UNIVERSITÀ DEGLI STUDI DI PADOVA

Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

Corso di Laurea Triennale in Matematica

$C^{1,\gamma}$-regularity for (Λ, r_0)-perimeter minimizers

Relatore: Prof. Roberto Monti

Laureanda: Irene Maniglia
Matricola: 2020606

Anno Accademico 2022/2023

15 dicembre 2023

Contents

Introduction

The aim of this thesis is to prove the $C^{1,\gamma}$-regularity of (Λ, r_0)-perimeter minimizers for all $\gamma \in (0, 1/2)$.

We are discussing a particular instance of the broader issue of regularity of area minimizing rectifiable currents. The field was developed mainly by:

- Ennio De Giorgi with his paper *Frontiere orientate di misura minima* (1960-61);

- Frederick Justin Almgren Jr. with his paper *Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two* (1983).

In the presentation of the topics we will follow the structure of the proof as in the book *Sets of Finite Perimeter and Geometric Variational Problems, An Introduction to Geometric Measure Theory* by Francesco Maggi, drawing mostly from Part III. In the book, the partial regularity theory diverges from De Giorgi's original approach, drawing inspiration instead from the contributions of Almgren, alongside several other influential authors such as Allard, Bombieri, Federer, Schoen, Simon, and more in the study of area minimizing currents and stationary varifolds. The resulting proofs only rely on direct comparison arguments and on geometrically viewable constructions.

Chapter 1 introduces the concept of perimeter, offering an implicit definition independent of boundary considerations. It proceeds to define the focal subject of study: (Λ, r_0)-perimeter minimizers. Additionally, a fundamental quantity, the cylindrical excess

$$e(E, x, r, v) = \frac{1}{r^{n-1}} \int_{C(x,r,v)\cap\partial^* E} \frac{|\nu_E(y) - v|^2}{2} \mathcal{H}^{n-1}(y),$$

is introduced and its essential properties briefly discussed, given its pivotal role across subsequent proofs.

Then, we state and prove the height bound Lemma 1.8, a crucial proposition enabling us to control the boundary's height within a cylinder with the cylindrical excess observed on a larger cylinder raised to the power of $1/2(n-1)$ multiplied by the radius of the cylinder. This proof entails considerable length and necessitates two intermediary lemmas.

Then the chapter presents the Lipschitz Lemma 1.9, which ensures the existence of a Lipschitz function, denoted as u, that covers a significant part of $C(x_0, r) \cap \partial E$, showing near-harmonic behavior. The first part of the lemma establishes that the difference between $C(x_0, r) \cap \partial E$ and the set $x_0 + \{(z, u(z)) : z \in D_r\}$ is limited by a constant times r^{n-1} multiplied by $e_n(x_0, 9r)$. The second part demonstrates that the quantity $\frac{1}{r^{n-1}} \int_{D_r} |\nabla' u|^2$ is bounded by a constant relative to the excess on a larger radius cylinder. The chapter concludes stating the reverse Poincaré inequality.

In Chapter 2, two useful lemmas for harmonic functions are verified initially. Subsequently, attention is directed towards another pivotal lemma: the excess improvement by tilting Theorem 2.3. This lemma establishes the existence (given $\alpha < 1/72$) of $v_0 \in \mathbb{S}^{n-1}$ such that $e(x_0, \alpha r, v_0) \leq C(n)(\alpha^2 e(x_0, r, v) + \alpha \Lambda r)$.

Chapter 3 culminates with the proof of the $C^{1,\gamma}$-regularity Theorem: initially we employ the excess improvement by tilting Theorem 2.3 to establish the Lemma 3.11 for the regularity theorem. This lemma is then utilized iteratively in conjunction with the height bound Lemma 1.8 and with the Lipschitz approximation Theorem 1.9 to conclude the regularity theorem.

Chapter 1

Notation and preliminary results

In this chapter we present the key premises to prove the $C^{1,\gamma}$-regularity. In the first section we introduce perimeter, (Λ, r_0)-perimeter minimizers, the excess and its properties and the lower density estimate Theorem. In the second section we prove in detail the height bound Lemma 1.8. In the third section we state without proof the Lipschitz approximation Lemma 1.9.

1.1 Perimeter and excess

We want to give a definition of perimeter and of sets of (locally) finite perimeter. The topological boundary of a set is a notoriously bizarre object, that often has a much more complicated structure than the set itself. For this reason, our definition of "perimeter" is implicit and does not call in the boundary. The idea behind the definition of the perimeter is the divergence theorem on E open set of class C^1, so that:

$$\begin{aligned} P(E) &= \sup\left\{\int_{\partial E} T(x)\cdot\nu_E\, d\mathcal{H}^{n-1} : T\in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \sup_{\mathbb{R}^n}|T|\le 1\right\} \\ &= \sup\left\{\int_E \operatorname{div} T(x)\,dx : T\in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \sup_{\mathbb{R}^n}|T|\le 1\right\}. \end{aligned}$$

Definition 1.1 (Sets of locally finite perimeter)**.** *Let $E\subset\mathbb{R}^n$ be a Lebesgue measurable set, we say that E is a set of locally finite perimeter in $\mathbb{R}^n$ if for every compact set $K\subset\mathbb{R}^n$ it holds*

$$\sup\left\{\int_E \operatorname{div} T(x)\,dx : T\in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt} T\subset K,\ \sup_{\mathbb{R}^n}|T|\le 1\right\} < \infty. \tag{1.1}$$

If this quantity is bounded independently of K, we call E a set of finite perimeter in $\mathbb{R}^n$.

Definition 1.2 (Gauss-Green measure and its existence)**.** *Let $E\subset\mathbb{R}^n$ be a Lebesgue measurable set. Then E is a set of locally finite perimeter if and only if there exists* *an $\mathbb{R}^n$-valued Radon measure on $\mathbb{R}^n$ μ_E such that*

$$\int_E \operatorname{div} T(x)\, dx = \int_{\mathbb{R}^n} T \cdot d\mu_E, \quad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n). \tag{1.2}$$

Moreover, E is set of finite perimeter if and only if $|\mu_E|(\mathbb{R}^n) < \infty$. We call such μ_E the Gauss-Green measure of E.

Proof. See [**MAG**], Proposition 12.1, pp.122-123.

Definition 1.3 (Perimeter)**.** *Given a set $E \subset \mathbb{R}^n$ of locally finite perimeter we define the relative perimeter of E in $F \subset \mathbb{R}^n$, and the perimeter of E, as*

$$P(E; F) = |\mu_E|(F), \quad P(E) = |\mu_E|(\mathbb{R}^n). \tag{1.3}$$

We will usually work with reduced boundaries, as they encapsulate more efficiently the structure of the set underneath.

Definition 1.4 (Reduced boundary)**.** *Given a set $E \subset \mathbb{R}^n$ of locally finite perimeter the reduced boundary $\partial^* E$ is the set of those $x \in \mathrm{spt}\mu_E$ such that: $\lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))}$ exists and belongs to S^{n-1}.*

Definition 1.5 (Measure-theoretic outer unit normal)**.** *Given a set $E \subset \mathbb{R}^n$ of locally finite perimeter we may define $\nu_E : \partial^* E \to \mathbb{S}^{n-1}$ by setting:*

$$\nu_E(x) = \lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))}, \quad x \in \partial^* E.$$

Now we introduce local perimeter minimizers.

Definition 1.6 (Local perimeter minimizer)**.** *Given an open set A and a set of locally finite perimeter $E \subset \mathbb{R}^n$, we say that E is a local perimeter minimizer at scale r_0 in A if whenever $x \in A$ and $E\Delta F \subset\subset B(x, r_0) \cap A$,*

$$P(E; A) \leq P(F; A).$$

Finally we introduce (Λ, r_0)-perimeter minimizers: sets such that the volume change produced by any variation F of E that is compactly supported in a ball of radius r_0 is controlled by Λ multiplied by the volume of the symmetric difference between E and F.

Definition 1.7 ((Λ, r_0)-perimeter minimizer)**.** *Given an open set A and a set of locally finite perimeter $E \subset \mathbb{R}^n$, $n \geq 2$, we say that E is a (Λ, r_0)-perimeter minimizer in A if there exist two constants Λ and r_0 with $0 \leq \Lambda < \infty$ and $r_0 > 0$ such that*

$$P(E; B(x,r)) \leq P(F; B(x,r)) + \Lambda |E\Delta F|,$$

whenever $E\Delta F \subset\subset B(x,r) \cap A$ and $r < r_0$.

Observe that:

- if E is a (Λ, r_0)-perimeter minimizer with $\Lambda = 0$ then E is trivially a local perimeter minimizer at scale $r \quad \forall r < r_0$.

- if E is a (Λ, r_0)-perimeter minimizer up to modification on sets of measure zero we have
$$\mathrm{spt}\mu_E = \partial^* E = \partial E.$$

  Proof. E is a local perimeter minimizer for the previous observation and the condition holds because of [**MAG**], Remark 15.3, pp.167-168.

Let $C(x, r, v)$ be cylinder of axis passing through x parallel to v and radius r, and $\nu_E(y)$ is the vector normal to ∂E at y.

Definition 1.8 (Cylindrical and spherical excess)**.** *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. The cylindrical excess of E at the point $x \in \partial E$, at scale $r > 0$, and with respect to the direction $v \in \mathbb{S}^{n-1}$, is defined as*

$$\begin{aligned} e(E, x, r, v) &= \frac{1}{r^{n-1}} \int_{C(x,r,v)\cap\partial^* E} \frac{|\nu_E(y) - v|^2}{2} d\mathcal{H}^{n-1}(y) \\ &= \frac{1}{r^{n-1}} \int_{C(x,r,v)\cap\partial^* E} (1 - (\nu_E(y) \cdot v)) d\mathcal{H}^{n-1}. \end{aligned} \tag{1.4}$$

The spherical excess of E at the point $x \in \partial E$ and the scale $r > 0$ is similarly defined as

$$e(E, x, r) = \min_{v\in S^{n-1}} \frac{1}{r^{n-1}} \int_{B(x,r)\cap\partial^* E} \frac{|\nu_E(y) - v|^2}{2} d\mathcal{H}^{n-1}(y). \tag{1.5}$$

The cylindrical excess measures the discrepancy between the actual perimeter of E and the perimeter of the hyperplane passing through x and orthogonal to v and the spherical excess works in a similar way. Now we prove three fundamental properties of excess:

- when a set is scaled by a factor r and translated, the excess remains unchanged;

- the excess of a set at a smaller scale s is bounded by the excess at a larger scale r scaled by a factor of $\left(\frac{r}{s}\right)^{n-1}$;

- the excess in a direction v can be bounded by a constant times the excess at a larger scale $\sqrt{2}r$ in a direction v_0 plus the squared difference between v and v_0.

Proposition 1.1 (Scaling of the excess). *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter, $x \in \partial E$, $r > 0$, $v \in \mathbb{S}^{n-1}$, then (with $E_{x,r} = \frac{E-x}{r}$)*

$$\begin{aligned} e(E, x, r, v) &= e(E_{x,r}, 0, 1, v), \\ e(E, x, r) &= e(E_{x,r}, 0, 1). \end{aligned} \tag{1.6}$$

Proof. We observe that for the first formula we have

$$\begin{aligned} e(E,x,r,v) &= \frac{1}{r^{n-1}}\int_{B(x,r)\cap\partial^*E}(1-v\cdot\nu_E)d\mathcal{H}^{n-1}(y) \\ &= \frac{|\mu_E|(C(x,r,v)) - v\cdot\mu_E(C(x,r,v)}{r^{n-1}}. \end{aligned}$$

So it is clear that (see [**MAG**], Lemma 15.11, p.171 for further details)

$$e(E,x,r,v) = |\mu_{Ex,r}|(C(0,1,v)) - v\cdot\mu_{Ex,r}(C(0,1,v) = e(E_{x,r},0,1,v).$$

Similarly for the spherical excess we have

$$\begin{aligned} e(E,x,r) &= \min_{v\in\mathbb{S}^{n-1}}\frac{|\mu_E|(C(x,r,v)) - v\cdot\mu_E(C(x,r,v)}{r^{n-1}} \\ &= \min_{v\in\mathbb{S}^{n-1}}\frac{|\mu_E|(C(x,r,v)) - |\mu_E(C(x,r,v)|}{r^{n-1}}. \end{aligned}$$

and we conclude. □

Proposition 1.2 (Excess at different scales). *Let $E\subset\mathbb{R}^n$ be a set of locally finite perimeter, $x\in\partial E$, $r>s>0$, $v\in\mathbb{S}^{n-1}$, then*

$$e(E,x,s,v)\le\left(\frac{r}{s}\right)^{n-1}e(E,x,r,v). \tag{1.7}$$

Proof. Trivial.

Proposition 1.3 (Excess and changes of direction). *For $n\ge 2$, there exists a constant $C(n)$ such that if E is a (Λ,r_0)-perimeter minimizer in the open set $A\subset\mathbb{R}$ with $\Lambda r_0\le 1$, then for $x\in A\cap\partial E$, $B(x,2r)\subset\subset A$ and $v,v_0\in\mathbb{S}^{n-1}$ it holds*

$$e(E,x,r,v)\le C(n)\Big(e(E,x,\sqrt{2}r,v_0)+|v-v_0|^2\Big). \tag{1.8}$$

Proof. Since $\frac{|\nu_E-v|^2}{2}\le|\nu_E-v_0|^2+|v-v_0|^2$ and $C(x,r,v)\subset C(x,\sqrt{2}r,v_0)$,

$$\begin{aligned} e(E,x,r,v) &\le \frac{2}{r^{n-1}}\int_{C(x,\sqrt{2}r,v_0)\cap\partial^*E}\frac{|\nu_E(y)-v_0|^2}{2}d\mathcal{H}^{n-1}(y) \\ &\quad+\frac{2}{r^{n-1}}\int_{C(x,\sqrt{2}r,v_0)\cap\partial^*E}\frac{|v_0-v|^2}{2}d\mathcal{H}^{n-1}(y) \\ &= \frac{2}{r^{n-1}}\int_{C(x,\sqrt{2}r,v)\cap\partial^*E}\frac{|\nu_E(y)-v_0|^2}{2}d\mathcal{H}^{n-1}(y)+|v-v_0|^2\frac{P(E;C(x,r,v))}{r^{n-1}}. \end{aligned}$$

So, since (by [**MAG**], Remark 21.12, p.283) $\frac{P(E;C(x,r,v))}{r^{n-1}}\le C(n)$, we deduce that

$$e(E,x,r,v)\le 2e(E,x,\sqrt{2}r,v_0)+C(n)|v-v_0|^2\le C(n)\Big(e(E,x,\sqrt{2}r,v_0)+|v-v_0|^2\Big).$$

□

Theorem 1.4 (Lower density estimate). *If $n \geq 2$, there exists a positive constant $c(n)$ such that if E is a (Λ, r_0)-perimeter minimizer in the open set $A \subset \mathbb{R}^n$ and $\Lambda r_0 \leq 1$, then*

$$c(n)r^{n-1} \leq P(E, B(x,r)). \tag{1.9}$$

Proof. See [**MAG**], Theorem 21.11, pp.282-284.

We now state Campanato's criterion, a cornerstone in the regularity theory for variational problems because it characterizes Hölder continuity in terms of the uniform decay of certain integral averages. We shall use this criterion in Section 3.2.

We set

$$(u)_{x,r} = \frac{1}{|B \cap B(x,r)|} \int_{B\cap B(x,r)} u, \quad x \in B, r > 0.$$

Theorem 1.5 (Campanato's criterion). *If $n \geq 1$, $p \in [1, \infty)$, $\gamma \in (0,1]$, then there exists a constant $C(n,p,\gamma)$ such that if $u \in L^p(B)$, and there exists a constant κ such that*

$$\left(\frac{1}{r^n} \int_{B\cap B(x,r)} |u - (u)_{x,r}|^p \right)^{1/p} \leq \kappa r^\gamma$$

holds true, then there exists a function $\bar{u} : B \to \mathbb{R}$ with $\bar{u} = u$ a.e. on B and

$$|\bar{u}(x) - \bar{u}(y)| \leq C(n,p,\gamma)\kappa |x-y|^\gamma, \quad \forall x \in B.$$

Proof. See [**MAG**], Theorem 6.1, pp.64-65.

1.2 The height bound

We now begin to analyze some first consequences of a small cylindrical excess assumption. We aim to prove the height bound Lemma, a fundamental estimate relating the height of a perimeter minimizer to his cylindrical excess. We divide the proof in several lemmas, the starting point is small-excess position Lemma 1.6, which we prove by a compactness argument, then in excess measure Lemma 1.15 these geometric properties are combined with the divergence theorem to introduce the notion of excess measure. Finally combining the two together we prove the height bound Lemma 1.8

Let q denote the projection of $\mathbb{R}^{n-1} \times \mathbb{R}$ on $\mathbb{R}$, $(x_1, x_2, ..., x_n) \mapsto x_n$ and p denote the projection of $\mathbb{R}^{n-1} \times \mathbb{R}$ on $\mathbb{R}^{n-1}$, $(x_1, x_2, ..., x_n) \mapsto (x_1, ..., x_{n-1})$.

Lemma 1.6 (Small-excess position). *For $n \geq 2$ and $t_0 \in (0,1)$, there exists a positive constant $w(n, t_0)$ such that:*

if E is a (Λ, r_0)-perimeter minimizer in C_2 with $\Lambda r_0 \leq 1$, $0 \in \partial E$, and $e_n(E, 0, s) \leq w(n, t_0)$, then

$$|qx| < t_0, \quad \forall x \in C \cap \partial E, \tag{1.10}$$

$$|\{x \in C \cap E : qx > t_0\}| = 0,$$

$$|\{x \in C \setminus E : qx < -t_0\}| = 0.$$

Proof. Let us argue by contradiction. Let us consider a sequence $\{E_h\}_{h\in\mathbb{N}}$ of a (Λ, r_0)-perimeter minimizers in C_2 such that $\Lambda r_0 \leq 1$, $\lim_{h\to\infty} e(E_h, 0, 2, e_n) = 0$, $0 \in \partial E_h$ $\forall h \in \mathbb{N}$ and at least *one* of the following conditions holds true for infinitely many $h \in \mathbb{N}$:

$$\begin{aligned} \text{either} \quad & |\{x \in C \cap \partial E_h : t_0 \leq |qx| \leq 1\}| \neq \emptyset, \\ \text{or} \quad & |\{x \in C \cap E_h : qx > t_0\}| > 0, \\ \text{or} \quad & |\{x \in C \setminus E_h : qx < -t_0\}| > 0. \end{aligned} \tag{1.11}$$

By properties of sequences of (Λ, r_0)-perimeter minimizers, there exists a set of finite perimeter $F \in C_{5/3}$, which is a (Λ, r_0)-perimeter minimizers in $C_{5/3}$, such that $0 \in \partial F$ and, up to extracting subsequences, $E_h \cap C_{5/3} \to F$. Because $C_{4/3}$ is a compact subset of $C_{5/3}$ he lower semicontinuity of the excess implies that

$$e(F, 0, 4/3, e_n) \leq \liminf_{h\to\infty} e(F, 0, 4/3, e_n) \leq \left(\frac{3}{2}\right)^{n-1} \lim_{h\to\infty} e(E_h, 0, 2, e_n) = 0.$$

Since $0 \in \partial F$ and $e(F, 0, 4/3, e_n) = 0$, we deduce that $F \cap C_{4/3}$ is equivalent to $C_{4/3} \cap \{qx < 0\}$. If (1.11.1) were valid for infinitely many values of $h \in \mathbb{N}$, we would have that up to extracting a subsequence, we may construct $\{x_h\}_{h\in\mathbb{N}}$ with $x_h \in C \cap \partial E_h$, $t_0 \leq |qx_h| \leq 1$ and $x_h \to x_0$, for some $x_0 \in \overline{C} \cap \partial F$. In particular, we would have that $C_{4/3} \cap \partial F \cap \{|qx| \geq t_0\} = \emptyset$, which contradicts the formula above.

So it exists $h_0 \in \mathbb{N}$ such that $\forall h \geq h_0$ holds $\{x \in C \cap \partial E_h : t_0 \leq |qx| \leq 1\} = \emptyset$ and $|\mu_{C\cap E_h}| = |\mu_C| \llcorner E_h^{(1)} + |\mu_{E_h}| \llcorner (C \cup \{\nu_{E_h} = \nu_C\})$.

We thus find that, for $h \geq h_0$ we have $|\mu_{C\cap E_h}|(\{x \in C : t_0 < |qx| < 1\}) = 0$. With $1_{C\cap E_h}$ constant on $\{x \in C : t_0 < qx < 1\}$ and, for the same reason, $1_{C\cap E_h}$ constant on $\{x \in C : -t_0 > qx > -1\}$ (possibly with a different constant).
By $E_h \cap C_{5/3} \to F$, necessarily $1_{C\cap E_h} = 0$ a.e. on $\{x \in C : t_0 < qx < 1\}$, and $1_{C\cap E_h} = 1$ a.e. on $\{x \in C : -t_0 > qx > -1\}$.
In particular this contradicts both (1.11.2) and (1.11.3).

□

Lemma 1.7 (Excess measure). *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter with $0 \in \partial E$, and such that, for some $t_0 \in (0, 1)$,*

$$|qx| < t_0, \quad \forall x \in C \cap \partial E \tag{1.12}$$

$$|\{x \in C \cap E : qx > t_0\}| = 0, \tag{1.13}$$

$$|\{x \in C \setminus E : qx < -t_0\}| = 0, \tag{1.14}$$

then (set $M = C \cap \partial^* E$), for every Borel set $G \subset D$, $\varphi \in C_c^0(D)$ and $t \in (-1, 1)$ we *have that*

$$\mathcal{H}^{n-1}(G) \leq \mathcal{H}^{n-1}(M \cap p^{-1}(G)), \tag{1.15}$$

$$\mathcal{H}^{n-1}(G) = \int_{M\cap p^{-1}(G)} (\nu_E \cdot e_n)\, d\mathcal{H}^{n-1}, \tag{1.16}$$

$$\int_D \varphi = \int_M \varphi(px)(\nu_E(x) \cdot e_n)\, d\mathcal{H}^{n-1}(x), \tag{1.17}$$

$$\int_{E_t\cap D} \varphi = \int_{M\cap\{qx>t\}} \varphi(px)(\nu_E(x) \cdot e_n)\, d\mathcal{H}^{n-1}(x), \tag{1.18}$$

and the set function

$$\begin{aligned}\zeta(G) &= P(E; C \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G) \\ &= \mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G), \quad G \subset \mathbb{R}^{n-1},\end{aligned}$$

defines a Radon measure on $\mathbb{R}^{n-1}$ concentrated on D.
The measure ζ is called the excess measure of E over D, since $\zeta(D) = e_n(1)$.

Remark (on excess measure lemma)**.** *The lower bound (1.15) ensures that $C \cap \partial^* E$ "leaves no holes" over D. If $e_n(1)$ is small, the upper bound $\zeta(G) \leq \zeta(D) = e_n(1)$, implies that $C \cap \partial^* E$ is "almost flat" over D, which means that*

$$\mathcal{H}^{n-1}(G) \leq \mathcal{H}^{n-1}(C \cap \partial^* E \cap p^{-1}(G)) \leq \mathcal{H}^{n-1}(G) + e_n(1),$$

for every Borel set $G \subset D$. In a similar way, starting from 1.18 we see that, for every $t \in (-1, 1)$,

$$\mathcal{H}^{n-1}(E_t \cap D) \leq \mathcal{H}^{n-1}(M \cap \{qx > t\} \leq \mathcal{H}^{n-1}(E_t \cap D) + e_n(1).$$

Proof. By a standard approximation argument one can prove that (1.18) implies (1.16), which then implies (1.15).

We explicitly prove only (1.17) and (1.18).

By a density argument we can assume that $\varphi \in C^1_c(D)$. We have

$$\begin{aligned}\mathcal{H}^{n-1}(\partial^* E \cap (\partial D_r \times \mathbb{R})) &= 0 \text{ for a.e. } r \in (0, 1) \\ \mathcal{H}^{n-1}(E \cap (D \times \{s\})) &= 0 \text{ for a.e. } s \in (t_0, 1) \\ \mathcal{H}^{n-1}(E \cap (D \times \{t\})) &= 0 \text{ for a.e. } t \in (-1, -t_0).\end{aligned} \tag{1.19}$$

We let $r \in (0, 1)$ and $s \in (t_0, 1)$ satisfy (1.19.2) and (1.19.3). And we define a set of finite perimeter (setting $t \in (-1, s)$) $F = E \cap (D_r(t, s))$. If we set $\nu(x) = \frac{px}{|px|}$ for every $x \in \mathbb{R}^n$ such that $px \neq 0$, then

$$\begin{aligned}\mu_F =& \mu_E \llcorner (D_r \times (t, s)) + \mu_{D_r\times(t,s)} \llcorner (E) \\ =& \mu_E \llcorner (D_r \times (t, s)) + e_n \mathcal{H}^{n-1}(D_r \times \{s\}) \\ &+ v\mathcal{H}^{n-1}(\partial D_r \times (t, s)).\end{aligned} \tag{1.20}$$

So we find that

$$e_n \cdot \mu_F = (e_n \cdot \nu_E)\mathcal{H}^{n-1} \llcorner (\partial^* E \cap (D_r \times (t,s)) - \mathcal{H}^{n-1} \llcorner (E \cap (\partial D_r \times \{t\})). \quad (1.21)$$

So, given $\varphi \in C_c^1(D)$ we can define the vector field $T(x) = \varphi(px)e_n$, $x \in \mathbb{R}^n$, which is such that $T \in C^1(\mathbb{R}^n; \mathbb{R}^n)$. Since $\mathrm{div}T = 0$, the divergence theorem applied on F combined with (1.21) implies

$$\int_{E \cap (D_r \times \{t\})} \varphi(px)\ d\mathcal{H}^{n-1}(x) = \int_{\partial^* E \cap (D_r \times (t,s))} \varphi(px)(e_n \cdot \nu_E(x))\ d\mathcal{H}^{n-1}(x).$$

Now we let $r \to 1^-$ and set $s \to 1^-$, that is

$$\begin{aligned} \int_{E_t \cap D} \varphi &= \int_{E \cap (D \times \{t\})} \varphi(px)\ d\mathcal{H}^{n-1}(x) \\ &= \int_{\partial^* E \cap (D \times (t,1))} \varphi(px)(e_n \cdot \nu_E(x))\ d\mathcal{H}^{n-1}(x) \\ &= \int_{M \cap \{qx > t\}} \varphi(px)(e_n \cdot \nu_E(x))\ d\mathcal{H}^{n-1}(x). \end{aligned} \quad (1.22)$$

This proves (1.18). Finally, we let $t \to (-1)^+$, and by (1.19), we prove (1.17).

□

Theorem 1.8 (The height bound). *Given $n \geq 2$, there exist positive constants $\varepsilon_0(n)$ and $C_0(n)$ such that if E is a (Λ, r_0)-perimeter minimizer in $C(x_0, 4r_0)$ with*

$$\Lambda r_0 \leq 1, \qquad x_0 \in \partial E, \qquad e_n(x_0, 4r_0) \leq \varepsilon_0(n),$$

then

$$\sup\left\{ \frac{|qy - qx|}{r_0} : y \in C(x_0, r_0) \cap \partial E \right\} \leq C_0(n) e_n(x_0, 4r_0)^{1/2(n-1)}. \quad (1.23)$$

Proof. Step one: scaling. Without loss of generality (replacing E with $E_{x_0, 2r_0}$) we can reduce to the following: given a $(\Lambda, 1/2)$-perimeter minimizer E in C_2, with $\frac{\Lambda}{2} \leq 1$, $0 \in \partial E$, $e_n(2) \leq \varepsilon_0(n)$, we want to prove that $|qx| \leq c(n)e_n(2)^{1/2(n-1)}$ $\forall x \in C_{1/2} \cap \partial E$ where we are setting $e_n(s) = e(E, 0, s, e_n) > 0$. We assume $\varepsilon_0(n) \leq \omega(n, 1/4)$ with $\omega(n, 1/4)$ as in small-excess position Lemma 1.6 and set $M = C \cap \partial E$, then we deduce that $|qx| \leq \frac{1}{4} \quad \forall x \in M$.

Step two: tools. Let $f : \mathbb{R} \to \{0, \mathcal{H}^{n-1}(M)\}$ be $f(t) = \mathcal{H}^{n-1}(M \cap \{qx > t\})$. Since the function is clearly decreasing we can define $t_0 \in \mathbb{R}$ to be

$$\begin{aligned} f(t) &\leq \frac{\mathcal{H}^{n-1}(M)}{2}, \quad \text{if } t \in [t_0, \infty) \\ f(t) &\geq \frac{\mathcal{H}^{n-1}(M)}{2}, \quad \text{if } t \in (-\infty, t_0) \end{aligned} \quad (1.24)$$

(interpreted as "the median value for the n^{th} coordinate of M") and let t_1 be such that $t_1 > t_0$ and

$$f(t_1) \geq \sqrt{e_n(2)} \tag{1.25}$$

(it is easy to prove such t_1 exists).

Step three: $t_1 - t_0 \leq c(n) e_n(2)^{1/2(n-1)}$. By remark for excess measure Lemma (1.2) for every $t \in (-1, 1)$ we have

$$0 \leq \mathcal{H}^{n-1}(M \cap \{qx > t\}) - \mathcal{H}^{n-1}(E_t \cap D) \leq 2^{n-1} e_n(2) \tag{1.26}$$

and so by 1.26 and 1.25, for $\varepsilon_0(n)$ small enough we have

$$\begin{aligned} \mathcal{H}^{n-1}(E_t \cap D) &\geq \mathcal{H}^{n-1}(M \cap \{qx > t\}) - 2^{n-1} e_n(2) \geq \sqrt{e_n(2)} - 2^{n-1} e_n(2) \\ &\geq c(n) \sqrt{e_n(2)} \end{aligned}$$

and so

$$\int_{t_0}^{t_1} \mathcal{H}^{n-1}(E_t \cap D)^{(n-2)/(n-1)} \, dt \geq c(n)(t_1 - t_0) \sqrt{e_n(2)}^{(n-2)/(n-1)}. \tag{1.27}$$

Now thanks to the slicing formula ([**MAG**], 18.25, p.225) with $g = 1_C$ and by Hölder inequality

$$\begin{aligned} \int_{-1}^{1} \mathcal{H}^{n-2}(\partial^* E_t \cap D) \, dt &= \int_{-1}^{1} \mathcal{H}^{n-2}((\partial^* E \cap C)_t) \, dt \\ &= \int_M \sqrt{1 - (\nu_E \cdot e_n)^2} \, d\mathcal{H}^{n-1} \\ &\leq \sqrt{2} \int_M \sqrt{1 - (\nu_E \cdot e_n)} \, d\mathcal{H}^{n-1} \\ &\leq \sqrt{2\mathcal{H}^{n-1}(M)} \sqrt{\int_M 1 - (\nu_E \cdot e_n) \, d\mathcal{H}^{n-1}} \\ &\leq C(n) \sqrt{e_n(2)}. \end{aligned} \tag{1.28}$$

So the relative isoperimetric inequality ([**MAG**], 12.45, p.143) can be applied (with the disk D and the set of finite perimeter $E_t \cap D$), giving

$$\mathcal{H}^{n-2}(\partial^* E_t \cap D) = P(E_t \cap D; D) \geq c(n) \mathcal{H}^{n-1}(E_t \cap D)^{(n-2)/(n-1)}, \tag{1.29}$$

and the equation above (1.28) becomes

$$\int_{t_0}^{t_1} \mathcal{H}^{n-1}(E_t \cap D)^{(n-2)/(n-1)} \, dt \leq C(n) \sqrt{e_n(2)}; \tag{1.30}$$

so, by 1.27 and 1.30 we have

$$c(n)(t_1 - t_0) \sqrt{e_n(2)}^{(n-2)/(n-1)} \leq C(n) \sqrt{e_n(2)}$$

which implies

$$t_1 - t_0 \leq c(n)e_n(2)^{1/2(n-1)}. \tag{1.31}$$

Step four: $qy - t_1 \leq c(n)e_n(2)^{1/2(n-1)}$. If $y \in C_{1/2} \cap \partial E$ and $qy > t_1$ then $B(y, qy - t_1)$ and $qy - t_1 < \frac{1}{2}$. So the lower density esitmate Theorem 1.4 implies

$$\begin{aligned} c(n)(qy - t_1^*)^{n-1} &\leq P(E; B(y, qy - t_1)) \\ &\leq \mathcal{H}^{n-1}(M \cap \{qx > t_1\}) \\ &= f(t_1) \leq \sqrt{e_n(2)} \end{aligned} \tag{1.32}$$

and so

$$qy - t_1 \leq c(n)e_n(2)^{1/2(n-1)}. \tag{1.33}$$

Step five: conclusion. By combining (1.31) and (1.33) we have

$$qy - t_0 \leq c(n)e_n(2)^{1/2(n-1)}. \tag{1.34}$$

By a similar argument

$$t_0 - qy \leq c(n)e_n(2)^{1/2(n-1)}, \tag{1.35}$$

and this allows us to conclude.

□

1.3 The Lipschitz approximation theorem

This section concludes the chapter with the statement of Lipschitz approximation Lemma 1.9, this lemma guarantees the existence of a Lipschitz function u that covers a substantial portion of $C(x_0, r) \cap \partial E$ in relation to the excess, while also exhibiting a behavior that is nearly harmonic. Precisely, the first statement would be that the measure of the symmetric difference between $C(x_0, r) \cap \partial E$ and $x_0 + \{(z, u(z)) : z \in D_r\}$ is controlled up to a constant by r^{n-1} multiplied by $e_n(x_0, 9r)$ and the second would be that the quantity $\frac{1}{r^{n-1}} \int_{D_r} |\nabla' u|^2$ is controlled up to a constant by the excess on a cylinder of bigger radius.

We set

$$M = C(x_0, r) \cap \partial E, \qquad M_0 = \{y \in M : \sup_{0<s<8r} e_n(y, s) \leq \delta_0(n)\}.$$

Theorem 1.9 (Lipschitz approximation theorem). *There exist positive constants $C_1(n)$, $\varepsilon_1(n)$, and $\delta_0(n)$ such that given $E \subset \mathbb{R}^n$ a (Λ, r_0)-perimeter minimizer in $C(x_0, 9r)$ with*

$$\Lambda r_0 \leq 1, \qquad 9r < r_0, \qquad x_0 \in \partial E, \qquad e_n(x_0, 9r) \leq \varepsilon_1(n),$$

then there exists a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ with

$$\sup_{\mathbb{R}^{n-1}} \frac{|u|}{r} \leq C_1(n)e_n(x_0, 9r)^{1/2(n-1)}, \qquad Lip(u) \leq 1, \tag{1.36}$$

such that if we set $\Gamma = x_0 + \{(z, u(z)) : z \in D_r\}$*, it holds*

$$M_0 \subset M \cap \Gamma, \tag{1.37}$$

and

$$\frac{\mathcal{H}^{n-1}(M\Delta\Gamma)}{r^{n-1}} \le C_1(n)e_n(x_0, 9r). \tag{1.38}$$

Moreover, u *is "almost harmonic" in* D_r*, so that for every* $\varphi \in C_c^1(D_r)$

$$\frac{1}{r^{n-1}} \int_{D_r} |\nabla' u|^2 \, dx' \le C_1(n)e_n(x_0, 9r), \tag{1.39}$$

$$\frac{1}{r^{n-1}} \left| \int_{D_r} \nabla' u \cdot \nabla' \varphi \, dx' \right| \le C_1(n) \sup_{D_r} |\nabla' \varphi| \big(e_n(x_0, 9r) + \Lambda r\big). \tag{1.40}$$

Proof. We omit the proof due to space constraints, the proof is in [**MAG**], Theorem 23.7, pp.309-319.

In order to prove excess improvement by tilting Theorem 2.3 we shall need a reverse height bound, in which the excess in controlled through a sort of L^2-height. We now define the flatness $f(E, x, r, v)$ that measures the L^2-average distance distance of $\partial^* E$ from the family of hyperplanes $\{y : (y - x) \cdot v = c\}$ $(c \in \mathbb{R})$ in the cylinder $C(x, t, v)$.

Definition 1.9 (Cylindrical flatness)**.** *Let* $E \subset \mathbb{R}^n$ *be a set of locally finite perimeter, the cylindrical flatness of* E *at* $x \in \mathbb{R}^n$ *with respect to* $v \in \mathbb{S}^{n-1}$ *at scale* $r > 0$ *is*

$$f(E, x, r, v) = \inf_{c \in \mathbb{R}} \frac{1}{r^{n-1}} \int_{C(x,r,v) \cap \partial^* E} \frac{|(y - x) \cdot v - c|^2}{r^2} d\mathcal{H}^{n-1}(y).$$

We now are able to provide the required bound on the excess in terms of the flatness. In the statement, $\omega(n, t)$ denotes the constant introduced in small-excess position Lemma 1.6.

Theorem 1.10 (Reverse Poincaré inequality). *There exists a positive constant* $C(n)$ *with the following property. If* E *is a* (Λ, r_0)*-perimeter minimizer in* $C(x_0, 4r, v)$ *with*

$$\Lambda r_0 \le 1, \qquad x_0 \in \partial E, \qquad 4r < r_0, \qquad e(E, x_0, 4r, v) \le \omega\Big(n, \frac{1}{8}\Big),$$

then

$$e(E, x_0, r, v) \le C(n)\Big(f(E, x_0, 2r, v) + \Lambda r\Big).$$

Proof. See [**MAG**], Theorem 24.1, pp.320-336.

Chapter 2

Harmonic approximation and excess improvement

The aim of this chapter is to prove excess improvement by tilting Theorem 2.3, that (if $\alpha < 1/72$) establishes the existence of $v_0 \in \mathbb{S}^{n-1}$ such that $e(x_0, \alpha r, v_0) \leq C(n)(\alpha^2 e(x_0, r, v) + \alpha\Lambda r)$. This theorem and the harmonic approximation Lemma 2.2 play a crucial role in proving the regularity theorem. The proof of excess improvement by tilting Theorem 2.3 will follow this rough sketch:

- we prove Lemma 2.1;
- we prove the harmonic approximation Lemma 2.2;
- we use the "almost harmonicity" of u on $D(px_0, r)$ from the Lipschitz approximation Lemma 1.9 and harmonic approximation Lemma 2.2 to find an harmonic function v on $D(px_0, r)$ which is L^2-close to u and such that $\int_{D(px_0,r)} |\nabla' u|^2 \, dx$ is controlled by $e(E, x_0, r, v)$;
- we use the harmonicity of v and Lemma 2.1 to prove that setting $w(x) = v(0) + (0) \cdot x$ we have
$$\frac{1}{\omega_n(\alpha r)^{n-1}} \int_{D(px_0,\alpha r)} \frac{|v-w|^2}{\alpha^2} \, dx C(n)\alpha^2 \int_{D(px_0,r)} |\nabla' v|^2 \, dx;$$
- when we set $v_0 = (-\nabla' v_0, 1)/|(-\nabla' v_0, 1)|$ we can use that v is L^2-close to u and that the graph of u is $\mathcal{H}^{n-1}$-close to $C(x_0, r) \cap \partial E$, to prove that $\alpha^2 \int_{D(px_0,r)} |\nabla' v|^2 \, dx$ controls $f(E, x_0, \alpha r, v_0)$;
- by *step one* and *step three* we have thus proved that
$$f(E, x_0, \alpha r, v_0) \leq C(n)\alpha^2 e(E, x_0, r, v)$$
and so by the reverse Poincaré inequality
$$e(E, x_0, \alpha r, v) \leq C(n)\Big(\alpha^2 e(E, x_0, r, v) + \alpha\Lambda r\Big).$$
We will prove the two lemmas in Section 2.1 while in Section 2.2 we will state and prove excess improvement by tilting Theorem 2.3.

2.1 Two lemmas on harmonic functions

In this section we prove the two properties of harmonic functions that we are going to exploit. First of all, let us recall that if v is harmonic in B, then, by an application of the divergence theorem, the mean value property holds true. In the following lemma we provide a bound on the $L^\infty(B_{1/2})-$norm of the Hessian of v in terms of the $L^2(B)$-norm of ∇v.
We set $\omega_n = |B(0,1)|$ in $\mathbb{R}^n$.

Lemma 2.1. *Let v be harmonic in B and $w(x) = v(0) + \nabla v(0) \cdot x \quad \forall x \in B$, then*

$$\sup_{B(0,\alpha)} |v - w| \leq C(n)\alpha^2 ||\Delta v||_{L^2(B)}, \qquad \forall \alpha \in (0, 1/2].$$

In particular,

$$\frac{1}{\omega_n \alpha^n} \int_{B(0,\alpha)} \frac{|v-w|^2}{\alpha^2}\, dx \leq C(n)\alpha^2 \int_B |\nabla v|^2\, dx. \tag{2.1}$$

Proof. Select $e \in \mathbb{S}^{n-1}$ and let $|x| > 1/2$. Then $e \cdot \nabla v$ is harmonic in B, so by the mean-value property for $r < 1/4$ we have that,

$$\begin{aligned}
|e \cdot \nabla v(x)| =& \left| \int_{B(x,r)} e \cdot \nabla v\, dx \right| = \frac{C(n)}{r^n} \left| \int_{\partial B(x,r)} v(e \cdot \nu_{B(x,r)})\, d\mathcal{H}^{n-1}(y) \right| \\
&\leq \frac{C(n)}{r^n} \int_{\partial B(x,r)} |v(y)|\, d\mathcal{H}^{n-1}(y) \\
&\leq \frac{C(n)}{r^n} \int_{\partial B(x,r)} \left| \fint_{\partial B(y,r)} v(z) dz \right| d\mathcal{H}^{n-1}(y) \\
&\leq \frac{C(n)}{r^n} \frac{1}{\omega_n r^n} \int_{\partial B(x,r)} \int_{\partial B(y,r)} |v(z)|\, dz\, d\mathcal{H}^{n-1}(y) \\
&\leq \frac{C(n)}{r^{2n}} \int_{B(x,2r)} (2r)^{n-1} |v(z)|\, dz \\
&\leq \frac{C(n)}{r^{n+1}} \int_{\partial B(x,2r)} |v(y)|\, d\mathcal{H}^{n-1}(y).
\end{aligned} \tag{2.2}$$

In particular

$$\sup_{B_{1/2}} |\nabla v| \leq C(n) ||v||_{L^2(B)}. \tag{2.3}$$

This inequality, applied to $e \cdot \nabla v$ in place of v, leads to

$$\sup_{B_{1/2}} |\nabla^2 v| \leq C(n) ||\nabla v||^2_L. \tag{2.4}$$

So by Taylor's formula, for every $x \in B$ there exists $t \in (0,1)$ such that $|v(x) - w(x)| \leq C|\nabla^2 v(tx)||x|^2$, and this concludes. ☐

Lemma 2.2 (Harmonic Approximation). *If $\tau > 0$ there exists $\sigma > 0$ with the following property. If $u \in W^{1,2}(B)$ is such that*

$$\int_B |\nabla u|^2 \, dx \leq 1, \qquad \left| \int_B \nabla u \cdot \nabla \varphi \, dx \right| \leq \sup_B |\nabla \varphi| \sigma, \qquad \forall \varphi \in C_c^\infty(B),$$

then there exists a harmonic function v on B such that

$$\int_B |\nabla v|^2 \, dx \leq 1, \qquad \int_B |v - u|^2 \, dx \leq \tau.$$

Proof. By contradiction assume that there exist $\tau > 0$ and a sequence $\{u_h\}_{h \in H} \subset W^{1,2}(B)$, such that

$$\int_D |\nabla u_h|^2 \, dx \leq 1, \qquad \left| \int_D \nabla u_h \cdot \nabla \varphi \, dx \right| \leq \frac{||\nabla \varphi||_{L^\infty(B)}}{h}, \qquad \forall \varphi \in C_c^\infty(B),$$

but for every harmonic function v such that $\int_B |\nabla v| \, dx \leq 1$ we have

$$\int_B |u_h - v|^2 \, dx \geq \tau > 0.$$

Let $c_h = \oint_B u_h \, dx$, then by the Poincaré inequality we have $||u_h - c_h||_{L^2(B)} \leq C(n)$. Thus the sequence $\{w_h\}_{h \in H}$, $w_h = u_h - c_h$, is bounded in $W^{1,2}(B)$. In particular, up to extracting a subsequence, we may assume $w_h \to w$ in $L^2(B)$ for some $w \in W^{1,2}(B)$ such that $||\nabla w||_{L^2(B)} \leq 1$. We find a contradiction by showing that w is harmonic. Indeed, when $\varphi \in C_c^\infty(B)$ we have that

$$\left| \int_B \nabla w \cdot \nabla \varphi \, dx \right| \leq \left| \int_B \nabla (w - w_h) \cdot \nabla \varphi \, dx \right| + \frac{\sup_B |\nabla \varphi|}{h},$$

where $\int_B \nabla(w - w_h) \cdot \nabla \varphi \, dx = -\int_B (w - w_h) \Delta \varphi \, dx$. So by the Hölder inequality, and letting $h \to \infty$, we obtain a contradiction. □

2.2 Excess improvement by tilting

We now finally prove the excess improvement by tilting Theorem 2.3, that is the existence (given $\alpha < 1/72$) of $v_0 \in \mathbb{S}^{n-1}$ such that $e(x_0, \alpha r, v_0) \leq C(n)(\alpha^2 e(x_0, r, v) + \alpha \Lambda r)$.

In the statement we directly set

$$e(E, x, r, v) = e(x, r, v), \quad 0 \in \partial E, \quad r > 0, \quad v \in \mathbb{S}^{n-1}, \quad e_n(s) = e(0, s, e_n).$$

Theorem 2.3 (Excess improvement by tilting). *Given $\alpha \in (0, 1/72)$, there exist positive constants $\varepsilon_2(n, \alpha)$ and $C_2(n)$ with the following property. If E is a (Λ, r_0)-perimeter minimizer in $C(x_0, r, v)$ with*

$$\Lambda r_0 \leq 1, \qquad r_0 < r, \qquad x_0 \in \partial E, \qquad e(x_0, r, v) + \Lambda r \leq \varepsilon_2(n, \alpha),$$

then there exists $v_0 \in \mathbb{S}^{n-1}$ such that

$$e(x_0, \alpha r, v_0) \leq C_2(n)(\alpha^2 e(x_0, r, v) + \alpha \Lambda r).$$

Proof. Step one: By replacing E with $E_{x_0,r/9}$, and considering a rotation taking v into e_n, we can assume E to be a (Λ', r_0')-perimeter minimizer in C_9 satisfying:

$$\Lambda' = \frac{\Lambda r}{9}, \qquad r_0' = \frac{9r_0}{r} > 9, \qquad \Lambda' r_0' \leq 1, \qquad x_0 \in \partial E.$$

Also, for $s > 0$, if

$$e_n(9) + \Lambda r \leq \varepsilon_2(n, \alpha), \tag{2.5}$$

and given $\alpha \in (0, 1/72)$, we thus aim to find positive constants $\varepsilon_2(n, \alpha)$ and $C_2(n)$ such that the validity of (2.5) implies the existence of $v_0 \in \mathbb{S}^{n-1}$ with

$$e(0, 9\alpha, v_0) \leq C_2(n)(\alpha^2 e_n(9) + \alpha \Lambda r). \tag{2.6}$$

Step two: Let $\varepsilon_0(n)$, $\varepsilon_1(n)$, and $C_1(n)$ be constants from the height bound Lemma 1.8 and Lipschitz approximation Theorem 1.9. Assuming $\varepsilon_2(n, \alpha) \leq \min\{\varepsilon_0(n), \varepsilon_1(n)\}$, we find a function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\mathrm{Lip}(u) \leq 1$ such that, setting $\Gamma = \{(z, u(z)) : z \in D\}$ the following inequalities hold true:

$$\mathcal{H}^{n-1}(M \Delta \Gamma) \leq C_1(n) e_n(9), \tag{2.7}$$

$$\sup_{\mathbb{R}^{n-1}} |u| \leq C_1(n) e_n(9)^{1/2(n-1)},$$

$$\sup\{|qu| : y \in M\} \leq C_1(n) e_n(9)^{1/2(n-1)}, \tag{2.8}$$

$$\int_D |\nabla' u|^2 d\mathcal{H}^{n-1} \leq C_1(n) e_n(9),$$

$$\left| \int_D \nabla' u \cdot \nabla' \varphi \right| \leq C_1(n), \sup_D |\nabla' \varphi| \{e_n(9) + \Lambda r\}$$

for every $\varphi \in C_c^1(D)$. If we set

$$\beta = C_1(n)\{e_n(9) + \Lambda r\} \qquad \text{and} \qquad u_0 = \frac{u}{\sqrt{\beta}},$$

then $u_0 \in W^{1,2}(D)$ with

$$\int_D |\nabla' u_0|^2 \leq 1, \qquad \left|\int_D \nabla' u \cdot \nabla' \varphi\right| \leq ||\nabla'\varphi||_{L^\infty}(D)\sqrt{\beta}, \qquad \forall \varphi \in C^1_c(D).$$

By the harmonic approximation Lemma 2.2, for every $\tau > 0$ there exists $\sigma(\tau) > 0$ such that if

$$\sqrt{\beta} \leq \sigma(\tau), \tag{2.9}$$

then there exists a harmonic function v_0 on D with

$$\int_D |\nabla' v_0|^2 \leq 1, \qquad \int_D |v_0 - u_0|^2 \leq \tau.$$

Therefore the function $v = \sqrt{\beta} v_0$ is harmonic on D and such that

$$\int_D |\nabla' v|^2 \leq \sqrt{\beta}, \qquad \int_D |v - u|^2 \leq \beta\tau.$$

As $36\alpha > 1/2$, by Lemma 2.1, if we set $w(z) = v(0) + \nabla v(0) \cdot z$ $(z \in D)$, then

$$\sup_{D_{36\alpha}} |v - w| \leq C(n)(36\alpha)^2 ||\nabla' v||_{L^2(D)} \leq C(n)\alpha^2\sqrt{\beta}.$$

So we easily find that

$$\frac{1}{\alpha^{n+1}} \int_{D_{36\alpha}} |u - w|^2 \leq C(n)\left(\frac{\tau}{\alpha^{n+1}} + \alpha^2\right)\beta. \tag{2.10}$$

We apply this argument with $\tau = \alpha^{n+3}$: provided $\varepsilon_2(n, \alpha)$ is such that

$$\sqrt{C_1(n)\varepsilon_2(n, \alpha)} \leq \sigma\alpha^{n+3}, \tag{2.11}$$

then, $\sqrt{\beta} \leq \sigma(\tau)$ holds true with $\tau = \alpha^{n+3}$ and (2.10) takes the form

$$\frac{1}{\alpha^{n+1}} \int_{D_{36\alpha}} |u - w|^2 \leq C(n)\alpha^2\beta. \tag{2.12}$$

We now relate the left-hand side with the excess of E at scale 18α with respect to the direction v_0 given by

$$v_0 = \frac{(-\nabla' v(0), 1)}{\sqrt{1 + |\nabla' v|^2}}.$$

In this way we shall be able to deduce by (2.12) and the reverse Poincaré inequality 1.10 that

$$e(0, 9\alpha, v_0) \leq C_2(n)(C(n)\alpha^2\beta + 9\alpha\Lambda). \tag{2.13}$$

We will prove this in the following steps.

<u>*Step three:*</u> We claim that if (2.11) and $\beta^{1/(n-1)} \leq \alpha^{n+3}$ are in force (recall that $\beta = C_1(n)(e_n(9) + \Lambda r)$) and we set $c = \frac{v(0)}{\sqrt{1+|\nabla' v(0)|^2}}$, then

$$\frac{1}{\alpha^{n+1}} \int_{M \cap C_{36\alpha}} |v_0 \cdot y - c|^2 d\mathcal{H}^{n-1}(y) \leq C(n)\alpha^2\beta. \tag{2.14}$$

In order to prove (2.14) we decompose M by the graph Γ. On the one hand, by (2.12) and thanks to the fact that $Lip(u) \leq 1$ we find that

$$\begin{aligned} &\frac{1}{\alpha^{n+1}} \int_{M\cap\Gamma\cap C_{36\alpha}} |v_0 \cdot y - c|^2 d\mathcal{H}^{n-1}(y) \\ &\leq \frac{1}{\alpha^{n+1}} \int_{D_{36\alpha}} \frac{|u-w|^2}{\sqrt{1+|\nabla' v|^2}} \sqrt{1+|\nabla' v|^2} d\mathcal{H}^{n-1}(y) \leq C(n)\alpha^2\beta. \end{aligned} \tag{2.15}$$

On the other hand,

$$\begin{aligned} &\int_{(M\setminus\Gamma)\cup C_{36\alpha}} |v_0 \cdot y - c|^2 d\mathcal{H}^{n-1}(y) \\ &\leq 2\int_{(M\setminus\Gamma)\cup C_{36\alpha}} |qy|^2 + |v(0) + \nabla' v(0) \cdot py|^2 d\mathcal{H}^{n-1}(y) \\ &\leq 4\mathcal{H}^{n-1}(M \setminus \Gamma)(\sup_{y\in M} |qy|^2 + |v(0)|^2 + |\nabla' v(0)|^2) \\ &\leq C(n)\beta\big(\beta^{1/(n-1)} + |v(0)|^2 + |\nabla' v(0)|^2\big), \end{aligned} \tag{2.16}$$

where in the last inequality we have applied (2.7) and (2.8). By the mean value property of v we trivially find $|v(0)|^2 \leq C(n)\int_D |v|^2$ so that

$$\begin{aligned} |v(0)|^2 + |\nabla' v(0)|^2 \leq C(n)\int_D |v|^2 &\leq C(n)\Big(\int_D |u-v|^2 + \int_D u^2\Big) \\ &\leq C(n)\Big(\alpha^{n+3}\beta + \beta^{1/(n-1)}\Big). \end{aligned} \tag{2.17}$$

Combining (2.16) and (2.18), we finally deduce

$$\begin{aligned} &\int_{(M\setminus\Gamma)\cup C_{36\alpha}} |v_0 \cdot y - c|^2 d\mathcal{H}^{n-1}(y) \leq \\ &\qquad \leq C(n)\Big(\alpha^{n+3}\beta^2 + \beta^{1+[1/(n-1)]}\Big) \leq C(n)\alpha^{n+3}\beta, \end{aligned} \tag{2.18}$$

which, together with (2.15), gives us (2.14).

<u>*Step four:*</u> We show that, if $\varepsilon_2(n,\alpha)$ is suitably small, then $e(0, 36\alpha, v_0) \leq w(n, \frac{1}{8})$ here $w(n, t_0)$ denotes the constant of Lemma 1.6 and Theorem 1.10.

Indeed, by excess and change of direction Lemma 1.3

$$e(0, 36\alpha, v_0) \leq C(n)\Big(e_n(36\sqrt{2}\alpha) + |v_0 - e_n|^2\Big).$$

By definition of v_0 and (2.18), we may roughly estimate that

$$|v_0 - e_n|^2 \leq C|\nabla' v(0)|^2 \leq C(n)\beta^{1/(n-1)},$$

while $e_n(36\sqrt{2}\alpha) \leq (9/36\sqrt{2}\alpha)^{n-1} e_n(9)$ by the scaling of the excess Proposition 1.4. Hence,

$$e(0, 36\alpha, v_0) \leq C_3(n, \alpha)\beta^{1/(n-1)},$$

and so $e(0, 36\alpha, v_0) \leq w(n, \frac{1}{8})$ follows provided $\varepsilon_2(n, \alpha)$ is small enough with respect to the constant $C_3(n, \alpha)$ appearing in this last inequality.

Step five: By step four we are now in the position to apply the reverse Poincaré inequality 1.10. Indeed, E is a (Λ', r_0')-perimeter minimizer in $C_{(0,36\alpha,v_0)}$, with $\Lambda' r_0' \leq 1$, $0 \in \partial E$, $36\alpha < r_0'$. Therefore by 1.10 and since $\Lambda' = \Lambda r_9$, we find

$$e(0, 9\alpha, v_0) \leq C(n)(f(0, 18\alpha, v_0) + \Lambda' 9\alpha) = C(n)(f(0, 18\alpha, v_0) + \alpha\Lambda r).$$

At the same time, by (2.14) we have

$$f(0, 18\alpha, v_0) \leq 2^{n+1} f(0, 36\alpha, v_0) \leq C(n)\alpha^2(e_n(9) + \Lambda r),$$

and thus (2.13) is proved, as desired. □

Chapter 3

Regularity theorem

In this chapter we conclude the proof of the $C^{1,\gamma}$-regularity theorem for $\gamma \in (0, 1/2)$ and (Λ, r_0)-perimeter minimizers.

3.1 A lemma for the regularity theorem

Lemma 3.1 (for the regularity theorem). *For $\gamma \in (0, 1/2)$ there exist positive constants $\alpha_1(n, \gamma) < 1$, $\varepsilon_5(n, \gamma)$ and $C_6(n, \gamma)$ with the following property. Let E be a (Λ, r_0)-perimeter minimizer in $C(x_0, r, v)$ and set*

$$e^*(x, s, v) = \max\left\{e(x, s, v), \frac{\Lambda s}{\alpha_1^{n-1+2\gamma}}\right\}, \qquad x \in \mathbb{R}^n, s > 0.$$

If we have $\Lambda r_0 \leq 1$, $r < r_0$, $x_0 \in \partial E$, and $e^*(x_0, r, v) \leq \varepsilon_5(n, \gamma)$, then there exists $v_0 \in S^{n-1}$ such that

$$e^*(x_0, \alpha_1 r, v_0) \leq \alpha_1^{2\gamma} e^*(x_0, r, v), \tag{3.1}$$

$$|v_0 - v|^2 \leq C_6(n, \gamma) e^*(x_0, r, v). \tag{3.2}$$

Proof. We define $\alpha_1(n, \gamma)$ and $\varepsilon_5(n, \gamma)$:

$$\alpha_1(n, \gamma) = \min\left\{\frac{1}{72}, \left(\frac{1}{2C_2(n)}\right)^{1/(1-2\gamma)}\right\}, \quad \varepsilon_5(n, \gamma) = \frac{\varepsilon_2(n, \alpha_1)\alpha_1^{n-1+2\gamma}}{2},$$

where ε_2 and C_2 are the constants introduced in excess improvement by tilting 2.3. Let us now prove 3.1. Taking into account that, by $2\gamma < 1$,

$$\frac{\Lambda \alpha_1 r}{\alpha_1^{n-1+2\gamma}} \leq \alpha_1 e^*(x_0, r, v), \tag{3.3}$$

we only have to show the existence of $v_0 \in \mathbb{S}^{n-1}$ such that

$$e(x_0, \alpha_1 r, v_0) \leq \alpha_1^{2\gamma} e^*(x_0, r, v). \tag{3.4}$$

If $\Lambda r \geq e(x_0, r, v)$, then this is a trivial consequence of the scaling of the excess Proposition 1.4,

$$e(x_0, \alpha_1 r, v) \leq \frac{e(x_0, r, v)}{\alpha_1^{n-1}} \leq \alpha_1^{2\gamma} \frac{\Lambda r}{\alpha_1^{n-1+2\gamma}} \leq \alpha_1^{2\gamma} e^*(x_0, r, v), \tag{3.5}$$

claim (3.1) holds with $v = v_0$, and claim (3.2) follows immediately. If, instead,

$$\Lambda r \leq e(x_0, r, v), \tag{3.6}$$

then we notice that, by our choice of $\varepsilon_5(n, \gamma)$, so by excess improvement by tilting 2.3 we find $v_0 \in \mathbb{S}^{n-1}$ such that

$$\begin{aligned} e(x_0, \alpha_1 r, v_0) &\leq C_2(n)\Big(\alpha_1^2 e(x_0, r, v) + \alpha_1 \Lambda r\Big) \leq C_2(n)(\alpha_1^2 + \alpha_1) e(x_0, r, v) \\ &\leq 2C_2(n)\alpha_1 e(x_0, r, v) \leq 2C_2(n)\alpha_1 e^*(x_0, r, v) \\ &\leq \alpha_1^{2\gamma} e^*(x_0, r, v), \end{aligned} \tag{3.7}$$

where in the last inequality we have exploited the definition of α_1. This concludes the proof of (3.1). Concerning the proof of (3.2), as already noticed, we may directly assume that (3.6), and thus (3.7) holds true. By integrating $|v_0 - v|^2 \leq 2(|v_0 - v_E|^2 + |v_E - v|^2)$ over $C(x, \alpha_0 r, v_0) \cap \partial^* E$ we find that:

$$\begin{aligned} &\frac{P(E; C(x, \alpha_0 r, v_0))}{(\alpha_0 r)^{n-1}} |v_0 - v|^2 \leq \\ &\leq 4e(x_0, \alpha_0 r, v_0) + \frac{2}{(\alpha_0 r)^{n-1}} \int_{C(x,\alpha_0 r, v_0)\cap\partial^* E} |v_E - v|^2 d\mathcal{H}^{n-1}. \end{aligned} \tag{3.8}$$

In turn, by the lower density estimate (1.15) and by (3.1)

$$|v_0 - v|^2 \leq C(n)\Big(e(x_0, \alpha_1 r, v_0) + \frac{1}{(\alpha_1 r)^{n-1}} \int_{C(x,\alpha_0 r, v_0)\cap\partial^* E} |v_E - v|^2 d\mathcal{H}^{n-1}\Big).$$

Since $\alpha_0 \leq \frac{1}{\sqrt{2}}$, we have $C(x_0, \alpha_1 r, v_0) \subset B(x_0, r) \subset C(x_0, r, v)$, and thus

$$\frac{1}{(\alpha_1 r)^{n-1}} \int_{C(x,\alpha_1 r, v_0)\cap\partial^* E} |v_E - v|^2 d\mathcal{H}^{n-1} \leq \frac{2}{\alpha_1^{n-1}} e(x_0, r, v).$$

In conclusion if $C(n)$ is as showed before, then it holds

$$|v_0 - v|^2 \leq C(n)\Big(e(x_0, \alpha_1 r, v_0) + \frac{1}{\alpha_1^{n-1}} e(x_0, r, v_0)\Big) \leq C(n)\Big(\alpha_1^{2\gamma} + \frac{2}{\alpha_1^{n-1}}\Big) e^*(x_0, r, v_0).$$

And (3.2) follows with $C_6(n, \gamma) = C(n)\Big(\alpha_1^{2\gamma} + \frac{2}{\alpha_1^{n-1}}\Big)$.

□

3.2 Regularity theorem

In this chapter we finally prove the $C^{1,\gamma}$-regularity theorem for (Λ, r_0)-perimeter minimizers if $\gamma \in (0, 1/2)$. We shall use the height bound Lemma 1.8, Lipschitz approximation Theorem 1.9, excess improvement by tilting Theorem 2.3 and Campanato's criterion Theorem 1.5.

Theorem 3.2 ($\mathbf{C^{1,\gamma}}$-regularity theorem for $(\boldsymbol{\Lambda}, \mathbf{r_0})$-perimeter minimizers). *For $\gamma \in (0, 1/2)$ there exist positive constants $\varepsilon_6(n, \gamma)$ and $C_8(n, \gamma)$ with the following property. If E is a (Λ, r_0)-perimeter minimizer in $C(x_0, 9r)$ with*

$$\Lambda r_0 \leq 1, \qquad 9r < r_0, x_0 \in \partial E, \qquad e_n(x_0, 9r) + \Lambda r \leq \varepsilon_6(n, \gamma).$$

then there exists a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ with

$$\sup_{\mathbb{R}^{n-1}} \frac{|u|}{r} \leq C_1(n) e_n(x_0, r, v)^{1/2(n-1)}, \quad Lip(u) \leq 1, \tag{3.9}$$

such that

$$C(x_0, r) \cup \partial E = x_0 + \{(z, u(z)) : z \in D_r\}, \tag{3.10}$$

$$C(x_0, r) \cup E = x_0 + \{(z, t) : z \in D_r, -r < t < u(z)\}. \tag{3.11}$$

In fact, $u \in C^{1,\gamma}(D(px_0, r))$, and for every $z, z' \in D(px_0, r)$ and $x, y \in C(x_0, r) \cap \partial E$ we have

$$|\nabla' u(z) - \nabla' u(z')| \leq C_8(n, \gamma)\big(e_n(x_0, 9r) + \Lambda r\big)^{1/2} \left(\frac{|z - z'|}{r}\right)^{\gamma}, \tag{3.12}$$

$$|\nu_E(x) - \nu_E(y)| \leq C_8(n, \gamma)\big(e_n(x_0, 9r) + \Lambda r\big)^{1/2} \left(\frac{|x - y|}{r}\right)^{\gamma}. \tag{3.13}$$

Proof. *Step one:* Given $\gamma \in (0, 1/2)$, we show the existence of a constant $C = C(n, \gamma)$ such that: if $\varepsilon_5(n, \gamma)$ denotes the constant of Lemma 3.1 and if $e_n^*(x_0, 9r) \leq \left(\frac{8}{9}\right)^{n-1} \varepsilon_5(n, \gamma)$, then for every $x \in C(x_0, r) \cap \partial E$ there exists $v(x) \in \mathbb{S}^{n-1}$ such that

$$e^*(x, s, v(x)) \leq C\left(\frac{s}{t}\right)^{2\gamma} e_n^*(x_0, 9r), \forall s \in (0, 4r), \tag{3.14}$$

$$|v(x) - e_n|^2 \leq C e_n^*(x_0, 9r), \tag{3.15}$$

$$e_n^*(x, s) \leq C e_n^*(x_0, 9r), \forall s \in (0, 8r). \tag{3.16}$$

First we prove (3.15). Indeed, let us fix $x \in C(x_0, r) \cap \partial E$, and set $t = 8r$ for brevity. By excess at different scales Proposition 1.2 we immediately find

$$e^*(x, t, v(x)) \leq \left(\frac{9}{8}\right)^{n-1} e_n^*(x_0, 9r) \leq \varepsilon_5(n, \gamma). \tag{3.17}$$

Thus we can apply the Lemma 3.1 to E at the point x and at the scale t, and correspondingly find a direction $v_1 = v_1(x) \in \mathbb{S}^{n-1}$ such that

$$\begin{aligned} e^*(x, \alpha t, v_1) &\leq \alpha^{2\gamma} e_n^*(x,t), \\ |v_1 - e_n|^2 &\leq C e_n^*(x,t), \end{aligned} \tag{3.18}$$

where $\alpha = \alpha(n,\gamma)$ and $C = C(n,\gamma)$ are as in Lemma 3.1 . Since $\alpha \leq 1$ we see that $e^*(x, \alpha t, v_1) \leq \varepsilon_3(n,\gamma)$.

In particular we can apply Lemma 3.1 again to E at the point x, but this time at the smaller scale αt. Iterating, we prove the existence of a sequence of vectors $v_h = v_h(x) \in \mathbb{S}^{n-1}$ such that

$$\begin{aligned} & e^*(x, \alpha^h t, v_h) \leq \alpha^{2\gamma h} e_n^*(x,t), \\ & |v_h - v_{h-1}|^2 \leq C e^*(x, \alpha^{h-1} t, v_{h-1}) \leq C \alpha^{2\gamma(h-1)} e_n^*(x,t), \end{aligned} \tag{3.19}$$

for every $h \in \mathbb{N}$ where we have set $v_0 = e_n$. So if $j \geq h \geq 1$ then

$$|v_j - v_{h-1}| \leq \sum_{k=h}^{j} |v_k - v_{k-1}| \leq \sqrt{C e_n^*(x,t)} \sum_{k=h}^{\infty} \alpha^{\gamma(k-1)} = \frac{\sqrt{C e_n^*(x,t)}}{1-\alpha^\gamma} \alpha^{\gamma(h-1)}. \tag{3.20}$$

Hence there exists $v(x) = \lim_{j\to\infty} v_j(x)$. Moreover, if we set $h = 1$ and let $j \to \infty$ by (3.17) and the formula above we find that, for a constant $C(n,\gamma)$,

$$|v(x) - e_n|^2 \leq C(n,\gamma) e_n^*(x_0, 9r), \tag{3.21}$$

which is (3.15).

We now prove 3.14. Since $s \in (0, t/2)$, there exists $h \geq 0$ such that $\alpha^{h+1} t \leq \sqrt{2} s \leq \alpha^h t$. In particular by excess and changes of direction Proposition 1.3 and excess at different scales Proposition 1.2.

$$\begin{aligned} e^*(x, s, v(x)) &\leq C(n) \Big(e^*(x, \sqrt{2} s, v_h) + |v(x) - v_h|^2 \Big) \\ &\leq C(n) \left(\Big(\frac{\alpha^h t}{s} \Big)^{n-1} e^*(x, \alpha^h t, v_h) + |v(x) - v_h|^2 \right), \end{aligned} \tag{3.22}$$

where for the first term by (3.15) we have

$$\begin{aligned} \Big(\frac{\alpha^h t}{s} \Big)^{n-1} e^*(x, \alpha^h t, v_h) &\leq \frac{C(n)}{\alpha^{n-1}} \alpha^{2\gamma h} e_n^*(x,t) \leq \frac{C(n)}{\alpha^{n-1+2\gamma}} \Big(\frac{s}{t} \Big)^{2\gamma} e_n^*(x,t) \\ &\leq C(n,\gamma) \Big(\frac{s}{t} \Big)^{2\gamma} e_n^*(x_0, 9r) \end{aligned} \tag{3.23}$$

while for the second term by (3.20) we have

$$|v(x) - v_h|^2 \leq C(n,\gamma)\alpha^{2\gamma h} e_n^*(x_0, 9r) \leq C(n,\gamma)\Big(\frac{s}{t}\Big)^{2\gamma} e_n^*(x_0, 9r). \tag{3.24}$$

so combining the two together

$$e^*(x, s, v(x)) \leq C(n,\gamma)\Big(\frac{s}{t}\Big)^{2\gamma} e_n^*(x_0, 9r), \quad \forall s \in (0, t/2), \tag{3.25}$$

which is (3.14).

We finally prove (3.16). If $s \in (0, t/4)$, then by excess and changes of direction Proposition 1.3 we deduce

$$e_n^*(x, s) \leq C(n)\Big(e^*(x, \sqrt{2}s, v(x)) + |v(x) - e_n|^2\Big), \tag{3.26}$$

then with (3.14) on the second term and excess at different scales Proposition 1.2 with (3.25) on the first term we have

$$e_n^*(x, s) \leq C(n,\gamma) e_n^*(x_0, 9r), \tag{3.27}$$

if otherwise $s \in (t/4, t)$, then, by $C(x, s) \subset C(x_0, 9r)$,

$$e_n^*(x, s) \leq \Big(\frac{9r}{s}\Big)^{n-1} e_n^*(x_0, 9r) \leq \Big(\frac{9}{2}\Big)^{n-1} e_n^*(x_0, 9r). \tag{3.28}$$

We have thus achieved the proof of (3.16).

Step two: Now we prove (3.9), (3.10) and (3.11). We define

$$\varepsilon_4(n,\gamma) = \min\Big\{\varepsilon_0(n), \varepsilon_1(n), \Big(\frac{8}{9}\Big)^{n-1}\varepsilon_3(n,\gamma), \frac{\delta_0(n)}{C_4(n,\gamma)}\Big\}; \tag{3.29}$$

where $\varepsilon_0(n)$ is constant introduced in the height bound, $\varepsilon_1(n)$ and $\delta_0(n)$ come from Lipschitz approximation Theorem 1.9, and $\varepsilon_5(n,\gamma)$ comes from Lemma 3.1. Since $e_n(x_0, 9r) \leq \varepsilon_4(n,\gamma)$ we have $e_n(x_0, 9r) \leq \big(\frac{8}{9}\big)^{n-1}\varepsilon_3(n,\gamma)$, so we are in the situation analyzed in *step one*.

First we prove (3.9).
So if $M_0 = \big\{x \in C(x_0, r) \cap \partial E : \sup_{0<s<8r} e_n(x, s) \leq \delta_0(n)\big\}$, by (3.16) we have $M_0 = C(x_0, r) \cap \partial E$, and since $e_n(x_0, 9r) \leq \varepsilon_1(n)$ we have that there exists a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}^n$ such that $M_0 = C(x_0, r) \cap \partial E \subset x_0 + \{(z, u(z)) : z \in \mathbb{R}^n\}$ and so (3.9) holds true.

Now we prove (3.10).
By Lipschiz graph criterion $C(x_0, r) \cap \partial E \subset x_0 + \{(z, u(z)) : z \in D_r\}$, so (3.10) holds true.

Finally we prove (3.10).
From step three in the proof of Lipschitz approximation Lemma 1.9 we see

that $\nu_E(x) = \frac{(-\nabla' u(px),1)}{\sqrt{1+|\nabla' u(px)|^2}}$, for $\mathcal{H}^{n-1}$-a.e. $x \in C(x_0,r) \cap \partial E$, so by small-excess position Lemma 1.6 and (3.10) follows that (3.11) holds true.

Step three: We now prove that $\forall z \in D(px_0, r)$ and $s \in (0, r)$,

$$\frac{1}{s^{n-1}} \int_{D(z,s)} |\nabla' u - (\nabla' u)_{z,s}|^2 \le C(n,\gamma) \Big(\frac{s}{r}\Big)^{2\gamma} e_n(x_0, 9r), \tag{3.30}$$

where $(\nabla' u)_{z,s}$ denotes the mean value of $\nabla' u$ on $D(z,s)$.

To this end, let us first notice that by (3.15), up to further decreasing $\varepsilon_3(n,\gamma)$, we can assume that $qv(x) \le \frac{1}{\sqrt{2}}$, $\forall x \in C(x_0,r) \cap \partial E$. In particular, the set inclusion $C(x,s) \subset C(x, 2s, v(x))$ will hold whenever $x \in C(x_0,r) \cap \partial E$ and $s > 0$. We can also define a vector field $\tau : C(x_0,r) \cap \partial E \to \mathbb{R}^{n-1}$ by setting

$$\tau(x) = -\frac{pv(x)}{qv(x)}, \quad x \in C(x_0,r) \cap \partial E, \tag{3.31}$$

so that for every $x \in C(x_0,r) \cap \partial E$,

$$pv(x) = \frac{-\tau(x)}{\sqrt{1+|\tau(x)|^2}}, \quad qv(x) = \frac{1}{\sqrt{1+|\tau(x)|^2}}, \quad |\tau(x)| \le 1.$$

If $z \in D(px_0, r)$, $s < r$, $x = (z, u(z))$, then $x \in C(x_0,r) \cap \partial E$, and

$$\begin{aligned}(2s)^{n-1} e(x, 2s, v(x)) &\ge \int_{C(x,s)\cap\partial^* E} \frac{|\nu_E - v(x)|^2}{2} d\mathcal{H}^{n-1} \\ &\ge \int_{D(z,s)} \left| \frac{\nabla' u}{\sqrt{1+|\nabla' u|^2}} - \frac{\tau(px)}{\sqrt{1+|\tau(px)|^2}} \right|^2 \sqrt{1+|\nabla' u|^2} \\ &\quad + \int_{D(z,s)} \left| \frac{1}{\sqrt{1+|\nabla' u|^2}} - \frac{1}{\sqrt{1+|\tau(px)|^2}} \right|^2 \sqrt{1+|\nabla' u|^2}.\end{aligned}$$

And from the above chain of inequalities and the fact that $|\tau(px)| \le 1$, we infer that

$$\begin{aligned}\int_{D(z,s)} |\nabla' u - (\nabla' u)_{z,s}|^2 = \inf_{\zeta \in \mathbb{R}^n} \int_{D(z,s)} |\nabla' u - \zeta|^2 &\le \int_{D(z,s)} |\nabla' u - \tau(px)|^2 \\ &\le 2 \int_{D(z,s)} \left| \frac{\nabla' u - \tau(px)}{\sqrt{1+|\tau(px)|^2}} \right|^2 \sqrt{1+|\nabla' u|^2} \\ &\le C(n) s^{n-1} e(x, 2s, v(x))\end{aligned}$$

which by (3.14) implies (3.30).

Step four: We now prove (3.2). From (3.30) and by Campanato's criterion Theorem 1.5 applied to $\nabla' u$ we immediately deduce (3.12).

Since $v \in \mathbb{R}^{n-1} \mapsto \frac{(-v,1)}{\sqrt{1+|v|^2}} \in \mathbb{R}^n$ defines a Lipschitz map on $\mathbb{R}^{n-1}$, we easily deduce from (3.1) that, if $x, y, \in C(x_0, r) \cap \partial E$, then

$$
\begin{aligned}
|\nu_E(x) - \nu_E(y)| \le C|\nabla' u(px) - \nabla' u(py)| &\le C e_n(x_0, 9r)^{1/2} \Big(\frac{|px - py|}{r}\Big)^{\gamma} \\
&\le C(n,\gamma) e_n(x_0, 9r)^{1/2} \Big(\frac{|x - y|}{r}\Big)^{\gamma}.
\end{aligned}
$$

□

Bibliography

[1] Francesco Maggi. *Sets of Finite Perimeter and Geometric Variational Problems, An Introduction to Geometric Measure Theory.* Cambridge University Press, 2012.

[2] Richard Schoen & Leon Simon. *A New Proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals.* Indiana University Mathematics Journal, May–June, 1982, Vol. 31, No. 3 (May–June, 1982), pp. 415-435.

[3] Ennio De Giorgi. *Frontiere orientate di misura minima. Seminario di Matematica.* Scuola Normale Superiore di Pisa, 1960-1961.

[4] Frederick Justin Almgren Jr. *Q-valued functions minimizing Dirichlet's integral and the regularity of area minimizing rectifiable currents up to codimension two.* Bulletin of the American Mathematical Society, New Series, 8 (2), pp. 327–328.