ω be an open subset of $\mathbb{R}^n$. We say that Ω satisfies the *interior sphere condition at a point* $x_0 \in \partial\Omega$ if there exist a point $x \in \Omega$ and a radius $r > 0$ such that

$$x_0 \in \partial B(x, r[, \; B(x, r] \setminus \{x_0\} \subset \Omega.$$

Theorem 1.1.3 (**Hopf Lemma, first version**). *Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $u : D \to \mathbb{R}$, $D \supset \Omega$, be a solution to* (1.4)*, where the coefficients $a_{i,j}$, b_i of the operator L are assumed uniformly bounded. Suppose that there exist $M \in \mathbb{R}$ and $P \in \partial\Omega \cap D$ such that $u \leq M$ in Ω, u is continuous at P and $u(P) = M$. Then, if Ω respects the interior sphere condition at P and u admits outer partial derivative at P with respect to a direction $\nu \in \mathbb{R}^n$, one has*

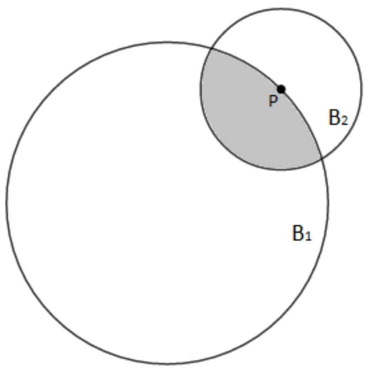
$$\frac{\partial u}{\partial \nu}(P) > 0,$$

unless u is constant on the connected component whose boundary contains P.

Proof. Let B_1 be an open ball such that $P \in \partial B_1$, $\overline{B_1} \setminus \{P\} \subset \Omega$. Set $r_1 > 0$ the radius of B_1, and consider the ball centered at P and of radius $r_2 = r_1/2$. Consider the map z defined at (1.5) and pick again an $\alpha > 0$ in order that $Lz > 0$. We can consider

$$w \equiv u + \epsilon z:$$

thanks to Maximum Principle 1.1.1, if u is nonidentically M in the connected component, then $u < M$ in B_1 and on $\partial B_1 \setminus \{P\}$. Let's choose $\epsilon > 0$ small enough in order that $w \leq M$ on $\partial B_2 \cap \overline{B_1}$: then $w \leq M$ on the boundary of the grey region in the figure below.



Being $Lw > 0$ in this region, the maximum needs to be attained on the boundary, and so necessarily at P (because $w(P) = M$). Therefore

$$\frac{\partial w}{\partial \nu}(P) = \frac{\partial u}{\partial \nu}(P) + \epsilon \frac{\partial z}{\partial \nu}(P) \geq 0,$$

where, observed that ν points to the outside of B_1, the last inequality is a direct consequence of Lemma 1.1.3. In order to conclude, it sufficies to show that $\frac{\partial z}{\partial \nu}(P) < 0$: this fact follows immediately using the definition of z and the fact that $n \cdot \nu > 0$, where n is the outer, unit normal to ∂B_1 at P. **QED**

Like the Maximum Principle, also the Hopf Lemma can be generalized.

Theorem 1.1.4 (**Hopf Lemma, second version**). *Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $u : D \to \mathbb{R}$, $D \supset \Omega$, be a solution to* (1.6)*, where the coefficients $a_{i,j}$, b_i, h of the operator $L+h$ are uniformly bounded and $h \leq 0$. Suppose that there exist $M \geq 0$ and $P \in \partial\Omega \cap D$ such that $u \leq M$ in Ω, u is continuous at P and $u(P) = M$. Then, if Ω respects the interior sphere condition at P and u admits outer partial derivative at P with respect to a direction $\nu \in \mathbb{R}^n$, one has*

$$\frac{\partial u}{\partial \nu}(P) > 0,$$

unless u is constant on the connected component whose boundary contains P.

Again, omitting the assumption about the nonpositivity of h, Hopf Lemma fails to hold: to see this, it sufficies to consider the same counterexample exploited in the past section, set $P = 0$ and vary suitably the domain of definition of the function (picking for example $\Omega \equiv \{x_n > 0\}$).

1.1.3 A slight generalization of the Hopf Lemma and the Maximum Principle

In the second chapter we will need a Maximum Principle and a Hopf Lemma holding for uniformly elliptic operators with arbitrary zero order term: in this sense, a slight generalization to this case can be done, as we show in this section. It's anyway clear that the generality of the previous versions of the Maximum Principle and the Hopf Lemma will be necessarily lost (recall the counterexample used in the second section). Let the hypotheses of Theorem 1.1.2 be in force, up to the assumption $h \leq 0$, and suppose additionally that $M = 0$. Up to translations, let $P = 0$, and set

$$\begin{aligned} &\widetilde{L} \equiv L + h \\ &v \equiv e^{-\alpha x_1} u, \ \alpha > 0 \text{ constant to be decided.} \end{aligned} \tag{1.7}$$

We note that, setting

$$L_0 \equiv \sum_{i,j} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i (b_i + 2\alpha a_{i,1}) \frac{\partial}{\partial x_i},$$

one has

$$\begin{aligned} 0 \leq \widetilde{L}u &= e^{\alpha x_1} L_0 v + \widetilde{L}(e^{\alpha x_1}) v = \\ &= e^{\alpha x_1} L_0 v + e^{\alpha x_1} v (a_{1,1}\alpha^2 + b_1 \alpha + h), \end{aligned}$$

and so $0 \leq L_0 v + v(a_{1,1}\alpha^2 + b_1\alpha + h)$. Set $g_\alpha \equiv a_{1,1}\alpha^2 + b_1\alpha + h$: choosing α big enough, g_α is nonnegative, and thus $L_0 v \geq 0$, because $v \leq 0$. If $u(P) = 0$, then $v(P) = 0$ and thus the first version of the Maximum Principle allows us to deduce that $v \equiv 0$, namely $u \equiv 0$.

Similarly, skipping the assumption about the nonpositivity of h in Theorem 1.1.4, let $M = 0$, $P = 0$ (without loss of generality) and define v like we did in (1.7). The computations just made ensure that $\frac{\partial v}{\partial \nu}(P) > 0$, for any outer direction ν. Therefore, being

$$\frac{\partial u}{\partial \nu}(P) = \frac{\partial v}{\partial \nu}(P),$$

we conclude that $\frac{\partial u}{\partial \nu}(P) > 0$, unless u (or equivalently v) is identically zero. Thus we have proved the following third version of the Maximum Principle.

Theorem 1.1.5 (**Maximum Principle, third version**). *Let Ω be an open connected subset of $\mathbb{R}^n$ and $u \in \mathcal{C}^2(\Omega)$ satisfy the differential inequality* (1.6) *in Ω, where the coefficients $a_{i,j}$, b_i, h of the operator $L+h$ are assumed uniformly bounded. Then, if $u \leq 0$ in Ω and $u(P) = 0$, for some $P \in \Omega$, one has*

$$u \equiv 0 \text{ in } \Omega.$$

We have also a third version of the Hopf Lemma.

Theorem 1.1.6 (**Hopf Lemma, third version**). *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $u : D \to \mathbb{R}$, $D \supset \Omega$, be a solution to* (1.6)*, where the coefficients $a_{i,j}$, b_i, h of the operator $L + h$ are assumed uniformly bounded. Suppose that $u \leq 0$ in Ω, and there exists $P \in \partial\Omega \cap D$ such that u is continuous at P and $u(P) = 0$. Then, if Ω respects the interior sphere condition at P and u admits outer partial derivative at P with respect to a direction $\nu \in \mathbb{R}^n$, one has*

$$\frac{\partial u}{\partial \nu}(P) > 0,$$

unless u is constant on the connected component whose boundary contains P.

1.2 Remarks about the notions of solution and weak solution

Definition 1.2.1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open subset and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function. Consider the equation

$$-\Delta u = f(x, u) \text{ in } \Omega. \tag{1.8}$$

We say that a map $u \in \mathcal{C}^2(\Omega)$ is a:

(i) *subsolution* to equation (1.8) in Ω provided that

$$-\Delta u \leq f(x, u) \text{ in } \Omega;$$

(ii) *supersolution* to equation (1.8) in Ω provided that

$$-\Delta u \geq f(x, u) \text{ in } \Omega;$$

(iii) *solution* to equation (1.8) in Ω if it's both a subsolution and a supersolution, namely if

$$-\Delta u = f(x, u) \text{ in } \Omega.$$

We now aim to weaken the notions given above. From now on, we denote by $\mathcal{C}^k_c(\Omega)_{\geq 0}$, $k \geq 0$, the class of those maps of class $\mathcal{C}^k$ compactly supported in Ω and nonnegative there.

With respect to the setting of Definition 1.2.1, let u be a subsolution (resp. supersolution, solution) to (1.8). Then, for any $\zeta \in \mathcal{C}^\infty_c(\Omega)_{\geq 0}$, one has:

$$\int_\Omega (\Delta u + f(x, u))\zeta \,\mathrm{d}x \geq 0 \ (\leq 0, = 0). \tag{1.9}$$

A straightforward computation guarantees that, for any $\phi \in \mathcal{C}^1(\Omega), \psi \in \mathcal{C}^2(\Omega)$, the following identity holds

$$\operatorname{div}(\phi \nabla \psi) = \nabla \phi \cdot \nabla \psi + \phi \Delta \psi. \tag{1.10}$$

Thus $\zeta \Delta u = u \Delta \zeta + \operatorname{div}(\zeta \nabla u) - \operatorname{div}(u \nabla \zeta)$ (use the relation (1.10) first with $\phi = \zeta$, $\psi = u$, then with $\phi = u$, $\psi = \zeta$ and finally subtract the second identity to the first one). Applying the Divergence Theorem to $\zeta \nabla u$ and $u \nabla \zeta$ and observing that these two vector fields are both compactly supported in Ω, we infer

$$\int_\Omega \zeta \Delta u \,\mathrm{d}x = \int_\Omega u \Delta \zeta \,\mathrm{d}x.$$

Then, by (1.9), we deduce that

$$\int_\Omega (u \Delta \zeta + f(x, u)\zeta) \,\mathrm{d}x \geq 0 \ (\leq 0, = 0). \tag{1.11}$$

In particular, (1.9) holds true if and only if (1.11) holds true: the asset given by the second formulation is that does not require any regularity condition on u, and then can be exploited to generalize the notion of subsolution (resp. supersolution, solution). We have then justified the following

Definition 1.2.2. A map $u \in L^1_{loc}(\Omega)$ such that $f(\cdot, u(\cdot)) \in L^1_{loc}(\Omega)$ is said to be a:

(i) *weak subsolution* to equation (1.8) in Ω if, for every $\zeta \in \mathcal{C}^\infty_c(\Omega)_{\geq 0}$,

$$\int_\Omega (u\Delta\zeta + f(x,u)\zeta)\,\mathrm{d}x \geq 0;$$

(ii) *weak supersolution* to equation (1.8) in Ω if, for every $\zeta \in \mathcal{C}^\infty_c(\Omega)_{\geq 0}$,

$$\int_\Omega (u\Delta\zeta + f(x,u)\zeta)\,\mathrm{d}x \leq 0;$$

(iii) *solution* to equation (1.8) in Ω if, for every $\zeta \in \mathcal{C}^\infty_c(\Omega)_{\geq 0}$,

$$\int_\Omega (u\Delta\zeta + f(x,u)\zeta)\,\mathrm{d}x = 0.$$

Note that nothing changes substituting $\mathcal{C}^\infty_c(\Omega)_{\geq 0}$ with $\mathcal{C}^k_c(\Omega)_{\geq 0}$, $k \geq 0$, or even with $\mathcal{C}^k_c(\Omega)$ in (iii). In addition, the Fundamental Lemma of the Calculus of Variations ensures that, if $f(\cdot, u(\cdot))$ is continuous in Ω, a function u of $\mathcal{C}^2$ class is a subsolution (resp. supersolution, solution) in Ω if and only if it's a weak subsolution (resp. supersolution, solution) in Ω.

If $f \equiv 0$ we adopt a more specific terminology.

Definition 1.2.3. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open subset. A map $u \in \mathcal{C}^2(\Omega)$ is said to be *subharmonic* (resp. *superharmonic*, *harmonic*) in Ω provided that it is a subsolution (resp. supersolution, solution) to the Laplace Equation

$$-\Delta u = 0 \text{ in } \Omega. \tag{1.12}$$

Definition 1.2.4. Let Ω in $\mathbb{R}^n$, $n \geq 2$, be an open subset. A map $u \in L^1_{loc}(\Omega)$ is said to be *weakly subharmonic* (resp. *weakly superharmonic*, *weakly harmonic*) in Ω provided that it is a weak subsolution (resp. weak supersolution, weak solution) to equation (1.12).

Even if we will not use these facts, it's anyway better to recall that:

(a) harmonic functions are of class $\mathcal{C}^\infty$;

(b) any weakly harmonic map coincides a.e. with a harmonic map (Weyl's Lemma).

Let's start giving two properties which will turn out to be useful.

Lemma 1.2.1. The following statements hold:

(i) if u is a (weak) subsolution to equation (1.8) in Ω with $f \leq 0$, then u is (weakly) subharmonic;

(ii) if u is a (weak) supersolution to equation (1.8) in Ω with $f \geq 0$, then u is (weakly) superharmonic.

The following Lemma will be often used along the third chapter in the particular case in which ϕ is the reflection in a hyperplane like $\gamma \cdot x = \lambda$, for some unit vector γ.

Lemma 1.2.2. Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$, $n \geq 2$, be two open subsets and $f : (\Omega \cup \widetilde{\Omega}) \times \mathbb{R} \to \mathbb{R}$ be a function. Suppose that $u \in L^1_{loc}(\Omega \cup \widetilde{\Omega})$ is a weak subsolution to

$$-\Delta u = f(x,u), \text{ in } \Omega$$

and a weak supersolution to

$$-\Delta u = f(x,u), \text{ in } \widetilde{\Omega}.$$

Let $\phi \in \mathcal{C}^2(\Omega, \widetilde{\Omega})$ be a diffeomorphism such that $\mathrm{Jac}(\phi)^2$ coicides with the identical matrix, $\mathrm{div}(e_i \cdot \mathrm{Jac}(\phi)) \equiv 0$, for any i, and assume that

$$f(\phi(x), u(\phi(x))) \geq f(x, u(x)), \text{ for a.e. } x \in \Omega.$$

Then $u \circ \phi - u$ is a weakly superharmonic map in Ω.

Proof. For any $\zeta \in \mathcal{C}_c^\infty(\Omega)_{\geq 0}$, we have

$$\int_\Omega (u\Delta\zeta + f(x,u)\zeta)\,\mathrm{d}x \geq 0.$$

Furthermore

$$\int_\Omega (u(\phi(x))\Delta\zeta(x) + f(\phi(x), u(\phi(x)))\zeta(x))\,\mathrm{d}x = \int_{\widetilde{\Omega}} (u(y)\Delta\zeta(\phi^{-1}(y)) + f(y, u(y))\zeta(\phi^{-1}(y)))\,\mathrm{d}y \leq 0,$$

because the conditions on ϕ ensure that $\Delta\zeta(\phi^{-1}(y)) = \Delta\widetilde{\zeta}(y)$, where we set $\widetilde{\zeta}(y) \equiv \zeta(\phi^{-1}(y))$, $\widetilde{\zeta} \in \mathcal{C}_c^\infty(\widetilde{\Omega})_{\geq 0}$. Therefore

$$\int_\Omega ((u(\phi(x)) - u(x))\Delta\zeta(x))\,\mathrm{d}x \leq \int_\Omega ((u(\phi(x)) - u(x))\Delta\zeta(x) + (f(\phi(x), u(\phi(x))) - f(x, u(x))))\,\mathrm{d}x \leq 0,$$

and we conclude. **QED**

The following result and the further Corollary stated below are very important: in fact, developing the reflection technique, they permit to overcome the problem of the presence of the singularity at the origin using a weak notion of solution.

Lemma 1.2.3. Let $u \in \mathcal{C}^2(B(0,2[\setminus\{0\})$, $u \geq 0$, solve the following equation

$$-\Delta u = g(u) \text{ in } B(0,2[\setminus\{0\} \tag{1.13}$$

in dimension $n \geq 3$. Assume that:

(i) $g(t) \geq 0$, as $t \geq 0$;

(ii) $\liminf_{t\to\infty} \dfrac{g(t)}{t^p} > 0$, for some $p \geq \dfrac{n}{n-2}$.

Then $u \in L^p(B(0,1[)$, $g(u) \in L^1(B(0,1[)$ and u is a weak solution to equation (1.13) in $B(0,1[$.

Proof. Let $k > \max_{\partial B(0,1[} u$, and pick a nonincreasing map $\phi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\phi(t) = \begin{cases} 1 & \text{if } t < k \\ 0 & \text{if } t \geq 2k \end{cases}$$

and set $\Phi(t) := \int_0^t \phi(\tau)\,\mathrm{d}\tau$. Moreover, fixed $0 < \epsilon < 1/2$ arbitrarily, let $\eta \equiv \eta(|x|)$ be a radial function such that

$$\eta(r) = \begin{cases} 0 & \text{if } 0 \leq r < \epsilon \\ 1 & \text{if } r \geq 2\epsilon. \end{cases} \tag{1.14}$$

We may consider $\psi(x) \equiv \phi(u(x))\eta(|x|)$, as $x \in B(0,2[$, because η is identically zero in $B(0,\epsilon[$, and then we have

$$\int_{B(0,1[} \nabla u \cdot \nabla\psi\,\mathrm{d}x = \int_{B(0,1[} \mathrm{div}(\psi\nabla u)\,\mathrm{d}x - \int_{B(0,1[} \psi\Delta u\,\mathrm{d}x = \int_{B(0,1[} \psi g(u)\,\mathrm{d}x + \int_{\partial B(0,1[} \frac{\partial u}{\partial\nu} d\sigma.$$

Now, observing that $\phi' \leq 0$ and $\nabla\eta \equiv 0$ out of $B(0,1[$, one has:

$$\int_{B(0,1[} \nabla u \cdot \nabla\psi \,\mathrm{d}x = \int_{B(0,1[} \eta\phi'(u)|\nabla u|^2 \,\mathrm{d}x + \int_{B(0,1[} \nabla\Phi(u)\cdot\nabla\eta \,\mathrm{d}x \leq$$
$$\leq \int_{\partial B(0,1[} \Phi(u)\nabla\eta\cdot\hat{n} d\sigma - \int_{B(0,1[} \Phi(u)\Delta\eta \,\mathrm{d}x = -\int_{B(0,1[} \Phi(u)\Delta\eta \,\mathrm{d}x = \mathcal{O}(\epsilon^{n-2}) \text{ as } \epsilon \to 0^+,$$

up to a good choice of the map η. Then

$$\int_{B(0,1[} \eta\phi(u)g(u) \,\mathrm{d}x + \int_{\partial B(0,1[} \frac{\partial u}{\partial \nu} d\sigma = \mathcal{O}(\epsilon^{n-2})$$

and letting $\epsilon \to 0^+$ we get

$$\int_{B(0,1[} \phi(u)g(u) \,\mathrm{d}x \leq -\int_{\partial B(0,1[} \frac{\partial u}{\partial \nu} d\sigma;$$

in particular being $\phi(u) \geq 0$, $g(u) \geq 0$, we have

$$\int_{B(0,1[\cap\{u<k\}} g(u) \,\mathrm{d}x \leq -\int_{\partial B(0,1[} \frac{\partial u}{\partial \nu} d\sigma$$

Letting $k \to \infty$ we infer that $g(u)$ is $L^1(B(0,1[)$. For u, we consider two cases:

(a) u is bounded in $B(0,1[$: in this case $u \in L^p(B(0,1[)$ needs to hold;

(b) u is not bounded in $B(0,1[$: we note that assumption (ii) guarantees that, as $M > 0$ is big enough, one has

$$\inf_{s\geq M} \frac{g(s)}{s^p} \in]\frac{l}{2}, \frac{3}{2}l[,$$

where $l := \liminf_{t\to\infty} \dfrac{g(t)}{t^p} > 0$. Then, as $u(x) \geq M$,

$$\frac{g(u(x))}{u(x)^p} \geq \inf_{\{u\geq M\}} \frac{g(u)}{u^p} \geq \inf_{s\geq M} \frac{g(s)}{s^p} \geq \frac{l}{2},$$

and then $u(x)^p \leq \dfrac{2}{l} g(u(x))$. Thus

$$\int_{B(0,1[} u(x)^p \,\mathrm{d}x = \int_{\{u<M\}\cap B(0,1[} u(x)^p \,\mathrm{d}x + \int_{\{u\geq M\}\cap B(0,1[} u(x)^p \,\mathrm{d}x \leq M^p\omega_n + \frac{2}{l}\int_{B(0,1[} g(u) \,\mathrm{d}x < \infty.$$

In order to conclude, it sufficies to demonstrate that

$$\int_{B(0,1[} u\Delta\zeta + g(u)\zeta \,\mathrm{d}x = 0, \text{ for any } \zeta \in \mathcal{C}_c^\infty(B(0,1[).$$

By hypoteses, u is a classical solution on $B(0,2[\setminus\{0\}$ and then is a classical (hence weak) solution on $B(0,1[\setminus\{0\}$. We immediately note that $\eta\zeta \in \mathcal{C}_c^\infty(B(0,1[\setminus\{0\})$, hence

$$0 = \int_{B(0,1[} u\Delta(\eta\zeta) + g(u)\eta\zeta \,\mathrm{d}x = \int_{B(0,1[} \eta(u\Delta\zeta + g(u)\zeta) \,\mathrm{d}x + \int_{B(0,1[} u(\zeta\Delta\eta + 2\nabla\eta\cdot\nabla\zeta) \,\mathrm{d}x.$$

Thus we infer:

$$\Big|\int_{B(0,1[} \eta(u\Delta\zeta + g(u)\zeta) \,\mathrm{d}x\Big| \leq \int_{B(0,1[} u|\zeta\Delta\eta + 2\nabla\eta\cdot\nabla\zeta| \,\mathrm{d}x \leq \frac{C}{\epsilon^2}\int_{B(0,2\epsilon[\setminus B(0,\epsilon[} u \,\mathrm{d}x \leq$$
$$\leq \frac{\widetilde{C}}{\epsilon^2}\,\epsilon^{n(1-\frac{1}{p})}\,\|u\|_{L^p(B(0,2\epsilon[)} = \mathcal{O}(\epsilon^{n-2-\frac{n}{p}})\|u\|_{L^p(B(0,2\epsilon[)} \text{ as } \epsilon \to 0^+,$$

and we conclude the proof. **QED**

Observe that actually u is a weak solution in $B(0,2[$ whole. To prove so, we can argue in the following way: first we observe that u is of class $\mathcal{C}^2$ in the open $B(0,2[\smallsetminus B(0,1]$, and $g(u)$ is necessarily continuous there (being Δu continuous), so surely u and $g(u)$ are both locally integrable in the open annulus $B(0,2[\smallsetminus B(0,1]$, and then are locally integrable in $B(0,2[$ whole. Then let $\zeta \in \mathcal{C}_c^\infty(B(0,2[)$, and suppose that, as $\epsilon > 0$, $\rho_\epsilon \in \mathcal{C}^\infty(B(0,2[)$ is a radial function such that $0 \le \rho_\epsilon \le 1$ and

$$\begin{aligned}&\rho_\epsilon \equiv 1 \text{ in } B(0,1-\epsilon] \cup (B(0,2[\smallsetminus B(0,1+\epsilon[)\\ &\rho_\epsilon \equiv 0 \text{ in } B(0,1+\epsilon/2] \smallsetminus B(0,1-\epsilon/2[\\ &|\nabla\rho_\epsilon| \le C/\epsilon,\ |\Delta\rho_\epsilon| \le C/\epsilon^2.\end{aligned}$$

Then, for any $\epsilon > 0$, by the fact that u is a (classical) solution in $B(0,2[\smallsetminus B(0,1[$ and a weak solution in $B(0,1[$,

$$\begin{aligned}0 &= \int_{B(0,2[} (u\Delta(\rho_\epsilon\zeta) + g(u)(\rho_\epsilon\zeta))\,\mathrm{d}x =\\ &= \int_{B(0,2[} \rho_\epsilon(u\Delta\zeta + g(u)\zeta)\,\mathrm{d}x + \int_{B(0,1+\epsilon]\smallsetminus B(0,1-\epsilon[} (2u\nabla\zeta\cdot\nabla\rho_\epsilon + u\zeta\Delta\rho_\epsilon)\,\mathrm{d}x.\end{aligned}$$

We note that the first term converges to $\int_{B(0,2[}(u\Delta\zeta + g(u)\zeta)\,\mathrm{d}x$ as $\epsilon \to 0^+$ and the second one is bounded by $C(u,\zeta)(\epsilon^{n-1} + \epsilon^{n-2})$ in modulus, and so shrinks to 0 as $\epsilon \to 0^+$. This is enough to conclude that u weakly solves the equation in the open ball of radius 2 too. So we've proved the following, stronger result.

Corollary 1.2.1. Let $u \in \mathcal{C}^2(B(0,1[\smallsetminus\{0\})$, $u \ge 0$ solve the following equation

$$-\Delta u = g(u) \text{ in } B(0,1[\smallsetminus\{0\} \tag{1.15}$$

in dimension $n \ge 3$. Assume that:

(i) $g(t) \ge 0$, as $t \ge 0$;

(ii) $\liminf_{t\to\infty} \dfrac{g(t)}{t^p} > 0$, for some $p \ge \dfrac{n}{n-2}$.

Then $u \in L^1_{loc}(B(0,1[)$, $g(u) \in L^1_{loc}(B(0,1[)$, and u weakly solves equation (1.15) (in $B(0,1[)$.

1.3 A Maximum Principle for weakly subharmonic functions

In this section we demonstrate a Maximum Principle for weakly subharmonic functions. This result will be used in the proof of the Reflection Theorem: we will exploit the whole power of this Maximum Principle, in particular the fact that it doesn't require the continuity of the weakly subharmonic function. We start with the following sort of Comparison Principle for weakly subharmonic maps.

Lemma 1.3.1. Let u be a weakly subharmonic and upper semicontinuous function in a open Ω, $x_0 \in \Omega$ and $r > 0$ be such that $B(x_0,r] \subset \Omega$. Then

$$u(x_0) \le \fint_{\partial B(x_0,r[} u\,\mathrm{d}\sigma$$

Proof. We give the proof only in a easier setting, i.e. assuming u continuous. Let ϕ be a mollifier and, for any $\epsilon > 0$, consider

$$u_\epsilon(x) := (u\chi_\Omega) * \phi_\epsilon(x) = \int_\Omega u(y)\phi_\epsilon(x-y)\,\mathrm{d}y.$$

We have $\|u_\epsilon - u\|_{L^\infty(B(x_0,r])} \to 0$ as $\epsilon \to 0^+$. The weak subharmonicity of u ensures that

$$-\Delta u_\epsilon(\cdot) = -\int_\Omega (u(y)\Delta\phi_\epsilon(\cdot - y))\,\mathrm{d}y \le 0,$$

and then that u_ϵ is subharmonic. For $0 < \rho \le r$, let

$$\Phi(\rho) \equiv \fint_{\partial B(x_0,\rho[} u_\epsilon(x)\,\mathrm{d}\sigma(x) = \fint_{\partial B(0,1[} u_\epsilon(x_0 + \rho y)\,\mathrm{d}\sigma(y):$$

Φ is continuous and, as $0 < \rho < r$, by the Divergence Theorem,

$$\begin{aligned}\Phi'(\rho) &= \fint_{\partial B(0,1[} \nabla u_\epsilon(x_0 + \rho y)\cdot y\,\mathrm{d}\sigma(y)\\ &= \fint_{\partial B(x_0,\rho[} \nabla u_\epsilon(x)\cdot \frac{x - x_0}{\rho}\,\mathrm{d}\sigma(x)\\ &= \fint_{B(x_0,\rho[} \Delta u_\epsilon(x)\,\mathrm{d}x \ge 0\end{aligned}$$

Then $u_\epsilon(x_0) = \lim_{\rho\to 0^+}\Phi(\rho) \le \lim_{\rho\to r^-}\Phi(\rho) = \fint_{\partial B(x_0,r[} u_\epsilon\,\mathrm{d}\sigma$ and thus, given $\delta > 0$, picking $\epsilon > 0$ small enough,

$$u(x_0) \le u_\epsilon(x_0) + \delta \le \fint_{\partial B(x_0,r[} u_\epsilon\,\mathrm{d}\sigma + \delta \le \fint_{\partial B(x_0,r[} u\,\mathrm{d}\sigma + 2\delta,$$

and the arbitrariness of δ allows us to conclude. **QED**

Clearly a dual version of the result above can be stated for u weakly superharmonic, observing that $-u$ is weakly subharmonic.

The proof of the general statement is less easy, and requires a precise study of the Lebesgue set of a weakly subharmonic function: in other words, one first proves that Lemma 1.3.1 holds true under the further assumption that the Lebesgue set of u is Ω whole, and then shows that the Lebesgue set of a weakly subharmonic function coincides exactly with Ω. All this machinery is exhaustively exposed in [9]: in particular Lemma 1.3.1 turns out to be an immediate consequence of Theorem 4.3 and Theorem 1.2 of that work used in tandem.

We are finally ready to demonstrate the following

Theorem 1.3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be an open, connected subset and u be a weakly subharmonic, upper semicontinuous map in Ω. If $M \in \mathbb{R}$ is such that $u \le M$ in Ω, and there exists a point $x_0 \in \Omega$ in which $u(x_0) = M$, then u is identically M in Ω.*

Proof. Let $r > 0$ be such that $B(x_0, r] \subset \Omega$. Then

$$M = u(x_0) \le \fint_{\partial B(x_0,r[} u\,\mathrm{d}\sigma,$$

and thus u has to be identically M on $\partial B(x_0, r[$: as a matter of fact, by upper semicontinuity of u, if there exists $x^* \in \partial B(x_0, r[$ such that $u(x^*) < M$, then there is a neighbourhood U of x^* in $\partial B(x_0, r[$ such that as $x \in U$ one has $u(x) < (u(x^*) + M)/2$. Thus

$$\begin{aligned}M = u(x_0) &\le \fint_{\partial B(x_0,r[} u\,\mathrm{d}\sigma\\ &= |\partial B(x_0,r[|^{-1}\Big(\int_{\partial B(x_0,r[\setminus U} u\,\mathrm{d}\sigma + \int_U u\,\mathrm{d}\sigma\Big)\\ &\le |\partial B(x_0,r[|^{-1}\left(|\partial B(x_0,r[\setminus U|\,M + |U|\,\frac{u(x^*)+M}{2}\right)\\ &< |\partial B(x_0,r[|^{-1}\left(|\partial B(x_0,r[\setminus U|\,M + |U|\,M\right)\\ &= M,\end{aligned}$$

and this is a contradiction. Applying this argument to all the radii $0 < \rho < r$ and using the equality $u(x_0) = M$, we infer that u is identically M in the ball $B(x_0, r]$. A chaining argument and the connectedness of Ω permit to conclude. **QED**

1.4 An estimate for the solution to a Dirichlet Problem

The aim of this part is very technical: we want to estimate the Schauder norm $\|\cdot\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})}$ of the solution to a homogeneous Dirichlet Problem associated to the Poisson Equation $-\Delta u = f$ by the $L^2(\Omega)$ norm of f. Such an estimate will be used in the third chapter in order to prove an Extension Lemma (Lemma 3.3.3).

We begin recalling some facts, first of all the theory about the homogeneous Dirichlet Problem for the Poisson Equation, and second a regularity result. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open, bounded subset. We denote by $H^1(\Omega)$ the Sobolev space $W^{1,2}(\Omega)$, and we set $H_0^1(\Omega)$ the enclosure of the space of the test functions $\mathcal{C}_c^\infty(\Omega)$ in $H^1(\Omega)$. Given $f \in L^2(\Omega)$, consider the homogeneous Dirichlet Problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.16}$$

For such a problem one can formulate a very natural notion of weak solution (such that, in particular, this new defintion in an "average" of the notions of classical solution and weak solution to $-\Delta u = f$).

Definition 1.4.1. A function $u \in H^1(\Omega)$ is said to be a $H_0^1(\Omega)$-*weak solution* to problem (1.16) provided that $u \in H_0^1(\Omega)$ and

$$\int_\Omega Du \cdot Dv \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x,$$

for every $v \in H_0^1(\Omega)$.

We immediately observe that if u is a $H_0^1(\Omega)$-weak solution to (1.16) then u is a weak solution to $-\Delta u = f$. As a matter of fact, for any $\zeta \in \mathcal{C}_c^\infty(\Omega)$, taken a smooth open subset $\Omega' \subset\subset \Omega$ [2] such that $\operatorname{supp}\zeta \subset \Omega'$, one has

$$\begin{aligned} &\int_\Omega (u\Delta\zeta + f\zeta) \, \mathrm{d}x = \int_{\Omega'} (u\Delta\zeta + f\zeta) \, \mathrm{d}x = \text{(Trace Theorem)} \\ &= -\int_{\Omega'} Du \cdot D\zeta \, \mathrm{d}x + \int_{\partial\Omega'} (Tu)(D\zeta \cdot \nu) \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\Omega'} fz \, \mathrm{d}x \\ &= -\int_{\Omega'} Du \cdot D\zeta \, \mathrm{d}x + \int_{\Omega'} f\zeta \, \mathrm{d}x = 0. \end{aligned}$$

One can prove the following, exhaustive

Theorem 1.4.1. *For $f \in L^2(\Omega)$, the Dirichlet boundary value problem* (1.16) *has exactly one $H_0^1(\Omega)$-weak solution.*

We recall that the scalar product

$$\langle f, g \rangle := \int_\Omega (fg + Df \cdot Dg) \, \mathrm{d}x$$

gives $H^1(\Omega)$ a Hilbert Space structure, and consequently, by closedness, also

$$(H_0^1(\Omega), \langle \cdot, \cdot \rangle)$$

[2] With this notation we mean that Ω' is a bounded open subset such that $\overline{\Omega'} \subset \Omega$.

is a Hilbert Space. The *Poincaré's Inequality* ensures also that the form in $H_0^1(\Omega)$

$$\langle f,g\rangle_* := \int_\Omega Df\cdot Dg\,\mathrm{d}x$$

is a scalar product that induces a norm $\|\cdot\|_*$ equivalent (just in $H_0^1(\Omega)$) to the norm $\|\cdot\| = \|\cdot\|_{H^1(\Omega)}$ associated to $\langle\cdot,\cdot\rangle$. Therefore $(H_0^1(\Omega),\langle\cdot,\cdot\rangle_*)$ is an Hilbert Space, and the Hilbert structures

$$(H_0^1(\Omega),\langle\cdot,\cdot\rangle) \qquad\qquad (H_0^1(\Omega),\langle\cdot,\cdot\rangle_*)$$

are equivalent in the sense that, for a suitable constant $C^*\equiv C^*(\Omega)$, one has

$$\|f\|_* = \Big(\int_\Omega |Df|^2\,\mathrm{d}x\Big)^{1/2} \le \Big(\int_\Omega (|f|^2+|Df|^2)\,\mathrm{d}x\Big)^{1/2} = \|f\| \le \sqrt{C_*^2+1}\Big(\int_\Omega |Df|^2\,\mathrm{d}x\Big)^{1/2} = \sqrt{C_*^2+1}\|f\|_*,$$

for all $f\in H_0^1(\Omega)$. Then, by definition of $H_0^1(\Omega)$-weak solution, we deduce the following estimate

$$\begin{aligned}\|Du\|_{L^2(\Omega)} = \|u\|_* = \sup_{\substack{v\in H_0^1(\Omega)\\ \|v\|_*\le 1}}\langle u,v\rangle_* = \sup_{\substack{v\in H_0^1(\Omega)\\ \|v\|_*\le 1}}\int_\Omega Du\cdot Dv\,\mathrm{d}x &= \sup_{\substack{v\in H_0^1(\Omega)\\ \|v\|_*\le 1}}\int_\Omega fv\,\mathrm{d}x \\ &\le \sup_{\substack{v\in H_0^1(\Omega)\\ \|v\|_{L^2(\Omega)}\le C^*}}\int_\Omega fv\,\mathrm{d}x \\ &\le \sup_{\substack{v\in L^2(\Omega)\\ \|v\|_{L^2(\Omega)}\le C^*}}\int_\Omega fv\,\mathrm{d}x = C^*\|f\|_{L^2(\Omega)}.\end{aligned} \tag{1.17}$$

Moreover we have

$$\|u\|_{L^2(\Omega)} \le C^*\,\|u\|_* \le C^{*2}\,\|f\|_{L^2(\Omega)}.$$

We now introduce a regularity estimate. We recall that, given a Lipschitz open bounded subset $\Omega\subset\mathbb{R}^n$, $n\ge 2$, the *Morrey Space* of exponents $1\le p<\infty$, $\lambda\ge 0$ is defined as

$$L^{p,\lambda}(\Omega) := \left\{u\in L^p(\Omega)\text{ s.t. }\sup_{\substack{x_0\in\Omega\\ \rho>0}}\rho^{-\lambda}\int_{\Omega\cap B(x_0,\rho[}|u|^p\,\mathrm{d}x<\infty\right\}.$$

The Morrey Space $L^{p,\lambda}(\Omega)$ can be endowed with the norm defined by

$$\|u\|^2_{L^{p,\lambda}(\Omega)} := \sup_{\substack{x_0\in\Omega\\ \rho>0}}\rho^{-\lambda}\int_{\Omega\cap B(x_0,\rho[}|u|^p\,\mathrm{d}x.$$

Moreover in local sense, for any open subset Ω, we set

$$L^{p,\lambda}_{loc}(\Omega) := \left\{u\in L^p_{loc}(\Omega)\text{ s.t. }u|_{\Omega'}\in L^{p,\lambda}(\Omega'),\text{ for any Lipschitz open subset }\Omega'\subset\subset\Omega\right\}.$$

We have the following important regularity result.

Theorem 1.4.2. *Given an open subset $\Omega\subset\mathbb{R}^n$, $n\ge 2$, let $u\in W^{1,2}_{loc}(\Omega)$ be a weak solution to*

$$-\Delta u = f,$$

where $f\in L^{2,\lambda_\beta}(\Omega)$, $\lambda_\beta = 2\beta+n-2$, for some $0<\beta<1$. Then Du is locally β-Hölder continuous. More specifically, given a sequence $\hat\Omega\subset\subset\widetilde\Omega\subset\subset\Omega$ such that $\widetilde\Omega$ is Lipschitz, the following estimate holds

$$|Du:\hat\Omega|_\beta \le C(\|Du\|_{L^2(\widetilde\Omega)} + \|f\|_{L^{2,\lambda_\beta}(\widetilde\Omega)}), \tag{1.18}$$

for a suitable constant $C\equiv C(\beta,n,\hat\Omega,\widetilde\Omega,\Omega)$.

For a proof of this result (possibly in a more general setting and with quite different assumptions), one can see for example [10].

Let's assume now that $u \in \mathcal{C}^{1,\beta}(\overline{\Omega})$ is a $H_0^1(\Omega)$-weak solution to (1.16), and let

$$f \in L^\infty(\Omega) \subset L^2(\Omega).$$

Then, for any $0 < \lambda < n$, fixed $\rho_0 > 0$ arbitrarily, one has:

(1) if $\rho \geq \rho_0$, $\rho^{-\lambda} \int_{\Omega \cap B(x_0,\rho[} |f|^2 \, \mathrm{d}x \leq \rho_0^{-\lambda} \|f\|^2_{L^2(\Omega)}$;

(2) if $\rho \leq \rho_0$, $\rho^{-\lambda} \int_{\Omega \cap B(x_0,\rho[} |f|^2 \, \mathrm{d}x \leq \rho^{n-\lambda} \|f\|^2_{L^\infty(\Omega)} \leq \rho_0^{n-\lambda} \|f\|^2_{L^\infty(\Omega)}$.

Therefore in particular

$$f \in \bigcap_{0<\lambda<n} L^{2,\lambda}(\Omega).$$

Fixed $\delta_1 > 0$ arbitrarily, pick $\hat{\Omega} \equiv \hat{\Omega}_{\delta_1}$ in order that

$$|Du : \Omega|_\beta \leq |Du : \hat{\Omega}|_\beta + \delta_1.$$

By estimates (1.17), (1.18), for $0 < \beta < 1$, up to a good choice of $C \equiv C(\beta, n, \delta_1, \Omega)$, we infer that

$$|Du : \hat{\Omega}|_\beta \leq C(\|f\|_{L^2(\Omega)} + \|f\|_{L^{2,\lambda_\beta}(\Omega)}).$$

Then, for $\delta_2 > 0$, taking $\rho_0 = \delta_2$ and supposing that $\|f\|_{L^2(\Omega)} \leq \delta_2^{n/2}$ in the estimate written some lines ago, we obtain

$$|Du : \Omega|_\beta \leq C(\beta, n, \delta_1, \Omega)[\delta_2^{n/2} + (\|f\|_{L^\infty(\Omega)} + 1)\delta_2^{\frac{n-\lambda_\beta}{2}}] + \delta_1 = \Xi_{\beta,n,\Omega,\|f\|_{L^\infty(\Omega)}}(\delta_1, \delta_2).$$

Exploiting the inequality above, we can show the following result.

Theorem 1.4.3. *Fixed $c \neq 0$ and a open nonempty subset $A \subset B(0,1[$, consider the boundary value problem*

$$\begin{aligned} -\Delta w &= c\chi_A \quad \text{in } B(0,1[\\ w &= 0 \quad \text{on } \partial B(0,1[. \end{aligned} \tag{1.19}$$

Then the only $H_0^1(\Omega)$-weak solution w to (1.19) is of class $\mathcal{C}^{1,\beta}(B(0,1])$, for any $0 < \beta < 1$. Moreover, there exists $\sigma \equiv \sigma(\beta, n, c) > 0$ with the property that, since $|A| \leq \sigma$, it follows

$$\|w\|_{\mathcal{C}^{1,\beta}(B(0,1])} \leq 1.$$

Proof. Assume that $w \in \mathcal{C}^{1,\beta}(B(0,1])$ is the $H_0^1(B(0,1[)$-weak solution to 1.19. Then w is continuous up to the boundary of the unit ball, is non identically zero, but is null on the boundary. Therefore there is a point $x_0 \in B(0,1[$ in which w attains its maximum or its minimum (if $c > 0$ or if $c < 0$ respectively). Then by differentiability

$$Dw(x_0) = 0.$$

Thus, fixed also a point $x_1 \in \partial B(0,1[$ and assuming that $|A| \leq \delta_2^n / |c|^2$, we have:

$$\begin{aligned} &|Dw : B(0,1[|_\beta \leq \Xi_{\beta,n,c}(\delta_1, \delta_2) \equiv \Xi_{\beta,n,B(0,1[,c}(\delta_1, \delta_2) \\ &|Dw(x)| \leq |Dw(x) - Dw(x_0)| \leq 2^\beta \Xi_{\beta,n,c}(\delta_1, \delta_2) \\ &|w(x)| \leq |w(x) - w(x_1)| \leq 2^{\beta+1} \Xi_{\beta,n,c}(\delta_1, \delta_2). \end{aligned}$$

By definition of $\Xi_{\beta,n,c}(\delta_1,\delta_2)$, taking for example $\delta_1 = 1/2$, choosing a good δ_2 (function of β, n and c) and setting $\sigma(\beta,n,c) = \delta_2^n/|c|^2$, we infer that

$$\|w\|_{\mathcal{C}^{1,\beta}(\overline{\Omega})} \leq 1,$$

and the thesis follows. To conclude, it sufficies to demonstrate that that the unique $H_0^1(B(0,1[)$-weak solution of (1.19) is of class $\mathcal{C}^{1,\beta}(B(0,1])$. This fact is an immediate consequence of the following

Theorem 1.4.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open, bounded, connected subset, and $f \in L^p(\Omega)$, for some $1 \leq p < \infty$. Then, denoted by S_n the fundamental solution to the Laplace Operator in dimension n, the* Newtonian Potential

$$\mathcal{N}f(x) \equiv \int_\Omega S_n(x-y)f(y)\,\mathrm{d}y,\ x \in \Omega,$$

is of class $W^{2,p}(\Omega)$ and

$$\Delta(\mathcal{N}f) = f \ \text{a.e. in } \Omega.$$

Furthermore, there exists a constant $C_{CZ} \equiv C_{CZ}(p,n)$, dependent only on p and n, such that the so called Calderon-Zygmund Inequality *holds true:*

$$\|D^2\mathcal{N}f\|_{L^p(\Omega)} \leq C_{CZ}\|f\|_{L^p(\Omega)}.$$

A proof of this result can be found in [5, p. 230].

The smoothness of the boundary of the ball allows to apply the well known Sobolev inclusions, deducing that, if $p > n$

$$\mathcal{N}f \in W^{1,p}(B(0,1[) \hookrightarrow \mathcal{C}^{1,1-n/p}(B(0,1])$$

continuously. Let $f \equiv c\chi_A$: it's clear that the only $H_0^1(B(0,1[)$-weak solution to (1.19) is

$$w \equiv w_0 - \mathcal{N}f,$$

where w_0 is the unique solution to

$$\begin{aligned} -\Delta w &= 0 && \text{in } B(0,1[\\ w &= \mathcal{N}f && \text{on } \partial B(0,1[. \end{aligned}$$

It's known that the Schauder regularity of w_0 coincides with the Schauder regularity of the boundary datum, and therefore it's clear that, if $p > n$, $w \in \mathcal{C}^{1,1-n/p}(B(0,1])$. From the fact that

$$\chi_A \in \bigcap_{p>n} L^p(B(0,1[),$$

it follows that w is of class $\mathcal{C}^{1,\beta}(B(0,1])$, for all $0 < \beta < 1$, and we conclude. **QED**

Chapter 2

Symmetry properties via Reflection Method

In this chapter we analyze into detail work [2], studying a first, quite standard reflection technique. Using this technique, we demonstrate an interesting symmetry property for solutions of equations like $-\Delta u + b(x)\partial_\gamma u + f(u) = 0$: this property turns out to ensure the radiality of the solutions to

$$\begin{aligned} -\Delta u = f(u) \quad & \text{in } B(0,R[\\ u = 0 \quad & \text{on } \partial B(0,R[. \end{aligned}$$

From now on, let the dimension $n \geq 2$.

2.1 A reflection technique for bounded, smooth open subsets

Let $\Omega \subset \mathbb{R}^n$ be a smooth (at least $\mathcal{C}^2$) open, bounded, connected subset. Given a unit vector $\gamma \in \mathbb{S}^{n-1}$, let T_λ be the hyperplane $\gamma \cdot x = \lambda$, for $\lambda \in \mathbb{R}$, and, given $x \in \mathbb{R}^n$, denote by x_λ the reflection of x in T_λ. As λ big, T_λ does not intersect $\overline{\Omega}$. Let's move T_λ letting λ to decrease: from a suitable value of λ on, T_λ intersects Ω and selects an open cap

$$\Sigma(\lambda) := \Omega \cap \{\gamma \cdot x > \lambda\}.$$

Set $\Sigma'(\lambda)$ the reflection of $\Sigma(\lambda)$ in the plane T_λ. At the beginning, $\Sigma'(\lambda)$ is contained in Ω; then $\Sigma'(\lambda)$ will remain in Ω until to one of the following two conditions occurs:

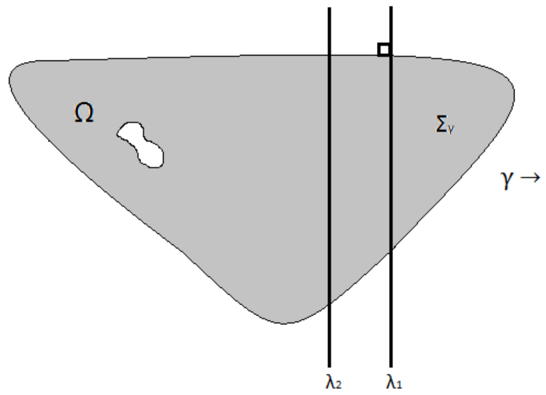
(a) $\Sigma'(\lambda)$ becomes internally tangent to $\partial\Omega$ at some point P not contained in T_λ;

(b) T_λ assumes a position orthogonal to $\partial\Omega$ at some point.

We denote by λ_1 the first value of λ such that one of the two positions (a) or (b) above is reached, and we set

$$\begin{aligned} \Sigma(\lambda_1) &:= \Sigma_\gamma \\ \Sigma'(\lambda_1) &:= \Sigma'_\gamma. \end{aligned}$$

We call Σ_γ the *maximal cap* associated to γ. Note that Σ'_γ is contained in Ω. The conditions (a), (b) above are necessary, but one can immediately see that are not sufficient to guarantee that, for $\lambda < \lambda_1$, Σ'_λ fails to be contained in Ω (see for example the open Ω represented in the figure at the beginning of the next page).



Let $\lambda_2 \le \lambda_1$ be the supremum of the values λ such that $\Sigma'(\lambda) \nsubseteq \Omega$: $\Sigma(\lambda_2)$ is said to be the *optimal cap* associated to γ.

2.2 The symmetry property for solutions to some elliptic equations and related consequences

We are now enabled to state and prove a symmetry result for solutions to elliptic equations. Given a smooth open, bounded, connected subset Ω, let $u \in \mathcal{C}^2(\Omega)$ solve

$$\Delta u + b_1(x)u_{x_1} + f(u) = 0 \text{ in } \Omega, \tag{2.1}$$

where $b_1 \in \mathcal{C}^0(\overline{\Omega})$, $f \in \mathcal{C}^1(\mathbb{R})$. Let $\gamma = (1, 0, ..., 0)$ and $\Sigma := \Sigma_\gamma$ be the maximal cap associated to γ. The correspondent hyperplane T_{λ_1} coincides with $x_1 = \lambda_1$ and, setting

$$\lambda_0 := \max_{x \in \overline{\Omega}} x_1,$$

we have $\lambda_1 < \lambda_0$. We introduce the following hypotheses on u:

$$\begin{aligned} &u > 0 \text{ in } \Omega \\ &u \in \mathcal{C}^2(\overline{\Omega} \cap \{x_1 > \lambda_1\}) \\ &u = 0 \text{ on } \partial\Omega \cap \{x_1 > \lambda_1\}. \end{aligned} \tag{2.2}$$

Given $x \in \mathbb{R}^n$, we further denote by x_λ the reflection of x in T_λ.

Theorem 2.2.1. *Let u like in* (2.1) *and suppose that u satisfies the assumptions* (2.2). *Assume that $b_1 \ge 0$ in $\Sigma \cup \Sigma'$. Then, for any $\lambda_1 < \lambda < \lambda_0$, one has*

$$u_{x_1} < 0 \textit{ and } u(x) < u(x_\lambda), \textit{ for all } x \in \Sigma(\lambda).$$

Thus $u_{x_1} < 0$ in Σ and in addition, if u_{x_1} vanishes on $T_{\lambda_1} \cap \Omega$, u has to be symmetric with respect to T_{λ_1}, $\Omega = \Sigma \cup \Sigma' \cup (T_{\lambda_1} \cap \Omega)$ and $b_1 \equiv 0$.

The proof will be given in the next section. First we prove some consequences of the theorem above.

Corollary 2.2.1. Let $u \in \mathcal{C}^2(B(0,R])$, $u \geq 0$, solve

$$\begin{aligned} -\Delta u = f(u) \quad & \text{in } B(0,R[\\ u = 0 \quad & \text{on } \partial B(0,R[, \end{aligned}$$

where $f \in \mathcal{C}^1(\mathbb{R})$. Then $u(x) \equiv u^*(|x|)$ is radial and

$$\frac{du^*}{dr} < 0 \text{ in }]0,R[.$$

Proof. Applying the previous theorem, we infer that $u_{x_1} < 0$, for $x_1 > 0$, for any choiche of the x_1 axis. This fact implies that $u_{x_1} > 0$ for $x_1 < 0$, and by continuity necessarily $u_{x_1} \equiv 0$ on $x_1 = 0$. The second part of the same theorem guarantees that u is symetric with respect to x_1 and so by arbitrariness of of the choice of the axis u is necessarily radial, and furthermore

$$\frac{du^*}{dr} < 0 \text{ in }]0,R[.$$

QED

Corollary 2.2.2. Let $u \in \mathcal{C}^2(B(0,R] \setminus B(0,R'])$, $u \geq 0$, be a solution to

$$\begin{aligned} -\Delta u = f(u) \quad & \text{in } B(0,R[\setminus B(0,R'] \\ u = 0 \quad & \text{on } \partial B(0,R[, \end{aligned}$$

where $f \in \mathcal{C}^1(\mathbb{R})$. Then $\partial_r u < 0$ in $[\frac{R+R'}{2}, R[$.

Proof. Also in this case we can choose the direction x_1 arbitrarily, namely associating it to whatever unit vector γ. The previous theorem ensures that $\gamma \cdot \nabla u < 0$. Observing that the union of the caps Σ_γ is exactly the ring of radii $(R+R')/2$, R, and that "by structure" of the annulus $\partial_r u$ cannot vanish on $\partial B(0, \frac{R+R'}{2}[$, we conclude. **QED**

In particular, we have the following further consequence.

Corollary 2.2.3. Let $u \in \mathcal{C}^2(B(0,R] \setminus \{0\})$, $u \geq 0$ solve

$$\begin{aligned} -\Delta u = f(u) \quad & \text{in } B(0,R[\setminus \{0\} \\ u = 0 \quad & \text{on } \partial B(0,R[, \end{aligned}$$

where $f \in \mathcal{C}^1(\mathbb{R})$. Then $\partial_r u < 0$ in $]\frac{R}{2}, R[$.

This result is very weak, and, we can say, its weakness justifies the necessity of a new, less standard reflection method, which will be described in the following chapter. Roughly speaking, the presence of the hole at the center of the unit ball makes fail the argument used in Corollary 2.2.1, and so permits to infer only a poor result, as Corollary 2.2.3 is.

2.3 Proof of the Theorem 2.2.1 on the symmetry property

Before moving on to expose the proof of the theorem, it's better to state and prove two Lemmas. Until to the end of the section, Ω is assumed to be a smooth open, bounded, connected subset. We set $\nu \equiv (\nu_1, ..., \nu_n)$ the outer unit normal vector field on $\partial\Omega$. Moreover, in addition to the hypotheses used in the following statements, we assume that u is like in (2.1) and that u respects assumptions (2.2).

Lemma 2.3.1. Let $x_0 \in \partial\Omega$ be a point in which $\nu_1(x_0) > 0$. For $\epsilon > 0$, let

$$\Omega_\epsilon := \Omega \cap B(x_0, \epsilon[$$

and $u \in \mathcal{C}^2(\overline{\Omega_\epsilon})$ be such that

$$u > 0 \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega \cap B(x_0, \epsilon[.$$

Then there exists a $\delta \equiv \delta_\epsilon > 0$ such that $u_{x_1} < 0$ in Ω_δ.

Proof. Being $u > 0$ in Ω, $u = 0$ on $\partial\Omega \cap B(x_0, \epsilon[$, it follows that $u_\nu \leq 0$ on $\partial\Omega \cap B(x_0, \epsilon[$, and then that $u_{x_1} \leq 0$ there (it sufficies to pick $\epsilon > 0$ enough small to ensure that $\nu_1 > 0$ on $\partial\Omega \cap B(x_0, \epsilon[$ whole). By contradiction, let the thesis be false. Then one can find a sequence $\{x^j\}_{j\geq 1}$ such that $x^j \to x_0$, $u_{x_1}(x^j) \geq 0$, for every j. For j big enough, the line

$$\{x^j + te_1 | t \geq 0\}$$

intersects $\partial\Omega$ in at least one point y^j in which necessarily u_{x_1} has to be nonpositive. For ϵ small enough, applying the Lagrange Mean Value Theorem on any segment $[x^j, y^j]$, we deduce that

$$u_{x_1}(x_0) = 0 \geq u_{x_1,x_1}(x_0).$$

Let $f(0) \geq 0$. Then u solves

$$\Delta u + b_1 u_{x_1} + f(u) - f(0) \leq 0 \text{ in } \Omega_\epsilon,$$

or equivalently, thanks to the Lagrange Theorem, fixed $c_1 \equiv c_1(x)$ suitably,

$$\Delta u + b_1 u_{x_1} + c_1 u \geq 0.$$

Applying to $-u$ Hopf Lemma 1.1.6 already proved, we find

$$u_\nu(x_0) < 0 \Rightarrow u_{x_1}(x_0) < 0,$$

but this is a contradiction. If instead $f(0) < 0$, another contradiction occurs. We argue as follows: let $u_1, ..., u_n$ be an orthonormal basis with associated coordinates $y_1, ..., y_n$ such that $\nu \cdot u_i > 0$, for any $1 \leq i \leq n$. With the same argument used to demonstrate that $u_{x_1}(x_0) = 0 \geq u_{x_1,x_1}(x_0)$, one shows that $u_{y_i}(x_0) = 0$, $u_{y_i,y_i}(x_0) \leq 0$, for any i. But therefore

$$0 \geq \Delta u = -f(u) > 0 \text{ at } x_0,$$

the Laplace Operator not changing passing to $y_1, ..., y_n$, and the reductio ad absurdum is concluded. **QED**

Lemma 2.3.2. Suppose that, for some $\lambda_1 \leq \lambda < \lambda_0$,

$$u_{x_1}(x) \leq 0,\ u(x) \leq u(x_\lambda), \text{ for any } x \in \Sigma(\lambda),$$

but $u(x)$ is not identically $u(x_\lambda)$ in $\Sigma(\lambda)$. Then if $b_1 \geq 0$ in $\Sigma(\lambda) \cup \Sigma'(\lambda)$, one has

$$u(x) < u(x_\lambda), \text{ for any } x \in \Sigma(\lambda),\ u_{x_1}(x) < 0, \text{ for any } x \in \Omega \cap T_\lambda.$$

Proof. Consider $v(x) \equiv u(x_\lambda)$ for $x \in \Sigma'(\lambda)$ (note that $x_\lambda \in \Sigma(\lambda)$). The map v respects $v_{x_1} \geq 0$ and solves

$$\Delta v(x) - b_1(x_\lambda)v_{x_1}(x) + f(v(x)) = 0.$$

Holding also (2.1), one infers that

$$\Delta v(x) - b_1(x_\lambda)v_{x_1}(x) + f(v(x)) - \Delta u(x) - b_1(x)u_{x_1}(x) - f(u(x)) = 0, \text{ in } \Sigma'(\lambda),$$

and so we deduce that

$$\Delta(v-u)(x) + b_1(x)(v-u)_{x_1}(x) + f(v(x)) - f(u(x)) = (b_1(x_\lambda) + b_1(x))v_{x_1}(x) \geq 0.$$

Now the assumptions under which we are working ensure us that in $\Sigma'(\lambda)$ the function $w \equiv v - u$ is nonpositive but not identically zero. Moreover, applying the mean integral Theorem to the inequality just inferred, we obtain that for a suitable map $c \equiv c(x)$

$$\Delta w + b_1(x)w_{x_1} + c(x)w \geq 0 \text{ in } \Sigma'(\lambda).$$

By $w \equiv 0$ in $T_\lambda \cap \Omega$, hence on $\partial\Sigma'(\lambda)$ whole, using Maximum Principle 1.1.5 and Hopf Lemma 1.1.6, we deduce that

$$w < 0 \text{ in } \Sigma'(\lambda),\ w_{x_1} > 0 \text{ on } T_\lambda \cap \Omega.$$

Consequently $-u_{x_1} = w_{x_1}/2 > 0$. **QED**

We are now ready to prove Theorem 2.2.1.

Proof (of Theorem 2.2.1). By Lemma 2.3.1, it follows that, as λ close to λ_0, $\lambda < \lambda_0$, it holds

$$u_{x_1}(x) < 0,\ u(x) < u(x_\lambda), \text{ for every } x \in \Sigma(\lambda). \tag{2.3}$$

Let $\mu < \lambda_0$ be the critical value such that, as $\lambda < \mu$, (2.3) no longer holds, and for $\lambda = \mu$ one has

$$u_{x_1}(x) < 0,\ u(x) \leq u(x_\mu), \text{ for any } x \in \Sigma(\mu).$$

Let's show that $\mu = \lambda_1$.
Suppose that $\mu > \lambda_1$. Therefore, for any $x_0 \in \partial\Sigma(\mu) \setminus T_\mu$,

$$x_0^\mu \in \Omega.$$

Because $0 = u(x_0) < u(x_0^\mu)$, we deduce that $u(x)$ is not identically $u(x_\mu)$ in $\Sigma(\mu)$. Hence we can apply Lemma 2.3.2 achieving that

$$u(x) < u(x_\mu) \text{ in } \Sigma(\mu),\ u_{x_1} < 0 \text{ on } \Omega \cap T_\mu.$$

Thus (2.3) holds for $\lambda = \mu$ too. Since $u_{x_1} < 0$ on $\Omega \cap T_\mu$, using Lemma 2.3.1, it follows that

$$u_{x_1} < 0 \text{ in } \Omega \cap \{x_1 > \mu - \epsilon\}, \text{ for some } \epsilon > 0. \tag{2.4}$$

By definition of μ, we deduce that what we are going to describe has to hold: there exist a sequence $\{\lambda^j\}_{j\geq 1}$ such that $\lambda_1 < \lambda^j \nearrow \mu$ and another sequence $\{x_j\}_{j\geq 1}$ such that

$$x_j \in \Sigma(\lambda^j) \text{ and } u(x_j) \geq u(x_j^{\lambda^j}).$$

Up to subsequences, by compactness, $x_j \to x$, for a suitable $x \in \overline{\Sigma(\mu)}$. Therefore

$$x_j^{\lambda^j} \to x_\mu \text{ and } u(x) \geq u(x_\mu). \tag{2.5}$$

By the validity of (2.3) as $\lambda = \mu$, it has to hold

$$x \in \partial\Sigma(\mu).$$

If x didn't belong to T_μ, then x would lay on $\partial\Omega$, and hence $x_\mu \in \Omega$,

$$0 = u(x) < u(x_\mu),$$

but this is a contradiction, by relation (2.5). Therefore $x \in T_\mu$ and $x_\mu = x$. Now, as j big enough, the segment joining x_j and $x_j^{\lambda^j}$ belongs to Ω and, by the Lagrange's Theorem, it has to contain a point y_j such that $u_{x_1}(y_j) \geq 0$. This fact contradicts (2.4), because y_j converges to x. Thus $\mu = \lambda_1$ and (2.3) holds, for any $\lambda_0 > \lambda > \lambda_1$. By continuity, one also deduces that, in Σ,

$$\begin{aligned} &u_{x_1} < 0 \\ &u(x) \leq u(x_{\lambda_1}). \end{aligned}$$

To complete the proof, suppose that u_{x_1} vanishes at some point of $\Omega \cap T_{\lambda_1}$. By Lemma 2.3.2, it follows that $u(x) \equiv u(x_{\lambda_1})$ in Σ. From $u(x) = 0$, for every $x \in \partial\Omega \cap \{x_1 \geq \lambda_1\}$, it follows that $u(x_{\lambda_1}) = 0$ and finally we infer that

$$\Omega = \Sigma \cup \Sigma' \cup (\Omega \cap T_{\lambda_1}).$$

Lastly, let $b_1 > 0$ at some point of Ω, that we may assume not contained in T_{λ_1} (by continuity). Thanks to the assumptions (2.2) and to the symmetry of u with respect to T_{λ_1} just proved, we note that

$$b_1(x)u_{x_1} = b_1 u_{x_1}(x_{\lambda_1}).$$

If $x \in \Sigma$, or similarly in Σ', the left hand side is negative, instead the right hand side is nonnegative. Thus a contradiction occurs, and b_1 is necessarily identically zero. **QED**

Chapter 3

The singular Yamabe Equation

We now enter the core of the thesis: in this chapter we study into detail work [1]. In particular, we need to develop a more sophisticated reflection technique that is, citing textually the work of Caffarelli, Gidas and Spruck, a «"measure theoretic" variation» of the more basic reflection method introduced and exploited in the previous chapter. The aim of this chapter is to furnish a complete characterization of the solutions to the Yamabe Equation in the punctured ball. To do so, we will need to develop a sophisticated machinery, boosting and trying to generalize to the limiting exponent $\alpha = \frac{n+2}{n-2}$ the techniques and the arguments used by Gidas and Spruck in [11] to classify the singular solutions to equations like $-\Delta u = u^{\alpha}$, for $1 < \alpha < \frac{n+2}{n-2}$. We begin with an exhaustive study of the radial solutions to the Yamabe Equation in the punctured ball: such an analysis is fundamental because permits to formulate the classification result contained in the last section. In the second section we first exploit the Kelvin Transform in order to change the formulation of the problem: roughly speaking, instead of studying an equation in the punctured ball, we deal with an equation defined in a neighbourhood of infinity. Second, we prove some general decay estimates: these estimates represent the fundamental assumption under which we will work along the whole chapter. The subsequent section is devoted to the proof of the Reflection Theorem, and to understand what is the asymptotic symmetry and how can be deduced by the Reflection Theorem. The final section is dedicated to analyze the applications of the theoretical tools developed in the previous parts: in particular, the last result of the chapter contains the classification of the solutions to the singular Yamabe Equation which represents the gist of the thesis. From now on, the dimension n is assumed to be ≥ 3.

3.1 Radial solutions to the Yamabe Equation

We start giving an exhaustive survey regarding the radial solutions to the Yamabe Equation, fully classifying them. This study is preliminary to the more difficult issue of classify the singular solutions to the Yamabe Equation that we will study in the final part of this chapter, proving that, around zero, any singular solution to the Yamabe Equation is "close to" a radial, singular solution to the same equation.

We want to find the radial, $\mathcal{C}^2$ solutions to the Yamabe Equation in the punctured ball, namely to the following equation:

$$-\Delta u = u^{\frac{n+2}{n-2}}, \text{ in } B(0,1[\setminus\{0\}. \tag{3.1}$$

To do so, we first recall the formula for the Laplace Operator in spherical coordinates. Denoted by $\mathbb{S} := \partial B(0,1[$, let $\Omega \subset \mathbb{R}^n \setminus \{0\}$ be an open subset, $u \in \mathcal{C}^2(\Omega)$ and consider the representation of u in spherical coordinates, that is

$$v(r,\theta) \equiv u(r\theta),$$

for $(r,\theta) \in \{(|x|, \frac{x}{|x|}) \mid x \in \Omega\} \subset]0,+\infty[\times\mathbb{S}$. Then we recall that

$$\Delta u(x) = \frac{\partial^2 v}{\partial r^2}(|x|, \frac{x}{|x|}) + \frac{n-1}{|x|}\frac{\partial v}{\partial r}(|x|, \frac{x}{|x|}) + \frac{1}{|x|^2}\Delta_{\mathbb{S}} v(|x|, \frac{x}{|x|}), \tag{3.2}$$

for any $x \in \Omega$, where $\Delta_{\mathbb{S}}$ denotes the Laplace-Beltrami Operator on the unit sphere.

Let $\Omega = B(0,R[\setminus B(0,r]$, for $0 \le r < R \le \infty$ arbitrarily chosen, and suppose that $u = \phi(|x|)$, $\phi \in \mathcal{C}^2(]r,R[)$, is radially symmetric. Then formula (3.2) yields

$$\Delta u(x) = \phi''(|x|) + \frac{n-1}{|x|}\phi'(|x|), \text{ for } x \in \Omega.$$

So the analysis of the radial solutions to (3.1) can be reduced to the study of an ordinary differential equation: in fact, it's clear that $u \in \mathcal{C}^2(B(0,1[\setminus\{0\})$ is a radial solution to (3.1) if and only if, taken $\phi \in \mathcal{C}^2(]0,1[)$ such that $u(x) = \phi(|x|)$, ϕ solves

$$\phi''(r) + \frac{n-1}{r}\phi'(r) + \phi(r)^{\frac{n+2}{n-2}} = 0, \text{ in }]0,1[. \tag{3.3}$$

In order to study this ODE, we may adopt the following ansatz [3]

$$\begin{aligned} \psi(t) &\equiv r^{\frac{n-2}{2}}\phi(r) \\ t &= -\ln(r),\ t > 0. \end{aligned} \tag{3.4}$$

Such an approach permits to deduce that ψ solves the second order ODE

$$\psi'' - \left(\frac{n-2}{2}\right)^2 \psi + \psi^{\frac{n+2}{n-2}} = 0, \text{ in }]0,+\infty[, \tag{3.5}$$

and conversely that if ψ solves (3.5), then ϕ is a solution to (3.3). We immediately note that (3.5) is the Newton Equation associated to a conservative field with potential

$$U(\psi) \equiv -\frac{1}{2}\left(\frac{n-2}{2}\right)^2\psi^2 + \frac{n-2}{2n}\psi^{\frac{2n}{n-2}}.$$

It's well known that the total energy defined by

$$E(\psi,\psi') \equiv K(\psi') + U(\psi) = \frac{1}{2}(\psi')^2 - \frac{1}{2}\left(\frac{n-2}{2}\right)^2\psi^2 + \frac{n-2}{2n}\psi^{\frac{2n}{n-2}},$$

i.e. the sum of the kynetic energy and the potential energy is a *prime integral* for equation (3.5), namely is constant along its solutions. These physical considerations allow then to infer that if ψ solves (3.5) then ψ solves

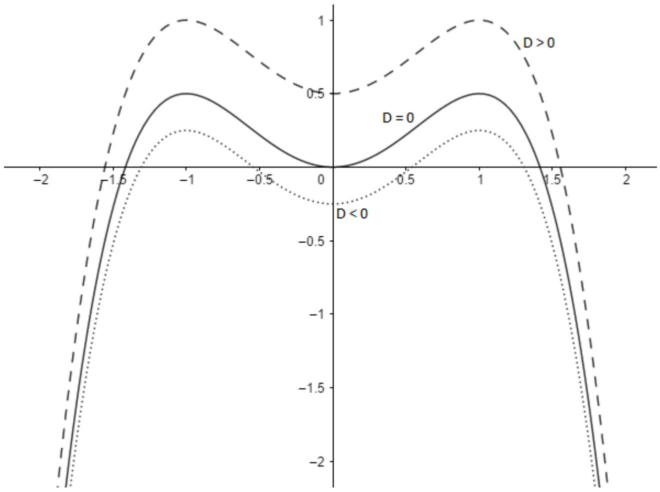
$$(\psi')^2 = \left(\frac{n-2}{2}\right)^2\psi^2 - \frac{n-2}{n}\psi^{\frac{2n}{n-2}} + D, \tag{3.6}$$

for a suitable value of the total energy $E = D/2$. Our purpose is now to analyze the $\mathcal{C}^2$ solutions ψ to (3.6) as $D \in \mathbb{R}$.

For $\psi \in \mathbb{R}$, consider the even, $\mathcal{C}^1(\mathbb{R})$ function

$$A_D(\psi) \equiv -2U(|\psi|) + D = \left(\frac{n-2}{2}\right)^2|\psi|^2 - \frac{n-2}{n}|\psi|^{\frac{2n}{n-2}} + D.$$

[3] substitution (3.4) is known as *Emden-Fowler substitution.*



As $\psi > 0$, $A'_D(\psi) = 2\left(\frac{n-2}{2}\right)^2 \psi - 2\psi^{\frac{n+2}{n-2}}$, and so $A'_D(\psi) \geq 0$ if and only if

$$\psi \leq \psi^* \equiv \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}},$$

and $A'_D(0) = 0$. Therefore ψ^* is a maximum and, being

$$A_0(\psi^*) = \frac{2}{n}\left(\frac{n-2}{2}\right)^n,$$

in order that (3.6) is well defined we need to impose the following constraint on D:

$$D \geq -\frac{2}{n}\left(\frac{n-2}{2}\right)^n.$$

Let

$$P_D := A_D^{-1}(]0, +\infty[):$$

by evenness, we deduce that as $D = 0$

$$P_0 =]a_0, b_0[\cup]-b_0, -a_0[, \text{ where } a_0 := 0,\ b_0 := \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{2}}.$$

Furthermore, if $0 > D > -\frac{2}{n}\left(\frac{n-2}{2}\right)^n$, then

$$P_D =]a_D, b_D[\cup]-b_D, -a_D[, \text{ where } a_0 < a_D < b_D < b_0,$$

and if $D > 0$, $P_D =]-b_D, b_D[$, $b_D > b_0$. In addition, for any $D > -\frac{2}{n}\left(\frac{n-2}{2}\right)^n$,

$$Z_D := A_D^{-1}(\{0\}) = \partial P_D,$$

and any element of Z_D is a constant solution to (3.6).

We immediately note that ψ solves (3.6) if and only if

$$\psi \in \overline{P_D}$$
$$\psi' \in \{\pm\sqrt{A_D(\psi)}\},$$

for any time, and so, for $D > -\frac{2}{n}\left(\frac{n-2}{2}\right)^n$, we want to study separately the following ODEs

$$\psi' = \sqrt{A_D(\psi)} \qquad (3.7a) \qquad\qquad \psi' = -\sqrt{A_D(\psi)} \qquad (3.7b)$$

for $\psi \in P_D$. The map $\sqrt{A_D}$ is locally Lipschitz in P_D, and thus we can apply the local existence and uniqueness result to both (3.7a), (3.7b). Moreover the escape from compact subsets Theorem guarantees that any maximal solution to a Cauchy Problem associated to (3.7a) or (3.7b) needs to be defined on $\mathbb{R}$ whole [4]. We finally observe that the constant solutions to (3.7a), (3.7b) are exactly the elements of Z_D. Therefore, for $D > -\frac{2}{n}\left(\frac{n-2}{2}\right)^n$, for every $t_0 \in \mathbb{R}$, $\psi_0 \in P_D$, the following Cauchy problems

$$\begin{cases} \psi' = \sqrt{A_D(\psi)} \\ \psi(t_0) = \psi_0 \end{cases} \qquad (3.8a) \qquad\qquad \begin{cases} \psi' = -\sqrt{A_D(\psi)} \\ \psi(t_0) = \psi_0 \end{cases} \qquad (3.8b)$$

have exactly one (maximal) solution defined on $\mathbb{R}$ and, observed that $\sqrt{A_D}$ is of class $\mathcal{C}^1$, such a maximal solution needs to be of class $\mathcal{C}^2$. More precisely, the following result holds.

Lemma 3.1.1. Let ψ be the maximal solution to Cauchy problem (3.8a) (resp. (3.8b)), for $t_0 \in \mathbb{R}$, $\psi_0 \in P_D$. Then:

(a) $-\psi$ solves (3.8b) (resp. (3.8a)) for initial data t_0, $-\psi_0 \in P_D$, and $\widetilde{\psi}(t) \equiv \psi(-t)$ solves (3.8b) (resp. (3.8a)) for initial data $-t_0$, ψ_0;

(b) ψ' never vanishes, and so ψ is is monotone. Moreover, for any time, ψ belongs to the connected components of P_D which contains ψ_0;

(c) ψ converges to a finite limit as $t \to \pm\infty$, and in particular:

(c1) if $D \le 0$,

$$\lim_{t\to+\infty} \psi(t) = \begin{cases} \frac{1}{2}(1+\operatorname{sgn}(\psi_0))b_D + \frac{1}{2}(\operatorname{sgn}(\psi_0)-1)a_D & \text{if } \psi' > 0 \\ \frac{1}{2}(1+\operatorname{sgn}(\psi_0))a_D + \frac{1}{2}(\operatorname{sgn}(\psi_0)-1)b_D & \text{if } \psi' < 0 \end{cases}$$

$$\lim_{t\to-\infty} \psi(t) = \begin{cases} \frac{1}{2}(1+\operatorname{sgn}(\psi_0))a_D + \frac{1}{2}(\operatorname{sgn}(\psi_0)-1)b_D & \text{if } \psi' > 0 \\ \frac{1}{2}(1+\operatorname{sgn}(\psi_0))b_D + \frac{1}{2}(\operatorname{sgn}(\psi_0)-1)a_D & \text{if } \psi' < 0 \end{cases};$$

(c2) if $D > 0$,

$$\lim_{t\to\pm\infty} \psi(t) = \begin{cases} \pm b_D & \text{if } \psi' > 0 \\ \mp b_D & \text{if } \psi' < 0. \end{cases}$$

Finally, if $\psi_0 \in Z_D$, the constant $\psi \equiv \psi_0$ is the unique solution to both (3.8a) and (3.8b).

Proof. The first point is an immediate consequence of the definitions of Cauchy problems (3.8a) and (3.8b). Let for example ψ solve (3.8a), and assume that ψ' vanishes at some time τ. Then necessarily $\psi(\tau) = c$, for some $c \in Z_D$, and then c is a constant solution (3.6). The Comparison Principle for ODEs ensures then that $\psi \equiv c$ in $[\tau, +\infty[$. From point (a) it follows that $\widetilde{\psi}$ solves (3.7b) and then $\widetilde{\psi} \equiv c$ in $[-\tau, +\infty[$, namely $\psi \equiv c$ in $]-\infty, \tau]$. Thus ψ is identically c, but this is

[4] This aspect is meaningful becuase ensures that any radial solution to the Yamabe Equation in the punctured ball can be naturally extended to $\mathbb{R}^n \setminus \{0\}$.

a contradiction, because $c \neq \psi_0$. Thus ψ' needs to be always positive or negative, and then to be monotone. Furthermore the continuity of ψ guarantees that ψ lays in the connected component of P_D which contains ψ_0, and so the proof of (b) is concluded. We now characterize this connected component, denoted by P_D^0:

(1) if $D \leq 0$,

$$P_D^0 = \begin{cases}]a_D, b_D[& \text{if } \psi_0 > 0 \\ ]-b_D, -a_D[& \text{if } \psi_0 < 0; \end{cases}$$

(2) if $D > 0$, $P_D^0 = P_D =]-b_D, b_D[$.

The monotonicity of ψ implies that:

$$\lim_{t \to +\infty} \psi(t) = \begin{cases} S & \text{if } \psi' > 0 \\ s & \text{if } \psi' < 0 \end{cases}$$

$$\lim_{t \to -\infty} \psi(t) = \begin{cases} s & \text{if } \psi' > 0 \\ S & \text{if } \psi' < 0 \end{cases},$$

where $S := \sup \psi$, $s := \inf \psi$, $S, s \in \overline{P_D^0}$. Observed that clearly $-\infty < s < S < +\infty$, we infer that ψ' needs to go to zero as $|t| \to \infty$. Thus necessarily $S, s \in Z_D$, more specifically

$$S, s \in \partial P_D \cap \overline{P_D^0} = \partial P_D^0,$$

and (c) follows from the characterization of P_D^0 operated before. Finally, the last statement follows since the fact that, if $\psi_0 \in Z_D$, the existence of a nonconstant solution to (3.8a) or (3.8b) would violate the classification for $\psi_0 \in P_D$ just demonstrated. **QED**

The main consequence of this result is that we can formulate a existence and "semi-uniqueness" result for equation (3.6).

Theorem 3.1.1. *We have:*

(a) if $D > -\frac{2}{n}\left(\frac{n-2}{2}\right)^n$, *then for every* $t_0 \in \mathbb{R}$, $\psi_0 \in P_D$, *the following Cauchy Problem*

$$\begin{cases} (\psi')^2 = \left(\frac{n-2}{2}\right)^2 \psi^2 - \frac{n-2}{n} \psi^{\frac{2n}{n-2}} + D \\ \psi(t_0) = \psi_0 \end{cases} \tag{3.9}$$

has exactly two $\mathcal{C}^2$ *solutions* ψ_1, ψ_2, *the first solving* (3.8a), *the second one solving* (3.8b). *Moreover, if* $\psi_0 \in Z_D$, *then* $\psi \equiv \psi_0$ *is the unique solution to* (3.9)*;*

(b) if $D = -\frac{2}{n}\left(\frac{n-2}{2}\right)^n$, *then* $\psi \equiv \pm\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}$ *are the unique solutions to equation* (3.6).

Proof. Let I be an interval and $\psi : I \longrightarrow \mathbb{R}$ be a solution to (3.9) for $t_0 \in \mathbb{R}$, $\psi_0 \in P_D$. Suppose that $\psi'(t_0) > 0$ (if $\psi'(t_0) < 0$ the proof is very similar). Then we can consider the maximal interval $]a, b[\subset I$ such that

$$t_0 \in]a, b[$$
$$\psi' > 0 \text{ in }]a, b[.$$

If $]a,b[=I$ then we deduce that ψ solves (3.8a), and we conclude. If $]a,b[\subsetneq I$, then ψ' necessarily vanishes at either a or b, but this is a contradiction because $\psi_0\in P_D$ and Lemma 3.1.1 ensures that ψ cannot vanish. Finally, if $\psi_0\in Z_D$, if there existed a nonconstant solution to (3.9), then there would be $\widetilde{\psi_0}\in P_D$ such that ψ is a solution to (3.9) for initial data t_0, $\widetilde{\psi_0}$, but this is a contradiction by the characterization done above. Point (b) follows from the fact that, for $D=-\frac{2}{n}\left(\frac{n-2}{2}\right)^n$ the function A_D (defined before) is nonnegative only at the points $\pm\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}$, and we conclude. **QED**

Now we can prove the following classification result for the radial solutions to the Yamabe Equation in the punctured ball.

Theorem 3.1.2. *Let $u(x)=\phi(|x|)$ be a radial, C^2 solution to equation* (3.1) *(the Yamabe Equation in the punctured ball). Then one of the following conditions holds true:*

(a) $\phi(r)=\pm\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}} r^{\frac{2-n}{2}}$, *for* $r>0$;

(b) $\phi(r)=r^{\frac{2-n}{2}}\psi_{D,c,i}(-\ln r)$, *for* $r>0$, *where* $\psi_{D,c,i}$ *is the unique solution to*

$$\begin{cases} \psi'=(-1)^i\sqrt{A_D(\psi)} \\ \psi(0)=c, \end{cases}$$

for some $D>-\frac{2}{n}\left(\frac{n-2}{2}\right)^n$, $c\in\overline{P_D}$, $i\in\{0,1\}$.

In particular any radial solution to (3.1) *can be naturally extended to a radial solution to the Yamabe Equation in* $\mathbb{R}^n\setminus\{0\}$.

Proof. By substitution (3.4), we know that ϕ needs to coincide with $r^{\frac{2-n}{2}}\psi(r)$, for a suitable solution ψ to (3.6). Point (a) corresponds to the limiting situation in which $D=-\frac{2}{n}\left(\frac{n-2}{2}\right)^n$. Instead at point (b) we classify all the solutions to (3.6) under the assumption $D>-\frac{2}{n}\left(\frac{n-2}{2}\right)^n$ using Theorem 3.1.1 (we fix the initial time $t_0=0$ [5], we let the initial value vary in $\overline{P_D}$, and we choose the sign of the derivative of the solution). **QED**

Finally we want to answer two questions: which are the radial solutions to (3.1) that are nonnegative? Which are the radial solutions to (3.1) that are not singular at the origin?

Corollary 3.1.1. We have:

(a) the nonnegative, radial solutions to (3.1) are exactly

$$\phi(r)=\left(\frac{n-2}{2}\right)^{\frac{n-2}{2}} r^{\frac{2-n}{2}}$$
$$\phi(r)=r^{\frac{2-n}{2}}\psi_{D,c,i}(-\ln r),$$

for $0\geq D>-\frac{2}{n}\left(\frac{n-2}{2}\right)^n$, $c\in P_D\cap[0,+\infty[$, $i\in\{0,1\}$;

[5] Note that this is an arbitary choice: we could select whatever real value t_0.

(b) the radial solutions to (3.1) that are bounded around the origin are exactly

$$\begin{aligned}&\phi \equiv 0\\ &\phi(r) = r^{\frac{2-n}{2}}\psi_{D,c,i}(-\ln r);\end{aligned}$$

for $D = 0$ and either $c \in P_D \cap [0, +\infty[$, $i = -1$, or $c \in P_D \cap] - \infty, 0]$, $i = 1$.

Proof. The first statement is a straightforward consequence of all the theory developed up to now. To show (b), it's enough to observe that since the boundedness of ϕ around zero it follows that

$$\psi(t) \equiv e^{-\frac{n-2}{2}t}\phi(e^{-t})$$

needs to shrink to zero as $t \to +\infty$. Therefore the only possibility is that (b) holds. **QED**

In [8], one can find the precise computation of the radial, nonsingular solutions to the Yamabe Equation: the idea is to prove that, up to a further substitution, the solutions to equation (3.6) for $D = 0$ are exactly the solutions to the *Euler Equation*

$$\rho^2\theta''(\rho) + r\theta'(\rho) = \theta(\rho).$$

Before concluding the section, we introduce the following notation which will be used at the end of the chapter. We set

$$\Phi(D) = \begin{cases} \{r^{\frac{2-n}{2}}\psi_{D,c,i}(-\ln r) \text{ s.t. } c \in \overline{P_D},\ i \in \{0,1\}\} & \text{if } D > -\dfrac{2}{n}\left(\dfrac{n-2}{2}\right)^n \\ \{\pm\left(\dfrac{n-2}{2}\right)^{\frac{n-2}{2}} r^{\frac{2-n}{2}}\} & \text{if } D = -\dfrac{2}{n}\left(\dfrac{n-2}{2}\right)^n. \end{cases}$$

In other words, $\Phi(D)$ is the collection of the radial solutions that have energy $E = D/2$. Moreover we denote by $\Phi^+(D)$ the family formed by the nonnegative elements of $\Phi(D)$ (classified in the previous Corollary).

3.2 A first approach to the classification problem for the singular Yamabe Equation

In this section we develop a first analysis of the classification issue. First we try to understand how the Kelvin Transform comes into play in the study of a singular solution to a Yamabe-type Equation, and then we aim to prove some general decay estimates valid for the Kelvin Transform of the singular solution. These estimates will be strongly used along the work, representing the basis for the validity of the Reflection Theorem.

3.2.1 The role of the Kelvin Transform

In order to study a nonnegative, smooth solution to an equation

$$-\Delta u = g(u), \text{ in } B(0,1[\setminus\{0\} \tag{3.10}$$

with a nonremovable singularity at the origin, we will make use of the Kelvin Transform with respect to a point z close to the origin: in such a way, we transform u into a map v with a good behaviour at infinity and with a singularity far from the origin.

Given $z \in \mathbb{R}^n$, for $r > 0$, we set

$$\Phi_{z,r} : \mathbb{R}^n \setminus \{z\} \longrightarrow \mathbb{R}^n \setminus \{z\},\ \Phi_{z,r}(y) \equiv z + \frac{r^2}{|y-z|^2}(y - z):$$

$\Phi_{z,r}$ is a $\mathcal{C}^\infty$ diffeomorphism of $\mathbb{R}^n \setminus \{z\}$ in itself; moreover, $\Phi_{z,r}$ is involutive, that is

$$\Phi_{z,r} \circ \Phi_{z,r} = id_{\mathbb{R}^n \setminus \{z\}}.$$

We recall the following

Definition 3.2.1. Given $z \in \mathbb{R}^n$, let $A \subset \mathbb{R}^n \setminus \{z\}$ and $u : A \to \mathbb{R}$ be a function. For $r > 0$, we define the (z,r)-*Kelvin Transform* of u as the map

$$\mathcal{K}_{z,r}u : \Phi_{z,r}(A) \longrightarrow \mathbb{R}$$

defined by $(\mathcal{K}_{z,r}u)(y) \equiv |y-z|^{2-n}u(z + \frac{r^2}{|y-z|^2}(y-z)) = |y-z|^{2-n}u(\Phi_{z,r}(y))$.

We immediately note that $\mathcal{K}_{z,r}(\mathcal{K}_{z,r}u) = r^{2(2-n)}u$. Let $A \subset \mathbb{R}^n \setminus \{z\}$ be an open and u be a $\mathcal{C}^2(A)$ map: being $\Phi_{z,r}$ a smooth diffeomorphism, $\Phi_{z,r}(A)$ is an open and $\mathcal{K}_{z,r}u$ is $\mathcal{C}^2(\Phi_{z,r}(A))$. With a straightforward computation one can verify that

$$\Delta(\mathcal{K}_{z,r}u)(y) = r^4|y-z|^{-(n+2)}\Delta u(z + \frac{r^2}{|y-z|^2}(y-z)) = r^4|y-z|^{-(n+2)}\Delta u(\Phi_{z,r}(y)), \tag{3.11}$$

for any $y \in \Phi_{z,r}(A)$. In particular, u is harmonic if and only if $\mathcal{K}_{z,r}u$ is harmonic.

Let $u \geq 0$ be a singular, $\mathcal{C}^2$ solution to the equation (3.10) introduced at the beginning of the chapter. Given $z \in B(0,1[$, consider

$$v(x) \equiv (\mathcal{K}_{z,1}u)(x+z) = |x|^{2-n}u(z + \frac{x}{|x|^2}). \tag{3.12}$$

We note that the singularity of u at the origin is sent in $\overline{x} := -z/|z|^2$: this is a singularity for v as far from the origin as z is close to 0. Relation (3.11) guarantees that

$$\Delta v(x) = |x|^{-(n+2)}\Delta u(z + \frac{x}{|x|^2})$$

and hence that

$$-\Delta v(x) = |x|^{-(n+2)}g(u(z + \frac{x}{|x|^2})) = |x|^{-(n+2)}g(|x|^{n-2}v(x));$$

equivalently, in $\Phi_{z,1}(B(0,1[\setminus\{0,z\}) - z$, the following equation holds

$$-\Delta v(x) = f(|x|, v(x)), \tag{3.13}$$

where $f(r,v) \equiv r^{-(n+2)}g(r^{n-2}v)$. Therefore if u solves (3.10), then v solves (3.13) (in the open set $\Phi_{z,1}(B(0,1[\setminus\{0,z\}) - z)$. In the following, we will need another property of the Kelvin Transform: the Kelvin Transform preserves the notion of weak solution too. Let's try to understand more precisely what this expression does mean. Assume that the map g fulfills the two assumptions (i), (ii) of Lemma 1.2.3: thanks to Corollary 1.2.1, we know that the map u is a weak solution to equation (3.10) in the entire ball $B(0,1[$. We claim that map v as defined in (3.12) weakly solves equation (3.13) in $\Omega := \Phi_{z,1}(B(0,1[\setminus\{z\}) - z$ (note that we are including the singularity $-z/|z|^2$ of v). To prove so, we first observe that, if $K' := K - z$ is a compact in Ω, then

$$\begin{aligned}
\int_{K'} |f(|x|, v(x))|\,\mathrm{d}x &= \int_{K'} |x|^{-(n+2)}|g(u(\Phi_{z,1}(x+z)))|\,\mathrm{d}x \\
&= \int_{K} |x'-z|^{-(n+2)}|g(u(\Phi_{z,1}(x')))|\,\mathrm{d}x' \\
&= \int_{\Phi_{z,1}(K)} |\Phi_{z,1}(x'')-z|^{-(n+2)}|g(u(x''))||det(\mathrm{Jac}(\Phi_{z,1})(x''))|\,\mathrm{d}x'' \\
&\leq C(K)\int_{\Phi_{z,1}(K)} |g(u(x''))|\,\mathrm{d}x'' < \infty, \text{ because } g(u(\cdot)) \in L^1_{loc}(B(0,1[),
\end{aligned}$$

and similarly

$$\int_{K'} |v(x)| \, \mathrm{d}x \le C(K) \int_{\Phi_{z,1}(K)} |u(x)| \, \mathrm{d}x < \infty, \text{ because } u(\cdot) \in L^1_{loc}(B(0,1[).$$

Therefore both v and $f(|\cdot|, v(\cdot))$ lay in $L^1_{loc}(\Omega)$. Pick now $\zeta \in \mathcal{C}^\infty_c(\Omega)$, and, as $\epsilon > 0$, consider a function $\rho_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $0 \le \rho_\epsilon \le 1$ and

$$\begin{aligned} &\rho_\epsilon \equiv 0 \text{ in } B(-z/|z|^2, \epsilon/2] \\ &\rho_\epsilon \equiv 1 \text{ in } \mathbb{R}^n \smallsetminus B(-z/|z|^2, \epsilon[\\ &|\nabla \rho_\epsilon| \le C/\epsilon, \ |\Delta \rho_\epsilon| \le C/\epsilon^2. \end{aligned}$$

Now v is a solution to (3.13) in Ω and thus, for every $\epsilon > 0$, one has

$$\begin{aligned} 0 &= \int_\Omega (v\Delta(\rho_\epsilon \zeta) + f(|x|, v(x))(\rho_\epsilon \zeta)) \, \mathrm{d}x \\ &= \int_\Omega \rho_\epsilon (v \Delta \zeta + f(|x|, v(x))\zeta) \, \mathrm{d}x + \int_{B(-z/|z^2|, \epsilon] \smallsetminus B(-z/|z^2|, \epsilon/2[} (2v \nabla \zeta \cdot \nabla \rho_\epsilon + v \zeta \Delta \rho_\epsilon) \, \mathrm{d}x. \end{aligned}$$

Observing that as $\epsilon \to 0^+$ the first summand converges to

$$\int_\Omega (v \Delta \zeta + f(|x|, v(x))\zeta) \, \mathrm{d}x,$$

and the second one goes to 0 (both by dominated convergence), one deduces that what we claimed above holds true.

So let's resume what we have proved up to now: if u is a smooth solution to

$$-\Delta u = g(u), \text{ in } B(0,1[\smallsetminus \{0\},$$

then by Corollary 1.2.1 u is a weak solution to the same equation in the whole ball $B(0,1[$, and the function defined by

$$v(x) \equiv (\mathcal{K}_{z,1} u)(x+z) = |x|^{2-n} u(z + \frac{x}{|x|^2})$$

is a solution in $\Phi_{z,1}(B(0,1[\smallsetminus \{0,z\}) - z$, and a weak solution in $\Phi_{z,1}(B(0,1[\smallsetminus \{z\}) - z$ to equation

$$-\Delta v(x) = f(|x|, v(x)),$$

where $f(r,v) \equiv r^{-(n+2)} g(r^{n-2} v)$. The idea of all the next work is to analyze the behaviour of v (instead of u) in order to classify the solutions to equation (3.10) in the punctured ball.

3.2.2 Decay estimates

In the setting of the previous subsection, we observe that, as $|x| \to \infty$, one has

$$\begin{aligned} &f(|x|, v(x)) = \mathcal{O}(|x|^{-(n+2)}) \\ &v(x) = \mathcal{O}(|x|^{2-n}), \end{aligned}$$

because of the continuity of u at z (and trivially of g at $u(z)$). Actually, using the Taylor expansion, we may prove more precise decay estimates.

Lemma 3.2.1. For any $1 \le i, j \le n$, the solution v to equation (3.13) defined at (3.12) satisfies the following asymptotic expansions:

$$v(x) = \frac{1}{|x|^{n-2}}\left(a_0 + \sum_{k=1}^{n} a_k \frac{x_k}{|x|^2}\right) + \mathcal{O}(|x|^{-n}) \tag{3.14}$$

$$\frac{\partial v}{\partial x_i}(x) = -(n-2)a_0 \frac{x_i}{|x|^n} + \mathcal{O}(|x|^{-n}) \tag{3.15}$$

$$\frac{\partial^2 v}{\partial x_i \partial x_j}(x) = \mathcal{O}(|x|^{-n}) \tag{3.16}$$

as $|x| \to \infty$, where $a_0 := u(z)$, $a_k := \dfrac{\partial u}{\partial x_k}(z)$.

Proof. Let's start proving (3.14): as $|x| \to \infty$

$$|x|^{n-2} v(x) = u(z + \frac{x}{|x|^2}) = u(z) + \nabla u(z) \cdot \frac{x}{|x|^2} + \mathcal{O}(|x|^{-2}),$$

and then

$$v(x) = \frac{1}{|x|^{n-2}}\left(u(z) + \sum_i \frac{\partial u}{\partial x_i}(z)\, \frac{x_i}{|x|^2}\right) + \mathcal{O}(|x|^{-n}).$$

Furthermore we observe that

$$\begin{aligned}
&(n-2)|x|^{n-4} x_i\, v(x) + |x|^{n-2} \frac{\partial v}{\partial x_i}(x) = \\
&= -\frac{2}{|x|^4} \sum_j \frac{\partial u}{\partial x_j}\Big(z + \frac{x}{|x|^2}\Big) x_i x_j + \frac{1}{|x|^2} \frac{\partial u}{\partial x_i}\Big(z + \frac{x}{|x|^2}\Big) \\
&= -\frac{2x_i}{|x|^4} \sum_j \left(\frac{\partial u}{\partial x_j}(z) + \mathcal{O}(|x|^{-1})\right) x_j + \frac{1}{|x|^2}\left(\frac{\partial u}{\partial x_i}(z) + \mathcal{O}(|x|^{-1})\right).
\end{aligned}$$

Therefore from (3.14) it follows that

$$\begin{aligned}
(n-2)\frac{1}{|x|^2} x_i\, u(z) + \frac{(n-2)x_i}{|x|^4} \sum_j \frac{\partial u}{\partial x_j}(z)\, x_j + |x|^{n-2} \frac{\partial v}{\partial x_i}(x) &= \\
= -\frac{2x_i}{|x|^4} \sum_j \frac{\partial u}{\partial x_j}(z)\, x_j + \frac{1}{|x|^2} \frac{\partial u}{\partial x_i}(z) + \mathcal{O}(|x|^{-3}),&
\end{aligned}$$

that is exactly (3.15). A similar computation using (3.14) and (3.15) permits to prove the validity of (3.16). **QED**

These estimates turn out to be very important, assuring the validity of two crucial Lemmata which we are going to prove in the next section.

3.3 The Reflection Theorem

In this section we will prove the Reflection Theorem, but first we need some preliminary Lemmata whose proofs are strongly based on the results given in the past sections. We use the following notation: given a point $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \equiv \mathbb{R}^n$, we indicate by

$$x_\lambda := (x', 2\lambda - x_n)$$

the reflection of x across the hyperplane $x_n = \lambda$, as $\lambda \in \mathbb{R}$.

3.3.1 Some preliminary results

The following, first preliminary result will be directly involved at the beginning of the proof of the Reflection Theorem: in fact, it permits to start the reflection process, representing a sort of weak, more general version of the Reflection Theorem.

Lemma 3.3.1. Let v be a function of class $\mathcal{C}^2$ defined in an open neighbourhood of infinity. If v satisfies the asymptotic expansions (3.14), (3.15), (3.16) for some $a_0, a_1, ..., a_n \in \mathbb{R}$, $a_0 > 0$, then there exist two constants $\overline{\lambda}, \overline{R} > 0$ such that, from $\lambda \geq \overline{\lambda}$, it follows that

$$v(x) > v(x_\lambda), \text{ for } x_n < \lambda,\ |x| > \overline{R}.$$

Proof. Expansions (3.14), (3.15) ensure respectively that

$$v(x) - v(x_\lambda) = a_0\left(\frac{1}{|x|^{n-2}} - \frac{1}{|x_\lambda|^{n-2}}\right) + \sum_{j=1}^{n-1} a_j x_j \left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right) + \\ + a_n \frac{(x - x_\lambda)_n}{|x|^n} + a_n (x_\lambda)_n \left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right) + \mathcal{O}(|x|^{-n}),$$

$$\frac{\partial v}{\partial x_n}(x) = -(n-2) a_0 \frac{x_n}{|x|^n} + \mathcal{O}(|x|^{-n}), \tag{3.17}$$

where, thanks to $x_n < \lambda$, we have $|x_\lambda| = \sqrt{|x'|^2 + (2\lambda - x_n)^2} = \sqrt{|x'|^2 + 4\lambda^2 - 4\lambda x_n + x_n^2} \geq |x|$, so

$$\frac{1}{|x_\lambda|^n} = \mathcal{O}(|x|^{-n}).$$

Now if $|x_\lambda| \geq 2|x|$, then

$$v(x) - v(x_\lambda) \geq \frac{1}{2} a_0 \frac{1}{|x|^{n-2}} + \mathcal{O}(|x|^{-(n-1)}) \geq \frac{a_0}{4|x|^{n-2}} > 0,$$

for $|x|$ big. If instead $|x_\lambda| < 2|x|$, as $|x| \geq 1$, the following relations hold

$$\frac{1}{|x|^{n-2}} - \frac{1}{|x_\lambda|^{n-2}} = \frac{1}{|x|^{n-2}} - \frac{1}{|x_\lambda|^{n-3}} \frac{1}{|x_\lambda|} \geq \frac{1}{|x|^{n-3}}\left(\frac{1}{|x|} - \frac{1}{|x_\lambda|}\right) = \frac{|x_\lambda| - |x|}{|x|^{n-2}|x_\lambda|} \geq \frac{1}{2}\frac{|x_\lambda| - |x|}{|x|^{n-1}}$$

$$\left|\sum_{j=1}^{n-1} a_j x_j \left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right)\right| \leq \frac{c|x|}{|x|^{n-1}}\left(\frac{1}{|x|} - \frac{1}{|x_\lambda|}\right) = \frac{c}{|x|^{n-2}}\left(\frac{1}{|x|} - \frac{1}{|x_\lambda|}\right) \leq c\frac{|x_\lambda| - |x|}{|x|^n}$$

$$\frac{|(x - x_\lambda)_n|}{|x|^n} = \frac{2(\lambda - x_n)}{|x|^n} = \frac{1}{2\lambda}\frac{|x_\lambda|^2 - |x|^2}{|x|^n} \leq \frac{3}{2\lambda}\frac{|x_\lambda| - |x|}{|x|^{n-1}},$$

where the last inequality follows from $|x_\lambda|^2 - |x|^2 = (|x_\lambda| + |x|)(|x_\lambda| - |x|) \leq 3|x|(|x_\lambda| - |x|)$. We also have

$$\begin{aligned} |(x_\lambda)_n|\left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right) &\leq |x_\lambda|\left(\frac{1}{|x|^n} - \frac{1}{|x_\lambda|^n}\right) \\ &\leq \frac{c}{|x|^{n-2}}\left(\frac{1}{|x|} - \frac{1}{|x_\lambda|}\right) \\ &\leq c\frac{|x_\lambda| - |x|}{|x|^n}, \end{aligned} \tag{3.18}$$

up to a suitable choice of a constant $c > 0$ depending on a_j, n. Thus, for λ, $|x|$ big, one finds

$$v(x) - v(x_\lambda) \geq c_1 \frac{|x_\lambda| - |x|}{|x|^{n-1}} - \frac{c_2}{|x|^n},$$

for two suitable constants $c_1, c_2 > 0$. As a matter of fact, for $\lambda, |x|$ big, we have

$$\begin{aligned} v(x) - v(x_\lambda) &\geq \frac{1}{4} \frac{|x_\lambda| - |x|}{|x|^{n-1}} - 2c \frac{|x_\lambda| - |x|}{|x|^n} + \mathcal{O}(|x|^{-n}) \\ &= \frac{1}{4} \frac{|x_\lambda| - |x|}{|x|^{n-1}} \left(1 - 8c \frac{|x_\lambda| - |x|}{|x|^n} \frac{|x|^{n-1}}{|x_\lambda| - |x|}\right) + \mathcal{O}(|x|^{-n}) \\ &\geq \frac{1}{8} \frac{|x_\lambda| - |x|}{|x|^{n-1}} - \frac{c_2}{|x|^n}. \end{aligned}$$

Now we have two alternatives:

(a) $|x_\lambda| - |x| > \frac{c_2}{c_1} \frac{1}{|x|}$: then we trivially obtain $v(x) - v(x_\lambda) > 0$;

(b) $|x_\lambda| - |x| \leq \frac{c_2}{c_1} \frac{1}{|x|}$: this is equivalent to

$$\sqrt{1 + \frac{4\lambda^2 - 4\lambda x_n}{|x|^2}} \leq \frac{c_2}{c_1} \frac{1}{|x|^2} + 1,$$

or, equivalently,

$$\frac{|x|^2 + 4\lambda^2 - 4\lambda x_n}{|x|^2} \leq \frac{A}{|x|^4} + \frac{B}{|x|^2} + 1 \iff$$

$$\iff 4\lambda(\lambda - x_n) \leq \frac{A}{|x|^2} + B + 1 \iff x_n \geq \lambda - \frac{C}{\lambda}, \text{ for some } C > 0.$$

Thus by (3.17), for $|x|, \lambda$ big, we infer

$$\begin{aligned} \frac{\partial v}{\partial x_n}(x) &\leq -(n-2)\, a_0\, (\lambda - \frac{C}{\lambda}) \frac{1}{|x|^n} + \frac{C'}{|x|^n} \\ &= -(n-2)\, a_0\, \lambda + (n-2) \frac{a_0}{|x^n|} \frac{C}{\lambda} + \frac{C'}{|x|^n} \\ &< 0, \text{ along the segment } \{(x', (1-t)x_n + t(x_\lambda)_n) \mid 0 \leq t \leq 1\}, \end{aligned}$$

and then $v(x) > v(x_\lambda)$.

QED

The following technical Lemma will help us to find the contradiction needed to conclude the proof of the Reflection Theorem.

Lemma 3.3.2. Let $v \in \mathcal{C}^2(\mathbb{R}^n \setminus B(0,R])$ be a positive solution to

$$-\Delta v = F(x), \text{ in } |x| > R,$$

and assume that v respects the expansions (3.14), (3.15), (3.16). Suppose that, as $x_n > 0, |x| > R$:

(i) the relations below hold

$$\begin{aligned} v(x', x_n) &\le v(x', -x_n), \text{ for any } (x', x_n) \\ v(x', x_n) &\ne v(x', -x_n), \text{ at some point;} \end{aligned}$$

(ii) $F(x', x_n) \le F(x', -x_n)$, for any (x', x_n).

Then there exist $\epsilon > 0$, $S > R$ such that the following statements hold:

(1) $\dfrac{\partial v}{\partial x_n} < 0$ in $|x_n| < \epsilon$, $|x| > S$;

(2) $v(x', x_n) < v(x', 2\lambda - x_n) = v(x_\lambda)$ in $x_n > \epsilon/2$, $|x| > S$, for $0 < |\lambda| < \alpha\epsilon$, where $0 < \alpha < 1/2$ is small enough.

Proof. Let $w(x) \equiv v(x', -x_n) - v(x', x_n) \ge 0$, for $|x| > R$, $x_n \ge 0$: w is of class $\mathcal{C}^2$ up to the boundary, and furthermore

$$\begin{aligned} \Delta w(x) &= \Delta v(x', -x_n) - \Delta v(x', x_n) \\ &= F(x', x_n) - F(x', -x_n) \le 0. \end{aligned}$$

Surely w admits the outer partial derivative along $-e_n$ on $|x| > R$, $x_n = 0$, and accordingly, thanks to Hopf Lemma 1.1.3, $\partial w/\partial(-e_n) < 0$ on $|x| > R$, $x_n = 0$, because w is null there and is not constant by hypothesis. Thus

$$\frac{\partial w}{\partial x_n} > 0, \text{ in } |x| > R,\ x_n = 0.$$

Moreover, thanks to Maximum Principle 1.1.1, being $w \ge 0$ in $|x| > R$, $x_n > 0$ and nonconstant there, w cannot vanish on $|x| = R+1$, $x_n > 0$. The compactness of $|x| = R+1$, $x_n \ge 0$ allows then to deduce that

$$w(x) \ge \frac{k\, x_n}{|x|^n} = \frac{k\, x_n}{(R+1)^n},$$

on $|x| = R+1$, $x_n > 0$, for k small enough. Now $\dfrac{x_n}{|x|^n}$ is harmonic out of the origin and then, applying Maximum Principle 1.1.1 again, we infer

$$w(x) > \frac{k\, x_n}{|x|^n},$$

in $|x| > R+1$, $x_n > 0$. Then it follows that:

$$\begin{aligned} w_{x_n}(x', 0) &= \lim_{x_n \to 0^+} \frac{w(x', x_n) - w(x', 0)}{x_n} \\ &= \lim_{x_n \to 0^+} \frac{w(x', x_n)}{x_n} \ge \lim_{x_n \to 0^+} \frac{k\, x_n}{|x|^n} \frac{1}{x_n} = \frac{k}{|x'|^n} \end{aligned}$$

and thus

$$v_{x_n}(x', 0) \le -\frac{k}{2|x'|^n},$$

as $|x'| > R+1$. Now, given $h > 0$, being v two times differentiable,

$$v_{x_n}(x', h) - v_{x_n}(x', 0) = v_{x_n x_n}(x', h^*)h,$$

for a suitable $0 \le h^* \le h$, for every $|x'| > R$. The decay estimate (3.16) ensures that

$$v_{x_n x_n}(x) = \mathcal{O}(|x|^{-n}),$$

where $x = (x', h)$, and then, as $|x'|$ big, for a suitable constant $C > 0$, one has

$$\begin{aligned} v_{x_n}(x', h) &\le v_{x_n}(x', 0) + \frac{C\,|h|}{|x|^n} \\ &\le -\frac{1}{2}\frac{k}{|x'|^n} + \frac{C\,|h|}{|x|^n} \le -\frac{1}{4}\frac{k}{|x'|^n}, \end{aligned}$$

up to the observation that $|x| \ge |x'|$ and the a choice of h such that $|h| \le k/4C \equiv \epsilon$. Then (1) holds true.

Let's now demonstrate (2). We note that, as $|x| > R+1$, $|\lambda|$ small,

$$v(x', 2\lambda - x_n) - v(x', -x_n) = v_{x_n}(x', \mu - x_n)2\lambda,$$

for some μ, and using (3.15), for $x_n > 0$ and $|x|$ big,

$$\begin{aligned} v_{x_n}(x', \mu - x_n) &= -(n-2)\,a_0\,\frac{\mu - x_n}{|x|^n} + \mathcal{O}(|x|^{-n}) \\ &\ge -\frac{c}{|x|^n}(x_n + c), \text{ for some } c > 0. \end{aligned}$$

Thus $v(x', 2\lambda - x_n) - v(x', x_n) \ge -\dfrac{c\,\lambda}{|x|^n}(x_n + c)$, and so

$$\begin{aligned} & v(x', 2\lambda - x_n) - v(x', x_n) = \\ &= (v(x', -x_n) - v(x', x_n)) + (v(x', 2\lambda - x_n) - v(x', -x_n)) \\ &\ge \frac{k\,x_n}{|x|^n} - \frac{c\,|\lambda|\,(x_n + c)}{|x|^n} = \frac{(k - c|\lambda|)\,x_n - c|\lambda|}{|x|^n} \\ &\ge \frac{k\epsilon/4 - c|\lambda|}{|x|^n}, \end{aligned}$$

taking $x_n > \epsilon/2$, $|\lambda| < \dfrac{k}{4\,c}\epsilon$, and we conclude. **QED**

We recall that

$$F(x) \equiv f(|x|, v(x)) = |x|^{-(n+2)}\,g(|x|^{n-2}v(x)),$$

where $v \ge 0$ is the solution (3.12). Assume that, given $\lambda \ge 0$, $v(x_\lambda) \ge v(x)$, as $x_n > \lambda$. We want to impose conditions on g in order to have

$$F(x_\lambda) \ge F(x). \tag{3.19}$$

So let's introduce the following two assumptions:

$$g \text{ is nondecreasing in } \mathbb{R} \tag{3.20}$$

$$t^{-\frac{n+2}{n-2}}\,g(t) \text{ is nonincreasing in } \mathbb{R}. \tag{3.21}$$

Therefore, as $x_n > \lambda$, setting $s := |x_\lambda|^{n-2}v(x)$, $t := |x|^{n-2}v(x)$, we have:

$$\begin{aligned} F(x_\lambda) &= |x_\lambda|^{-(n+2)}\,g(|x_\lambda|^{n-2}v(x_\lambda)) \\ &\ge |x_\lambda|^{-(n+2)}\,g(|x_\lambda|^{n-2}v(x)) \\ &= v(x)^{\frac{n+2}{n-2}}\,s^{-\frac{n+2}{n-2}}\,g(s) \\ &\ge v(x)^{\frac{n+2}{n-2}}\,t^{\frac{n+2}{n-2}}\,g(t) \\ &\dots \\ &\ge |x|^{-(n+2)}\,g(|x|^{n-2}v(x)) = F(x), \end{aligned}$$

that is what we wanted to achieve. Note that the inequality (3.19) holds in particular as we treat the Yamabe Equation, namely for $g(t) \equiv t^{\frac{n+2}{n-2}}$.

We now pass to state an prove an Extension Lemma which will be involved in the final part of the proof of the Reflection Theorem.

Lemma 3.3.3 (**Extension Lemma**)**.** Given $0 < \beta < 1$, let

$$v \in \mathcal{C}^{1,\beta}(B(0,2] \setminus B(0,1[),\ v > 0$$

be a weak, nonnegative supersolution to

$$-\Delta v = f(x,v), \text{ in } 1 \le |x| \le 2,$$

where $f(x,\cdot) \in L^\infty_{loc}(\mathbb{R})$, $f(\cdot,v) \in L^\infty(B(0,2[)$, for any $x \in B(0,2]$, $v \in \mathbb{R}$. In addition, assume that there exist $0 < \delta_0 < 1$, $M > 0$ such that

$$\delta_0 \le v \le 1/\delta_0,\ \|v\|_{\mathcal{C}^{1,\beta}(B(0,2]\setminus B(0,1[)} \le M.$$

Then there exists a value $\sigma \equiv \sigma(\beta, n, \delta_0, M, f) > 0$ such that, if $\varnothing \ne A \subset B(0,1[$ is open and $|A| \le \sigma$, then v can be extended to a map $\overline{v} \in \mathrm{Lip}(B(0,2[)$ which satisfies the properties below:

(a) $\delta_0/2 \le \overline{v} < 3/\delta_0$ in $B(0,2[$;

(b) $\overline{v}$ is a weak supersolution to

$$-\Delta\overline{v} = f(x,\overline{v}), \text{ in } \{1 < |x| < 2\} \cup A;$$

(c) $\overline{v}_\nu \ge M + 1$ on $\partial B(0,1[$.

Roughly speaking, the Extension Lemma says that a supersolution in the ring $1 \le |x| \le 2$ can be extended to a supersolution on a sufficiently small measure open subset contained in the unit ball.

Proof (of the Extension Lemma). We first extend v to $B(0,1]$ choosing $\widetilde{v} \in \mathcal{C}^{1,\beta}(B(0,1])$ such that

$$\begin{aligned}
&\delta_0/2 \le \widetilde{v} \le 2/\delta_0\\
&|\Delta\widetilde{v}| \le C(\delta_0, M) \equiv C, \text{ for a suitable constant } C\\
&\widetilde{v} = v,\ \widetilde{v}_\nu \ge M + 2 \text{ on } \partial B(0,1[
\end{aligned}$$

(such a map $\widetilde{v}$ can be built setting $\widetilde{v} \equiv \widetilde{v}_0 + \rho$, where $\widetilde{v}_0$ is the harmonic extension of $v|_{\partial B(0,1[}$ to the whole unit ball, and ρ is a suitable radial corrector). By Theorem 1.4.3, we know that, for any $0 < \beta < 1$, there is $\sigma \equiv \sigma(\beta, n, \delta_0, M, f)$ such that, since $|A| \le \sigma$, it follows that the only $H^1_0(B(0,1[)$-weak solution to

$$\begin{cases}
-\Delta w = (C + \sup\limits_{\substack{x\in B(0,1]\\ 0\le t\le 3/\delta_0}} |f(x,t)|)\chi_A & \text{in } B(0,1[\\
w = 0 & \text{on } \partial B(0,1[.
\end{cases}$$

is of class $\mathcal{C}^{1,\beta}(B(0,1])$ and $\|w\|_{\mathcal{C}^{1,\beta}(B(0,1])} \le 1$. By superharmonicity we have $w \ge 0$, and so

$$\delta_0/2 \le \widetilde{v} + w \le 2/\delta_0 + 1 < 3/\delta_0.$$

Thus, by construction, we also infer that

$$\begin{cases}
-\Delta(\widetilde{v} + w) \ge \sup\limits_{\substack{x\in B(0,1]\\ \delta_0/2\le t\le 3/\delta_0}} |f(x,t)| \ge f(x, \widetilde{v} + w) & \text{in } A\\
(\widetilde{v} + w)_\nu \ge M + 1 & \text{on } \partial B(0,1[.
\end{cases}$$

Therefore, observed that $\widetilde{v} + w$ is Lipschitz, the choice $\overline{v} \equiv \widetilde{v} + w$ turns out to be good. **QED**

3.3.2 The Reflection Theorem: statement and proof

We are now ready to prove the following

Theorem 3.3.1 (**Reflection Theorem**). *Let $v > 0$ be a weak solution to*

$$-\Delta v = f(x, v) \equiv F(x),$$

in $\{|x| \geq 1\}$ [6], where $f : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that $f(x, \cdot) \in L^\infty_{loc}(\mathbb{R})$, for any $x \in B(0, 2]$, $f(\cdot, v) \in L^\infty(B(0, 2[)$, for any $v \in \mathbb{R}$, and $f \geq 0$ as $v \geq 0$. Assume that:

(i) v is of class $\mathcal{C}^2$ in $\{1 \leq |x| \leq 2\} \cup \{|x| > R\} \cup \{x_n > 1\}$ (the grey region in the next figure), for some $R > 2$, and v is lower semicontinuous;

(ii) v satisfies the asymptotic expansions (3.14), (3.15), (3.16);

(iii) provided that $x_n > \lambda > 0$ and $v(x) \leq v(x_\lambda)$, $F(x) \leq F(x_\lambda)$ holds true;

(iv) there exist δ_0, $C > 0$ such that, for some $0 < \beta < 1$,

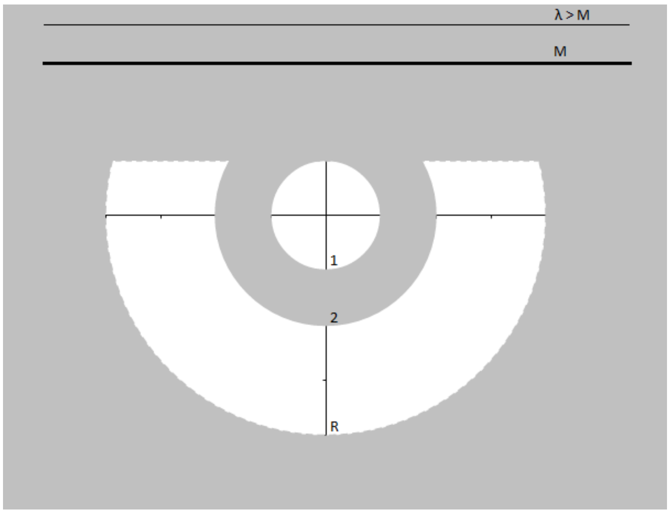
$$0 < \delta_0 \leq v|_{B(0,2[\smallsetminus B(0,1]} \leq \frac{1}{\delta_0}, \ \|v\|_{\mathcal{C}^{1,\beta}(B(0,2[\smallsetminus B(0,1])} \leq C;$$

(v) there exists an open $A' \subset \{(x', 0) \mid |x'| < 1\}$ such that, taken the value σ given by the Extension Lemma, one has $|A'| < \sigma/2$ and there exists $M > 2$ such that, if $x = (x', x_n)$, for $x' \notin A'$, $|x'| < 1$ and $x_n \geq M$, then

$$v(x) \leq \frac{\delta_0}{4}.$$

Therefore, denoting by $\overline{v}$ the extension of v whose existence is ensured by Extension Lemma for $A = \{x = (x', x_n) \mid x' \in A', |x| < 1\}$,

$$\overline{v}(x) \leq \overline{v}(x_\lambda), \ as \ x_n > \lambda \geq M.$$



<u>*Proof.*</u> Lemma 3.3.1 ensures us that there exist $\overline{\lambda}$, $\overline{R} > R$ such that, for $\lambda \geq \overline{\lambda}$,

$$\overline{v}(x) < \overline{v}(x_\lambda), \text{ for } x_n > \lambda, \ |x_\lambda| > \overline{R}.$$

[6] In other words, we require that there exists a $\delta > 0$ such that v is a weak solution to the equation in the open set $\{|x| > 1 - \delta\}$.

Observing that v is positive and lower semicontinuous in $\{|x| \geq 1\}$, v attains a positive minimum in $\{1 \leq |x| \leq \overline{R}\}$, and so we may consider a radius $\widetilde{R} > \overline{R}$ such that

$$|x| \geq \widetilde{R} \Rightarrow |\overline{v}(x)| \leq (\min_{1 \leq |x| \leq \overline{R}} \overline{v}(x))/2,$$

where such an $\widetilde{R}$ exists because $\overline{v}$ shrinks to 0 as $|x|$ goes to infinity. Now, for $\lambda \geq \max\{\overline{\lambda}, \widetilde{R}\}$,

$$\overline{v}(x) < \overline{v}(x_\lambda), \text{ for } x_n > \lambda,\ |x_\lambda| > \overline{R},$$

thanks to what we observed above. Furthermore, whenever $|x_\lambda| \leq \overline{R}$, we have

$$\overline{v}(x_\lambda) \geq \min_{1 \leq |x| \leq \overline{R}} \overline{v}(x) > \max_{|x| \geq \widetilde{R}} |\overline{v}(x)| \geq \max_{|x| \geq \overline{\lambda}} |\overline{v}(x)| \geq \overline{v}(x).$$

Therefore as λ_0 big enough

$$\overline{v}(x) \leq \overline{v}(x_\lambda), \text{ for } x_n > \lambda \geq \lambda_0. \tag{3.22}$$

It remains to show that we can take $\lambda_0 = M$. To prove so, the idea is to demonstrate that the set of the $\lambda \geq M$ such that relation (3.22) holds is both open and closed in $[M, +\infty[$. We start proving that it's open. To do so, let $\overline{\lambda} > M$ be such that

$$\overline{v}(x) \leq \overline{v}(x_{\overline{\lambda}}), \text{ for } x_n > \overline{\lambda}.$$

We want to show that there exists a neighbourhood U of $\overline{\lambda}$ in $[M, +\infty[$ such that $\lambda \in U$ implies

$$\overline{v}(x) \leq \overline{v}(x_\lambda), \text{ for } x_n > \lambda.$$

By contradiction, let this proposition be false. Then there are a sequence $\{\lambda_j\}_{j \geq 1} \subset]M, +\infty[$ and a sequence $\{x^j\}_{j \geq 1}$ of points such that

$$\lambda_j \to \overline{\lambda},\ x_n^j > \lambda_j,\ \overline{v}(x^j) > \overline{v}(x^j_{\lambda_j}).$$

We claim that one may extract a subsequence from $\{x^j\}_{j \geq 1}$ converging to a point $\overline{x}$ with $\overline{x}_n \geq \overline{\lambda}$. It sufficies to show that $\{x^j\}_{j \geq 1}$ has to be bounded. If it were false, one could find a subsequence $\{x^{j_k}\}_{k \geq 1}$ such that $|x^{j_k}| \to \infty$. Referring our notations to Lemma 3.3.2 (extended to the case of $\overline{\lambda}$ general, not necessarily zero), choosing $\overline{k}$ big enough, one can suppose

$$|\lambda_{j_k} - \overline{\lambda}| < \epsilon/4,\ |x^{j_k}| > S, \text{ as } k \geq \overline{k}.$$

There are two cases:

(a) up to a further subsequence,

$$\lambda_{j_k} < x_n^{j_k} < \lambda_{j_k} + \frac{3}{4}\epsilon, \text{ if } k \geq \overline{k}.$$

Therefore, using (1) of Lemma 3.3.2, one deduces that

$$\overline{v}(x^{j_k}) < \overline{v}(x^{j_k}_{\lambda_{j_k}}),$$

and this is a contradiction;

(b) definitively $x_n^{j_k} \geq \lambda_{j_k} + \frac{3}{4}\epsilon$. By construction, $x_n^{j_k} > 0$ and then, by (2) of Lemma 3.3.2, one has

$$\overline{v}(x^{j_k}) < \overline{v}(x^{j_k}_{\lambda_{j_k}}),$$

and a further contradiction occurs.

Thus $\{x^j\}_{j\geq 1}$ has to be bounded, and so to admit a converging subsequence $\{x^{j_k}\}_{k\geq 1}$ whose limit is a suitable point $\overline{x}$. Clearly

$$\begin{aligned}\overline{x}_n &= \overline{x}_n - x_n^{j_k} + x_n^{j_k} \geq -|\overline{x}_n - x_n^{j_k}| + x_n^{j_k} = \\ &= -|\overline{x}_n - x_n^{j_k}| + x_n^{j_k} - \lambda_{j_k} + \lambda_{j_k} - \overline{\lambda} + \overline{\lambda} \\ &\geq -|\overline{x}_n - x_n^{j_k}| - |\lambda_{j_k} - \overline{\lambda}| + \overline{\lambda} \to \overline{\lambda}.\end{aligned}$$

We have three possibilities:

(a) $\overline{x}_n = \overline{\lambda}$ and $\overline{v}_{x_n}(\overline{x}) \geq 0$;

(b) $\overline{x}_n > \overline{\lambda}$ and $|\overline{x}_{\overline{\lambda}}| \geq 1$;

(c) $\overline{x}_n > \overline{\lambda}$ and $|\overline{x}_{\overline{\lambda}}| < 1$.

In all these cases, consider $w(x) \equiv \overline{v}(x_{\overline{\lambda}}) - \overline{v}(x)$, for $x_n > \overline{\lambda}$: w is nonnegative and weakly superharmonic, thanks to Lemma 1.2.2, in $\{x_n > \overline{\lambda}\} \cap |x_{\overline{\lambda}}| > 1 - \delta$, for a $\delta > 0$ small enough; furthermore the lower semicontinuity of $\overline{v}(x_{\overline{\lambda}})$ ensures that w is lower semicontinuous and that $w(\overline{x}) = 0$, because

$$\begin{aligned}0 \leq w(\overline{x}) &= \overline{v}(\overline{x}_{\overline{\lambda}}) - \overline{v}(\overline{x}) \\ &\leq \liminf_{k\to\infty} [\overline{v}(x^{j_k}_{\lambda_{j_k}}) - \overline{v}(x^{j_k})] + [\overline{v}(x^{j_k}) - \overline{v}(\overline{x})] \\ &= \liminf_{k\to\infty} \overline{v}(x^{j_k}_{\lambda_{j_k}}) - \overline{v}(x^{j_k}) \leq 0.\end{aligned}$$

If (a) holds true, then, up to a suitable choice of $r > 0$ in order that $1 < \overline{\lambda} - r < 2$, we have

$$w \in \mathcal{C}^2(\{\overline{\lambda} \leq x_n \leq \overline{\lambda} + r\})$$

and $\overline{x} \in \{x_n = \overline{\lambda}\}$. Now w is nonidentically zero in such a strip, because for $x_n \geq \overline{\lambda} > M$, as $x' \notin A'$, $|x'| < 1$, $\overline{v}(x) \leq \delta_0/4$, but in the annulus of radii 1, 2 we have $\overline{v} \geq \delta_0$ (by hypotheses). Being $w(\overline{x}) = 0$, thanks to Hopf Lemma 1.1.3, $\dfrac{\partial w}{\partial(-e_n)}(\overline{x}) < 0$ and thus

$$-2\overline{v}_{x_n}(\overline{x}) = w_{x_n}(\overline{x}) > 0,$$

and this is a contradiction.

If (b) holds true, then the lower semicontinuity and the weak superharmonicity of w allow to apply the Maximum Principle for weakly subharmonic maps 1.3.1 to $-w$ in

$$\{x \mid x_n > \overline{\lambda}, |x_{\overline{\lambda}}| > 1 - \delta\}.$$

Observing that like in the case (a), w cannot be identically zero, we deduce that $w(\overline{x}) = 0$ is a contradiction, because $\overline{x}$ lays in the interior.

Finally if (c) holds true, then $\overline{x}' \in A'$ because otherwise

$$w(\overline{x}) = \overline{v}(\overline{x}_{\overline{\lambda}}) - \overline{v}(\overline{x}) \geq \frac{\delta_0}{2} - \frac{\delta_0}{4} = \frac{\delta_0}{4} > 0,$$

and this is a contradiction. Thus picking $r > 0$ such that $B_{\mathbb{R}^{n-1}}(\overline{x}', r[\subset A'$, we may apply to $-w$ the Maximum Principle 1.3.1 for weakly subharmonic functions in

$$\{x \mid |x_{\overline{\lambda}}| < 1, x_n > \overline{\lambda}\} \cap B_{\mathbb{R}^{n-1}}(\overline{x}', r[\times \mathbb{R},$$

because, thanks to the Extension Lemma and Lemma 1.2.2, w has to be weakly superharmonic. As in (b), this is a contradiction.

Therefore we proved that the set of the $\lambda \geq M$ such that relation (3.22) holds is open in $[M, +\infty[$. Let now $\{\lambda_j\}$ be a sequence in $[M, +\infty[$ converging to $\overline{\lambda}$: we want to prove that, for all x such that $x_n < \overline{\lambda}$,

$$\overline{v}(x) \geq \overline{v}(x_\lambda).$$

To do so, it's sufficient to note that since $x_n < \overline{\lambda}$ it follows that, as $\overline{j}$ is big enough, $x_n < \lambda_j$, for $j \geq \overline{j}$, and

$$\overline{v}(x_{\lambda_j}) \to \overline{v}(x_{\overline{\lambda}}).$$

Thus the set of the $\lambda \geq M$ such that (3.22) is valid is both open and closed in $[M, +\infty[$, and so it necessarily coincides with $[M, +\infty[$. This concludes the proof. **QED**

We have the following two remarks about the Reflection Theorem:

(1) M plays the role of λ_1 in Theorem 2.2.1;

(2) assumption (v) is the strongest: from now on, a direction τ along which (v) holds will be called *admissible* (in the statement we tacitly assumed that $\tau = e_n$. This is general up to an orthogonal change of coordinates).

3.4 Estimate of the measure of the set of the admissible directions

The objective of this section is to estimate the (surface) measure of the collection of nonadmissible directions τ (seen as unit vectors contained in the unit sphere). To do so, we need to introduce two assumptions on a map v defined in the complement of the unit ball:

(a) $v \geq 0$ is weakly superharmonic and lower semicontinuous in $\mathbb{R}^n \setminus B(0,1]$;

(b) $\int_{\mathbb{R}^n \setminus B(0,1]} v^p/|x|^\beta < c_0 < \infty$, for some $p \geq 1$, $\beta < n$.

From now on, we will work under these two assumptions. Note that both (a) and (b) are surely satisfied by a map v which respects the hypotheses of the Reflection Theorem: as a matter of fact (a) trivially holds, and the validity of (b) is ensured by the decay estimate

$$v = \mathcal{O}(|x|^{2-n}) \text{ as } |x| \to \infty.$$

Let's start with some notations: given a unit vector τ, we set

$$\Gamma(\tau) := \{\lambda\tau + \mu u \mid \lambda \geq 0,\ |\mu| < 3,\ u \in \tau^\perp,\ |u| = 1.\}$$
$$\Gamma_k(\tau) := \Gamma(\tau) \cap (B(0, 2^{k+1}[\setminus B(0, 2^k]),\ \text{as } k \geq 0;$$

we denote by P_τ^k the orthogonal projection along the direction τ on the hyperplane $x \cdot \tau = 2^k$.

Definition 3.4.1. Given $k \geq 0$, $\mu > 0$ the (k,μ)*-exceptional set* is defined by

$$A(k,\mu) := \{\tau \mid |P_\tau^k(\{v(x) > \mu\} \cap \Gamma_k(\tau))|_{n-1} > \mu\}.$$

We already state the crucial, main result of the section.

Theorem 3.4.1. *There exists a constant $C > 0$ independent of μ and k such that*

$$|A(k,\mu)| \leq \frac{C}{\mu^2} 2^{-\frac{k}{p(n-\beta)}}.$$

In particular, for $\mu = 2^{-\delta k}$, $\delta = \frac{n-\beta}{3p}$, $A(k) := A(k, 2^{-\delta k})$, one has

$$|A(k)| \leq C\, 2^{-\delta k}. \tag{3.23}$$

The proof of this result is not trivial and will be given later. We first explain how to use such a result for our study, showing that it's precisely what we need. Let $k_0 \geq 0$ be an integer and τ be a unit vector not contained in $\bigcup_{k\geq k_0} A(k)$: then by definition, for any $k \geq k_0$, one has

$$|P_\tau^k(\{v(x) > 2^{-\delta k}\} \cap \Gamma_k(\tau))|_{n-1} \leq 2^{-\delta k}.$$

Then let

$$B'_\tau := \bigcup_{k\geq k_0} [-2^k\tau + P_\tau^k(\{v(x) > 2^{-\delta k}\} \cap \Gamma_k(\tau))]:$$

we observe that $B'_\tau \subset \{x \cdot \tau = 0\} \cap B(0,3[$ and

$$|B'_\tau|_{n-1} \leq \sum_{k\geq k_0} 2^{-\delta k} = c^*\, 2^{-\delta k_0}, \text{ where } c^* = \frac{2^\delta}{2^\delta - 1}.$$

Moreover, if $x' \notin B'_\tau$, $x' \in B(0,3[$, setting

$$\begin{aligned} x &:= x' + \alpha\tau,\ \alpha \geq 2^{k_0} \\ k_\alpha &:= \max\{k \geq 1 \mid 2^k \leq \alpha\} \geq k_0, \end{aligned}$$

one has

$$v(x) \leq 2^{-\delta k_\alpha} (\leq 2^{-\delta k_0}),$$

because if $v(x) > 2^{-\delta k_\alpha}$ then $x \in \{v(x) > 2^{-\delta k_\alpha}\} \cap \Gamma_{k_\alpha}(\tau)$ and so $x' \in B'_\tau$, and this is a contradiction. Thus, if $k_0 > \max\{\delta^{-1}\log_2(\frac{2c^*}{\sigma}), \delta^{-1}\log_2(\frac{4}{\delta_0})\}$ and $\tau \notin \bigcup_{k\geq k_0} A(k)$, then τ is an sdmissible direction: in fact, by construction, setting

$$A'_\tau := B'_\tau \cap B(0,1[,$$

A'_τ is open (in $x \cdot \tau = 0$) and $|A'_\tau|_{n-1} < \sigma/2$. Moreover, if $x' \notin A'_\tau$, $x' \in B(0,1[$, since $x = x' + x \cdot \tau$, $x \cdot \tau \geq M = 2^{k_0}$, it follows that

$$v(x) \leq 2^{-\delta k_0} \leq \frac{\delta_0}{4}.$$

We now have to demostrate Theorem 3.4.1.

Proof (of Theorem 3.4.1). We estimate the measure of $A(k,\mu)$ by covering it with a union of spherical caps $D(\tau_i)$, $1 \leq i \leq m$, centered at a unit vector τ_i and of radius $C2^{-k}$, where C is a constant independent of k chosen in order that the measure of the radial projection of $P_\tau^k(\Gamma_k(\tau_i))$ onto the unit sphere coincides with the measure of $D(\tau_i)$, fork big. We have

$$\begin{aligned} |A(k,\mu)| &\leq \sum_{i=1}^m |D(\tau_i)| && (3.24)\\ &\leq C(2^{-k})^{n-1} \sum_{i=1}^m |P_{\tau_i}^k(\Gamma_k(\tau_i))| && (3.25)\\ &\leq C(2^{-k})^{n-1}\frac{1}{\mu} \sum_{i=1}^m |P_{\tau_i}^k(\{v(x) > \mu\} \cap \Gamma_k(\tau))|. && (3.26) \end{aligned}$$

We want to estimate the right hand side of the relation above by the average of v. So let w be the capacitory potential of $2^{1-k}E$ in $U := B(0,8[\setminus B(0,1]$ where

$$E := \bigcup_{i=1}^m \{v(x) > \mu\} \cap \Gamma_k(\tau_i):$$

by definition, w is harmonic in $U \setminus E$, $w = 1$ on E and $w = 0$ on $2U$.By the Maximum Principle 1.1.1 we know

$$v(2^{k-1}x) \geq \mu w$$

in U. By definition of capacitory potential, we know that the capacity is given by the following formula (here ν is the interior normal)

$$\operatorname{cap}(2^{1-k}E) = \int_U |\nabla w|^2 \,\mathrm{d}x = \int_{\partial U} w_\nu \,\mathrm{d}\sigma. \tag{3.27}$$

Moreover, we have

$$\frac{n-2}{R^{n-1}} \int_{\partial B(0,R[} w = \left(1 - \frac{1}{R^{n-2}}\right) \int_{\partial B(0,1[} w_\nu \qquad \text{for } 1 < R < 2, \tag{3.28}$$

$$\frac{n-2}{R^{n-1}} \int_{\partial B(0,R[} w = \left(\frac{1}{R^{n-2}} - \frac{1}{8^{n-2}}\right) \int_{\partial B(0,8[} w_\nu \qquad \text{for } 4 < R < 8. \tag{3.29}$$

Using (3.27) and (3.28), (3.29) we obtain the following estimate:

$$\int_U |\nabla w|^2 \,\mathrm{d}x \leq c \fint_U w \,\mathrm{d}x \leq \frac{c}{\mu} \fint_{B(0,2^{k+2}[\setminus B(0,2^{k-1}[} v \,\mathrm{d}x. \tag{3.30}$$

Denote by $\overline{P^k_\tau}$ the orthogonal projection P^k_τ composed on the left with the radial projection onto the unit sphere. By construction, $\overline{P^k_\tau}(\Gamma_k(\tau))$ is essentially $D(\tau)$ for large k. Given a point Q in $\overline{P^k_\tau}(E)$, let $\hat{Q}$ be the first point of $(\overline{P^k_\tau})^{-1}(Q)$ on the section curve $\widetilde{\gamma} = 2^{1-k}\gamma$ sitting over Q. We have

$$1 = \int_{\widetilde{\gamma}} \frac{\mathrm{d}}{\mathrm{d}s} w \,\mathrm{d}s.$$

Integrating over $\overline{P^k_\tau(E)}$ one achieves

$$|\overline{P^k_\tau}(E)| \leq \int_{\overline{P^k_\tau}(E)} \int_{\widetilde{\gamma}} |\nabla w| \,\mathrm{d}s \,\mathrm{d}r,$$

and thus by Hölder's Inequality

$$|\overline{P^k_\tau}(E)| \leq \int_{D(\tau)} \int_{\widetilde{\gamma}} |\nabla w|^2 \,\mathrm{d}s \,\mathrm{d}r. \tag{3.31}$$

The disks $D(\tau_i)$ can be chosen in order to have finite overlapping and so by (3.24), (3.31) we deduce

$$|A(k,\mu)| \leq \frac{c}{\mu} \int_U |\nabla w|^2 \,\mathrm{d}x.$$

Exploiting (3.30), we infer

$$|A(k,\mu)| \leq \frac{c}{\mu^2} \fint_{R_k} v \,\mathrm{d}x, \tag{3.32}$$

where $R_k \coloneqq B(0, 2^{k+2}[\setminus B(0, 2^{k-1}]$. We now use assumptions (a), (b) written at the beginning, obtaining

$$\begin{aligned} \fint_{R_k} v \,\mathrm{d}x &\leq \frac{c}{2^{nk}} \int_{R_k} v \,\mathrm{d}x \\ &\leq \frac{c}{2^{nk}} 2^{nk(1-1/p)} 2^{k\beta/p} \left(\int_{\mathbb{R}^n \setminus B(0,1]} \frac{v^p}{|x|^\beta} \,\mathrm{d}x \right) \\ &\leq c 2^{\frac{k(\beta-n)}{p}}. \end{aligned} \tag{3.33}$$

Combining (3.32) and (3.33) we deduce that

$$|A(k,\mu)| \leq \frac{c}{\mu^2} 2^{\frac{k(\beta-n)}{p}},$$

that is exactly what we aimed to show. **QED**

3.5 The asymptotic symmetry

The last step before considering the applications of the Reflection Theorem is to understand how it allows to infer the asymptotic symmetry, and what this terminology would mean.

Given a unit vector τ, we set

$$x_\lambda \equiv x + (2\lambda - x\cdot\tau)\tau,\ x \in \mathbb{R}^n,$$

the reflection of x in the hyperplane $z\cdot\tau = \lambda$, as $\lambda \in \mathbb{R}$. Moreover, in order to have a less heavy notation, we set $\mathbb{S} := \partial B(0,1[$ $(R\mathbb{S} = \partial B(0,R[)$.

Theorem 3.5.1. *Let v be a scalar function defined in $\mathbb{R}^n \setminus B(0,1[$ with the property that, for some $M > 0$, $\mathcal{A} \subset \mathbb{S}$ measurable,*

$$v(x) \le v(x_\lambda),\ \textit{provided that}\ x\cdot\tau > \lambda \ge M,\ \textit{as}\ \tau \in \mathcal{A}.$$

Then there are two constants $\epsilon_0 > 0$, $C > 0$, both independent of M, such that, since $|\mathbb{S}\setminus\mathcal{A}| = |\mathbb{S}| - |\mathcal{A}| \le \epsilon_0$ it follows that

$$v(x) \ge v(y),\ \textit{whenever}\ |x| > 1,\ |y| \ge |x| + CM.$$

We first note that the proof turns out to be trivial as $\mathcal{A} = \mathbb{S}$. In fact, let $x, y \in \mathbb{R}^n$ and suppose that $|y| \ge |x| + CM$, for some constant $C > 0$ to be found. Then, choosing $\tau = \frac{y-x}{|y-x|}$, $\lambda = \frac{y-x}{|y-x|}\cdot\frac{x+y}{2}$, one has

$$y = x_\lambda,\ y\cdot\tau \ge \lambda.$$

Let's look for a condition on C in order to achieve $\lambda > M$:

$$\begin{aligned}\lambda = \frac{y-x}{|y-x|}\cdot\frac{x+y}{2} &= \frac{1}{2|y-x|}(|y|^2 - |x|^2)\\ &= \frac{1}{2|y-x|}(|y|-|x|)(|y|+|x|)\\ &\ge \frac{CM}{2|y-x|}(|y|+|x|).\end{aligned}$$

This last quantity is bigger or equal than M if and only if $C(|y|+|x|) \ge 2|y-x|$. It then sufficies to pick $C \ge 2$. Therefore, as $|y| \ge |x| + 2M$, thanks to the assumption done before, $v(x) \ge v(y)$, and so the proof (in this simple case) is completed.

Our purpose is now to show the result above in the general case: the argument used above cannot work, because $\frac{y-x}{|y-x|}$ could be not contained in $\mathcal{A}$.

Proof (of Theorem 3.5.1). We start with some notations. Given $z \in \mathbb{R}^n$, let Γ_z be the cone with vertex at the origin, axis $-z$ and aperture $\pi/4$, and let $\mathcal{C}_z$ be the cone with vertex at z, axis $-z$ and aperture $\pi/4$ $(\mathcal{C}_z = z + \Gamma_z)$.

Fix now a point $x \in \mathbb{R}^n$ and set $R := |x| + 2M$. We look at those points $z \in R\mathbb{S}$ which can be obtained by the reflection of x in a plane Π_τ with normal $\tau \in \mathbb{S}\cap\Gamma_x$ and such that Π_τ separates z from $B(0,M[$. If $\tau \in \mathcal{A}$, we say that such a point z is admissible for x, and we then consider the collection of the admissible points for x

$$\mathcal{A}_x := \{z \in R\mathbb{S} \mid z = x + 2(\lambda - x\cdot\tau)\ \text{with}\ z\cdot\tau > \lambda,\ \tau \in \mathbb{S}\cap\Gamma_x\cap\mathcal{A}\}.$$

We immediately note that, denoted by x' the orthogonal projection of x onto the hyperplane $z\cdot\tau = 0$, as $\tau \in \mathbb{S}\cap\Gamma_x$, one has $|x|^2 = |x'|^2 + (x\cdot\tau)^2$. Thus, if $z = x + 2(\lambda - x\cdot\tau)\tau$, $z \in R\mathbb{S}$, then

$$\begin{aligned}|z|^2 &= |x' - (x\cdot\tau)\tau + 2\lambda\tau|\\ &= |x'|^2 + (2\lambda - x\cdot\tau)^2 = |x|^2 + 4\lambda(\lambda - x\cdot\tau) = R^2,\end{aligned}$$

and so $R^2 - |x|^2 \le 4\lambda(\lambda + |x|)$, equivalently

$$(2\lambda + |x|)^2 \ge R^2.$$

Therefore $2\lambda \ge R - |x| = 2M$, and thus $\lambda \ge M$. Hence, by assumption, as $z \in \mathcal{A}_x$,

$$v(z) \le v(x).$$

Let now y be a point such that $|y| \ge |x| + CM$, $C > 0$ to be decided. We may similarly consider the set of the admissible points for y

$$\mathcal{A}_y := \{z \in R\mathbb{S} \mid z = y + 2(\lambda - y \cdot \tau)\tau \text{ with } y \cdot \tau > \lambda,\ z \cdot y < 0,\ -\tau \in \mathbb{S} \cap \Gamma_y,\ \tau \in \mathcal{A}\}.$$

For the moment assume also that the angle $\alpha(x, y)$ between x and y is small enough, for example it sufficies that $\alpha(x, y) < \pi/8$. We aim to find a condition on C ensuring that, since

$$z = y + 2(\lambda - y \cdot \tau)\tau,\ z \in R\mathbb{S},\ -\tau \in \mathbb{S} \cap \Gamma_y, \tag{3.34}$$

in particular since $z \in \mathcal{A}_y$, it follows that $\lambda \ge M$. We first note that, as z satisfies the assumptions (3.34) written above,

$$|z|^2 = |y|^2 + 4\lambda(\lambda - y \cdot \tau) = R^2,$$

and so $|y|^2 = R^2 + 4\lambda(y \cdot \tau - \lambda) > R^2$, $|y| > R = |x| + 2M$, namely $C > 2$ necessarily. We now look for a more restrictive condition on C. In order to have that the condition described just before holds true, it's enough to require that, as $-\tau \in \mathbb{S} \cap \Gamma_y$,

$$|y + 2(M - y \cdot \tau)\tau|^2 \ge R^2,$$

equivalently

$$|y|^2 + 4M(M - y \cdot \tau) \ge R^2. \tag{3.35}$$

In the particular case of $\tau = y/|y|$, we observe that

$$\begin{aligned}(3.35) &\Leftrightarrow |y|^2 + 4M^2 - 4M|y| \ge R^2 \\ &\Leftrightarrow (|y| - 2M)^2 \ge R^2 \\ &\Leftrightarrow |y| \ge |x| + 4M.\end{aligned}$$

So we can take $C = 4$ (but recall that we are working under the assumption $\alpha(x, y) < \pi/8$). Let's now treat the case of general τ:

$$(3.35) \Leftrightarrow |y|^2 + 4M^2 - 4M|y|(y/|y|) \cdot \tau) \ge R^2.$$

Being $(y/|y|) \cdot \tau = \cos(\alpha(y, \tau)) \le 1$, we deduce that

$$\begin{aligned}&|y|^2 + 4M^2 - 4M|y|((y/|y|) \cdot \tau) \ge \\ &\qquad \ge |y|^2 + 4M^2 - 4M|y|.\end{aligned}$$

So in order that (3.35) holds true, it sufficies to impose that $|y|^2 + 4M^2 - 4M|y| \ge R^2$, and we know that $C = 4$ guarantees the validity of this last inequality. Therefore in particular, as $|y| \ge |x| + 4M$, from

$$z = y + 2(\lambda - y \cdot \tau)\tau \in \mathcal{A}_y,$$

it follows $v(z) \ge v(y)$. It's now geometrically clear that any point $z \in \mathcal{C}_x \cap R\mathbb{S}$ can be written as

$$\begin{aligned}&z = x + 2(\lambda - x \cdot \tau)\tau \\ &\tau \in \mathbb{S} \cap \Gamma_x,\ \lambda \ge M,\end{aligned}$$

and similarly that every point $z \in \mathcal{C}_y \cap R\mathbb{S}$ such that $z \cdot y < 0$ can be represented by

$$z = y + 2(\lambda - x \cdot \tau)\tau$$
$$-\tau \in \mathbb{S} \cap \Gamma_y,\ \lambda \geq M.$$

Then, selecting $\epsilon_0 > 0$ small enough, $\mathcal{A}_x$ covers a portion of $\mathcal{C}_x \cap R\mathbb{S}$ as big as we require, and similarly $\mathcal{A}_y$ covers as a big part of $\{z \in \mathcal{C}_y \cap R\mathbb{S} \mid z \cdot y < 0\}$ as we like. Moreover the request on $\alpha(x,y)$ ensures that

$$\mu := |(\mathcal{C}_x \cap R\mathbb{S}) \cap \{z \in \mathcal{C}_y \cap R\mathbb{S} \mid z \cdot y < 0\}| > 0.$$

We then can impose that

$$|\mathcal{A}_x \cap (\mathcal{C}_x \cap R\mathbb{S}) \cap \{z \in \mathcal{C}_y \cap R\mathbb{S} \mid z \cdot y < 0\}| > \mu/2$$
$$|\mathcal{A}_y \cap (\mathcal{C}_x \cap R\mathbb{S}) \cap \{z \in \mathcal{C}_y \cap R\mathbb{S} \mid z \cdot y < 0\}| > \mu/2.$$

Thus there exists $z \in \mathcal{A}_x \cap \mathcal{A}_y$, and one necessarily has

$$v(y) \leq v(z) \leq v(x).$$

Finally it remains to prove the statement as no conditions on the angle $\alpha(x,y)$ are imposed. If $\alpha(x,y) \geq \pi/8$, we can build a finite sequence $z_1, ..., z_k$, $k \leq 9$, such that

$$z_1 = x,\ z_k = y,\ \alpha(z_i, z_{i+1}) \leq \pi/9 < \pi/8,\ |z_{i+1}| \geq |z_i| + 4M,$$

imposing $|y| \geq |x| + (9 \times 4)M$. **QED**

We are now ready to prove the following consequence.

Corollary 3.5.1. Let v respect the assumptions of the Theorem 3.5.1, and suppose in addition that v is nonnegative, weakly superharmonic and continuous. Then

$$v(x) = (1 + \mathcal{O}(|x|^{-1})) \inf_{|x|\mathbb{S}} v, \text{ as } |x| \to \infty. \tag{3.36}$$

Proof. By Theorem 3.5.1, we deduce that, as $|x| > 1$,

$$\sup_{(|x|+CM)\mathbb{S}} v \leq \inf_{|x|\mathbb{S}} \leq \sup_{|x|\mathbb{S}} v \leq \inf_{(|x|-CM)\mathbb{S}} v. \tag{3.37}$$

On the other hand, as $|y| \geq R_1$, $R_1 > 1$,

$$v(y) \geq \left(\frac{R_1}{|y|}\right)^{n-2} \inf_{R_1\mathbb{S}} v. \tag{3.38}$$

As a matter of fact, consider for $\epsilon > 0$

$$w_\epsilon(y) \equiv v(y) + \epsilon - \left(\frac{R_1}{|y|}\right)^{n-2} \inf_{R_1\mathbb{S}} v:$$

it's clear that $v \geq \inf_{R_1\mathbb{S}} v$ on $R_1\mathbb{S}$, and so that $w_\epsilon \geq 0$ on $R_1\mathbb{S}$. In addition, as $S \geq \frac{R_1}{\epsilon^{n-2}} \inf_{R_1\mathbb{S}}$, it also holds that $w_\epsilon \geq 0$ on $S\mathbb{S}$, because $v \geq 0$. Hence, being $|y|^{-(n-2)}$ harmonic, and so having that w_ϵ is weakly superharmonic and continuous, by the Maximum Principle 1.3.1 (applied to $-w_\epsilon$), we immediately deduce that $w_\epsilon \geq 0$ in $R_1 \leq |y| \leq S$. Letting $S \to \infty$, one infers that $w_\epsilon \geq 0$

in $|y| \geq R_1$, for any $\epsilon > 0$, and the arbitrariness of ϵ permits to deduce that (3.38) holds true. In particular, for $R_1 = |x| - CM$, $|y| = |x| + CM$,

$$\inf_{(|x|+CM)\mathbb{S}} v \geq \left(\frac{|x| - CM}{|x| + CM}\right)^{n-2} \inf_{(|x|-CM)\mathbb{S}} v,$$

and thus

$$\inf_{(|x|-CM)\mathbb{S}} v \leq (1 + \mathcal{O}(|x|^{-1})) \inf_{(|x|+CM)\mathbb{S}} v.$$

Thanks to the sequence of inequalities (3.37), we have that $v \leq \inf_{(|x|-CM)\mathbb{S}} v$, and so we may conclude. **QED**

Property (3.36) is the so called *asymptotic symmetry.*

3.6 Applications of the Reflection Theorem and the asymptotic symmetry

We are now ready to study the applications of the theory developed up to now.

3.6.1 A first, general result

We begin demonstrating a general result holding for equations like $-\Delta u = g(u)$ in $B(0,1[\smallsetminus\{0\}$.

Theorem 3.6.1. *Let $u \in \mathcal{C}^2(B(0,1[\smallsetminus\{0\}) \cap \mathcal{C}^{2,\alpha}(B(0,1] \smallsetminus B(0,r[)$, for some $0 < r < 1$, $u \geq 0$, be a smooth solution to*

$$-\Delta u = g(u) \text{ in } B(0,1[\smallsetminus\{0\}$$

with an isolated singularity at the origin. Assume that:

(I) $g(0) = 0$;

(II) g *is nondecreasing;*

(III) $t^{-\frac{n+2}{n-2}} g(t)$ *is nonincreasing;*

(IV) $g(t) \geq ct^p$ *for some* $p \geq \frac{n}{n-2}$, $c > 0$.

Then

$$u(x) = (1 + \mathcal{O}(|x|))\, m(|x|) \text{ as } x \to 0,$$

where, as $r > 0$, $m(r) \equiv ⨍_{\mathbb{S}} u(rz)\,\mathrm{d}\sigma(z) = ⨍_{r\mathbb{S}} u(w)\,\mathrm{d}\sigma(w)$ denotes the average of u on the sphere of radius r.

Proof. As we did in the first section of this chapter, let's transform u by the Kelvin Transform performed around a point z close to the origin: more specifically, for $\mu > 0$, we pick $z = \mu^{-1} e_n$ and we indicate by v_μ this transformation of u. Then the singularity of u at 0 is sent to the point $-\mu e_n$, that is a singularity for v_μ. The hypothesis (iv) ensures that g satisfies the two assumptions of Lemma 1.2.3, and thus, since the computation performed at the end of the first section of this chapter it follows that v_μ weakly solves equation

$$-\Delta v = f(|x|, v) \equiv F(x),$$

in $\mathbb{R}^n \smallsetminus B(0, 1 + 2\mu^{-1}]$, where $f(|x|, v) = |x|^{-(n+2)} g(|x|^{n-2} v(x))$. We now show that, replacing 1 with $1 + 2\mu^{-1}$, the five hypotheses of the Reflection Theorem are respected:

(i) it's enough to set $R = \mu + 1$ and assign a suitable value to v_μ at the singularity $-\mu e_n$, for example

$$v(-\mu e_n) := \liminf_{x \to -\mu e_n} v(x):$$

with such a choice, v_μ turns out to be weakly superharmonic and lower semicontinuous, and so $v(-\mu e_n) > 0$, because by assumption v_μ is nonidentically zero;

(ii) holds because of Lemma 3.2.1;

(iii) hypotheses (II), (III) coincide exactly with assumptions (3.20) and (3.21) respectively: we recall that these two assumptions were introduced in order that the map

$$F(x) = f(|x|, v(x)) = |x|^{-(n+2)}\, g(|x|^{n-2} v(x))$$

satisfies $F(x) \leq F(x_\lambda)$, for $\lambda \geq 0$, $x_n > \lambda$, provided that $v(x) \leq v(x_\lambda)$, that's precisely what is required in the assumption (iii) of the Reflection Theorem;

(iv) choosing suitably δ_0, $C > 0$ this assumptions follows: as matter of fact $v \in \mathcal{C}^{2,\alpha}(B(0,2[\smallsetminus B(0,1]))$, and $\mathcal{C}^{2,\alpha}(B(0,2] \smallsetminus B(0,1[) \hookrightarrow \mathcal{C}^{1,1}(B(0,2] \smallsetminus B(0,1[)$;

(v) v_μ is weakly superharmonic and lower semicontinuous in $|x| \geq 1 + 2\mu^{-1}$. Moreover the fact that $v_\mu = \mathcal{O}(|x|^{2-n})$ for $|x|$ big permits to infer that there exist $p \geq 1$ and $\beta < n$, both independet of μ, such that

$$\int_{\mathbb{R}^n \smallsetminus B(0,1+2\mu^{-1}[} v_\mu^p / |x|^\beta$$

is finite. Then assumptions (a) and (b) introduced at the beginning of the section 4.4 of the current chapter are fulfilled. Therefore Theorem 3.4.1 and the computations next performed allow to infer that that assumption (v) of the Reflection Theorem is satisfied, for any admissible direction τ.

Exploiting the notations of section 4.4, taken the value ϵ_0 met in Theorem 3.5.1, the estimate (3.23) ensures us that, picking an integer $k_0 > \delta^{-1} \log_2(\frac{C\,c^*}{\epsilon_0})$, the set of the nonadmissible directions has measure less than ϵ_0. Furthermore we know that $M = 2^{k_0}$ is a good choice for the M of the Reflection Theorem.

For every $\mu > 0$, let $\mathcal{A}_\mu$ be the collection of the admissible directions of v_μ, and, for $k \geq 1$, j integers, set

$$\mathcal{A}^* := \limsup_{k \to \infty} \mathcal{A}_k = \bigcap_{k \geq 1} \bigcup_{j \geq k} \mathcal{A}_j.$$

Let v be the Kelvin Transform of u with respect to the origin, and set $\mathcal{A}$ the collection of the admissible directions for v: it's clear that

$$\mathcal{A} \supset \mathcal{A}^*.$$

In fact, let $\tau \in \mathcal{A}^*$: then, for any $k \geq 1$, there exists $j_k \geq k$ such that $\tau \in \mathcal{A}_{j_k}$, namely such that

$$v_{j_k}(x) \leq v_{j_k}(x_\lambda) \text{ as } x \cdot \tau > \lambda \geq M,$$

and thus, letting k go to ∞ and exploiting the regularity of v, we find

$$v(x) \leq v(x_\lambda) \text{ as } x \cdot \tau > \lambda \geq M.$$

Therefore we have

$$
\begin{aligned}
|\mathcal{A}| \geq |\mathcal{A}^*| &= \int_{\mathbb{S}} \chi(\limsup_{k\to\infty} \mathcal{A}_k)\, \mathrm{d}\sigma = \\
&= \int_{\mathbb{S}} \limsup_{k\to\infty} \chi(\mathcal{A}_k)\, \mathrm{d}\sigma \geq \ \text{(Fatou's Lemma)} \\
&\geq \limsup_{k\to\infty} \int_{\mathbb{S}} \chi(\mathcal{A}_k)\, \mathrm{d}\sigma = \limsup_{k\to\infty} |\mathcal{A}_k| \geq |\mathbb{S}| - \epsilon_0.
\end{aligned}
$$

This estimate permits to apply Theorem 3.5.1 to v, and so to infer by Corollary 3.5.1 that

$$
v(x) = (1 + \mathcal{O}(|x|^{-1})) \inf_{|x|\mathbb{S}} v, \text{ as } |x| \to \infty,
$$

namely that as $|y| \to 0$

$$
u(y) = (1 + \mathcal{O}(|y|)) \inf_{|y|\mathbb{S}} u \leq (1 + \mathcal{O}(|y|))\, m(|y|),
$$

that is precisely what we aimed to show. **QED**

3.6.2 Application to the Yamabe Equation

From now on, we concentrate our attention on the Yamabe Equation, fully exploiting that $g(t) \equiv t^{(n+2)/(n-2)}$, and thus proving a more specific, strong classification result. To do so, we exploit the classification of the radial solutions performed in the first section and the general, less specific result proved in the previous subsection

Theorem 3.6.2. *Let $u > 0$ be a $\mathcal{C}^2$ solution to*

$$
-\Delta u = u^{\frac{n+2}{n-2}}, \text{ in } B(0,1[\smallsetminus\{0\}, \tag{3.39}
$$

with a nonremovable, isolated singularity at the origin. Then there are an asymptotic constant $0 > D_\infty \geq -\dfrac{2}{n}\left(\dfrac{n-2}{2}\right)^n$ *and a radial solution $\phi \equiv \phi(|x|) \in \Phi^+(D_\infty)$ such that*

$$
u(x) = (1 + o(1))\, \phi(|x|),
$$

as $x \to 0$.

Proof. We start showing that, if u solves (3.39), then

$$
m(r) = \mathcal{O}(r^{\frac{2-n}{2}}) \tag{3.40}
$$

$$
m'(r) = \mathcal{O}(r^{-\frac{n}{2}}), \tag{3.41}
$$

where we recall that $m(r) = ⨍_{\partial B(0,r[} u(z)\, \mathrm{d}\sigma_z = ⨍_{\partial B(0,1[} u(rw)\, \mathrm{d}\sigma_w$. Lemma 1.2.3 ensures that, if $0 < r < 1$, $u \in L^{\frac{n+2}{n-2}}(B(0,r[)$, and so, for η smooth, we have

$$
\begin{aligned}
\int_{B(0,r[} u\Delta\eta + \eta u^{\frac{n+2}{n-2}}\, \mathrm{d}x &= \lim_{\epsilon\to 0^+} \int_{B(0,r[\smallsetminus B(0,\epsilon]} u\Delta\eta - \eta\Delta u\, \mathrm{d}x \\
&= \int_{\partial B(0,r[} u\eta_\nu - \eta u_\nu\, \mathrm{d}\sigma - \lim_{\epsilon\to 0^+} \int_{\partial B(0,\epsilon[} u\eta_\nu - \eta u_\nu\, \mathrm{d}\sigma \\
&= \int_{\partial B(0,r[} u\eta_\nu - \eta u_\nu\, \mathrm{d}\sigma.
\end{aligned}
$$

Choosing $\eta \equiv r^2 - |x|^2$, we deduce that

$$\int_{B(0,r[} (r^2 - |x|^2)\, u^{\frac{n+2}{n-2}}\,\mathrm{d}x + 2r \int_{\partial B(0,r[} u\,\mathrm{d}\sigma = 2n \int_{B(0,r[} u\,\mathrm{d}x, \tag{3.42}$$

because $\Delta\eta = -2n$, and on $\partial B(0,r[$ η is null, $\eta_\nu = -2x \cdot \left(\frac{x}{r}\right) = -2r$. We note that, for $r > 0$, the following relation holds

$$\begin{aligned}
-m'(r) &= -\fint_{\partial B(0,1[} \nabla u(rw) \cdot w\,\mathrm{d}\sigma_w \\
&= -\fint_{\partial B(0,r[} \nabla u(z) \cdot \left(\frac{z}{r}\right) \mathrm{d}\sigma_z \\
&= -\fint_{\partial B(0,r[} u_\nu\,\mathrm{d}\sigma = \frac{1}{|\partial B(0,1[|\, r^{n-1}} \int_{B(0,r[} u^{\frac{n+2}{n-2}}\,\mathrm{d}x > 0,
\end{aligned}$$

and hence m decreases. Theorem 3.6.1 guarantees that

$$u(x) = (1 + \mathcal{O}(|x|))\, m(|x|)$$

as $x \to 0$, and so, by (3.42), one deduces that

$$\begin{aligned}
\int_{B(0,r[} (r^2 - |x|^2) u(x)^{\frac{n+2}{n-2}}\,\mathrm{d}x \geq \ &(\text{for } r > 0 \text{ small enough}) \\
&\geq \int_{B(0,r[} (r^2 - |x|^2) \frac{1}{2} m(|x|)^{\frac{n+2}{n-2}}\,\mathrm{d}x \\
&\geq \int_{B(0,r/2[} \frac{3}{8} r^2 m(|x|)^{\frac{n+2}{n-2}}\,\mathrm{d}x \\
&\geq C r^{n+2} m\left(\frac{r}{2}\right)^{\frac{n+2}{n-2}} \geq C r^{n+2} m(r)^{\frac{n+2}{n-2}},
\end{aligned}$$

where the last two inequalities follow from $m' < 0$. On the other hand we observe that, taking $r > 0$ small enough,

$$\int_{B(0,r[} (r^2 - |x|^2) u(x)^{\frac{n+2}{n-2}}\,\mathrm{d}x < 2n \int_{B(0,r[} u\,\mathrm{d}x \leq 3n \int_0^r m(t) t^{n-1}\,\mathrm{d}t.$$

Therefore, joining these two estimates, we infer that, for r sufficiently small,

$$\begin{aligned}
r^{n+2} m(r)^{\frac{n+2}{n-2}} &< C_1 \int_0^r m(t) t^{n-1}\,\mathrm{d}t \leq C_1\, r^{4\frac{n-1}{n+2}} \int_0^r m(t) t^{n-1-4\frac{n-1}{n+2}}\,\mathrm{d}t \\
&\leq C_1\, r^{4\frac{n-1}{n+2}} r^{\frac{4}{n+2}} \Big(\int_0^r m(t)^{\frac{n+2}{n-2}} t^{n-1} \Big)^{\frac{n-2}{n+2}} \\
&= C_1\, r^{\frac{4n}{n+2}} \Big(\int_0^r m(t)^{\frac{n+2}{n-2}} t^{n-1}\,\mathrm{d}t \Big)^{\frac{n-2}{n+2}} \\
&\leq C_1\, r^{\frac{4n}{n+2}} \Big(\int_0^R m(t)^{\frac{n+2}{n-2}} t^{n-1}\,\mathrm{d}t \Big)^{\frac{n-2}{n+2}},
\end{aligned}$$

for $r \leq R$. Therefore we obtain

$$\begin{aligned}
&\int_0^R r^{n-1} m(r)^{\frac{n+2}{n-2}}\,\mathrm{d}r \\
&\leq C_1 \int_0^R r^{\frac{n-6}{n+2}}\,\mathrm{d}r \times \Big(\int_0^R m(t)^{\frac{n+2}{n-2}} t^{n-1}\,\mathrm{d}t \Big)^{\frac{n-2}{n+2}} \\
&= C_2\, R^{\frac{2n}{n+2}} \Big(\int_0^R m(t)^{\frac{n+2}{n-2}} t^{n-1}\,\mathrm{d}t \Big)^{\frac{n-2}{n+2}},
\end{aligned}$$

equivalently

$$\int_0^R r^{n-1} m(r)^{\frac{n+2}{n-2}} \,\mathrm{d}r \le C_3 \, R^{\frac{n-2}{2}}.$$

Thus we infer that $R^n m(R)^{\frac{n+2}{n-2}} \le C_3 R^{\frac{n-2}{2}}$, namely

$$m(R) \le C_4 \, R^{-\frac{n-2}{2}},$$

that is exactly (3.40). Estimate (3.41) can be obtained substituting (3.40) in the formula for $-m'(r)$ demostrated before. Indeed

$$\begin{aligned} |m'(r)| = -m'(r) &\le \frac{C_5}{r^{n-1}} \int_{B(0,r[} m(|x|)^{\frac{n+2}{n-2}} \,\mathrm{d}x \\ &\le \frac{C_5}{r^{n-1}} \int_0^r m(t)^{\frac{n+2}{n-2}} t^{n-1} \,\mathrm{d}t \\ &\le \frac{C_6}{r^{n-1}} \int_0^r t^{\frac{n-2}{2}-1} \,\mathrm{d}t = \frac{C_7}{r^{n-1}} \, r^{\frac{n-2}{2}} = C_7 r^{-\frac{n}{2}}. \end{aligned}$$

Now consider the representation of u in spherical coordinates:

$$v(r,\theta) \equiv u(r\theta),$$

for $0 < r < 1$, $\theta \in \mathbb{S}$. By the substitution $\psi(t,\theta) \equiv r^{\frac{n-2}{2}} v(r,\theta)$, $t = -\ln(r)$ [7], using (3.2), we immediately deduce the validity of the following identity:

$$\psi_{tt} - \left(\frac{n-2}{2}\right)^2 \psi + \Delta_{\mathbb{S}} \psi + \psi^{\frac{n+2}{n-2}} = 0, \tag{3.43}$$

for $t > 0$, $\theta \in \mathbb{S}$. For $t = -\ln(r)$, set

$$\beta(t) \equiv \fint_{\mathbb{S}} \psi \,\mathrm{d}\theta = r^{\frac{n-2}{2}} \fint_{\mathbb{S}} v(r,\theta) \,\mathrm{d}\theta = r^{\frac{n-2}{2}} m(r).$$

We observe that

$$\begin{aligned} \psi(t,\theta) &= e^{-\frac{n-2}{2}t} v(e^{-t},\theta) \\ &= e^{-\frac{n-2}{2}t} m(e^{-t})(1 + \mathcal{O}(e^{-t})) \\ &= \beta(t)(1 + \mathcal{O}(e^{-t})), \end{aligned}$$

as $t \to \infty$, and also that

$$\beta' + \frac{n-2}{2}\beta = -e^{-\frac{n}{2}t} m'(e^{-t}) = -r^{\frac{n}{2}} m'(r) \ge 0.$$

Now estimate (3.40) applied to the definition of β ensures that $\beta = \mathcal{O}(1)$, as $t \to \infty$. Furthermore the identity just deduced yields

$$\beta' = \mathcal{O}(1),$$

as $t \to \infty$. To go further, we have to prove the following estimates

$$\frac{\partial}{\partial t}(\psi - \beta) = \beta \mathcal{O}(e^{-t}) \tag{3.44}$$

$$|\nabla_\theta(\psi - \beta)|(= |\nabla_\theta \psi|) = \beta \mathcal{O}(e^{-t}), \tag{3.45}$$

as $t \to \infty$. In order to demonstrate (3.44), (3.45), we need two preliminary results. The first one is a gradient estimate for the Poisson Equation that can be found in [5, p. 41].

[7] Note that this is exactly the Emden-Fowler substitution (3.4) introduced to study the radial solutions to the Yamabe Equation.

Theorem 3.6.3. *Let Ω be an open, connected subset and $u \in \mathcal{C}^2(\Omega)$ be a solution to the Poisson Equation $-\Delta u = f$ in Ω. Denoted by*

$$d_x := \mathrm{dist}(x, \partial\Omega),$$

for every $x \in \Omega$, one has

$$\sup_{x\in\Omega} d_x |Du(x)| \le C_P(\sup_{\Omega} |u| + \sup_{x\in\Omega} d_x^2 |f(x)|), \tag{3.46}$$

for any $x \in \Omega$, for a suitable constant $C_P \equiv C_P(n)$.

The second one is a sophisticated *Harnack-type Inequality* taken from [11, p. 539].

Theorem 3.6.4. *Given $r > 0$, let u be a positive, $\mathcal{C}^2(B(0,r[\smallsetminus\{0\})$ solution to*

$$-\Delta u = c(x)u, \text{ in } B(0,r[\smallsetminus\{0\},$$

where $c = hu^{\alpha-1}$, for some $1 < \alpha < (n+2)/(n-2)$. Suppose that $h(x)$ is a nonnegative, $\mathcal{C}^1(B(0,r[\smallsetminus\{0\})$ function such that, around the origin,

$$\begin{aligned} &c_1|x|^\sigma \le h(x) \le c_2|x|^\sigma \\ &|\nabla \ln(h)| \le c_3/|x|, \end{aligned}$$

for positive pure constants c_1, c_2, c_3, and for an arbitrary real value σ. Then there exists $\zeta_0 > 0$ such that, for all $0 < \zeta < \zeta_0$, and for any $0 < \epsilon \le r/2$, the following Harnack's Inequality holds:

$$\sup_{\epsilon\le|x|\le(1+\zeta)\epsilon} u(x) \le C_H \inf_{\epsilon\le|x|\le(1+\zeta)\epsilon} u(x),$$

where C_H does not depend on ϵ, ζ, u and h.

Let's continue with the proof of the Classification Theorem 3.6.2. Since Theorem 3.6.1 it follows that

$$-\Delta(u-m) = m^{\frac{n+2}{n-2}}\mathcal{O}(r)$$

in $\frac{1}{2}r < |x| < 2r$. Inequality (3.46) says us that

$$\begin{aligned} |\nabla(u-m)| &\le C_9\left(\frac{\sup_\Omega |u-m|}{r} + r\sup_\Omega m^{\frac{n+2}{n-2}}\mathcal{O}(r)\right) &(3.47)\\ &\le C_{10}(\sup_\Omega(m + r^2 m^{\frac{n+2}{n-2}})) &(3.48)\end{aligned}$$

on $\partial B(0,r[$, where r is taken small enough, $C_{10} \equiv C_{10}(n)$. Moreover, we note that u solves

$$-\Delta u = u^{\frac{n+2}{n-2}} = u^\beta u^{\alpha-1}u = c(x)u,$$

and then the assumptions of Theorem 3.6.4 are satisfied (it sufficies to set $h = u^\beta$, $c = hu^{\alpha-1}$, $\alpha+\beta = \frac{n+2}{n-2}$, $1 < \alpha < \frac{n+2}{n-2}$). Therefore, by (3.47) and the Harnack Inequality (applied in $r \le |x| \le (1+\zeta)r$), we infer

$$\begin{aligned} |\nabla(u-m)| &\le C_{11}(m(r) + r^2 m(r)^{\frac{4}{n-2}} m(r)) \\ &\le C_{12}\, m(r) \end{aligned}$$

on $\partial B(0,r[$, where the last inequality follows from $m = \mathcal{O}(r^{\frac{2-n}{2}})$, and C_{12} is independent of r, u and m, up to a choice of a sufficiently, definitively small $r > 0$. In particular we have shown that

$$\left|\frac{\partial}{\partial r}(v-m)\right| \le C_{13}\, m$$
$$|\nabla_\theta(v-m)| \le C_{14}\, rm,$$

where we recall that $v(r,\theta) = u(r\theta)$ is the spherical representation of u. From the fact that

$$\left|\frac{\partial}{\partial t}(\psi(t,\theta)-\beta(t))\right| \le C_{15}|\psi(t,\theta)-\beta(t)| + e^{-(\frac{n-2}{2}+1)t}\left|\frac{\partial}{\partial r}(v-m)(e^{-t},\theta)\right| = \beta\mathcal{O}(e^{-t}),$$
$$|\nabla_\theta(\psi-\beta)| = e^{-(\frac{n-2}{2}+1)t}|\nabla_\theta(v-m)(e^{-t},\theta)| \le C_{14}e^{-t}\beta = \beta\mathcal{O}(e^{-t}),$$

as $t\to\infty$, estimates (3.44), (3.45) follow.

Our aim is now to derive a energy estimate from identity (3.43), by multiplying by ψ_t and integrating. We start observing that, if $u(x) = \phi(|x|)$ is radial, then ψ doesn't depend on θ and $\psi(t) = r^{\frac{n-2}{2}}\phi(r)$, for $t = -\ln(r)$. The classification of the radial solutions to the Yamabe Equation operated at the beginning of the current chapter suggests us the following notion of energy for a generic singular solution u:

$$D(t) \equiv (\beta')^2 - \left(\frac{n-2}{2}\right)^2\beta + \frac{n-2}{2}\beta^{\frac{2n}{n-2}}.$$

If ψ is radial, $\psi = \beta$ and we know that the quantity D defined above is a prime integral. The idea is now that, if ψ is not radial, then D is no longer a prime integral, but a weaker identity holds: for $t \ge s$,

$$D(t) = D(s) + (\beta^2 + (\beta')^2)\mathcal{O}(e^{-s}) + \mathcal{O}(e^{-t}), \tag{3.49}$$

as $t,s\to\infty$. To prove so, it's sufficient to note that, multiplying identity (3.43) by $2\psi_t$ and integrating, one has

$$\int_{\mathbb{S}} \psi_t^2 - \left(\frac{n-2}{2}\right)^2\psi^2 + \frac{n-2}{n}\psi^{\frac{2n}{n-2}} - |\nabla_\theta\psi|^2\Bigg|_s^t \,\mathrm{d}\theta = 0.$$

Exploiting (3.44), (3.45), we achieve

$$D(t)-D(s) = (\beta')^2 - \left(\frac{n-2}{2}\right)^2\beta^2 + \frac{n-2}{n}\beta^{\frac{2n}{n-2}}\Bigg|_s^t = (\beta^2+(\beta')^2)\mathcal{O}(e^{-\tau})\Bigg|_s^t,$$

that is precisely the energy estimate (3.49), up to the remark that $\beta^2(t) + (\beta'(t))^2 = \mathcal{O}(1)$. Since (3.49), it follows that, for $k \ge 1$ integer,

$$D(k+1) - D(k) = \mathcal{O}(e^{-k}).$$

Therefore the sequence $\{D(k)\}_{k\ge1}$ fulfills the Cauchy Property and then converges to a limit D_∞. Substituting this value of the limit in (3.49), we infer that

$$D(s) = D_\infty + (\beta^2 + (\beta')^2)\mathcal{O}(e^{-s}),$$

namely that

$$(\beta')^2 = \left(\frac{n-2}{2}\right)^2\beta^2 - \frac{n-2}{n}\beta^{\frac{2n}{n-2}} + D_\infty + (\beta^2+(\beta')^2)\mathcal{O}(e^{-t}),$$

for t going to ∞. Since the fact that $(\beta^2+(\beta')^2)\mathcal{O}(e^{-t})$ shrinks to 0 as $t \to \infty$ and the nonnegativity of β, it follows that D_∞ needs tto respect the following, just met relation

$$0 \geq D_\infty \geq -\frac{2}{n}\left(\frac{n-2}{2}\right)^n.$$

If $D_\infty < 0$, considering the only radial solution $\phi(|x|) \in \Phi^+(D_\infty)$ corresponding to initial value $\psi(t_0) = \beta(t_0)$, and $\mathrm{sgn}(\psi) = \mathrm{sgn}(\beta)$, we conclude that the statement holds for such a choice of the radial solution. Instead, if $D_\infty = 0$, it's clear that β needs to go to 0 like $e^{-(n-2)t/2}$, and so that u is nonsingular: this remark allows to deduce that D_∞ cannot be 0 (because, by assumption, u has a nonremovable singularity at the origin), and thus the proof is concluded. **QED**

Bibliography

[1] Caffarelli L., Gidas B., Spruck J., *Asymptotic Symmetry and Local Behavior of Semilinear Elliptic Equations with Critical Sobolev Growth*, Comm. Pure Appl. Math., 42, 1989, pp. 271-297.

[2] Gidas B., Ni W.M., Niremberg L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys., 68, 1979, pp. 209-243.

[3] Mastrolia P., Rigoli M., Setti A., *Yamabe-type Equations on Complete, Noncompact Manifolds*, Springer, Basel, Heidelberg, New York, Dordrecht, London, 2012.

[4] Murray Protter H., Weinberger H. F., *Maximum Principles in Differential Equations*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1967.

[5] Gilbarg D., Trudinger N., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[6] Evans L. C., *Partial Differential Equations*, American Mathematical Society, Providence, Rhode Island, 1998.

[7] Evans L.C., Gariepy R.F., *Measure Theory and fine Properties of Functions*, CRC Press, Taylor & Francis Group, Boca Raton, London, New York, 1992.

[8] Fraccaroli M., *L'Equazione di Yamabe: analisi e soluzioni attraverso il Metodo delle Sfere Mobili e il Principio del Massimo*, Bachelor Degree Thesis, Università di Padova, September 26, 2014, Supervisor: Prof. Roberto Monti.

[9] Serrin J., *Weakly Subharmonic Function*, Bollettino dell'Unione Matematica Italiana, 9, 2011, pp. 347-361.

[10] Giquinta M., Martinazzi L., *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, Edizioni della Normale, Pisa, 2012.

[11] Gidas B., Spruck J., *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math., 34, 1981, pp. 525-598.