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DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

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Constrained variational problems in one-dimensional Sobolev spaces

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Introduction

Throughout the first 200 years of the development of the calculus of variations, following the work of Gauss, Steiner, Lord Kelvin, Dirichlet and Riemann, the classical approach involved looking for necessary conditions which would have to be satisfied by minimizers. This is what is now known as the *classical indirect approach*, and it is based on the somewhat optimistic idea that every minimization problem does in fact have a solution. An analysis of the necessary conditions (for example, the Euler equation which is to be satisfied by a sufficiently regular minimizer u) permits one to eliminate many candidates, however, without having proven the existence of a minimizer, one in general cannot identify a unique solution. These considerations, among many others, brought mathematicians to try to attack the minimum problem directly, by attempting to immediately prove the existence of a minimizer. This approach, which in turn would also yield existence theorems for solutions to Euler equations satisfying prescribed conditions, is usually referred to as the *direct method of the calculus of variations*, and it specifically originated from the work of Gauss, Lord Kelvin, Dirichlet and Riemann following their work on boundary value problems for the potential equation $\Delta u = 0$.

In the nineteenth century, this method was refined by Tonelli, who realized (expanding on the work of Baire and Lebesgue) that Ascoli-Arzelà's compactness theorem and Baire's semicontinuity concept could be transferred to the calculus of variations and used as tools to demonstrate the existence of minimizers of one-dimensional vairational integrals by means of direct methods. Under these assumptions, Tonelli worked inside the class of absolutely continuous functions defined on a closed and bounded interval $I \subseteq \mathbb{R}$. He popularised these ideas, which are nowadays very much a part of our mathematical culture, in a series of lectures and published papers during the first thirty years of the century, applying them to a wide range of variational problems [1].

In this thesis we will apply Tonelli's ideas, in the more modern setting of so-called Sobolev spaces, to minimum problems for variational integrals with an added constraint, given in the form of an integral of a specified Lagrangian. Specifically, we will first of all prove an existence result for solutions to minimum problems of specified type, then investigate the regularity properties of these extrema as a function of the given Lagrangians and finally distinguish between the behaviour of *regular* and *singular* extrema. We will see that the theory can be developed in all one-dimensional Sobolev spaces $H^{1,p}(I)$, with $p \geq 1$, and that the solutions actually inherit the regularity properties of the objective and constraint Lagrangians.

Conclusively, in the last chapter, we will investigate a specific parametric Dirichlet-energy constrained minimization problem of the aforementioned type and study the regularity of its solutions. This will prove to be particularly difficult for a certain choice of parameters, for which the question of whether a minimizer is necessarily regular with respect to the constraint remains open. This question turns out to be particularly interesting because, in case of an affirmative answer, it would provide an example of a "corner-type" extremal, in contraposition to results like those in [5] and [4] which explicitly prove the non-minimality of such functions in different contexts.

Contents

Chapter 1

Existence theory

Let $I = (a, b) \subseteq \mathbb{R}$ be a bounded open interval, $n \geq 1$ and $p \geq 1$. Given the lagrangians $G : \bar{I} \times \mathbb{R} \to \mathbb{R}^n$, $F : \bar{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with $G \in C^1(I \times \mathbb{R}, \mathbb{R}^n)$ and $F \in C^1(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, we respectively define the objective and constraint functionals

$$\mathcal{F} : H^{1,p}(I) \to \mathbb{R}, \qquad \mathcal{G} : H^{1,p}(I) \to \mathbb{R}^n$$

as

$$\mathcal{F}(u) := \int_a^b F(x, u(x), u'(x))dx, \qquad \mathcal{G}(u) := \int_a^b G(x, u(x))dx,$$

where $H^{1,p}(I)$ is a one-dimensional Sobolev space which will be rigorously defined in the next section.
The aim of this chapter is to prove the following theorem:

Theorem 1 (Tonelli's existence theorem)**.** *Let $p > 1$ and suppose that the Lagrangian $F \in C^1(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies the following conditions:*

1. *$F(x, z, \xi)$ is convex in ξ, meaning that $\xi \mapsto F(x, z, \xi)$ is convex $\forall (x, z) \in \bar{I} \times \mathbb{R}$;*
2. *$F(x, z, \xi)$ has polynomial growth, meaning $\exists\, c_0, c_1 \in \mathbb{R}$ such that given $(x, z) \in \bar{I} \times \mathbb{R}$,*
$$c_0|\xi|^p \leq F(x, z, \xi) \leq c_1(1 + |\xi|^p) \qquad \forall \xi \in \mathbb{R}; \tag{1.1}$$
3. *$C_p(\alpha, \beta) := \{u \in H^{1,p}(I) \mid u(a) = \alpha,\ u(b) = \beta \text{ and } \mathcal{G}(u) = 0\} \neq \emptyset$.*

Then there exists a minimizer of $\mathcal{F}$ under the constraint $\mathcal{G}(u) = 0$ in the class $C_p(\alpha, \beta)$, where $\alpha, \beta \in \mathbb{R}$ are fixed.

In the last section of this chapter we will prove that the same theorem extends to functions in $H^{1,1}(I)$ if we replace condition (2.) with

2. *$F(x, z, \xi)$ has superlinear growth, meaning there exists a function $\theta(\xi)$ such that*

$$\begin{cases} F(x, z, \xi) \geq \theta(\xi), & \forall (x, z, \xi) \in \bar{I} \times \mathbb{R} \times \mathbb{R} \\ \theta(\xi)/|\xi| \to \infty, & \text{as } |\xi| \to \infty. \end{cases} \tag{1.2}$$

This will require a more detailed weak compactness criterion, as $L^1(I)$ (and thus $H^{1,1}(I)$) is not reflexive.

1.1 Background

We give a brief overview of the main definitions and results (without proof) regarding Sobolev spaces in dimension 1 and absolutely continuous functions, which will be used throughout the entire thesis.

Definition 1 (Sobolev spaces in dimension 1)**.** *Given an open interval $I \subseteq \mathbb{R}$ and $p \geq 1$, we denote by X the linear subspace of $C^1(I)$ consisting of functions u for which*

$$||u||_{H^{1,p}(I)} := \left(\int_I (|u|^p + |u'|^p) dx \right)^{1/p} < \infty.$$

$|| \cdot ||_{H^{1,p}(I)}$ is a norm on X, and the completion of X with respect to this norm is denoted by $H^{1,p}(I)$ and referred to as a Sobolev space. If $p = 2$, it is conventional to denote $H^1(I) := H^{1,2}(I)$, which is also a Hilbert space.

Definition 2 (Absolutely continuous functions)**.** *A function $f : (a, b) \to \mathbb{R}$ is said to be absolutely continuous (in the sense of Vitali) if, for every $\epsilon > 0$, there is a $\delta > 0$ such that*

$$\sum_{i=1}^{N} (\beta_i - \alpha_i) < \delta \implies \sum_{i=1}^{N} |f(\beta_i) - f(\alpha_i)| < \epsilon$$

whenever $(\alpha_1, \beta_i), \ldots, (\alpha_N, \beta_N)$ are disjoint segments in (a, b). The class of absolutely continuous functions is denoted by $AC(a, b)$.

Theorem 2. *We have*

$$AC(a, b) = H^{1,1}(a, b).$$

More precisely, every $u \in AC(a, b)$ has an almost everywhere classical derivative u' which belongs to $L^1(a, b)$, and viewed as an element of L^1, u' is the weak derivative of u. Conversely, every $u \in H^{1,1}(a, b)$, modulo a modification on a set of measure zero, is an absolutely continous function. Finally, $u \in AC(a, b)$ if and only if u is almost everywhere differentiable in a classical sense, u' belongs to $L^1(a, b)$ and the fundamental theorem of calculus holds true, i.e. for all $x, y \in (a, b)$ we have

$$u(x) - u(y) = \int_y^x u'(t) dt.$$

Theorem 3. *Given an open interval $I \subseteq \mathbb{R}$, we have*

i) Every function in $H^{1,1}(I)$ is uniformly continuous in I, in particular

$$H^{1,1}(I) \subseteq C^0(\bar{I});$$

ii) if $u \in H^{1,p}(I), p > 1$ *then* $u \in C^{0,1-1/p}(I)$ *and, for all* $x, y \in \bar{I}$ *we have*

$$|u(x) - u(y)| \leq \left(\int_I |u'|^p dx \right)^{1/p} |x - y|^{1-1/p}.$$

1.2 Tonelli's semicontinuity Theorem

In order to prove Theorem 1 we need a semicontinuity result for $\mathcal{F}(u)$ under the assumption that F is convex in ξ:

Theorem 4 (Tonelli's semicontinuity theorem). *Let* $I = (a, b) \subseteq \mathbb{R}$ *be a bounded open interval and let* $F \in C^1(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *satisfy:*

1. *F is non-negative (or bounded below by an L^1 function);*

2. *$F(x, z, \xi)$ is convex in $\xi \in \mathbb{R}$ for all $(x, z) \in \bar{I} \times \mathbb{R}$.*

Then the functional $\mathcal{F}(u)$ *is sequentially weakly lower semicontinuous in* $H^{1,p}(I)$ *for all* $p \geq 1$*, meaning that if* $u_k \rightharpoonup u$ *weakly in* $H^{1,p}$*, then*

$$\mathcal{F}(u) \leq \liminf_{k \to \infty} \mathcal{F}(u_k). \tag{1.3}$$

The proof relies on three standard results from measure theory, which we state here (for proofs of these results, see [2]):

Theorem (Egorov). *Let* $f, f_k : I \to \mathbb{R}, k \in \mathbb{N}$ *be measurable functions such that* $f_k(x) \to f(x)$ *for a.e.* $x \in I$*. Then for all* $\epsilon > 0$*, there exists a compact set* $K \subseteq I$ *with* $\text{meas}(I \setminus K) < \epsilon$ *such that* $f_k \rightrightarrows f$ *on* K*.*

Theorem (Lusin). *Let* $f : I \to \mathbb{R}$ *be measurable. Then for all* $\epsilon > 0$ *there exists a compact set* $K \subseteq I$ *such that* $\text{meas}(I \setminus K) < \epsilon$ *and* $f : K \to \mathbb{R}$ *is continuous.*

Theorem (Absolute continuity of the Lebesgue integral). *Let* $f : I \to \mathbb{R}$ *be integrable. Then for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$A \subseteq I, \quad \text{meas}(A) < \delta \implies \int_A |f| < \epsilon.$$

Proof of Theorem 2. We first notice that it is sufficient to consider the case $p = 1$. In fact, if $p > 1$, since $\text{meas}(I) < \infty$ we have a continuous inclusion $H^{1,p}(I) \hookrightarrow H^{1,1}(I)$, which remains continuous if both spaces are equipped with their respective weak topologies. This means that if $\{u_k\}$ converges weakly to u in $H^{1,p}(I)$, it also converges weakly to u in $H^{1,1}(I)$.
At this point, let $\{u_k\} \subseteq H^{1,1}(I)$ be a sequence which converges weakly to u in $H^{1,1}(I)$. It is a consequence of the Banach-Steinhaus Theorem that the sequence is bounded in $H^{1,1}(I)$. Furthermore, the inclusion $H^{1,1}(I) \hookrightarrow L^1(I)$ is compact [Brezis, Theorem 8.8 page 213] so after possibly passing to a subsequence we can assume that $\{u_k\}$ converges

stongly to u in $L^1(I)$ hence almost everywhere.
At this point, assume $\mathcal{F}(u) < \infty$. Given an $\epsilon > 0$ we obtain $\delta > 0$ from the absolute continuity theorem above applied to $F(x, u, u')$, and applying Egorov's and Lusin's theorem we find a compact subset $K \subseteq I$ with $\text{meas}(I \setminus K) < \delta$ such that

$$u_k \rightrightarrows u \text{ in } K, \qquad u, u' \text{ are continuous in } K$$

and by construction

$$\int_K F(x, u, u')dx \geq \int_I F(x, u, u')dx - \epsilon.$$

Since F is convex in ξ and continuously differentiable we have

$$F(x, z, \xi_1) \geq F(x, z, \xi_2) + \frac{\partial F}{\partial \xi}(x, z, \xi_2)(\xi_1 - \xi_2), \quad \forall \xi_1, \xi_2 \in \mathbb{R}.$$

Applying this with $\xi_1 = u_k'$ and $\xi_2 = u_k$ we obtain

$$\begin{aligned}
\mathcal{F}(u_k) &\geq \int_K F(x, u_k, u_k')dx \\
&\geq \int_K \frac{\partial F}{\partial \xi}(x, u_k, u')(u_k' - u')dx + \int_K F(x, u_k, u')dx \\
&= \int_K F(x, u_k, u')dx + \int_K \frac{\partial F}{\partial \xi}(x, u, u')(u_k' - u')dx \\
&+ \int_K \left(\frac{\partial F}{\partial \xi}(x, u_k, u') - \frac{\partial F}{\partial \xi}(x, u, u') \right) (u_k' - u')dx.
\end{aligned}$$

Now because K is compact and $\partial_\xi F$ is continuous we have that $\partial_\xi F(\cdot, u(\cdot), u'(\cdot)) \in L^\infty(K)$ and thus

$$\int_K \frac{\partial F}{\partial \xi}(x, u, u')(u_k' - u')dx \to 0, \qquad \text{for } k \to \infty$$

by definition of weak convergence $u_k' \rightharpoonup u'$ in $L^1(K)$.
Furthermore, $(\partial_\xi F(x, u_k, u') - \partial_\xi(x, u, u'))_{k \in \mathbb{N}}$ converges uniformly to zero on K as $k \to \infty$. In fact, F is continuous in (x, z, ξ) by hypothesis and given $\epsilon > 0$ we have $|u_k(x) - u(x)| < \epsilon \; \forall x \in K$ for sufficiently large $k \in \mathbb{N}$. Since $(u_k' - u')_{k \in \mathbb{N}}$ is equibounded in $L^1(I)$ (follows from the weak convergence) we obtain

$$\begin{aligned}
&\left| \int_K \left(\frac{\partial F}{\partial \xi}(x, u_k, u') - \frac{\partial F}{\partial \xi}(x, u, u') \right) (u_k' - u')dx \right| \\
&\leq ||u_k' - u'||_{L^1(I)} \left|\left| \frac{\partial F}{\partial \xi}(x, u_k, u') - \frac{\partial F}{\partial \xi}(x, u, u') \right|\right|_{L^\infty(K)} \to 0, \quad \text{for } k \to \infty.
\end{aligned}$$

Using Fatou's lemma to exchange limes inferior and integration we obtain that for all

$\epsilon > 0$

$$\begin{aligned}\liminf_{k\to\infty} \mathcal{F}(u_k) &\geq \liminf_{k\to\infty} \int_K F(x, u_k, u')dx \\ &\geq \int_K F(x, u, u')dx \geq \int_I F(x, u, u')dx - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the Theorem follows.
Finally, if $\mathcal{F}(u) = \infty$, given $\epsilon > 0$ we can choose K compact like before such that

$$\int_K F(x, u, u')dx \geq \frac{1}{\epsilon}$$

holds. Reasoning exactly as above we arrive at

$$\liminf_{k\to\infty} \int_K F(x, u_k, u_k')dx \geq \int_K F(x, u, u')dx \geq \frac{1}{\epsilon}$$

so $\liminf_{k\to\infty} \mathcal{F}(u_k) = \infty$ as claimed. □

1.3 Tonelli's existence Theorem

We are now ready to prove Theorem 1. We will first prove it in $H^{1,p}(I)$ with $p > 1$, then introduce a weak compactness criterion in L^1 that will be used to prove the version in $H^{1,1}(I)$.

1.3.1 In $H^{1,p}(I), p > 1$

Proof of Theorem 1. Because of the polynomial growth in (1.1), the functional $\mathcal{F}$ is bounded from below by 0. Let $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathcal{C}_p(\alpha, \beta)$ be a minimizing sequence for $\mathcal{F}$, i.e. $\lim_{k\to\infty} \mathcal{F}(u_k) = \inf_{v\in\mathcal{C}_p(\alpha,\beta)} \mathcal{F}(v)$ (this is possible since $\mathcal{C}_p(\alpha, \beta)$ is non-empty by assumption).
If $\lim_{k\to\infty} \mathcal{F}(u_k) = +\infty$, then $\mathcal{F} \equiv +\infty$ on $\mathcal{C}_p(\alpha, \beta)$, therefore we may assume without loss of generality that $\lim_{k\to\infty} \mathcal{F}(u_k) < +\infty$.
At this point, we want to show that $\{u_k\}$ is bounded in $H^{1,p}(I)$. Since $H^{1,p}(I)$ is reflexive for $p > 1$, it is a consequence of Kakutani's Theorem that in this case we can extract a subsequence converging weakly in $H^{1,p}(I)$ to some u. By Tonelli's semicontinuity Theorem, we then see that u is a candidate minimizer for $\mathcal{F}$ in $\mathcal{C}_p(\alpha, \beta)$.
From the polynomial growth of F we deduce

$$\sup_{k\in\mathbb{N}} \int_I |u_k'|^p dx \leq \sup_{k\in\mathbb{N}} \frac{1}{c_1} \int_I F(x, u_k, u_k')dx = \frac{1}{c_1} \sup_{k\in\mathbb{N}} \mathcal{F}(u_k) < +\infty,$$

so u_k' are equibounded in $L^p(I)$. Furthermore, because I is bounded, we have $H^{1,p}(I) \subseteq AC(I)$ and thus

$$u_k(x) = \alpha + \int_a^x u_k'(s)ds \implies |u_k(x)| \leq |\alpha| + ||u_k'||_{L^p(I)} \operatorname{meas}(I)^{1/p'} \leq C||u_k'||_{L^p(I)}, \quad (1.4)$$

for some $C \in \mathbb{R}$ depending on α, I and p, for all $x \in I$. From this it follows that

$$\sup_{k\in\mathbb{N}} ||u_k||_{L^p(I)} \leq C \left(\sup_{k\in\mathbb{N}} ||u_k'||_{L^p(I)} \right) \operatorname{meas}(I)^{1/p} < \infty$$

and thus u_k is bounded in $H^{1,p}(I)$.
It remains to verify that $u \in \mathcal{C}_p(\alpha, \beta)$, i.e. $u(a) = \alpha, u(b) = \beta$ and $\mathcal{G}(u) = 0$. From eq. (1.4) it follows that u_k are equibounded, further we have

$$|u_k(y) - u_k(x)| \leq \int_x^y |u_k'(s)|ds \leq \left(\sup_{k\in\mathbb{N}} ||u_k'||_{L^p(I)} \right) |x-y|^{1/p'},$$

therefore the sequence is also equicontinuous. Since $\bar{I}$ is compact, it follows from the Ascoli-Arzelà compactness Theorem that, after passing to a subsequence, u_{k_i} converges uniformly to u on $\bar{I}$ which implies $u(a) = \alpha$ and $u(b) = \beta$.
The constraint follows since the u_{k_i} are equibounded, G is continuous and $\bar{I}$ is compact. This means that the family of functions $x \mapsto G(x, u_{k_i}(x))$ is equibounded and thus by dominated convergence

$$\mathcal{G}(u) = \lim_{k\to\infty} \mathcal{G}(u_k) = 0.$$

□

1.3.2 In $H^{1,1}(I)$

We recall the following definition regarding Radon measures:

Definition 3 (Radon measure and total variation)**.** *A Radon measure μ on a topological space (X, τ) is an outer measure that satisfies:*

1. *μ is outer regular with respect to the family of open sets, i.e.*

$$\mu(E) = \inf\{\mu(A) : A \in \tau, E \subseteq A\};$$

2. *$\mu(K) < \infty$ for all compact K;*

3. *for all $A \in \tau$ we have*

$$\mu(A) = \sup\{\mu(K) : K \text{ compact}, K \subseteq A\}.$$

Given a Radon measure μ_α, the total variation of μ_α is defined by

$$||\mu_\alpha|| := \sup\{\alpha(f) : f \in C_c^0,\ |f(x)| \leq 1\ \forall x \in X\}, \quad (1.5)$$

where α is a continuous linear mapping $C^0_c(X) \to \mathbb{R}$ of the form

$$\alpha(f) = \int f du_\alpha, \quad \forall f \in C^0_c(X). \tag{1.6}$$

Remark 1. *Riesz's Theorem identifies Radon measures with continuous linear functionals on $C^0_c(X)$, in the sense that every $\alpha \in \mathscr{L}(C^0_c(X), \mathbb{R})$ can be expressed as in eq.(1.5) for some Radon measure μ_α.*

In what follows we will also use the following result, which is a consequence of the Banach-Alaouglu theorem applied to the space of Radon measures equipped with its natural weak* topology:

Lemma 1. *The space of Radon measures with bounded total variation is a Banach space with the norm $||\cdot||$ in (1.5). Furthermore, from every subsequence $\{\mu_k\}_{k\in\mathbb{N}}$ of Radon measures with equibounded total variation we can extract a subsequence $\{\mu_{k_i}\}_{i\in\mathbb{N}}$ that converges weakly in the sense of measures to another Radon measure μ, i.e.*

$$\langle \mu_{k_i}, \phi\rangle := \int \phi d\mu \to \langle \mu, \phi\rangle, \quad \forall \phi \in C^0_c(X).$$

Remark 2. *Every function $u \in L^1(a,b)$ defines a Radon measure $u\ dx$ on (a,b). Thus we conclude that, after possibly passing to a subsequence, every bounded sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq L^1(a,b)$ converges in the sense of measures to some Radon measure μ, i.e.*

$$\int_a^b \phi u_k \, dx \to \langle \mu, \phi\rangle, \qquad \forall \phi \in C^0_c(a,b).$$

We are now ready to prove the following Theorem, which gives a sufficient condition for a bounded set in $L^1(a,b)$ to be sequentially weakly compact in $L^1(a,b)$:

Theorem 5. *Let $\Omega \subseteq \mathbb{R}$ be a bounded open set, and let $\{u_k\}_{k\in\mathbb{N}} \subseteq L^1(\Omega)$ satisfy*

(i) $\sup_{k\in\mathbb{N}} ||u_k||_{L^1(\Omega)} < \infty$*;*

(ii) the set functions

$$E \mapsto \int_E |u_k| \, dx, \quad E \subseteq \Omega \text{ measurable}$$

are equiabsolutely continuous, i.e. $\forall \epsilon > 0 \, \exists \delta > 0$ *such that*

$$\int_E |u_k| \, dx < \epsilon \quad \forall k \in \mathbb{N}, \ \forall E \subseteq \Omega \text{ with } \mathrm{meas}(E) < \delta.$$

Then there exists a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ that converges weakly in $L^1(\Omega)$.

Proof. As explained in Remark 2, property (i) implies that there exists a subsequence $\{u_{k_i}\}$ and a Radon measure μ_α such that $\langle u_{k_i} dx, \phi\rangle \to \alpha(\phi)$, for all $\phi \in C^0_c(\Omega)$. We will now show that the limit

$$\lim_{i\to\infty} \int_B u_{k_i} \, dx = \gamma(B) \tag{1.7}$$

exists for all measurable sets $B \subseteq \Omega$ by proving that $\{\int_B u_{k_i}\,dx\}$ is a Cauchy sequence. Since the characteristic function χ_B is measurable and bounded by 1, applying Lusin's Theorem we find a sequence $\{\phi_h\}_{h\in\mathbb{N}} \subseteq C^0_c(\Omega)$ such that $||\phi_h||_{L^\infty} \leq 1$ for all $h \in \mathbb{N}$ and $\phi_h(x) \to \chi_B(x)$ for a.e. $x \in \Omega$. In correspondence with the $\delta > 0$ given by condition *(ii)*, we find by Egorov's Theorem an open set $B_\delta \subseteq \Omega$ such that $\text{meas}(B_\delta) < \delta$ and $\phi_h \rightrightarrows \chi_B$ on $\Omega \setminus B_\delta$. Now we have

$$\begin{aligned}\left|\int_\Omega (u_{k_i} - u_{k_j})\chi_B\,dx\right| &\leq \left|\int_{B_\delta} (u_{k_i} - u_{k_j})(\chi_B - \phi_h)\,dx\right| + \left|\int_\Omega (u_{k_i} - u_{k_j})\phi_h\,dx\right| \\ &\quad + \left|\int_{\Omega\setminus B_\delta} (u_{k_i} - u_{k_j})(\chi_B - \phi_h)\,dx\right| \\ &\leq 2\int_{B_\delta} (|u_{k_i}| + |u_{k_j}|)\,dx + \sup_{\Omega\setminus B_\delta} |\chi_B - \phi_h| \int_\Omega (|u_{k_i}| + |u_{k_j}|)\,dx \\ &\quad + \left|\int_\Omega (u_{k_i} - u_{k_j})\phi_h\,dx\right|.\end{aligned}$$

For any $\epsilon > 0$ $\exists h_0 \in \mathbb{N}$ such that $\sup_{\Omega\setminus B_\delta} |\chi_B - \phi_h| < \epsilon$ $\forall h \geq h_0$ by uniform convergence of ϕ_h on $\Omega \setminus B_\delta$. Since $\phi_{h_0} \in C^0_c(\Omega)$, the sequence $\{\int_\Omega u_{k_i}\phi_{h_0}\,dx\}_{i\in\mathbb{N}}$ is a Cauchy sequence, and thus

$$\left|\int_\Omega (u_{k_i} - u_{k_j})\phi_{h_0}\,dx\right| < \epsilon$$

for all i, j larger than some $k_0(h_0, \epsilon) \in \mathbb{N}$. We therefore obtain that there exists a suitably large constant $K > 0$ such that

$$\left|\int_B (u_{k_i} - u_{k_j})\,dx\right| \leq 4\epsilon + 2\epsilon K + \epsilon = (5 + 2K)\epsilon \quad \forall i, j \geq k_0,$$

which proves (1.7).

At this point, we claim that actually $\gamma(B) = \alpha(B)$ for all measurable sets $B \subseteq \Omega$. Given $\epsilon > 0$, let $0 < \delta < \epsilon/2$ be given by the equiabsolute continuity property. since α is a Radon measure, there exist a compact set K and an open set U with $K \subseteq B \subseteq U$ satisfying $\alpha(B \setminus K), \alpha(U \setminus B) < \delta$ and $\text{meas}(U \setminus K) < \delta$. We may now take a $\phi \in C^0_c(\Omega)$ such that $0 \leq \phi \leq 1$, $\phi|_K \equiv 1$ and $\text{supp}\,\phi \subseteq U$. This yields

$$\left|\int_B u_{k_i}dx - \alpha(B)\right| \leq \left|\int_\Omega u_{k_i}(\chi_B - \phi)dx\right| + \left|\int_\Omega u_{k_i}\phi dx - \alpha(\phi)\right| + |\alpha(\phi) - \alpha(B)|.$$

Now

$$\left|\int_\Omega u_{k_i}(\chi_B - \phi)dx\right| \leq \int_{U\setminus K} |u_{k_i}| < \epsilon$$

by assumption *(ii)*,

$$\lim_{i\to\infty} \left|\int_\Omega u_{k_i}\phi dx - \alpha(\phi)\right| = 0$$

by definition of α and $|\alpha(\phi) - \alpha(B)| \leq \int_\Omega |\phi - \chi_B| d\alpha \leq \alpha(U \setminus K) < 2\delta < \epsilon$. Therefore the claim is verified, and in particular we deduce that α is absolutely continuous with respect to the Lebesgue measure. Hence, by the Radon-Nikodym theorem, it is represented by a function $u \in L^1(\Omega)$. Since step functions of measurable sets are dense in $L^\infty(\Omega)$, relation (1.7) together with a direct application of the dominated convergence theorem implies that $\{u_{k_i}\}$ converges weakly to u in $L^1(\Omega)$, and the theorem is proved. □

Remark 3. *The converse also holds, i.e. if $\{u_k\}_{k\in\mathbb{N}}$ converges weakly in $L^1(\Omega)$ then (i) and (ii) hold true, but we omit the proof. The full theorem, proving the equivalence between these two statements, is sometimes referred to as the Dunford-Pettis Theorem.*

We now prove a theorem which collects criteria for sequential weak compactness in $L^1(\Omega)$; to prove Tonelli's existence theorem in $H^{1,1}(I)$ we will in particular use the equivalence $(i_1) \iff (i_4)$.

Theorem 6. *Let $\mathcal{F} \subseteq L^1(\Omega)$. The following claims are equivalent:*

1. *$\mathcal{F}$ is sequentially weakly compact in $L^1(\Omega)$;*
2. *$\mathcal{F}$ is bounded in $L^1(\Omega)$ and the set functions*
$$E \mapsto \int_E |u|\,dx, \quad E \subseteq \Omega \text{ measurable}, \quad u \in \mathcal{F},$$
are equiabsolutely continuous;
3. *the functions $u \in \mathcal{F}$ are uniformly integrable, i.e.*
$$\lim_{c\to\infty} \int_{\{x\in\Omega:|u(x)|>c\}} |u(x)|dx = 0$$
uniformly for $u \in \mathcal{F}$;
4. *there exists a function $\Theta : (0,\infty) \to \mathbb{R}$ (which can be taken convex and increasing) such that*
$$\lim_{t\to\infty} \frac{\Theta(t)}{t} = \infty \quad \text{and} \quad \sup_{u\in\mathcal{F}} \int_\Omega \Theta(|u|)dx < \infty.$$

Proof. The equivalence $(i_1) \iff (i_2)$ is the content of the Dunford-Pettis Theorem.
$(i_3 \implies i_2)$: we suppose the functions in $\mathcal{F}$ are uniformly integrable. We have the elementary inequality

$$\int_E |u|dx \le c\,\text{meas}(E) + \int_{\{x\in\Omega:|u(x)|>c\}} |u(x)|dx, \quad \forall E \subseteq \Omega \text{ measurable}, \quad \forall u \in \mathcal{F}.$$

We can choose $c > 0$ such that the latter term is less than $\epsilon/2$. Choosing $E = \Omega$ we obtain equiboundedness in $L^1(\Omega)$ (since Ω is bounded), while choosing $\delta = \epsilon/(2c)$ we satisfy the equiabsolute continuity condition for all $\epsilon > 0$.
$(i_2 \implies i_3)$: Given $c > 0$ and $u \in F$, let $E := \{x \in \Omega : |u(x)| > c\}$. We have by

definition of E that $\text{meas}(E)c \leq ||u||_{L^1(E)} \leq \sup_{u\in\mathcal{F}} ||u||_{L^1(\Omega)}$. Given $\epsilon > 0$, choosing $c = (1/\delta)\sup_{u\in\mathcal{F}} ||u||_{L^1(\Omega)}$, where $\delta > 0$ is the number appearing in condition (i_2), we find

$$\int_{\{x\in\Omega:|u(x)|>c\}} |u(x)|dx \leq \epsilon, \quad \forall u \in \mathcal{F}.$$

$(i_4) \implies (i_3)$: Given a Θ with the required properties, let $M := \sup_{u\in\mathcal{F}} ||\Theta(|u|)||_{L^1(\Omega} < \infty$. Given $\epsilon > 0$, we choose $c > 0$ such that $\Theta(t)/t \geq M/\epsilon$ for all $t \geq c$. Then $|u| \leq \epsilon\Theta(|u|)/M$ on $\{x \in \Omega : |u(x)| > c\}$, and thus

$$\int_{\{x\in\Omega:|u(x)|>c\}} |u|dx \leq \frac{\epsilon}{M}\int_{\{x\in\Omega:|u(x)|>c\}} \Theta(|u|)dx \leq \epsilon \qquad \forall u \in \mathcal{F}.$$

$(i_3) \implies (i_4)$: We construct a function $\Theta : (0,\infty) \to \mathbb{R}$ of the form $\Theta(t) = \int_0^t g(s)ds$, where g is an increasing function with $g(0) = 0$ and $\lim_{s\to\infty} g(s) = \infty$ which assumes constant values on each interval $(n, n+1), n \in \mathbb{N}$. For $u \in \mathcal{F}, n \in \mathbb{N}$ we define

$$a_n(u) := \int_{\{x\in\Omega:|u(x)|\geq n\}} |u(x)|dx;$$

It follows that

$$\begin{aligned}
\int_\Omega \Theta(|u|)dx &= \int_\Omega \left(\int_0^{|u(x)|} g(s)ds\right) dx = \int_0^\infty \left(\int_{\{|u(x)|\geq s\}} g(s)dx\right) ds \\
&= \sum_{n=0}^\infty \int_n^{n+1} \left(\int_{\{|u(x)|\geq s\}} g(s)dx\right) ds = \sum_{n=0}^\infty \int_n^{n+1} (g_n \,\text{meas}\{|u(x)| \geq s\}) \\
&\leq \sum_{n=0}^\infty g_n \left(\int_{\{|u(x)|\geq n\}} |u(x)|dx\right) = \sum_{n=0}^\infty g_n a_n(u),
\end{aligned}$$

where we have used Fubini-Tonelli's theorem twice. At this point, using the assumption of uniform integrability we can choose integers $c_n \to \infty$ such that

$$\int_{\{x\in\Omega:|u(x)|\geq c_n\}} |u(x)|dx \leq 2^{-n}, \quad \forall u \in \mathcal{F},\ n \in \mathbb{N}.$$

By construction of the c_n, it follows that the numbers

$$\sum_{n=0}^\infty \sum_{k=c_n}^\infty a_k(u)$$

are uniformly bounded for $u \in \mathcal{F}$. Because this sum can be written in the form $\sum_k g_k a_k(u)$ (where $g_k = |\{n \in \mathbb{N} : c_n < k\}|$), we can find a sequence of values $g_n \to \infty$ such that Θ as constructed satisfies the required conditions. □

We are now ready to prove Theorem 1 in $H^{1,1}(I)$:

Proof of Theorem 1 in $H^{1,1}(I)$. By the superlinear growth of F, the functional $\mathcal{F}$ is bounded from below. Let $\{u_k\}$ be a minimizing sequence in $C_1(\alpha, \beta)$ (we may assume that $\inf_k \mathcal{F}(u_K) < +\infty$, otherwise $\mathcal{F}(u) \equiv +\infty$ on $C_1(\alpha, \beta)$). Using that F has superlinear growth we obtain that the sequence $\{u_k'\} \subseteq L^1(I)$ is bounded. Moreover, since the functions are absolutely continuous, the fundamental theorem of calculus holds true and thus it follows that $\{u_k\}$ is bounded in $H^{1,1}(I)$. By assumption on F,

$$\sup_k \int_I \theta(u_k')dx \leq \sup_k \int_I F(x, u_k, u_k')dx = \sup_k \mathcal{F}(u_k) < +\infty$$

so that the integrals $\{\int_I \theta(u_k')dx\}$ are equibounded. By Theorem 4, this is equivalent to saying that the family $\{u_k'\}$ is equiabsolutely integrable, i.e. the set functions $E \mapsto \int_E u_k' dx$ for $E \subseteq I$ measurable are equiabsolutely continuous.
From the $H^{1,1}(I)$- boundedness of $\{u_k\}$ we infer that (a suitable subsequence of) $\{u_k\}$ converges strongly in $L^1(I)$ to some function u (this is because $H^{1,1}(I) \hookrightarrow L^1(I)$ is compact for meas$(I) < \infty$, as we have already seen). Further, taking the equiabsolute integrability of $\{u_k'\}$ into account, it is a consequence of Ascoli-Arzelà's theorem that a subsequence of $\{u_k\}$ converges uniformly to u.
At this point, by Theorem 4, passing to another subsequence, $\{u_k'\}$ converges weakly in $L^1(I)$ to some $w \in L^1(I)$. We have the relations

$$-\int_I u_k \phi' dx = \int_I u_k' \phi dx \to \int_I w\phi \quad \text{for all } \phi \in C_c^1(I)$$

and

$$-\int_I u_k \phi' dx \to -\int_I u\phi' dx = \int_I u'\phi dx;$$

hence $u_k' \rightharpoonup u'$ in $L^1(I)$ and thus $u_k \rightharpoonup u$ in $H^{1,1}(I)$. Theorem 2 then yields

$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k).$$

Finally, u obtains values of α and β respectively at a and b because of the uniform convergence $u_k \rightrightarrows u$ on $\bar{I}$, and the constraint $G(u) = 0$ follows from the dominated convergence theorem as in the case of $H^{1,p}(I)$ with $p > 1$. We conclude that u is a minimizer of $\mathcal{F}$ in $C_1(\alpha, \beta)$.

□

Chapter 2

Regular Minimizers

In the following chapter we will prove a regularity result for $\mathcal{G}$-regular minimizers of $\mathcal{F}$ in $\mathcal{C}(\alpha, \beta)$. Specifically, we will prove the following Theorem:

Theorem 7. *Let $I = (a, b) \subseteq \mathbb{R}$ be a bounded open interval, $F \in C^1(\bar{I} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $F(x, z, \cdot) \in C^2(\mathbb{R})$ and $G \in C^1(\bar{I} \times \mathbb{R}, \mathbb{R}^n)$ satisfying the following conditions:*

1. *There exists $c \in \mathbb{R}$ such that*

$$F(x, z, \xi) \leq c(1 + |\xi|^2), \quad \forall (x, z, \xi) \in \bar{I} \times \mathbb{R} \times \mathbb{R}; \tag{2.1}$$

2. *There exists $c_3 \in \mathbb{R}$ such that for all $(x, z, \xi) \in \bar{I} \times \mathbb{R} \times \mathbb{R}$ we have*

$$\left|\frac{\partial F}{\partial z}(x, z, \xi)\right| + \left|\frac{\partial F}{\partial \xi}(x, z, \xi)\right| \leq c_3(1 + |\xi|); \tag{2.2}$$

3. *There exists $\delta > 0$ such that for all $(x, z, \xi) \in \bar{I} \times \mathbb{R} \times \mathbb{R}$*

$$\frac{\partial^2 F}{\partial \xi^2}(x, z, \xi) > \delta. \tag{2.3}$$

If $u \in AC(I)$ is a $\mathcal{G}$-regular minimizer for $\mathcal{F}$, then $u \in C^1(\bar{I})$. Furthermore, if F and G are of class C^k for some $2 \leq k \leq \infty$, then $u \in C^k(\bar{I})$.

2.1 Lagrange multipliers for $\mathcal{G}$-regular extrema

In this section we first introduce the concepts of $\mathcal{G}$-regular and $\mathcal{G}$-singular extrema and then prove the Lagrange multiplier Theorem for the former.

Definition 4 (Singular extremal)**.** *We say that $u \in AC(I)$ is a $\mathcal{G}$-singular extremal if for all $\psi_1, \ldots, \psi_n \in C_c^\infty(I, \mathbb{R})$ we have*

$$\det J_\tau \Psi(0) = 0,$$

where $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ *is given by* $\Psi(\tau) := \mathcal{G}(u + \sum_{i=1}^n \tau_i \psi_i)$ *and* J_τ *denotes the Jacobian matrix of* Ψ *with respect to* τ*.*

Remark 4 (Regular extremal)**.** *If* $u \in AC(I)$ *is a* $\mathcal{G}$*-regular extremal then* $\exists \psi_1, \dots, \psi_n \in C_c^\infty(I, \mathbb{R})$ *such that* $\det(J_\tau \Psi(0)) \neq 0$*. For* $i, j \in \{1, \dots, n\}$ *we have*

$$(J_\tau \Psi)_{i,j}(0) = \frac{\partial \Psi_i}{\partial \tau_j}(0) = \frac{\partial}{\partial \tau_j} \mathcal{G}_i \left(u + \sum_{k=1}^n \tau_k \psi_k \right) \Bigg|_{\tau=0} =$$
$$= \frac{\partial}{\partial \tau_j} \int_I G_i \left(x, u + \sum_{k=1}^n \tau_k \psi_k \right) dx \Bigg|_{\tau=0} .$$

Exchaning differentiation and integration by the Leibniz integral rule we obtain

$$(J_\tau \Psi)_{i,j}(0) = \int_I \frac{\partial G_i}{\partial z}(x, u) \psi_j dx. \tag{2.4}$$

This remark will be used to prove the following Theorem:

Theorem 8 (Lagrange multipliers)**.** *Let* $u \in AC(I)$ *be a* $\mathcal{G}$*-regular minimizer of* $\mathcal{F}$ *in* $\mathcal{C}(\alpha, \beta)$*. Then* $\exists \lambda_1, \dots, \lambda_n \in \mathbb{R}$ *such that*

$$\frac{d}{d\epsilon} \left(\int_I F(x, u + \epsilon\phi, u' + \epsilon\phi') dx - \sum_{i=1}^n \lambda_i \int_I G_i(x, u + \epsilon\phi) dx \right) \Bigg|_{\epsilon=0} = 0 \tag{2.5}$$

for all $\phi \in C_c^\infty(I)$ *(this is the weak Euler-Lagrange equation). Further, if* $u \in C^2(I)$ *it also satisfies the strong Euler-Lagrange equation:*

$$\frac{\partial F}{\partial z}(x, u, u') - \frac{d}{dx} \frac{\partial F}{\partial \xi}(x, u, u') - \sum_{i=1}^n \lambda_i \frac{\partial G_i}{\partial z}(x, u) = 0, \quad \forall x \in I. \tag{2.6}$$

Proof. We fix $0 < \epsilon_0, \tau_0 << 1$ and let $\mathcal{Q} := \{(\epsilon, \tau) \in \mathbb{R} \times \mathbb{R}^n : |\epsilon| < \epsilon_0, |\tau_i| < \tau_0 \ \forall i \in \{1, \dots, n\}\}$. Given $\phi \in C_c^\infty(I)$ and $\psi_1, \dots, \psi_n \in C_c^\infty(I)$ we define $\Phi : \mathcal{Q} \to \mathbb{R}$ and $\Psi : \mathcal{Q} \to \mathbb{R}^n$ as

$$\Phi(\epsilon, \tau) := \mathcal{F} \left(u + \epsilon\phi + \sum_{i=1}^n \tau_i \psi_i \right) \quad \text{and} \quad \Psi(\epsilon, \tau) := \mathcal{G} \left(u + \epsilon\phi + \sum_{i=1}^n \tau_i \psi_i \right)$$

respectively.
We have $\Psi(0, 0) = 0$ by assumption and, since u is a regular minimizer, there exist $\psi_1, \dots, \psi_n \in C_c^\infty(I)$ such that $J_\tau \Psi(0, 0)$ is invertible. Since $\Psi \in C^1(Q, \mathbb{R}^n)$, we apply the implicit function theorem and obtain a curve $\tau \in C^1((-\epsilon_0, \epsilon_0), \mathbb{R}^n)$ such that $\Psi(\epsilon, \tau(\epsilon)) =$

0, for $|\epsilon| < \epsilon_0$. From this we deduce

$$0 = \frac{d}{d\epsilon}\Psi(\epsilon, \tau(\epsilon))\bigg|_{\epsilon=0} = \int_I D_z G(x,u)\left(\phi + \sum_{k=1}^n \tau_k'(0)\psi_k\right)dx$$
$$\implies -\int_I D_z G(x,u)\phi dx = \sum_{k=1}^n \tau_k'(0)\int_I D_z G(x,u)\psi_k dx = J_\tau\Psi(0,0)\cdot\tau'(0),$$

where we used the definition of Ψ in the last equivalence. Using that $J_\tau\Psi(0,0)$ is invertible we get

$$\tau'(0) = -[J_\tau\Psi(0,0)]^{-1}\cdot\int_I D_z G(x,u)\phi dx, \tag{2.7}$$

or component-wise

$$\tau_i'(0) = -\sum_{i=1}^n M_{ik}\int_I \frac{\partial G_k}{\partial z}(y,u)\phi(y)dy, \qquad \forall i \in \{1,\dots,n\},$$

where $M := [J_\tau\Psi(0,0)]^{-1} \in \mathrm{GL}_n(\mathbb{R})$.
We now want to compute the derivative of $\Phi(\epsilon,\tau(\epsilon))$ with respect to ϵ. We let $\phi \in C_c^\infty(I)$ be arbitrary, by the assumption of minimality of u we have:

$$\begin{aligned}
0 &= \frac{d}{d\epsilon}\Phi(\epsilon,\tau(\epsilon))\bigg|_{\epsilon=0} \\
&= \int_I \frac{d}{d\epsilon}\left(F(x, u+\epsilon\phi + \sum_{i=1}^n \tau_i(\epsilon)\psi_i, u' + \epsilon\phi' + \sum_{i=1}^n \tau_i(\epsilon)\psi_i')\right)\bigg|_{\epsilon=0} dx \\
&= \int_I \left\{\frac{\partial F}{\partial z}(x,u,u')\left(\phi + \sum_{i=1}^n \tau_i'(0)\psi_i\right) + \frac{\partial F}{\partial \xi}(x,u,u')\left(\phi' + \sum_{i=1}^n \tau_i'(0)\psi_i'\right)\right\} dx \\
&=: I + II + III + IV
\end{aligned}$$

For term II we use relation (2.7):

$$\begin{aligned}
II &:= \int_I \frac{\partial F}{\partial z}(x,u,u')\sum_{i=1}^n \tau_i'(0)\psi_i(x)dx \\
&= -\int_I \frac{\partial F}{\partial z}(x,u,u')\sum_{i=1}^n\left(\sum_{k=1}^n M_{ik}\int_I \frac{\partial G_k}{\partial z}(y,u)\phi(y)dy\right)\psi_i(x)dx \\
&= -\sum_{i,k=1}^n M_{ik}\int_I \frac{\partial F}{\partial z}(x,u,u')\psi_i(x)\left(\int_I \frac{\partial G_k}{\partial z}(y,u)\phi(y)dy\right)dx \\
&= -\sum_{i,k=1}^n M_{ik}\int_I \frac{\partial G_k}{\partial z}(y,u)\phi(y)\left(\int_I \frac{\partial F}{\partial z}(x,u,u')\psi_i(x)dx\right)dy,
\end{aligned}$$

where we have used Fubini-Tonelli's theorem in the last line in order to switch the order of integration. Regarding IV, a similar computation yields

$$\begin{aligned} IV &:= \int_I \frac{\partial F}{\partial \xi}(x,u,u') \sum_{i=1}^n \tau_i'(0)\psi_i'(x)dx \\ &= -\sum_{i,k=1}^n M_{ik} \int_I \frac{\partial G_k}{\partial z}(y,u)\phi(y) \left(\int_I \frac{\partial F}{\partial \xi}(x,u,u')\psi_i(x)dx \right) dy. \end{aligned}$$

Adding everything back together using these identities we get, for all $\phi \in C_c^\infty(I)$,

$$\int_I \Bigg\{ \left(\frac{\partial F}{\partial z}(x,u,u') - \sum_{i,k=1}^n M_{ik} \frac{\partial G_k}{\partial z}(x,u) \int_I \left[\frac{\partial F}{\partial z}(y,u,u')\psi_i(y) + \frac{\partial F}{\partial \xi}(y,u,u')\psi_i'(y) \right] dy \right) \phi(x) \\ + \frac{\partial F}{\partial \xi}(x,u,u')\phi'(x) \Bigg\} = 0.$$

We can now define Lagrange multipliers (independent of ϕ for all $k \in \{1,\dots,n\}$) as

$$\lambda_k := \sum_{i=1}^n M_{ik} \int_I \left[\frac{\partial F}{\partial z}(y,u,u')\psi_i(y) + \frac{\partial F}{\partial \xi}(y,u,u')\psi_i'(y) \right] dy, \quad \forall k \in \{1,\dots,n\}.$$

In terms of Φ and Ψ, we have shown that

$$\partial_\epsilon \Phi(0,0) - \partial_\tau \Phi(0,0)^T [J_\tau \Psi(0,0)]^{-1} \partial_\epsilon \Psi(0,0) = 0,$$

where $\partial_\tau \Phi(0,0)^T [J_\tau \Psi(0,0)]^{-1} := \lambda \in \mathbb{R}^n$ is the vector of Lagrange multipliers. This proves that u satisfies the weak Euler-Lagrange equation in (2.5) associated to the minimization problem in question.
If $u \in C^2(I)$, we can perform an integration by parts for the term involving $\phi'(x)$. We thus obtain that for all $\phi \in C_c^\infty(I)$

$$\int_I \left\{ \frac{\partial F}{\partial z}(x,u,u') - \frac{d}{dx}\frac{\partial F}{\partial \xi}(x,u,u') - \sum_{k=1}^n \lambda_k \frac{\partial G_k}{\partial z}(x,u) \right\} \phi(x)dx = 0.$$

Since the equation holds for all $\phi \in C_c^\infty(I)$, we conclude that u satisfies the strong Euler-Lagrange equation in (2.6). □

2.2 Regularity results for $\mathcal{G}$-regular extrema

The aim of this section is to prove that a $\mathcal{G}$-regular extremal of the minimization problem

$$\min\{\mathcal{F}(u) \mid u \in AC(I), u(a) = \alpha, u(b) = \beta \text{ and } \mathcal{G}(u) = 0\} \tag{2.8}$$

inherits the regularity of F and G. We define the Lagrangian of the constrained problem $H : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$H(x, z, \xi) := F(x, z, \xi) - \sum_{k=1}^{n} \lambda_k G_k(x, z),$$

where $\lambda_1, \ldots, \lambda_n$ are the multipliers from Theorem 6. Further, we define the functional $\mathcal{H} : AC(I) \to \mathbb{R}$ as

$$\mathcal{H}(u) := \int_I H(x, u(x), u'(x))dx.$$

Remark 5. *The functional $\mathcal{H}$ is well defined for $u \in AC(I)$. In fact, since u and u' are measurable and $H : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, the composition $x \mapsto H(x, u(x), u'(x))$ is measurable. The bound in (2.1) guarantees the integrability of $x \mapsto F(x, u(x), u'(x))$ while the uniform continuity of u guarantees the integrability of $x \mapsto G(x, u(x))$, since $\bar{I}$ is compact.*

We are now ready to prove the first regularity result:

Proposition 1. *Under the assumption in Theorem 5, if $u \in AC(I)$ is a $\mathcal{G}$-regular minimizer of problem (2.8) then $u \in C^1(I)$.*

Proof. Firstly, we observe that the mappings $x \mapsto \frac{\partial H}{\partial z}(x, u(x), u'(x))$ and $x \mapsto \frac{\partial H}{\partial \xi}(x, u(x), u'(x))$ are measurable. As in Remark 5, since $\bar{I}$ is compact, they are in $L^1(I)$ by condition (2.2). The weak Euler-Lagrange equation (2.5) then reads

$$\int_I \left(\frac{\partial H}{\partial z}(x, u, u')\phi(x) + \frac{\partial H}{\partial \xi}(x, u, u')\phi'(x) \right) dx = 0, \quad \forall \phi \in C_c^\infty(I). \tag{2.9}$$

Integrating the first terms by parts we obtain

$$\begin{aligned}\int_I \frac{\partial H}{\partial z}(x, u, u')\phi(x)dx &= \left(\int_a^x \frac{\partial H}{\partial z}(s, u, u')ds \right) \phi(x)\Big|_{x=a}^{x=b} - \int_I \left(\int_a^x \frac{\partial H}{\partial u}(s, u, u')ds \right) \phi'(x)dx \\ &= - \int_I \left(\int_a^x \frac{\partial H}{\partial z}(s, u, u')ds \right) \phi'(x)dx,\end{aligned}$$

where we have used that ϕ has compact support. Equation (2.9) becomes

$$\int_I \left(\frac{\partial H}{\partial \xi}(x, u, u') - \int_a^x \frac{\partial H}{\partial z}(s, u, u')ds \right) \phi'(x)dx = 0, \quad \forall \phi \in C_c^\infty(I).$$

By the du Bois-Raymond Lemma (see [3]) we deduce that there exists a constant $c \in \mathbb{R}$ such that

$$\frac{\partial H}{\partial \xi}(x,u,u') = c + \int_a^x \frac{\partial H}{\partial z}(s,u,u')ds =: \pi(x) \quad \text{for a.e. } x \in I, \tag{2.10}$$

where $\pi(x) \in AC(I)$.
We now define a mapping $\Gamma : \bar{I} \times \mathbb{R} \times \mathbb{R} \to \bar{I} \times \mathbb{R} \times \mathbb{R}$ as $\Gamma(x,z,\xi) = (x,z,\partial_\xi H(x,z,\xi))$. Condition (2.3) implies that $\partial_\xi H(x,u,\mathbb{R}) = \mathbb{R}$ for all fixed $(x,z) \in \bar{I} \times \mathbb{R}$, therefore $\operatorname{Im}\Gamma = \bar{I} \times \mathbb{R} \times \mathbb{R}$. Further, for fixed $(x,z) \in \bar{I} \times \mathbb{R}$, the mapping $\xi \mapsto \partial_\xi H(x,z,\xi)$ is continuously differentiable and $\partial_\xi^2 H(x,z,\xi) > \delta > 0$ for all $\xi \in \mathbb{R}$ by condition (2.3), thus the inverse map $\partial_\xi H^{-1}(x,z,\cdot)$ exists and is C^1. Therefore, $\Gamma : \bar{I} \times \mathbb{R} \times \mathbb{R} \to \bar{I} \times \mathbb{R} \times \mathbb{R}$ is a C^1 diffeomorphism.
At this point, we define the curves

$$\sigma(x) := (x, u(x), u'(x)) \quad \text{and} \quad e(x) := (x, u(x), \pi(x)),$$

where σ is defined a.e. in $\bar{I}$, while e is defined for all $x \in \bar{I}$ since $\pi, u : \bar{I} \to \mathbb{R}$ are absolutely continuous. Identity (2.10) gives

$$\Gamma(\sigma(x)) = e(x) \quad \text{for a.e. } x \in \bar{I}. \tag{2.11}$$

Because $\operatorname{Im}\Gamma = \bar{I} \times \mathbb{R} \times \mathbb{R}$, $\Gamma^{-1}(e(x))$ is well-defined and continuous for all $x \in \bar{I}$. We deduce that there exists a function $v \in C(\bar{I})$ such that $\Gamma^{-1}(e(x)) = (x, u(x), v(x))$ for all $x \in \bar{I}$. Then (2.11) implies

$$(x, u(x), u'(x)) = (x, u(x), v(x)), \quad \text{for a.e. } x \in \bar{I},$$

so that in particular $u'(x) = v(x)$ a.e. in $\bar{I}$. Since u is absolutely continuous, we have

$$u(x) = u(a) + \int_a^x u'(s)ds = u(a) + \int_a^x v(s)ds,$$

and thus $u \in C^1(I)$. □

Let us now use this result to prove that a $\mathcal{G}$-regular minimizer of Problem (2.8) actually inherits the regularity properties of F and G.

Proof of Thm. 4. Let $u \in AC(I)$ be a $\mathcal{G}$-regular minimizer of Problem (2.8). We prove by induction that $u \in C^k(I)$ for all $k \in \mathbb{N}$ whenever F and G are of class C^k.
We have proved in Proposition 1 that $u \in C^1(I)$. Let us now show that we have $u \in C^2(I)$, which will be our base step.
To do this, let $P : \bar{I} \times \mathbb{R} \to \mathbb{R}$ be given by

$$P(x,\xi) := \frac{\partial H}{\partial \xi}(x, u(x), \xi) - \pi(x),$$

where $\pi(x)$ is as in (2.10). Since $u \in C^1(I)$, we have that both $(x,\xi) \mapsto \partial_\xi H(x, u(x), \xi)$

and $\pi : \bar{I} \to \mathbb{R}$ are of class C^1, so $P \in C^1(I \times \mathbb{R}, \mathbb{R})$. Moreover, still by relation (2.10), we have $P(x, u'(x)) = 0$ for all $x \in I$ and

$$\frac{\partial P}{\partial \xi}(x, \xi) = \frac{\partial^2 F}{\partial^2 \xi}(x, u(x), \xi) > 0, \quad \forall (x, \xi) \in I \times \mathbb{R},$$

by assumption (2.3). In particular we have that $\partial_\xi P(x, u') > 0$. By the implicit function theorem, there exists an open neighbourhood $U_x \subseteq I$ and a function $v \in C^1(U_x)$ such that $v(x) = u'(x)$ for all $x \in U_x$. As in the proof of Proposition 1, this implies $u \in C^2(U_x)$. However, $x \in I$ can be chosen arbitrarily, therefore $u \in C^2(I)$.
For the inductive step, we let $k \in \mathbb{N}, k \geq 2$ be arbitrary, $u \in C^k(I)$ and we show that $u \in C^{k+1}(I)$. Exactly the same as above we see that under these assumptions $P \in C^k(I \times \mathbb{R})$ and the implicit function theorem guarantees the existence of some $v \in C^k(U_x)$ such that $v(x) = u'(x)$ in some open neighbourhood $U_x \subseteq I$ of x. As before, this yields $u \in C^{k+1}(I)$ by the arbitrariness of x. This concludes the inductive step and the proof. $\square$

Chapter 3

Singular extrema for a Dirichlet energy minimization problem

In this chapter we turn to the study of the following minimization problem:

$$\min_{u \in H^1(I)} \{\mathcal{F}(u) : u(\pm 1) = 1 \text{ and } \mathcal{G}(u) = V_{h,k}\}, \tag{3.1}$$

where $I := [-1, 1] \subseteq \mathbb{R}$, $h, k \in \mathbb{N}_{\geq 1}$ and $V_{h,k} = \frac{4k^2}{(2k+1)(2kh+k+h)}$. The functionals $\mathcal{F}$ and $\mathcal{G}$ are defined by:

$$\mathcal{F}(u) := \frac{1}{2} \int_I |u'(x)|^2 dx, \quad \mathcal{G}(u) := \int_I \left(x^{2h} u(x) - \frac{u(x)^{2k+1}}{2k+1} \right) dx.$$

Remark 6. *The Lagrangian $F(\xi) = \frac{1}{2}|\xi|^2$ and the constraint $G(x, z) = x^{2h} z - z^{2k+1}/(2k+1)$ are of class C^∞. Further, $\mathcal{F}$ is well-defined for $u \in H^1(I)$ and $\mathcal{G}$ is well-defined because u is continuous.*

Remark 7. *the value $V_{h,k}$ might initially appear arbitrary; we will see that it is exactly the value of $\mathcal{G}(u)$ for a singular extremal u.*

Remark 8. *Note that there exists a minimizer $u \in H^1(I)$ of (3.1), since the conditions of Theorem 1 hold: F has quadratic growth and is convex.*

Our goal in this chapter is to determine (as a function of the parameters $h, k \in \mathbb{N}$) whether or not there exists singular minimizers of problem (3.1). If u is a singular extremal of (3.1), then by definition we have

$$\frac{d}{d\epsilon} \mathcal{G}(u + \epsilon \psi)|_{\epsilon=0} = 0,$$

for all $\psi \in C_c^\infty(\mathring{I})$. In our case, this implies

$$\frac{\partial}{\partial u} \left(x^{2h} u - \frac{u^{2k+1}}{2k+1} \right) = 0, \quad \forall x \in I,$$

which yields $u(x) = |x|^{h/k}$. We note that this is a smooth function for $h > k$, it is Lipschitz-continuous but not $C^1(I)$ if $h = k$ and not even Lipschitz if $h < k$.
Using that u is even on I we obtain

$$\begin{aligned}\mathcal{G}(u) &= 2\int_0^1 G(x, u(x))dx = 2\int_0^1 \left(x^{(2kh+h)/k} - \frac{x^{(2kh+h)/k}}{2k+1} \right) dx \\ &= \frac{4k^2}{(2k+1)(2hk+h+k)} = V_{h,k},\end{aligned}$$

which motivates the latter's definition.

3.1 $h \neq k$

In this case, we have the following

Theorem 9. *For all $h, k \in \mathbb{N}, h \neq k$, the minimizer $u \in H^1(I)$ of problem (3.1) is $\mathcal{G}$-regular.*

The proof of this theorem is described in detail in [5], in this section we limit ourselves to outlining the basic idea underlining the argument.
First of all, we note that the function u above is an even function for all choices of $h, k \in \mathbb{N}$, therefore we can restrict our attention to $I = [0, 1]$ and extend the result by symmetry. Given a $\delta > 0$, the idea is the following: let $\epsilon > 0$ and $\eta \in \mathbb{R}$ be fixed and define

$$u_\epsilon^\delta(x) := \begin{cases} \epsilon & : x \leq \delta \\ u(x) & : x \in [\delta, \frac{1}{2}] \\ m_1 x + d_1 & : x \in [\frac{1}{2}, \frac{1}{2} + |\eta|] \\ u(x) + \eta & : x \in [\frac{1}{2} + |\eta|, \frac{3}{4} - |\eta|] \\ m_2 x + d_2 & : x \in [\frac{3}{4} - |\eta|, \frac{3}{4}] \\ u(x) & : x \geq \frac{3}{4}, \end{cases}$$

where $m_1(\eta, h, k), d_1(\eta, h, k), m_2(\eta, h, k), d_2(\eta, h, k) \in \mathbb{R}$. In order for u_ϵ^η to be continuous we choose $\delta = \epsilon^{k/h}$; since u_ϵ^η is bounded and Lipschitz-continuous on I we have $u \in H^{1,\infty} \subseteq H^1(I)$.
At this point, it is possible to prove that, if $h \neq k$, there exists an $\eta(\epsilon) \in \mathbb{R}$ such that the constraint $\mathcal{G}(u_\epsilon^\eta) = V_{h,k}$ is satisfied. Finally, from the same calculations it follows that for $\epsilon > 0$ small enough we have $\mathcal{F}(u_\epsilon^\eta) - \mathcal{F}(u) < 0$; this means that u cannot be a minimizer of problem (3.1) and thus any such function must be $\mathcal{G}$-regular.

3.2 An open problem

We are left with the question of what happens if $h = k$. In this case, unfortunately, the construction described in the previous section fails, as the linear terms of a Taylor expansion cancel and it is impossible to find $\eta(\epsilon)$.

Another approach would be to compare analytically regular and singular extrema. By assumption $v \in H^1(I), F \in C^\infty(\mathbb{R})$ and $G \in C^\infty(I \times \mathbb{R}, \mathbb{R})$; further, F has quadratic growth, it satisfies $\partial_\epsilon F = \epsilon \leq (1+\epsilon)$ and $\partial_\epsilon^2 F = 1 > 0$ so all conditions of Theorem 7 hold. This means that a $\mathcal{G}$-regular minimizer v of (3.1) will actually satisfy $v \in C^\infty(I)$ and we can thus compute the associated Euler Lagrange equation. Specifically, there exists some $\lambda \in \mathbb{R}$ such that

$$\frac{\partial F}{\partial z}(x, v') - \frac{d}{dx}\frac{\partial F}{\partial \xi}(x, v') - \lambda \frac{\partial G}{\partial z}(x, v) = 0.$$

In our case, this equation leads to the following boundary value problem:

$$\begin{cases} u''(x) = \lambda v^{2k}(x) - \lambda x^{2h}, & \forall x \in I; \\ v(-1) = v(1) = 1. \end{cases} \tag{3.2}$$

Using the results from the previous chapter, we recall that the Lagrange multiplier λ can be expressed as $\lambda := \partial_\tau \Phi(0,0)^T \partial_\tau \Psi(0,0)^{-1}$, where $\Phi(\epsilon, \tau) = \mathcal{F}(v, \epsilon\phi + \tau\psi)$, $\Psi(\epsilon, \tau) = \mathcal{G}(v + \epsilon\phi + \tau\psi)$, $\phi \in C_c^\infty(I, \mathbb{R})$ is arbitrary and $\psi \in C_c^\infty(I, \mathbb{R})$ is such that $\partial_\tau \Psi(0,0) \neq 0$. Using the explicit formulas for F and G we arrive at

$$\lambda = \frac{\int_I v'(x)\psi'(x)dx}{\int_I (x^{2h} - v^{2k}(x))\psi(x)dx}.$$

At this point, one could express ψ (at least implicitly) as a function of v and, using the previous formula for λ, write (3.2) as a Cauchy problem only dependent on v. This would allow us, at least in theory, to compute the analytically regular extrema and compare their Dirichlet energy to that of our candidate minimizer. Unfortunately, the resulting expression turns out to be very complicated and of little practical use, therefore this approach has not been followed.

An alternative line of reasoning we attempted to follow involved studying how the functionals behaved as a result of perturbations to our candidate minimizer $u(x) = |x|$. This approach was motivated by the following observation: if we take $h = k = 1$ and consider the perturbed functional $|x| + w(x)$, where $w \in H^1(I)$, we obtain the first order variation

$$\begin{aligned} \mathcal{G}(|\cdot| + w) - \mathcal{G}(|\cdot|) &= -\frac{1}{3} + \int_I \left(x^2(|x| + w) - \frac{(|x| + w(x))^3}{3} \right) dx = \\ &= -\int_I \left(|x| w(x)^2 + \frac{w(x)^3}{3} \right) dx. \end{aligned} \tag{3.3}$$

The first term in this expression is always negative, while the second is of higher order in w. If we could establish a sufficiently strong bound on the second term $\int_I w^3/3$, uniformly for $w \in H^1(I)$ with $||w||_{H^1(I)} < \delta$, then we could prove that our candidate minizer u is an isolated point with respect to our constraint $\mathcal{G}$ in the Sobolev metric, and thus at least a local minimum. Unfortunately, this conclusion turns out to be false (and thus the approach is deemed ineffectual). In fact, by taking $w := -\epsilon\chi_{[-\delta,\delta]}$ with $\delta < 2/3\epsilon$, the term in (3.3) turns out positive. If we then convolve w with an appropriate regularisation kernel, we can make sure the resulting function is also in $H^1(I)$.

It is likely that this problem might require more advanced and refined techniques. That said, it is interesting to note that, in various different contexts, there have been results proven which explicitly disallow such “corner-type singularities” to be extrema of certain classes of variational problems. For example, in [5], the author proves the following result regarding constrained Dirichlet-energy minimization problems in $H^1(I)$:

Theorem 10. *Let $G \in C^\infty(I \times \mathbb{R}, \mathbb{R})$ be such that there exists $\delta \in (0, 1)$ with*

$$\partial_z^2 G(0, 0)\partial_z^2 G(\delta, \delta) < 0.$$

If $u(x) = |x|$ is a $\mathcal{G}$-singular extremal then u is not a solution of the minimization problem

$$\min_{v \in H^1(I)} \{\mathcal{F}(u) \ : \ u(\pm 1) = 1, \mathcal{G}(u) = \mathcal{G}(|\cdot|)\},$$

where $\mathcal{F}$ is the Dirichlet-energy functional.

To take another recent example, in [4] the author proves that length-minimizing curves with respect to a certain type of metric do not have corner-type singularities. In the case of curves, however, we are free to perturb candidate minimizers with more freedom, since we are not constrained to graphs of one-dimensional functions.

Bibliography

[1] G. Buttazzo, M. Giaquinta, and S. Hildebrandt. *One-dimensional Variational Problems: An Introduction*. Oxford lecture series in mathematics and its applications. Clarendon Press, 1998. ISBN: 9780198504658. URL: https://books.google.it/books?id=fursm_sGf_EC.

[2] G.B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts. Wiley, 2013. ISBN: 9781118626399. URL: https://books.google.it/books?id=wI4fAwAAQBAJ.

[3] I.M. Gelfand, S.V. Fomin, and R.A. Silverman. *Calculus of Variations*. Dover Books on Mathematics. Dover Publications, 2000. ISBN: 9780486414485. URL: https://books.google.it/books?id=YkFLGQeGRw4C.

[4] Eero Hakavuori and Enrico Le Donne. "Non-minimality of corners in subriemannian geometry". In: *Inventiones mathematicae* 206.3 (Apr. 2016), pp. 693–704. ISSN: 1432-1297. DOI: 10.1007/s00222-016-0661-9. URL: http://dx.doi.org/10.1007/s00222-016-0661-9.

[5] Stefan Strebel. *Singular extremals of constrained variational problems*. January 31, 2023.