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Plateau’s Problem: the Douglas–Radó Theorem

Laureanda:
Annalaura Pegoraro
2076334

Relatore:
Prof. Roberto Monti

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Chapter 1

Introduction

1.1 Motivation

An intuitive example of minimal surfaces is provided by soap films, which are formed by immersing a wire frame in a solution containing soap and water. Microscopically, you can think of a soap film as a thin layer of water trapped between two layers of soap molecules, where each layer behaves like an "elastic membrane" that opposes any increase in the area of the surface. In particular, they can be seen as a dense network of tiny springs pulling in all directions along the surface. The wire frame is fixed, so the film cannot contract indefinitely and disappear. However, the molecules in the interior can rearrange: as long as there exists a local deformation that decreases the area, and hence the "energy" of the surface, the film continues to move and adjust. When the film reaches a configuration in which any small local deformation would increase the area, i.e. the forces exerted by the molecules balance each other in all directions, the system is in equilibrium and the resulting surface is, in a variational sense, minimal. Mathematically, a surface is called *minimal* if it is a critical point of the area functional with respect to compactly supported deformations.

Let us fix a closed curve $\Gamma \subset \mathbb{R}^3$ as the boundary. Does there exist a surface bounded by Γ with minimal area? This is the question of *Plateau's Problem*. Although Lagrange initiated the study of minimal surfaces in the 18th century, the problem is named after Joseph Plateau, whose experiments with soap films provided the physical motivation for a rigorous mathematical formulation.

A fundamental answer to Plateau's problem was given in the 1930s by Jesse Douglas [2] and Tibor Radó [5], who independently proved the existence of a solution for sufficiently regular boundary curves. Their result, the Douglas–Radó theorem, and main subject of this thesis, contributed significantly to the development of the calculus of variations.

The exposition below follows Prof. Roberto Monti's lecture notes [4] and the first chapter of Struwe [7].

1.2 Problem Formulation and Statement of the Theorem

Consider the unit disk

$$D = \{z = u + iv \in \mathbb{C} : |z| < 1\},$$

and let $n \geq 3$ be a fixed dimension. Let $\gamma : \partial D \to \mathbb{R}^n$ be a C^1-regular simple curve, i.e. γ is injective of class C^1 and $\gamma'(z) \neq 0$, $\forall z \in \mathbb{C}$. We say γ is a *Jordan curve.* Let

$$\Gamma = \gamma(\partial D)$$

be the support of γ. Let $F \in C^2(D;\mathbb{R}^n) \cap C(\overline{D};\mathbb{R}^n)$ be such that $F(\partial D) = \Gamma$. We say

$$\Sigma = F(D)$$

is a *parametric* $2-$*surface* of class C^2 with $\partial\Sigma = \Gamma$.
For any $z = u + iv \in D$ we write

$$F_u = \frac{\partial F}{\partial u} \in T\Sigma,$$
$$F_v = \frac{\partial F}{\partial v} \in T\Sigma.$$

On $T\Sigma$ we introduce the the bilinear form

$$I = \begin{pmatrix} \langle F_u, F_u\rangle & \langle F_u, F_v\rangle \\ \langle F_v, F_u\rangle & \langle F_v, F_v\rangle \end{pmatrix}.$$

Note this is the first fundamental form of F with respect to local coordinates (u, v) on Σ. We denote the area of F by

$$\begin{aligned}\mathscr{A}(F) &= \int_D \sqrt{\det(I)} \;\; dudv \\ &= \int_D \sqrt{|F_u|^2|F_v|^2 - \langle F_u, F_v\rangle^2} \;\; dudv\end{aligned}$$

Furthermore, $\mathrm{tr}(I(z)) = |F_u(z)|^2 + |F_v(z)|^2$. Consider now the class

$$\mathscr{C}(\Gamma) = \{F \in C^2(D;\mathbb{R}^n) \cap C(\overline{D};\mathbb{R}^n) \mid F : \partial D \to \mathbb{R}^n \text{ is a homeomorphism}\}$$

The parametric Plateau problem consists in finding a minimizer of

$$\inf\{\mathscr{A}(F) : F \in \mathscr{C}(\Gamma)\}. \tag{1}$$

The natural idea to approach such a problem would be to pick a minimizing sequence $F_1, F_2, \ldots$ for the area and hope that a subsequence will converge to an F which gives a surface $\Sigma = F(D)$ of least area. However, $\mathscr{C}(\Gamma)$ is not compact and the area is invariant under changes of parameters: for any diffeomorphism $\varphi : \overline{D} \to \overline{D}$, we have

$$\mathscr{A}(F) = \mathscr{A}(F \circ \varphi)$$

So there is no hope of solving the minimum problem in this way. This motivates the introduciton of the following *energy functional.*

Definition 1.1. The *energy* of the map $F : D \to \mathbb{R}^n$, where $F = (F_1, \dots, F_n)$, is

$$\begin{aligned}\mathscr{E}(F) &= \frac{1}{2}\int_D \left(|F_u|^2 + |F_v|^2\right) \quad dudv \\ &= \frac{1}{2}\sum_{i=1}^{n}\int_D |\nabla F^i| \quad dudv\end{aligned}$$

Note that it could be $\mathscr{E}(F) = +\infty$.

There are a few useful results that link the area and energy functionals, which will lead us to Douglas' formulation of Plateau's problem.

Definition 1.2. The function $F : D \to \mathbb{R}^n$ is *weakly conformal* if we have:

i) $|F_u| = |F_v|$,

ii) $\langle F_u, F_v \rangle = 0$.

Note that it could be $F_u(z) = F_v(z) = 0$ at some $z \in D$. If $|F_u| = |F_v| \neq 0$, then F is *conformal.*

Lemma 1.3 (Energy dominates area)**.** *For every map $F : D \to \mathbb{R}^n$, we have $\mathscr{A}(F) \leq \mathscr{E}(F)$. Equality holds if and only if F is weakly conformal.*

Proof. We wish to prove

$$\det(I) = |F_u|^2|F_v|^2 - \langle F_u, F_v\rangle^2 \leq \left[\frac{1}{2}\left(|F_u|^2 + |F_v|^2\right)\right]^2 = \left[\frac{1}{2}\mathrm{tr}(I)\right]^2,$$

where the left hand-side is positive by Cauchy-Schwarz inequality. The above inequality is clearly true by the arithmetic-geometric mean inequality

$$|F_u|^2|F_v|^2 - \langle F_u, F_v\rangle^2 \leq |F_u|^2|F_v|^2 \leq \left[\frac{1}{2}\left(|F_u|^2 + |F_v|^2\right)\right]^2.$$

Moreover, equality holds if and only if $|F_u| = |F_v|$ and $F_u \cdot F_v = 0$, i.e. F is weakly conformal. It follows that

$$\mathscr{A}(F) = \int_D \sqrt{\det(I)} \;\; dudv \leq \frac{1}{2}\int_D \mathrm{tr}(I) \;\; dudv = \mathscr{E}(F),$$

and $\mathscr{A}(F) = \mathscr{E}(F)$ if and only if F is weakly conformal. In the latter case we have $\sqrt{\det(I)} = \frac{1}{2}\mathrm{tr}(I)$ on D. $\square$

Furthermore, the following $\varepsilon-$conformality result, which we are not going to prove, holds.

Theorem 1.4 (Morrey)**.** *For any F we have* $\inf_{F\in\mathscr{C}(\Gamma)} \mathscr{A}(F) = \inf_{F\in\mathscr{C}(\Gamma)} \mathscr{E}(F)$.

Douglas' idea was to solve the minimum problem for the energy, i.e. finding a minimizer of

$$\inf\{\mathscr{E}(F) : F \in \mathscr{C}(\Gamma)\}. \tag{2}$$

Once again, we may note that the situation is still non-compact, but $\mathscr{E}$ is invariant under a smaller class of diffeomorphisms of D. In fact,

$$\mathscr{E}(F\circ\varphi) = \mathscr{E}(F)$$

precisely when φ is a conformal map.

Proof. Let $\varphi : D \to D$, $(x, y) \mapsto (u(x, y), v(x, y))$ be a C^1 diffeomorphism and $G = F\circ\varphi$. Then,

$$\begin{aligned} J_\varphi &= \begin{pmatrix}\varphi_x & \varphi_y\end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}, \\ G_x &= (F_u\circ\varphi)u_x + (F_v\circ\varphi)v_x, \\ G_y &= (F_u\circ\varphi)u_y + (F_v\circ\varphi)v_y. \end{aligned}$$

Compute

$$\begin{aligned} \mathscr{E}(F\circ\varphi) &= \mathscr{E}(G) \\ &= \frac{1}{2}\int_D |G_x|^2 + |G_y|^2 \; dxdy \\ &= \frac{1}{2}\int_D |F_u\circ\varphi|^2|\nabla u|^2 + 2\langle F_u, F_v\rangle\langle\nabla u, \nabla v\rangle + |F_v\circ\varphi|^2|\nabla v|^2 \; dxdy. \end{aligned}$$

On the other hand, by a change of variable,

$$\begin{aligned} \mathscr{E}(F) &= \frac{1}{2}\int_D |F_u|^2 + |F_v|^2 \; dudv \\ &= \frac{1}{2}\int_D \left(|F_u\circ\varphi|^2 + |F_v\circ\varphi|^2\right)|\det J_\varphi| \;\; dxdy \end{aligned}$$

Therefore, $\mathscr{E}(F\circ\varphi) = \mathscr{E}(F)$ if and only if

$$\begin{cases} |\nabla u|^2 = |\nabla v|^2 = |\det J_\varphi| \neq 0 \\ \langle\nabla u, \nabla v\rangle = 0 \end{cases}$$

$$\begin{cases} |\varphi_x|^2 = |\varphi_y|^2 = |\det J_\varphi| \neq 0 \\ \langle \varphi_x, \varphi_y \rangle = 0 \end{cases}$$

if and only if φ is a conformal map $D \to D$. □

Note that the condition above is equivalent to the following

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad \text{or} \qquad \begin{cases} u_x = -v_y \\ u_y = v_x \end{cases}$$

Therefore, the set of diffeomorphisms under which $\mathscr{E}$ is invariant consists of two components: the group of *holomorphic* conformal diffeomorphisms of the disk, which preserve both angles and orientation, and the group of *antiholomorphic* conformal diffeomorphisms, which preserve angles while reversing orientation. Since the latter can be obtained from the former by complex conjugation, we will restrict our attention to the holomorphic case.
The group of holomorphic conformal diffeomorphisms $\varphi : D \to D$, known as the *Möbius group*, is given by the maps

$$\varphi(z) = e^{i\vartheta}\frac{z + z_0}{1 + \overline{z_0}z}, \qquad \text{where } z \in D,\ \vartheta \in [0, 2\pi),\ \text{and } z_0 \in D.$$

In particular, we still have some problems because the group loses compactness when $|z_0| \to 1$. For a detailed proof see [7]. We will deal with this issue in Section 2.2.

Definition 1.5. Let $F \in C^1(D; \mathbb{R}^n)$ be weakly conformal. The set

$$\mathrm{Sing}(F) = \{z \in D : |F_u(z)| = |F_v(z)| = 0\}$$

is the *singular set* of F.

Note that if $z \in \mathrm{Sing}(F)$, then $\Sigma = F(D)$ may have a singularity at the point $F(z) \in \Sigma$.

Finally, we are now ready to state the main theorem of the parametric Plateau problem.

Theorem 1.6 (Douglas-Radò)**.** *Let $\Gamma \subset \mathbb{R}^n$, $n \geq 2$, be a Jordan curve of class C^1. There exists $F \in \mathscr{C}(\Gamma)$ realizing the minimum*

$$\min\{\mathscr{A}(F) : F \in \mathscr{C}(\Gamma)\}$$

such that:

(1) $F \in C^\infty(D; \mathbb{R}^n)$ and $\Delta F = 0$, i.e. each coordinate of F is harmonic;

(2) $F : \partial D \to \Gamma$ is a homeomorphism, i.e. $F \in \mathscr{C}(F)$;

(3) F is weakly conformal;

(4) Sing(F) consists of isolated points of D.

To understand where Laplace's equation $\Delta F = 0$ comes from, let's fix a continuous parametrization $\gamma : \partial D \to \Gamma$ of Γ. By the theory of Partial Differential Equations, the *Dirichlet problem*

$$\begin{cases} \Delta F = 0 & \text{in } D \\ F|_{\partial D} = \gamma \end{cases} \tag{3}$$

has a unique solution $F \in C^\infty(D; \mathbb{R}^n) \cap C(\overline{D}, \mathbb{R}^n)$. In particular, given a solution, uniqueness can be proved via the Maximum Principle for Harmonic Functions. Note that it could be $\mathscr{E}(F) = +\infty$. However, since γ is of class C^1, we have $\mathscr{E}(F) < \infty$. In particular, if F is a solution to the Dirichlet problem, then F is a minimizer of $\mathscr{E}$. Indeed, we can write any function $H \in C^\infty(D; \mathbb{R}^n) \cap C(\overline{D}, \mathbb{R}^n)$ such that $H|_{\partial D} = \gamma$ as $H = F + G$, where $G \in C_c^\infty(D; \mathbb{R}^n)$ is a *bump function.* Then, by linearity of the gradient we have

$$\begin{aligned} \mathscr{E}(F) &= \frac{1}{2} \int_D |\nabla F + \nabla G|^2 \ dudv \\ &= \frac{1}{2} \int_D \left\{ |\nabla F|^2 + |\nabla G|^2 + 2 \sum_{i=1}^n \langle \nabla F^i, \nabla G^i \rangle \right\} \ dudv \end{aligned}$$

By a known identity

$$\langle \nabla F^i, \nabla G^i \rangle = \text{div } (G^i \, \nabla F^i) - G^i \underbrace{\Delta F^i}_{=0}.$$

Note that $G|_{\partial D} = 0$, thus by the Divergence Theorem

$$\int_D \text{div } (G^i \, \nabla F^i) \ dudv = \int_{\partial D} \langle G^i \, \nabla F^i, N \rangle \ d\mathscr{H}^1 = 0,$$

where N is the outward pointing unit normal. Hence, we get

$$\mathscr{E}(F+G) = \frac{1}{2} \int_D \left\{ |\nabla F|^2 + |\nabla G|^2 \right\} \ dudv \geq \mathscr{E}(F).$$

Chapter 2

Proof of the Theorem

2.1 Getting rid of the parametrization

After having shown that a minimizer exists and that it satisfies Laplace's equation, we must get rid of the specific parametrization γ of Γ. Given $\gamma \in C^1(\partial D; \Gamma)$ a homeomorphism, let F^γ be the solution to the Dirichlet problem 3. Let $(\gamma_j)_{j\in\mathbb{N}}$ be a sequence such that

$$\lim_{j\to\infty} \mathscr{E}(F^{\gamma_j}) = \overline{\mathscr{E}} = \inf\{\mathscr{E}(F) : F \in \mathscr{C}(\Gamma)\}.$$

Recall that $\overline{\mathscr{E}} < \infty$ because $\gamma \in C^1$. Assume that

$$\gamma_j \underset{j\to\infty}{\rightrightarrows} \gamma \in C(\partial D, \Gamma),$$

we will deal with this in Section 2.2. By the Maximum Principle for harmonic functions,

$$\max_{z\in\overline{D}} \left|F^j(z) - F^k(z)\right| = \max_{z\in\partial D} \left|\gamma_j(z) - \gamma_k(z)\right|,$$

from which we can deduce

$$F^{\gamma_j} \underset{j\to\infty}{\rightrightarrows} F \in C(\overline{D}, \mathbb{R}^n).$$

Note that harmonicity is preserved by uniform convergence, therefore $F \in C^\infty(D, \mathbb{R}^n)$ and $\Delta F = 0$ in D.
Note that $(\nabla F^{\gamma_j})_{j\in\mathbb{N}}$ is a uniformly bounded sequence in $L^2(D)$. By weak compactness of $L^2(D)$, we can find a weakly converging subsequence

$$\nabla F^{\gamma_j} \underset{j\to\infty}{\rightharpoonup} G \quad \text{in } L^2(D).$$

We now show that $G = \nabla F$. For $\varphi \in C_c^\infty(D)$ and each $\alpha = 1, \ldots, n$, we have

$$\int_D F_\alpha^{\gamma_j}\, \partial_i \varphi = -\int_D (\partial_i F_\alpha^{\gamma_j})\, \varphi.$$

Since $F_\alpha^{\gamma_j} \rightrightarrows F_\alpha$ uniformly and $\partial_i F_\alpha^{\gamma_j} \rightharpoonup G_{\alpha i}$ in $L^2(D)$, passing to the limit yields

$$\int_D F_\alpha \, \partial_i \varphi = - \int_D G_{\alpha i} \, \varphi, \quad \forall \, \varphi \in C_c^\infty(D), \; i = 1, 2.$$

Hence $G = \nabla F$ in the weak sense, and we can deduce

$$\nabla F^{\gamma_j} \underset{j \to \infty}{\rightharpoonup} \nabla F \quad \text{in } L^2(D).$$

Finally, this implies

$$\mathscr{E}(F) = \|\nabla F\|_{L^2}^2 \leq \liminf_{j \to \infty} \|\nabla F^{\gamma_j}\|_{L^2}^2 = \liminf_{j \to \infty} \mathscr{E}(F^{\gamma_j}) = \overline{\mathscr{E}}.$$

We will now prove that $F \in \mathscr{C}(\Gamma)$, so that we can conclude $\mathscr{E}(F) = \overline{\mathscr{E}}$. In fact, even though γ_j are homeomorphisms, the uniform limit γ may lose injectivity.

Lemma 2.1. *The minimizing F is a homeomorphism from ∂D to Γ.*

Proof. Assume by contradiction that there exist $z_1, z_2 \in \partial D$ such that $F(z_1) = F(z_2)$. Since the limiting curves γ_j are homeomorphisms, F must be constant on one of the two arcs $\widehat{z_1 z_2}$. Let's say it's equal to some constant vector $(a_1, ..., a_n) \in \mathbb{R}^n$. We wish to extend F harmonically beyond the unit disk. To do so, we want to apply Schwarz's Reflection Principle to the pull back of F under the Möbius map

$$T(z) = i \frac{1+z}{1-z},$$

which sends the unit disk to the real axis. Therefore we define $\widehat{F} : \mathbb{C} \to \mathbb{R}^n$ as

$$\widehat{F}_i(z) = \begin{cases} F_i(z) & z \in \overline{D} \\ 2a_i - F_i\left(\frac{z}{|z|^2}\right) & |z| > 1 \end{cases}, \qquad i = 1, ..., n.$$

Since F is constant on $\widehat{z_1 z_2}$, by applying Schwarz's Reflection Principle to the translated maps $G_i(z) = F_i(z) - a_i$, we obtain that $\widehat{F}$ is harmonic in a small open set $\mathcal{U}$ crossing $\widehat{z_1 z_2}$. This implies that the map $\widehat{F}_z : \mathcal{U} \to \mathbb{C}^n$ is holomorphic.

Moreover, $\widehat{F}_z = 0$ on $\widehat{z_1 z_2}$. By the uniqueness principle for analytic functions, $\widehat{F}_z = 0$ on $\mathcal{U}$. This implies that F must be constant on $D \cap \mathcal{U}$, hence on D. However, this is not possible because $F : \partial D \to \Gamma$ is surjective. □

2.2 Achieving compactness: the Courant–Lebesgue lemma

Now we must show that we can choose a minimizing sequence $(\gamma_j)_{j \in \mathbb{N}}$ that converges uniformly. As already highlighted, $\mathscr{C}(\Gamma)$ is not compact. We can avoid this problem by doing a *three points normalization* of the curves γ_i and then applying the

Courant–Lebesgue lemma.
It is well known that given $z_1, z_2, z_3 \in \partial D$ and $w_1, w_2, w_3 \in \partial D$ two triples of distinct points in ∂D taken with the same order, there exists a unique Möbius transformation $\varphi : D \to D$ such that $\varphi(z_k) = w_k$ for $k = 1, 2, 3$. Now fix three distinct points $p_1, p_2, p_3 \in \Gamma$. Since $\mathscr{E}(F \circ \varphi) = \mathscr{E}(F)$ for φ Möbius, we can assume that each curve γ^j of the minimizing sequence satisfies

$$\gamma^j \left(e^{i\frac{2k}{3}\pi}\right) = p_k, \qquad k = 1, 2, 3.$$

We will now need two lemmas.

Lemma 2.2 (Courant–Lebesgue)**.** *Fix a point $z \in \partial D$ and use polar coordinates centered at z. Let*

$$G(r, \vartheta) = F(z + re^{i\vartheta}),$$

where $r \in (0, 2)$ and $\vartheta \in \Theta_r = (\vartheta_-(r), \vartheta_+(r))$. For a fixed radius r we consider the curve $\vartheta \mapsto G(r, \vartheta)$. Then, for any $\delta \in (0, 1)$, there exists $r \in [\delta, \sqrt{\delta}]$ such that

$$\mathrm{Length}(G(r, \,\cdot\,)) \le \sqrt{\frac{4\pi \mathscr{E}(F)}{-\log(\delta)}}.$$

Proof. The energy $\mathscr{E}(F)$ in polar coordinates is

$$\begin{aligned}
\mathscr{E}(F) &= \frac{1}{2}\int_D |\nabla F|^2 \; dudv \\
&= \frac{1}{2}\int_\delta^2 \int_{\Theta_r} \left(|G_r|^2 + \frac{1}{r^2}|G_\vartheta|^2\right) \; d\vartheta \, rdr \\
&\ge \int_0^{\sqrt{\delta}} \frac{|\Theta_r|}{r} \fint_{\Theta_r} |G_\vartheta|^2 \; d\vartheta dr.
\end{aligned}$$

Note that by Holder's inequality,

$$\fint_{\Theta_r} |G_\vartheta| \; d\vartheta dr \le \left(\fint_{\Theta_r} |G_\vartheta|^2 \; d\vartheta dr\right)^{\frac{1}{2}}.$$

Hence,

$$\begin{aligned}
2\mathscr{E}(F) &\ge \int_\delta^{\sqrt{\delta}} \frac{|\Theta_r|}{r} \left(\fint_{\Theta_r} |G_\vartheta| \; d\vartheta\right)^2 dr \\
&= \int_\delta^{\sqrt{\delta}} \frac{1}{r|\Theta_r|} \left(\int_{\Theta_r} |G_\vartheta| \; d\vartheta\right)^2 dr \\
&\ge \frac{1}{\pi}\int_\delta^{\sqrt{\delta}} \frac{1}{r} \left(\int_{\Theta_r} |G_\vartheta| \; d\vartheta\right)^2 dr,
\end{aligned}$$

since $|\Theta_r| \leq \pi$. We can now choose $r \in [\delta, \sqrt{\delta}]$ such that

$$\text{Length}(G(r,\,\cdot\,)) = \int_{\Theta_r} |G_\vartheta| \; d\vartheta$$

is minimized, and we obtain

$$2\pi\mathscr{E}(F) \geq \text{Length}(G(r,\,\cdot\,))^2 \int_\delta^{\sqrt{\delta}} \frac{1}{r} \; dr = \log\left(\frac{1}{\sqrt{\delta}}\right) \; \text{Length}(G(r,\,\cdot\,))^2.$$

This concludes the proof. □

Lemma 2.3 (Equicontinuity)**.** *Let $\Gamma \subset \mathbb{R}^n$ be a Jordan curve of class C^1 and let $\gamma : \partial D \to \Gamma$ be a parametrization normalized by three points. Assume that $\mathscr{E}(F^\gamma) \leq \hat{\mathscr{E}} < \infty$. Then, for any $\varepsilon > 0$ there is $\delta = \delta(\hat{\mathscr{E}}) > 0$ such that*

$$|z - w| < \delta \implies |\gamma(z) - \gamma(w)| < \varepsilon$$

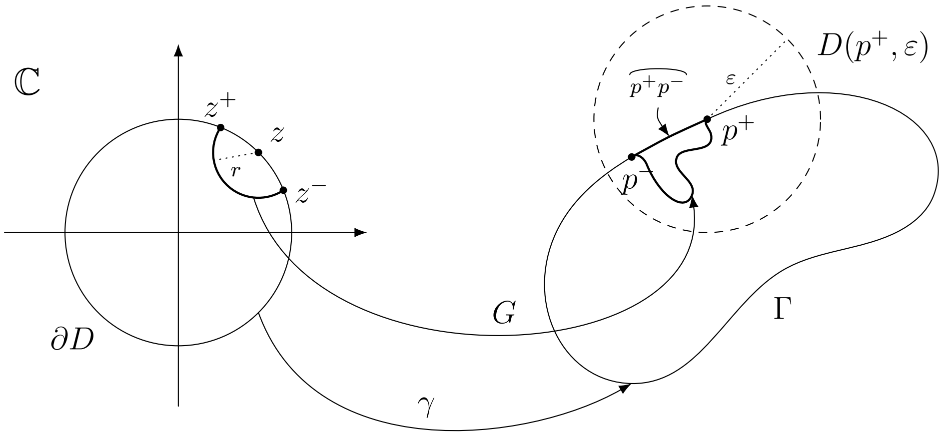
where $z, w \in \partial D$.

Proof. Let $p_1, p_2, p_3 \in \Gamma$ be the points of the normalization. We can choose $\varepsilon > 0$ such that $2\varepsilon < \min\{|p_1 - p_2|, |p_1 - p_3|, |p_2 - p_3|\}$.
Moreover, since Γ is a C^1-embedded curve in $\mathbb{R}^n$, there exists $\varepsilon_0 > 0$ such that $\forall p \in \Gamma$ the set $\Gamma \cap B(p, \varepsilon)$ is connected for all $0 < \varepsilon < \varepsilon_0$. Indeed, as a consequence of the Implicit Function Theorem, for each $p \in \Gamma$ there exists $\varepsilon_p > 0$ such that $\Gamma \cap B(p, \varepsilon_p)$ is the image of an interval under a C^1 embedding and, hence, it's connected. The balls $B(p, r_p/2)$ cover the compact set Γ, so finitely many suffice. Let $\varepsilon_0 = \min_i\{\varepsilon_{p_i}/2\} > 0$. Then for any $p \in \Gamma$ and any $0 < \varepsilon < \varepsilon_0$ we have $\Gamma \cap B(p, \varepsilon) \subset \Gamma \cap B(p_i, r_{p_i})$ for some i, and therefore $\Gamma \cap B(p, \varepsilon)$ is connected.

For $z \in \partial D$, let $\vartheta \mapsto G(r, \vartheta) = F(z + re^{i\vartheta})$, as previously defined. If we choose an appropriate $\delta \in (0, 1)$, by Lemma 2.2, there exists $r \in [\delta, \sqrt{\delta}]$ such that

$$\text{Length}(G(r,\,\cdot\,)) \leq \sqrt{\frac{4\pi\,\mathscr{E}(F)}{-\log(\delta)}} \leq \sqrt{\frac{4\pi\,\hat{\mathscr{E}}}{-\log(\delta)}} < \varepsilon.$$

Let $z^+, z^- \in \partial D$ be as in the picture below, and $p^+ = \gamma(z^+)$, $p^- = \gamma(z^-)$ be the extremal points of $G(r,\,\cdot\,)$. Let $\overset{\frown}{z^+z^-} = \partial D \cap B(z, r)$ and $\overset{\frown}{p^+p^-} \subset \Gamma$ be the arc from p^- to p^+ that is contained in $B(p^+, \varepsilon)$. We note that the definition is well-posed, because $\Gamma \cap B(p^+, \varepsilon)$ is connected and $p^- \in \Gamma \cap B(p^+, \varepsilon)$.
Now there are two cases: $\gamma(\overset{\frown}{z^+z^-}) = \overset{\frown}{p^+p^-}$ or $\gamma(\overset{\frown}{z^+z^-}) = \Gamma \setminus \overset{\frown}{p^+p^-}$. But $B(z, r)$ contains at most one of $z_k = e^{i\frac{2k\pi}{3}}$, $k = 1, 2, 3$, while $\Gamma \setminus \overset{\frown}{p^+p^-}$ contains at least two



of p_1, p_2, p_3. Thus, $\gamma(\widehat{z^+z^-}) = \widehat{p^+p^-}$.
Finally, let $w \in \partial D \cap B(z,\delta) \subset \partial D \cap B(z,r) = \widehat{z^+z^-}$. Then,

$$|\gamma(z) - \gamma(w)| \leq |\gamma(w) - p^+| + |p^+ - \gamma(w)| < 2\varepsilon,$$

because $\gamma(z), \gamma(w) \in \gamma(\widehat{z^+z^-}) \subset \widehat{p^+p^-} \subset B(p^+,\varepsilon)$. □

Going back to the minimizing problem, let $(\gamma^j)_{j\in\mathbb{N}}$ be a minimizing sequence for

$$\inf\{\mathscr{E}(F^\gamma) \mid \gamma : \partial D \to \Gamma \text{ homeomorphism}\}. \tag{4}$$

Then, we have $\mathscr{E}(F^{\gamma_j}) \leq \hat{\mathscr{E}} < \infty$ for all $j \in \mathbb{N}$. We can assume that each γ_j is normalized by the same three points. By Lemma 2.3 the sequence is equicontinuous. Finally, by Ascoli–Arzelà theorem, there exists a subsequence that converges uniformly.

Following from Section 2.1, we have proved the compactness of the set of functions $F \in C(\Gamma)$ such that $F\left(e^{i\frac{2k}{3}\pi}\right) = p_k,\ \ k = 1,2,3.$

2.3 F is weakly conformal

In the previous sections, we obtained a map $F : \overline{D} \to \mathbb{R}^n$ such that:

1) $F \in C^\infty(D,\mathbb{R}^n)$ and $\Delta F = 0$;

2) $F \in C(\overline{D},\mathbb{R}^n)$ and $F : (\partial D) \to \Gamma$ is a homeomorphism;

3) For any diffeomorphism $g : \overline{D} \to \overline{D}$ we have

$$\mathscr{E}(F) \leq \mathscr{E}(F \circ g) \tag{5}$$

Using this final observation, we will now prove that F is weakly conformal.

Lemma 2.4. *Let $F \in C^\infty(D;\mathbb{R}^n)$ be harmonic. Then the map $\Phi : D \to \mathbb{C}$*

$$\Phi(z) = |F_u(z)|^2 - |F_v(z)|^2 - 2i\langle F_u, F_v\rangle$$

is holomorphic.

Proof. Use the standard notation

$$\partial_z = \frac{1}{2}\left(\partial_u - i\partial_v\right),$$

$$\partial_{\bar z} = \frac{1}{2}\left(\partial_u + i\partial_v\right),$$

and recall that

$$\partial_z\partial_{\bar z} = \partial_u^2 + \partial_v^2 = \Delta.$$

Note that

$$\begin{aligned}(F_z)^2 :&= \langle F_z, F_z\rangle \\ &= \frac{1}{4}\langle F_u - iF_v, F_u - iF_v\rangle \\ &= \frac{1}{4}\left(|F_u|^2 - |F_v|^2 - 2i\langle F_u, F_v\rangle\right) \\ &= \frac{1}{4}\Phi\end{aligned}$$

Hence,

$$\Phi_{\bar z} = 4\,\partial_{\bar z}(F_z)^2 = 8\langle F_z, F_{z\bar z}\rangle = 2\langle F_z, \Delta F\rangle = 0.$$

□

The map F will be weakly conformal precisely when $\Phi = 0$.

Let $\tau : \overline{D} \to \mathbb{C}$ be a C^1 vector field such that $\tau(z) \in T_z\partial D$ for all $z \in \partial D$. Given $\varepsilon \in \mathbb{R}$, denote its flow by $\varphi(z,\varepsilon)$. Namely,

$$\begin{cases}\varphi(z,0) = z \\ \dfrac{\partial\varphi}{\partial\varepsilon}(z,\varepsilon) = \tau\left(\varphi(z,\varepsilon)\right)\end{cases}$$

We write $\varphi(z,\varepsilon) = \varphi_\varepsilon(z)$. Then, for any ε we have $\varphi_\varepsilon : \overline{D} \to \overline{D}$ and $\varphi_\varepsilon : \partial D \to \partial D$.

Lemma 2.5. *We have* $\dfrac{d\mathscr{E}(F\circ\varphi_\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = \displaystyle\int_D \Re(\Phi\cdot\tau_{\bar z})\ dudv.$

Proof. Consider $\mathbb{R}^2 \cong \mathbb{C}$, and write $\tau = (\tau_1, \tau_2)$. Recall $(\varphi_\varepsilon)^{-1} = \varphi_{-\varepsilon}$, so $\{\varphi_\varepsilon\}_\varepsilon$ is a family of C^1-diffeomorphisms of the unit disk D. At the first order we have

$$\varphi_\varepsilon = \mathrm{id} + \varepsilon\tau + O(\varepsilon^2).$$

Now, compute

$$\begin{aligned}
\mathscr{E}(F \circ \varphi_\varepsilon) &= \frac{1}{2}\int_D |\nabla(F \circ \varphi_\varepsilon)|^2 \;\; dxdy \\
&= \frac{1}{2}\int_D |(\nabla F \circ \varphi_\varepsilon) \cdot J_{\varphi_\varepsilon}|^2 \;\; dxdy \\
&= \frac{1}{2}\int_D \left|\nabla F \cdot (J_{\varphi_\varepsilon} \circ \varphi_\varepsilon^{-1})\right|^2 \left|\det(J_{\varphi_\varepsilon^{-1}})\right| \;\; dudv
\end{aligned}$$

where the last line is obtained by the change of variable $(u, v) = \varphi_\varepsilon(x, y)$.
By differentiating $\varphi_\varepsilon \circ \varphi_\varepsilon^{-1} = \mathrm{id}$, we obtain $(J_{\varphi_\varepsilon} \circ \varphi_\varepsilon^{-1}) \cdot J_{\varphi_\varepsilon^{-1}} = \mathbf{1}$, hence

$$\begin{aligned}
J_{\varphi_\varepsilon} \circ \varphi_\varepsilon^{-1} &= (J_{\varphi_\varepsilon^{-1}})^{-1} \\
&= \mathbf{1} + \varepsilon J_\tau + O(\varepsilon^2) \\
&= \mathbf{1} + \varepsilon \begin{pmatrix} \tau_u^1 & \tau_v^1 \\ \tau_u^2 & \tau_v^2 \end{pmatrix} + O(\varepsilon^2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\det(J_{\varphi_\varepsilon^{-1}}) = \det(J_{\varphi_{-\varepsilon}}) &= \det(\mathbf{1} - \varepsilon J_\tau + O(\varepsilon^2)) \\
&= 1 - \varepsilon\, \mathrm{tr}(J_\tau) + O(\varepsilon^2) \\
&= 1 - \varepsilon(\tau_u^1 + \tau_v^2) + O(\varepsilon^2).
\end{aligned}$$

Putting this together, we obtain

$$\begin{aligned}
\mathscr{E}(F \circ \varphi_\varepsilon) = \frac{1}{2}\int_D \big\{&|\nabla F|^2 + 2\varepsilon\left(|F_u|^2\tau_u^1 + |F_v|^2\tau_v^2 + \langle F_u, F_v\rangle(\tau_u^2 + \tau_v^1)\right) + \\
&- \varepsilon|\nabla F|^2(\tau_u^1 + \tau_v^2)\big\} \;\; dudv + O(\varepsilon^2)
\end{aligned}$$

Thus, we can differentiate at $\varepsilon = 0$ and get

$$\frac{d}{d\varepsilon}\mathscr{E}(F \circ \varphi_\varepsilon)\Big|_{\varepsilon=0} = \frac{1}{2}\int_D \left\{\left(|F_u|^2 - |F_v|^2\right)\left(\tau_u^1 - \tau_v^2\right) + 2\langle F_u, F_v\rangle\left(\tau_u^2 + \tau_v^1\right)\right\} \;\; dudv.$$

If we now reconsider $\mathbb{R}^2 \cong \mathbb{C}$ by letting $z = u + iv$, $\tau = \tau^1 + i\tau^2$, we can see that the integrand

$$\begin{aligned}
&\left(|F_u|^2 - |F_v|^2\right)\left(\tau_u^1 - \tau_v^2\right) + 2\langle F_u, F_v\rangle\left(\tau_u^2 + \tau_v^1\right) = \\
&= \Re\left\{2\big(\big(\tau_u^1 - \tau_v^2\big) + i\big(\tau_u^2 + \tau_v^1\big)\big)\big(|F_u|^2 - |F_v|^2 - 2i\langle F_u, F_v\rangle\big)\right\} \\
&= 2\;\Re\big(\tau_{\bar{z}} \cdot \Phi\big),
\end{aligned}$$

which concludes the proof. $\square$

If F has the minimizing property 5, then this lemma implies that

$$0 = \int_D \Re(\Phi \cdot \tau_{\bar{z}}) \; dudv.$$

Using $\Phi_{\bar{z}} = 0$, note that

$$\Phi \cdot \tau_{\bar{z}} = (\Phi \cdot \tau)_{\bar{z}}\,.$$

Since we don't know if $F \in C^1(\overline{D}, \mathbb{R}^n)$, Φ is not defined on ∂D, so we cannot apply the Divergence Theorem in the equation above. Noting that $\Phi \cdot \tau_{\bar{z}} \in L^1(D)$ and applying the Dominated Convergence theorem, we proceed as follows

$$\begin{aligned}
0 &= \Re\left(\int_D \Phi \cdot \tau_{\bar{z}} \; dudv\right) \\
&= \lim_{r \uparrow 1} \Re\left(\int_{D_r} \Phi \cdot \tau_{\bar{z}} \; dudv\right) \\
&= \Re\left(\lim_{r \uparrow 1} \int_{D_r} (\Phi \cdot \tau)_{\bar{z}} \; dudv\right) \\
&= \Re\left(\lim_{r \uparrow 1} \int_{\partial D_r} \frac{z}{|z|} \cdot \Phi(z) \cdot \tau(z) \; dH^1(z)\right),
\end{aligned}$$

where the last equality follows from the Divergence theorem.

Now we make a smart choice for τ. Recall the representation kernel for harmonic functions in the disk. If $h : D \to \mathbb{R}$ is harmonic and $|w| < r < 1$, then

$$\begin{aligned}
h(w) &= \frac{1}{2\pi r} \int_{\partial D_r} \frac{|z|^2 - |w|^2}{|z - w|^2}\, h(z)\, dH^1(z) \\
&= \int_{\partial D_r} K(w, z)\, h(z)\, dH^1(z),
\end{aligned}$$

with $K(w, z) = \dfrac{1}{2\pi|z|} \cdot \dfrac{|z|^2 - |w|^2}{|z - w|^2}$. Since the singularity at w can be cut off, we can choose

$$\tau(z) = K(w, z) \cdot (iz),$$

where we note that $K(w, z)$ is real, hence $\tau(z)$ is tangent do ∂D. We get

$$\begin{aligned}
0 &= \Re\left(\lim_{r \uparrow 1} \int_{\partial D_r} iz^2 \cdot \Phi(z) \cdot K(w, z) \; dH^1\right) \\
&= -\lim_{r \uparrow 1} \int_{\partial D_r} \Im(z^2 \Phi(z)) K(w, z) \; dH^1 \\
&= -\Im(w^2 \Phi(w)).
\end{aligned}$$

Hence $\Im(z^2\Phi(z)) = 0$ on D. Since $z^2\Phi(z)$ is holomorphic on D, this implies $z^2\Phi(z)$ is constant on D. In particular, $z^2\Phi(z) = 0$. We can conclude that $\Phi(z) = 0$ on D.

This proves that F is weakly conformal.

2.4 Conclusion

We are now ready to conclude the proof. Let $F \in C^\infty(D, \mathbb{R}^n) \cap C(\overline{D}, \mathbb{R}^n)$ be a solution to the minimizing problem for the energy. Since F is weakly conformal, we have

$$\mathscr{A}(F) = \mathscr{E}(F) = \inf_{F \in \mathscr{C}(F)} \mathscr{E}(F) = \inf_{F \in \mathscr{C}(F)} \mathscr{A}(F),$$

where the last equality is obtained by Morrey's result. Furthermore, recall that we have proved $F \in \mathscr{C}(\Gamma)$. Finally, we can rewrite the singular set as

$$\operatorname{Sing}(F) = \{z \in D : F_z = 0\}.$$

Noting that

$$\Delta F = 0 \iff \frac{\partial}{\partial \bar{z}} \frac{\partial F}{\partial z} = 0 \iff F_z \text{ is holomorphic}$$

and recalling that zeros of holomorphic functions are isolated, we can conclude that $\operatorname{Sing}(F)$ consists of isolated points.

Chapter 3

Final Considerations

The Douglas–Radó theorem solves Plateau's problem in a classical parametric setting. Given a Jordan curve $\Gamma \subset \mathbb{R}^n$ parametrized by a homeomorphism γ, we considered the class of sufficiently regular maps $F : D \to \mathbb{R}^n$ that agree with γ on the boundary. We finally proved that a minimizer of the area functional exists and that it is a harmonic and conformal immersion of the disk in $\mathbb{R}^n$.

However, this solution has limitations when compared to real physical soap films. We can observe that the Douglas–Radó problem only produces surfaces that are homeomorphic to a disk. In reality, soap films can form non-orientable surfaces: for instance, a soap film bounded by a looped wire can have the topology of a Möbius strip.
Another issue, involving self-intersections, is well illustrated by the "disk with a tongue" example in [1]. Consider a wire frame Γ that is essentially a round horizontal circle where a portion of the boundary folds inwards and crosses the interior to form a thin "tongue". The minimizing surface $F(D)$, obtained via parametrization, self-intersects: the "tongue" passes through the main disk and the two sheets simply cross each other, without any interaction, because the area functional does not see that the two pieces get close to each other. However, real soap films behave differently. Indeed, intersecting pieces tend to rearrange and merge, forming singular sets where three sheets meet (a Y-singularity). The parametric model does not naturally describe these types of singularities found in experiments.

These issues suggest looking at alternative formulations of Plateau's problem. Reifenberg's approach [6] uses closed sets, Hausdorff measure for area, and a *homological* boundary definition, which allows for arbitrary topology, including non-orientable films. The theory of *integral currents*, a geometric measure theoretic generalization of surfaces, is also a useful tool to study the existence and regularity of solutions to Plateau's problem, see [3], but excludes non-orientable solutions. Finally, recent developments involving *Almgren's minimal sets* with "sliding boundary conditions" allow the boundary points to "slide" along the wire. This seems to better model the physical behavior of soap films, including the formation of singularities, but many questions regarding existence and regularity remain open; see [1].

Bibliography

[1] Guy David, *Should we solve plateau's problem again?*, Advances in Analysis: The Legacy of Elias M. Stein (Charles Fefferman, Alexandru D. Ionescu, D. H. Phong, and Stephen Wainger, eds.), Princeton University Press, 01 2014.

[2] Jesse Douglas, *Solution of the problem of plateau*, Transactions of the American Mathematical Society **33** (1931), no. 1, 263–321.

[3] Herbert Federer and Wendell H. Fleming, *Normal and integral currents*, Annals of Mathematics **72** (1960), no. 3, 458–520.

[4] Roberto Monti, *Notes on the calculus of variations*, Lecture notes for the course of Calculus of Variations, Università degli Studi di Padova, second semester 2023.

[5] Tibor Radó, *On plateau's problem*, Annals of Mathematics **31** (1930), no. 3, 457–469.

[6] E. R. Reifenberg, *Solution of the plateau problem for m-dimensional surfaces of varying topological type*, Acta Mathematica **104** (1960), no. 1–2, 1–92.

[7] Michael Struwe, *Plateau's problem and the calculus of variations. (mn-35):*, Princeton University Press, 1989.