Università degli Studi di Padova

Department of Mathematics

Bachelor Thesis in Mathematics

An application of homogenization theory to a 2-dimensional periodic checkerboard

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Alla mia famiglia, per avermi sempre sostenuto,
och till min familj i Sverige, för att ni har fått mig att känna mig som er son och bror.

Abstract

Homogenization theory studies the relationship between microscopic and macroscopic scales in continuum mechanics and physics. This thesis focuses on partial differential equations with rapidly oscillating coefficients, which model composite materials with fine periodic structures.

We introduce the mathematical framework necessary for homogenization: weak convergence in Banach spaces, distributional convergence, and Sobolev spaces. These tools allow us to rigorously define weak solutions and study the limiting behavior of sequences of solutions as the microstructural scale parameter ε tends to zero.

The central result of this work is the explicit computation of the homogenized matrix for a two-dimensional periodic checkerboard structure composed of two phases with conductivities α and β. Using symmetry arguments and compactness methods, we prove that the homogenized matrix is $A^* = \sqrt{\alpha\beta}\,\mathrm{Id}$.

This example illustrates how homogenization theory "divides and conquers": the difficult resolution of the original oscillatory problem is replaced by computing the homogenized coefficients and solving a simpler limit equation.

Contents

List of Symbols

$\mathbb{R}$ Real numbers

$\mathbb{R}^n$ n-dimensional Euclidean space

$\mathbb{Z}$ Integers

Ω Open domain in $\mathbb{R}^n$

D Bounded open domain (spatial region)

Q Unit cell $Q =]0, \ell_1[\times \cdots \times]0, \ell_N[$, typically $Q =]0, 1[^2$

$\partial D, \partial\Omega$ Boundary of domain D or Ω

ε Small parameter representing the microstructural scale

Operators and Derivatives

∇u Gradient of u

$\operatorname{div} F$ Divergence of vector field F

$\operatorname{curl} F$ Scalar curl in 2D: $\partial_1 F_2 - \partial_2 F_1$

$\frac{\partial u}{\partial x_i}$ Weak partial derivative with respect to x_i

Convergence and Limits

$\rightharpoonup$ Weak convergence

$\overset{*}{\rightharpoonup}$ Weak-* convergence

$\to$ Strong convergence

1

1 - The Periodic Checkerboard

1.1 Introduction to the periodic checkerboard and the homogenized matrix

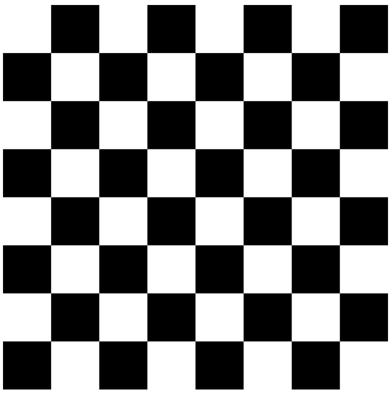

Figure 1.1: Periodic checkerboard structure

The following theorem will be the focus of our thesis:

Theorem. Given $Q = (0,1)^2$, let $a_{\text{per}}(x_1, x_2)$ be a function that is Q-periodic, piecewise constant, alternately taking the values $\alpha > 0$ and $\beta > 0$:

$$a_{\text{per}}(x_1, x_2) = \begin{cases} \alpha & \text{if } (x_1, x_2) \in \left]0, \frac{1}{2}\right[^2 \cup \left]\frac{1}{2}, 1\right[^2, \\ \beta & \text{otherwise.} \end{cases}$$

Let $u_\varepsilon \in H_0^1(D)$ be the solution to

$$\begin{cases} -\operatorname{div}\big(A_\varepsilon(x)\nabla u_\varepsilon(x)\big) = f(x) & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \partial D. \end{cases}$$

We show that, as $\varepsilon \to 0$, $u_\varepsilon \rightharpoonup u^*$ weakly in $H_0^1(D)$, where u^* is the solution to the homogenized problem

$$-\operatorname{div}(A^* \nabla u^*) = f \quad \text{in } D,$$

and the homogenized matrix is given by

$$A^* = \sqrt{\alpha\beta}\,\mathrm{Id}.$$

The main focus will be the calculation of the matrix A^* which describes the macroscopic properties of the composite material modeled by a_{per}. The parameter ε is needed to model the two scales of the problem, the microscopic and the macroscopic which embed different properties of the material. The main theorems and prepositions of this thesis come from *Homogenization Theory for Multiscale Problems: An Introduction* by Xavier Blanc and Claude Le Bris [3], the preliminary mathematical framework is taken from the work of Lawrence C. Evans and Haim Brezis [1, 2], while the textbook by Tartar and the one by Doina Cioranescu and Patrizia Donato have been consulted for their thorough treatment of homogenization theory [4, 5].

2

2 - Preliminaries

2.1 The diffusion equation, a physical approach

We will build a diffusion equation from its physical interpretation, to get some intuition on the problem. We will describe how a quantity (for example, heat) flows through a material. The microscopic framework will be presented later.

- Let $u(x) \in C^2(D)$ describe the density of some quantity in equilibrium, $x \in D \subset \mathbb{R}^n$.
- The gradient is the direction of local steepest increase:

$$\nabla u(x) \in \mathbb{R}^n.$$

- Since different directions may conduct differently, we have the matrix $A(x) \in M_n(\mathbb{R})^n$, with the coefficients $a^{ij} \in L^\infty(D)$ to model conductivity through the region:

$$A(x)\nabla u(x).$$

- In many instances it is physically reasonable to assume the flux F of our quantity is "proportional" to the gradient but points in the opposite direction (since the flow is from regions of higher to lower concentration):

$$F(x) = -A(x)\nabla u(x).$$

- The divergence of a vector field describes net outflow from a point. In a bounded region, if a quantity (the flux) moves outside, it decreases inside, and we equate that change to a source function $f \in L^2(D)$, $f(x) : D \to \mathbb{R}$ if present:

$$\mathrm{div}(F) = f(x).$$

So we obtain this partial differential equation, which will be the central focus of this thesis:

$$-\,\mathrm{div}(A(x)\nabla u(x)) = f(x).$$

We will need to add the Dirichlet's boundary condition $u = 0$ on ∂D.

2.2 Convergence: Weak, Strong, and in the Sense of Distributions

Quoting L. Tartar: "using weak convergence for relating different levels was not a new idea, and it was used implicitly each time a discrete distribution of masses (...) was replaced by a density of mass".

Instead of comparing functions pointwise, we compare their behavior against test functions, so in the sense of averages, and this makes the discrete and continuous indistinguishable. Let us now formally introduce this idea and the mathematical framework around it.

2.2.1 Definitions

Let X be a real Banach space.

(I) A bounded linear operator $u^* : X \to \mathbb{R}$ is called a bounded linear functional on X.

(II) We write X^* to denote the collection of all bounded linear functionals on X; X^* is the dual space of X.

(III) If $u \in X$, $u^* \in X^*$, we write

$$\langle u^*, u \rangle$$

to denote the real number $u^*(u)$. The symbol $\langle \cdot, \cdot \rangle$ denotes the pairing of X^* and X.

(IV) A Banach space is reflexive if $(X^*)^* = X$. Namely, this means that for each $u^{**} \in$

$(X^*)^*$, there exists $u \in X$ such that

$$\langle u^{**}, u^* \rangle = \langle u^*, u \rangle \qquad \text{for all } u^* \in X^*.$$

2.2.2 Weak Convergence

We say a sequence $\{u_k\}_{k=1}^{\infty} \subset X$ converges weakly to $u \in X$, written

$$u_k \rightharpoonup u,$$

if

$$\langle u^*, u_k \rangle \to \langle u^*, u \rangle$$

for each bounded linear functional $u^* \in X^*$.

If $u_k \to u$, then $u_k \rightharpoonup u$. Also, any weakly convergent sequence is bounded because of the uniform boundedness principle [2, p.32].

2.2.3 L^p Spaces

Let $(\Omega, \mathcal{M}, \mu)$ denote a measure space.
Let $p \in \mathbb{R}$ with $1 < p < \infty$; we set

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f \text{ is measurable and } |f|^p \in L^1(\Omega) \right\},$$

with the norm

$$\|f\|_{L^p} = \|f\|_p = \left(\int_\Omega |f(x)|^p \, d\mu \right)^{1/p}.$$

We also set

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable and there is a constant } C \\ \text{such that } |f(x)| \le C \text{ a.e. on } \Omega \end{array} \right\},$$

with the norm

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ C \mid |f(x)| \le C \text{ a.e. on } \Omega \}.$$

We will most often employ weak convergence ideas in the following context.
Take $U \subset \mathbb{R}^n = \Omega$ to be open, and assume $1 \leq p < \infty$. Then

$$\text{the dual space of } X = L^p(U) \text{ is } X^* = L^q(U),$$

where

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad 1 < q \leq \infty.$$

Namely, each bounded linear functional on $L^p(U)$ can be represented as

$$f \mapsto \int_U gf\,dx$$

for some $g \in L^q(U)$. Therefore

$$f_k \rightharpoonup f \qquad \text{weakly in } L^p(U)$$

means

$$\int_U gf_k\,dx \to \int_U gf\,dx \qquad \text{as } k \to \infty, \quad \text{for all } g \in L^q(U).$$

The convergence above does not imply that $f_k \to f$ pointwise or almost everywhere. It may very well be, for example, that the functions $\{f_k\}_{k=1}^{\infty}$ oscillate more and more rapidly as $k \to \infty$.

2.2.4 Convergence in the Sense of Distributions

Let $\Omega \subset \mathbb{R}^n$ be open and let $(f_k) \subset L^2(\Omega)$.
We say that f_k converges to f in the sense of distributions in Ω if

$$\lim_{k\to\infty} \int_\Omega f_k(x)\,\varphi(x)\,dx = \int_\Omega f(x)\,\varphi(x)\,dx \qquad \text{for every } \varphi \in C_c^\infty(\Omega).$$

Since $C_c^\infty(\Omega) \subset L^2(\Omega)$, we notice that weak convergence implies distributional convergence, but not the opposite.

2.2.5 Weak-* Convergence

We will also need the weak-* convergence in a very specific case.

Definition. Let $\{f_k\}$ be a bounded sequence in $L^\infty(\Omega)$. We say that

$$f_k \underset{*}{\rightharpoonup} f \quad \text{weakly-* in } L^\infty(\Omega)$$

if

$$\int_\Omega f_k \, \varphi \, dx \;\to\; \int_\Omega f \, \varphi \, dx, \qquad \text{for every } \varphi \in L^1(\Omega).$$

2.3 From Weak Convergence to Sobolev Spaces

We have introduced weak convergence and distributional convergence; however, a fundamental aspect remains undefined.

Recall our PDE from the introduction:

$$-\operatorname{div}(A(x)\nabla u(x)) = f(x).$$

This equation involves derivatives of the solution u. In the classical sense, this requires u to be twice differentiable. However, when dealing with:

- composite materials with discontinuous coefficients (like the checkerboard structure that will be defined ahead in our example),
- rapidly oscillating coefficients $A_\varepsilon(x)$,
- limits of sequences of solutions u^ε,

the weak limits of such sequences may not preserve classical derivatives.

2.3.1 Weak Derivatives

The idea behind weak derivatives is to transfer the derivative onto smooth test functions via integration by parts. This is exactly the technique used to derive the variational formulation of our PDE.

Definition (Weak Derivative). Let $\Omega \subset \mathbb{R}^N$ be open and let $u \in L^1_{\text{loc}}(\Omega)$. We say that $g_i \in L^1_{\text{loc}}(\Omega)$ is the weak partial derivative of u with respect to x_i, written $\frac{\partial u}{\partial x_i} = g_i$, if

$$\int_\Omega u(x)\,\frac{\partial \varphi}{\partial x_i}(x)\,dx = -\int_\Omega g_i(x)\,\varphi(x)\,dx \qquad \text{for all } \varphi \in C_c^\infty(\Omega).$$

Remarks:

- If u is classically differentiable, its classical derivative coincides with its weak derivative.
- The weak derivative, if it exists, is unique (up to sets of measure zero).
- A function may have a weak derivative even when it is not classically differentiable.

The definition above is precisely what appears in the definition of Sobolev spaces.

2.4 Sobolev Spaces

We do not need to delve deep into the properties of Sobolev spaces; it is sufficient to know that they are the natural setting that arises from the study of partial differential equations, such as the one presented earlier. Because of the definition of weak solutions and weak derivatives, they control simultaneously the function itself (in L^p) and its weak derivatives (also in L^p).

This thesis is not focused on finding the weak solution but rather on a different but somehow equivalent problem at a macroscopic level. A weak solution is obtained by writing the problem in its variational form, which is obtained by taking the differential equation, multiplying it by a test function, and integrating it by parts. The meaning of a weak solution is to satisfy the equation in an average (integral) sense.

The spaces where weak solutions exist are called Sobolev Spaces. Using the notation $\frac{\partial u}{\partial x_i}$ to denote the weak derivative defined above, we can now give the formal definition:

Definition. The Sobolev space $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, is defined as

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \frac{\partial u}{\partial x_i} \in L^p(\Omega) \text{ for } i = 1, 2, \dots, n \right\}.$$

Equivalently, expanding the definition of weak derivative:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \exists\, g_1, g_2, \dots, g_n \in L^p(\Omega) \text{ such that} \\ \displaystyle\int_\Omega u \,\frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i\, \varphi \quad \forall \varphi \in C_c^\infty(\Omega), \quad i = 1, 2, \dots, n \end{array} \right\}.$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega).$$

The space $H^1(\Omega)$ is equipped with the scalar product

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2} = \int_\Omega \left(uv + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) dx.$$

The associated norm is

$$\|u\|_{H^1} = \left(\|u\|_{L^2}^2 + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2}^2 \right)^{1/2}.$$

Definition. Let $1 \leq p < \infty$. $W_0^{1,p}(\Omega)$ denotes the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$. We set

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

H_0^1, equipped with the H^1 scalar product, is a Hilbert space. We denote by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$.

The space $H_0^1(\Omega)$ can be also definied as the set of functions in $H^1(\Omega)$ that vanish on the boundary. Although functions in Sobolev spaces are not necessarily continuous, the trace theorem allows us to define their boundary values[1, pp. 272-275, p. 315]. In this sense, every function in $H_0^1(\Omega)$ has boundary value zero, so they meet Dirichlet boundary conditions.

3

3 - Homogenization of a 2-Periodic Checkerboard

3.1 Q-Periodic

Definition. Let $Q = \,]0, \ell_1[\, \times \cdots \times \,]0, \ell_n[$ where $\ell_1, \ldots, \ell_n$ are given positive numbers and a a function defined a.e. on $\mathbb{R}^n$. The function a is called Q-periodic iff

$$a(x + k\ell_i e_i) = a(x) \quad \text{a.e. on } \mathbb{R}^n, \qquad \forall k \in \mathbb{Z}, \quad \forall i \in \{1, \ldots, n\},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Example. Let us now define $a_{\text{per}}(x_1, x_2)$ as a function that is Q-periodic, $Q = \,]0, 1[^2$, piecewise constant, alternately taking the values $\alpha > 0$ and $\beta > 0$:

$$a_{\text{per}}(x_1, x_2) = \begin{cases} \alpha & \text{if } (x_1, x_2) \in \,]0, \frac{1}{2}[^2 \cup \,]\frac{1}{2}, 1[^2, \\ \beta & \text{otherwise.} \end{cases}$$

3.2 Homogenization Problem

In our specific case we are going to be working in $\mathbb{R}^2$: $u^\varepsilon(x) = u^\varepsilon(x_1, x_2)$ is a scalar-valued function and $a_{\text{per}}(x) = a_{\text{per}}(x_1, x_2)$ is a periodic function, hence

$$A_\varepsilon(x) = A_{\text{per}}\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) = a_{\text{per}}\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \text{ Id.}$$

The matrix A_ε describes a composite material composed of two constituents, with respective coefficients α and β. Each block of constituent is called a phase, and has length and width $\varepsilon/2$. This material structure is referred to as "Periodic Checkerboard".

Let us point out that if a is Q-periodic, then a_ε is εQ-periodic. Moreover, as can be seen in the example above, the smaller ε is, the more rapid are the oscillations.

The coefficient $a_\varepsilon(x) = a_{\text{per}}(x/\varepsilon)$ should be interpreted as a rescaling of the fixed coefficient $a_{\text{per}}(x)$.

The guiding principle of homogenization theory is to embed the specific equation for a given small scale ε into a family of equations for a sequence of parameters ε vanishing in the limit. The hope is that considering the limit "$\varepsilon = 0$" yields a problem simpler to understand than the original problem at $\varepsilon \neq 0$ fixed. In the limit problem, the small scale would disappear as $a_{\text{per}} \to a^*$ (and hence $A_\varepsilon \to A^*$), where the whole difficulty hides behind the practical identification of A^*, called the homogenized matrix which, in our case, will have some "average" properties of the starting material.

For the original equation, the limit equation, as we shall see, takes the form

$$-\operatorname{div}(A^*(x)\nabla u^*(x)) = f(x).$$

To some extent, homogenization theory "divides and conquers". The difficult resolution of the original equation is traded for a supposedly simpler calculation of A^* and the resolution, also expected simpler, of the limit equation. This thesis is focused on the calculation of A^* in the case of the periodic checkerboard.

For $\varepsilon > 0$ the problem then reads as

$$\begin{cases} -\operatorname{div}(A_\varepsilon(x)\nabla u^\varepsilon(x)) = f(x) & \text{in } D \subset \mathbb{R}^d, \\ u^\varepsilon = 0 & \text{on } \partial D. \end{cases} \tag{1}$$

The matrix A_{per} has already been defined previously.

Let u^ε be the solution in H_0^1 of (1), which now becomes:

$$-\operatorname{div}\left(\begin{pmatrix} a_{\text{per}}\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) & 0 \\ 0 & a_{\text{per}}\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \end{pmatrix} \nabla u^\varepsilon(x_1, x_2)\right) = f. \tag{2}$$

This equation is posed with homogeneous Dirichlet boundary conditions. The homogenized problem is obtained by letting ε tend to 0.

3.2.1 Theoretical Instruments

To formally tackle the limit passage from the microscopic scale to the macroscopic average scale, it is no surprise that we have to calculate an average in some integral sense:

Proposition 1. Let b be a function in $L^\infty(\mathbb{R})$. Assume that it is periodic of period 1. Then the sequence of functions $b\left(\frac{\cdot}{\varepsilon}\right)$ converges weakly-* in L^∞ to the constant function, denoted by $\langle b \rangle$, called the average of b, and equal to

$$\langle b \rangle = \int_0^1 b(x)\, dx.$$

Proof. We want to show that, for any $v \in L^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} b\left(\frac{x}{\varepsilon}\right) v(x)\, dx \xrightarrow{\varepsilon \to 0} \left(\int_0^1 b(y)\, dy\right) \left(\int_{\mathbb{R}} v(x)\, dx\right).$$

We prove this for v being the characteristic function of an interval.

The density of piecewise constant functions in $L^1(\mathbb{R})$ then allows to conclude.

We need to prove that, for any $\alpha < \beta$,

$$\int_\alpha^\beta b\left(\frac{x}{\varepsilon}\right) dx \xrightarrow{\varepsilon \to 0} (\beta - \alpha) \int_0^1 b(y)\, dy.$$

We use the fact that b is periodic, and write

$$\int_\alpha^\beta b\Big(\frac{x}{\varepsilon}\Big)\, dx = \varepsilon \int_{\alpha/\varepsilon}^{\beta/\varepsilon} b(y)\, dy.$$

Let

$$m = \Big\lfloor \frac{\alpha}{\varepsilon} \Big\rfloor, \qquad n = \Big\lfloor \frac{\beta}{\varepsilon} \Big\rfloor.$$

We decompose the interval $[\alpha/\varepsilon, \beta/\varepsilon]$ as

$$[\alpha/\varepsilon, \beta/\varepsilon] = [\alpha/\varepsilon, m+1] \,\cup\, \bigcup_{k=m+1}^{n-1} [k, k+1] \,\cup\, [n, \beta/\varepsilon].$$

Therefore,

$$\int_{\alpha/\varepsilon}^{\beta/\varepsilon} b(y)\, dy = \int_{\alpha/\varepsilon}^{m+1} b(y)\, dy + \sum_{k=m+1}^{n-1} \int_k^{k+1} b(y)\, dy + \int_n^{\beta/\varepsilon} b(y)\, dy.$$

Since b is 1-periodic, we have

$$\int_k^{k+1} b(y)\, dy = \int_0^1 b(y)\, dy =: \langle b \rangle,$$

and thus

$$\sum_{k=m+1}^{n-1} \int_k^{k+1} b(y)\, dy = (n-m-1)\langle b \rangle.$$

Hence,

$$\int_\alpha^\beta b\Big(\frac{x}{\varepsilon}\Big)\, dx = \varepsilon(n-m-1)\langle b \rangle + \varepsilon \int_{\alpha/\varepsilon}^{m+1} b(y)\, dy + \varepsilon \int_n^{\beta/\varepsilon} b(y)\, dy.$$

We need to work on the last two terms. Since b is bounded, there exists $C > 0$ such that $|b(y)| \leq C$ for all y. Recalling how we defined m and n:

$$0 \leq (m+1) - \frac{\alpha}{\varepsilon} \leq 1, \qquad 0 \leq \frac{\beta}{\varepsilon} - n \leq 1,$$

so

$$\left|\int_{\alpha/\varepsilon}^{m+1} b(y)\,dy\right| \le C, \qquad \left|\int_{n}^{\beta/\varepsilon} b(y)\,dy\right| \le C.$$

Multiplying by ε, we obtain

$$\varepsilon\int_{\alpha/\varepsilon}^{m+1} b(y)\,dy = O(\varepsilon), \qquad \varepsilon\int_{n}^{\beta/\varepsilon} b(y)\,dy = O(\varepsilon).$$

It remains to show the relations of $\varepsilon(n-m-1)$ with $(\beta-\alpha)$.
To compare $\varepsilon(n-m-1)$ with $(\beta-\alpha)$, we use the identity $x = \lfloor x \rfloor + \{x\}$, where $0 \le \{x\} < 1$. Then

$$\frac{\beta}{\varepsilon} = n + \left\{\frac{\beta}{\varepsilon}\right\}, \qquad \frac{\alpha}{\varepsilon} = m + \left\{\frac{\alpha}{\varepsilon}\right\}.$$

Subtracting the two,

$$\frac{\beta-\alpha}{\varepsilon} = (n-m) + \left\{\frac{\beta}{\varepsilon}\right\} - \left\{\frac{\alpha}{\varepsilon}\right\}.$$

Hence,

$$\varepsilon(n-m-1) = (\beta-\alpha) - \varepsilon\left(1 + \left\{\frac{\beta}{\varepsilon}\right\} - \left\{\frac{\alpha}{\varepsilon}\right\}\right).$$

The fractional parts are bounded in $[0,1)$, the term in parentheses is bounded, so we can write

$$\varepsilon(n-m-1) = (\beta-\alpha) + O(\varepsilon).$$

Putting everything together, we conclude

$$\int_{\alpha}^{\beta} b\left(\frac{x}{\varepsilon}\right)\,dx = (\beta-\alpha)\langle b\rangle + 3O(\varepsilon).$$

□

3.3 Ellipticity Bound

3.3.1 Lax-Milgram Theorem

We need the Lax-Milgram Theorem to show existence and uniqueness of our solution [3, p.101, pp.427-430].

Theorem (Lax-Milgram) Let H be a Hilbert space, and let $B : H \times H \to \mathbb{R}$ be a continuous bilinear form. Assume that B is coercive, that is, it satisfies

$$B(u,u) \geq \mu \|u\|_H^2, \quad \forall u \in H, \quad \text{for some } \mu > 0$$

Consider $L : H \to \mathbb{R}$ a linear continuous map. Then there exists a unique $u \in H$ satisfying

$$B(u,v) = L(v), \forall v \in H.$$

Moreover, this solution satisfies

$$\|u\|_H \leq \frac{1}{\mu}\|L\|_{H'}.$$

We use Lax-Milgram theorem in the following framework. We recall that $H = H_0^1(D)$, equipped with the scalar product

$$(u,v) \mapsto \int_D \nabla u \cdot \nabla v.$$

is a Hilbert space. We define

$$B(u,v) = \int_D (A\nabla u) \cdot \nabla v,$$

which is a bilinear form. It is continuous because A is bounded, and coercive because A is coercive.

We also define

$$L(v) = \langle f, v \rangle_{H^{-1}(D),\, H_0^1(D)},$$

which, by definition, is linear and continuous over the Hilbert space H.

3.3.2 Ellipticity Bound

Let A_ε be a simmetric matrix, the function $x \to A_\varepsilon(x)$, $x \in D$ is uniformly elliptic over D if exists $0 < \mu, M < +\infty$, such that

$$\forall \varepsilon > 0,\ \forall x \in D,\ \forall \xi \in \mathbb{R}^n, \qquad \mu|\xi|^2 \leq (A_\varepsilon(x)\xi) \cdot \xi \leq M|\xi|^2. \tag{B1}$$

By the lower ellipticity bound in (B1), the matrix A^ε is coercive. Consequently,

$$\int_D A^\varepsilon \nabla u \cdot \nabla u \, dx \ \geq \ \mu \int_D |\nabla u|^2 \, dx.$$

Since $u \in H_0^1(D)$, using the Poincaré inequality [1, p.290] we can prove the coercivity of the associated bilinear form on $H_0^1(D)$, so Lax-Milgram theorem implies the existence of a unique solution $u \in H_0^1(D)$ for all $f \in H^{-1}(D)$ using the variational formulation of the problem.

The following result is stated with generality but directly applies to our case:

3.3.3 Lax-Milgram Corollary

Assume that A is bounded and coercive. Then, for all $f \in H^{-1}(D)$, there exists a unique solution $u \in H_0^1(D)$ to the weak formulation of

$$-\operatorname{div}(A\nabla u) = f,$$

posed in a bounded open set $D \subset \mathbb{R}^n$, with the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial D.$$

Moreover, this solution satisfies

$$\|u\|_{H_0^1(D)} \leq \frac{1}{\mu}\|f\|_{H^{-1}(D)}.$$

We refer to Evans [1, pp.313-343] and Blanc-Le Bris [3, p.101, pp. 427-430] for more details.

The main difficulty of letting $\varepsilon \to 0$ in equation (1) is that, in a general sense, the product of two weakly convergent sequences does not converge to the product of the weak limits. The ellipcity bound (B1) above enables the following propositions, whose proofs are based on a compactness argument [3, pp.102-109, 4]. We will assume in the following results that the matrices are symmetric, but the results can be generalized.

Proposition 2. Let D be a bounded open subset of $\mathbb{R}^d$ and $f \in H^{-1}(D)$. Assume that $A_\varepsilon \in L^\infty(\mathbb{R}^d)$ satisfies (B1), uniformly with respect to ε.
Then, there exists a matrix A^*, $u^* \in H_0^1(D)$, and a sequence $\varepsilon_k > 0$, $\varepsilon_k \xrightarrow[k\to\infty]{} 0$, such that:

$$\begin{aligned} u^{\varepsilon_k} &\underset{\varepsilon_k\to 0}{\rightharpoonup} u^* && \text{weakly in } H_0^1(D), \\ A_{\varepsilon_k}\nabla u^{\varepsilon_k} &\underset{\varepsilon_k\to 0}{\rightharpoonup} A^*\nabla u^* && \text{weakly in } L^2(D), \\ A_{\varepsilon_k}\nabla u^{\varepsilon_k}\cdot\nabla u^{\varepsilon_k} &\underset{\varepsilon_k\to 0}{\longrightarrow} A^*\nabla u^*\cdot\nabla u^* && \text{in the distribution sense in } D. \end{aligned}$$

Moreover,

$$\int_D A_{\varepsilon_k}\nabla u^{\varepsilon_k}\cdot\nabla u^{\varepsilon_k}\,dx \xrightarrow[\varepsilon_k\to 0]{} \int_D A^*\nabla u^*\cdot\nabla u^*\,dx,$$

and u^* is the solution in $H_0^1(D)$ of

$$-\operatorname{div}(A^*\nabla u^*) = f. \tag{3}$$

Remark. The matrix A^* satisfies the same bounds as described in (B1). The matrix A^* and the subsequence $\{\varepsilon_k\}$ do not depend on f; this describes how there exists an equivalent (homogenized) material to the one of the starting problem. The homogenized equation in general does not have to be of the same form as the original equation; the proposition above states that in this particular case, with those specific bounds, the limit equation does read as the original one. The results above do not give us a way to compute either A^* nor u^*; it just notes that the homogenized matrix exists. We are able to compute it only in specific cases.

3.3.4 Sketch of the Proof.

We give only a sketch of the proof of Proposition 2. For the details see [3, pp.102-109].

Step 1 - Defining the linear operators $R(f)$ and $S(f)$

Using the bound (B1) and applying Poincaré's inequality, we have that u^ε is bounded in

$H_0^1(D)$ uniformly with respect to ε.

From these bounds for A_ε and u^ε, it is then possible to extract a subsequence ε_k such that:

- $A^{\varepsilon_k}\nabla u^{\varepsilon_k}$ is weakly convergent to r^* in $L^2(D)$;
- u^{ε_k} is weakly convergent to u^* in $H_0^1(D)$;
- u^{ε_k} strongly convergent in $L^2(D)$ due to Rellich Theorem.

By applying a diagonal extraction process, we extract this subsequence.

We then introduce the following functions that are linear and continuos.

Let $S : H^{-1}(D) \to H_0^1(D)$ be defined letting and

$$u^* = S(f).$$

Also define $R : H^{-1}(D) \to L^2(D)^2$, with

$$R(f) = A^*\nabla S(f).$$

The remainder of the proof consists in showing that $r^* = A^*\nabla u^*$, that is $RS^{-1}(u^*) = A^*\nabla u^*$ provided that S is invertible.

Step 2 - Showing that S is invertible We apply Lax-Milgram theorem by writing the equation $S(f) = u^*$ as follows: for all $g \in H^{-1}(D)$,

$$\langle g, S(f)\rangle_{H^{-1},H_0^1} = \langle g, u\rangle_{H^{-1},H_0^1}.$$

Working again on (B1) we prove estimates that allow us to show that the left-hand side is bilinear, continuous, and coercive in $H^{-1}(D)$, while the right-hand side is clearly linear and continuous in $H^{-1}(D)$, proving this way that S is invertible with continuous inverse.

Step 3 - Pointwise Lower Bound on $RS^{-1}(v)\cdot\nabla v$ In this step, fixed $v \in H_1^0(D)$, we show

$$RS^{-1}(v)\cdot\nabla v \;\geq\; \mu\,|\nabla v|^2, \quad \text{almost everywhere in } D.$$

By using Step 1, we define

$$S^{-1}(v) = -\operatorname{div}\left(RS^{-1}(v)\right), \quad \text{and} \quad g = S^{-1}(v).$$

Next, let v^ε be a solution of

$$-\operatorname{div}(A_\varepsilon \nabla v^\varepsilon) = g, \quad v^\varepsilon \in H_0^1(D),$$

for a suitable subsequence of $\varepsilon > 0$.

By using the convergence results of subsequences for S and R from Step 1, the coerciveness of A_ε that comes from the left side of bound B1, the convexity of

$$v \mapsto \int_D |\nabla v|^2 \, dx,$$

and the weak convergence of v^{ε_k} and ∇v^{ε_k}, it can be shown that the pointwise lower bound holds.

Step 4 - Pointwise Upper Bound on $RS^{-1}(v) \cdot \nabla v$ By using similar arguments to Step 3, the right part of the bound (B1) and applying the Cauchy-Schwarz inequality in $\mathbb{R}^d$, we infer

$$|RS^{-1}(v)| \leq M|\nabla v|, \quad \text{almost everywhere in } D.$$

Step 5 - Existence of A^* Applying the Pointwise Upper Bound to $v - w$, we get that, for any open set $O \subset D$

$$\forall v, w \in H_0^1(D), \quad \nabla v = \nabla w \text{ a.e. in } O \implies RS^{-1}(v) = RS^{-1}(w) \text{ a.e. in } O.$$

We then prove that

$$\forall x \in O, \quad RS^{-1}(v)(x) = a^*(x)\,\xi,$$

where $\xi \mapsto RS^{-1}(v)$ is a linear map for all $\xi \in \mathbb{R}^d$.

The function $v(x) = (\xi \cdot x)\chi(x)$, satisfies $\nabla v = \xi \quad$ in O, where χ is a smooth cut-off function having compact support in D such that $\chi|_O = 1$

After showing that the map a^* is well-defined, we prove that

$$\forall v \in H_0^1(D), \quad RS^{-1}v = a^*\nabla v, \quad \text{almost everywhere},$$

by assuming v is piecewise affine and using the fact that the set of piecewise affine functions is dense in $H_0^1(D)$.

STEP 6 - CONCLUSION The last equality of Step 5 together with the definitions of the operators R and S, allow us to prove that

$$r^* = RS^{-1}(u^*) = a^* \nabla u^*.$$

This and the convergences of Step 1 allow us to pass to the limit in the weak formulation of

$$-\operatorname{div}\left(A_{\varepsilon_k} \nabla u^{\varepsilon_k}\right) = f,$$

thus proving

$$-\operatorname{div}\left(A^* \nabla u^*\right) = f.$$

To prove that

$$A_{\varepsilon_k} \nabla u^{\varepsilon_k} \cdot \nabla u^{\varepsilon_k} \underset{\varepsilon_k \to 0}{\longrightarrow} A^* \nabla u^* \cdot \nabla u^* \quad \text{in the distribution sense in } D,$$

we fix $\varphi \in C^\infty(\overline{D})$. Using $u^{\varepsilon_k}\varphi$ as a test function in the variational formulation satisfied by u^{ε_k} and we have the convergence.
We use the same argument and set φ=1 to prove

$$\int_D A_{\varepsilon_k} \nabla u^{\varepsilon_k} \cdot \nabla u^{\varepsilon_k}\, dx \underset{\varepsilon_k \to 0}{\longrightarrow} \int_D A^* \nabla u^* \cdot \nabla u^*\, dx,$$

□

To prove our main result, we need the following lemma to pass to the limit of a sequence of solutions which do not necessary converge to the homogenized solution u^*.

Lemma 1. Assume that the conditions of Proposition 2 are satisfied. Let $\widetilde{D} \subset D$ be an open set, $f \in H^{-1}(\widetilde{D})$ and let $v^{\varepsilon'}$ be a sequence in $H^1(\widetilde{D})$ such that

$$-\operatorname{div}\left(A_{\varepsilon'}\nabla v^{\varepsilon'}\right) = f \quad \text{in } \widetilde{D},$$

and

$$v^{\varepsilon'} \underset{\varepsilon'\to 0}{\rightharpoonup} v \quad \text{in } H^1(\widetilde{D}).$$

Then, we have

$$A_{\varepsilon'}\nabla v^{\varepsilon'} \underset{\varepsilon'\to 0}{\rightharpoonup} A^*\nabla v \quad \text{in } L^2(\widetilde{D}),$$

and hence

$$-\operatorname{div}\left(A^*\nabla v\right) = f \quad \text{in } \widetilde{D}.$$

For the proof of result above we refer to [3]. Now we get to the core of our thesis, which is how to calculate the homogenized matrix A^* in the case of the checkerboard.

Theorem 1. Given $Q = (0,1)^2$, let $a_{\text{per}}(x_1, x_2)$ be a function that is Q-periodic, piecewise constant, alternately taking the values $\alpha > 0$ and $\beta > 0$:

$$a_{\text{per}}(x_1, x_2) = \begin{cases} \alpha & \text{if } (x_1, x_2) \in]0, \frac{1}{2}[^2 \cup]\frac{1}{2}, 1[^2, \\ \beta & \text{otherwise.} \end{cases}$$

Let $u_\varepsilon \in H_0^1(D)$ be the solution to

$$\begin{cases} -\operatorname{div}\big(A_\varepsilon(x)\nabla u_\varepsilon(x)\big) = f(x) & \text{in } D, \\ u_\varepsilon = 0 & \text{on } \partial D. \end{cases}$$

We show that, as $\varepsilon \to 0$, $u_\varepsilon \rightharpoonup u^*$ weakly in $H_0^1(D)$, where u^* is the solution to the homogenized problem

$$-\operatorname{div}(A^*\nabla u^*) = f \quad \text{in } D,$$

and the homogenized matrix is given by

$$A^* = \sqrt{\alpha\beta}\,\mathrm{Id}.$$

Remark 1. The proof uses several specific properties of the problem:

- we work in dimension 2;
- we have exactly two phases (α and β);
- the squared checkerboard has symmetries.

Proof. Since A_ε satisfies the assumptions of Proposition 2, we have the existence of the homogenized matrix A^*. We will now explicitly find A^*, proving in this way its uniqueness and indipendence from the succession ε_k. We will use the following strategy: we will find a solution for the problem with periodic boundary conditions, defining a proper sequence of solutions that converges to it, then applying the previous lemma and prepositions we will pass to the limit and by applying the geometric symmetry of the problem and the fact that we work in dimension 2 we will be able to compute A^*. In the original problem (1), we replace the Dirichlet homogeneous condition $u^\varepsilon = 0$ by periodic boundary conditions. In other words, we look for a solution u^ε that is periodic of $\mathcal{Q}$ $=]0,1[^2$ and solution in the whole space $\mathbb{R}^2$. Such a solution is only defined up to the addition of a constant. To make it unique we would need to assume for instance that

$$\int_Q u^\varepsilon = 0.$$

The associated homogenized solution is, according to Proposition 2, the periodic solution u^*. Such a solution is again unique due to the condition

$$\int_Q u^* = 0.$$

Let $\lambda \in \mathbb{R}^2$ and let u_{per} be a function in $H^1(Q)$ satisfying periodic boundary conditions on the boundary of Q and

$$-\operatorname{div}(A_{\text{per}} \nabla u_{\text{per}}) = \operatorname{div}(A_{\text{per}} \lambda). \tag{4}$$

The existence of such a function u_{per} may be proved using the Lax–Milgram Theorem [1, pp.313-343].

Remark 2. Equation (4) can be rewritten as $\operatorname{div}(A_{\text{per}}(\nabla u_{\text{per}} + \lambda)) = 0$, which shows that the flux $A_{\text{per}} v_{\text{per}}$ is divergence-free, whereby we set

$$v_{\text{per}} = \nabla u_{\text{per}} + \lambda,$$

which is periodic too.

We note that $\langle v_{\text{per}} \rangle = \lambda$ because $\langle \nabla u_{\text{per}} \rangle = 0$, since u_{per} is periodic and the average of a periodic gradient vanishes:

$$\langle \nabla u_{\text{per}} \rangle = \left(\int_Q \partial_{x_1} u_{\text{per}}(x)\, dx, \int_Q \partial_{x_2} u_{\text{per}}(x)\, dx \right) = 0.$$

Indeed let's compute the first coordinate:

$$\int_0^1 \int_0^1 \partial_{x_1} u_{\text{per}}(x_1, x_2)\, dx_1\, dx_2 = \int_0^1 \big[u_{\text{per}}(1, x_2) - u_{\text{per}}(0, x_2) \big]\, dx_2 = 0,$$

since $u_{\text{per}}(1, x_2) = u_{\text{per}}(0, x_2)$ due to periodicity and the same applies for the second coordinate.

We now prove that

$$A^* \langle v_{\text{per}} \rangle = \langle A_{\text{per}} v_{\text{per}} \rangle. \tag{5}$$

In order to do so, we define the sequence

$$w^\varepsilon(x) = \lambda \cdot x + \varepsilon \, u_{\text{per}} \left(\frac{x}{\varepsilon} \right).$$

Applying Proposition 1, we deduce that the gradient

$$\nabla w^\varepsilon(x) = v_{\text{per}} \left(\frac{x}{\varepsilon} \right)$$

converges to $\langle v_{\text{per}} \rangle = \lambda$ weakly in $L^2(Q)$. Likewise, $u_{\text{per}}\left(\frac{x}{\varepsilon}\right)$ converges to $\langle u_{\text{per}} \rangle$ weakly in $L^2(Q)$. Thus, w^ε converges to $w_0(x) = \lambda \cdot x$ weakly in $H^1(Q)$.

We already noted that $\nabla w^\varepsilon(x) = v_{\text{per}}\left(\frac{x}{\varepsilon}\right)$, we then have

$$-\operatorname{div}(A_\varepsilon \nabla w^\varepsilon) = -\operatorname{div}\left(A_\varepsilon v_{\text{per}} \left(\frac{x}{\varepsilon} \right) \right) = 0,$$

which is equal to 0 as shown in the Remark, with a change of coordinates $x/\varepsilon = x'$. Then we can apply Lemma 1, which gives that $A_\varepsilon \nabla w^\varepsilon$ converges weakly to $A^* \nabla w_0 = A^* \lambda$.

On the other hand, applying Proposition 1 to the periodic function $A_{\text{per}} v_{\text{per}}$, we have that

$$A_\varepsilon \nabla w^\varepsilon = (A_{\text{per}} v_{\text{per}}) \left(\frac{x}{\varepsilon} \right)$$

weakly converges to $\langle A_{\text{per}} v_{\text{per}} \rangle$. This proves (5).

We have now proved that a homogenized matrix against an average periodic vector is equal to the average of the original matrix against the vector.

Now we will use particular geometric symmetries that characterize only this specific periodic composite material structure.

We return to the matrix A_{per} that we have defined above, and which models the checkerboard structure. Let σ be the rotation of angle $\pi/2$. For all $x \in Q$, we have

$$A_{\text{per}}(x) \cdot A_{\text{per}}(\sigma(x)) = \alpha\beta \, \mathrm{Id}. \tag{6}$$

Remark 3. Due to the checkerboard geometry, rotating by $\pi/2$ swaps the phases: if $a_{\text{per}}(x) =$

α, then $a_{\text{per}}(\sigma(x)) = \beta$, and vice versa. Thus $a_{\text{per}}(x) \cdot a_{\text{per}}(\sigma(x)) = \alpha\beta$ always, which gives (6) since $A_{\text{per}} = a_{\text{per}}\,\mathrm{Id}$.

This allows us to write

$$A_{\text{per}}(\sigma(x))v_{\text{per}}(\sigma(x)) = \alpha\beta\,(A_{\text{per}}(x))^{-1}v_{\text{per}}(\sigma(x)). \tag{6 bis}$$

By (5), we have

$$A^*\langle v_{\text{per}}\rangle = \langle A_{\text{per}}v_{\text{per}}\rangle = \langle (A_{\text{per}}v_{\text{per}}) \circ \sigma\rangle,$$

since σ does not change the average. And thus from (6 bis) we obtain,

$$A^*\langle v_{\text{per}}\rangle = \alpha\beta \left\langle A_{\text{per}}^{-1}v_{\text{per}} \circ \sigma \right\rangle.$$

Remark 4. The rotation σ preserves the integral average over $Q =]0,1[^2$: by change of variables we have $\langle f \circ \sigma\rangle = \langle f\rangle$ for any integrable function f.

Moreover, we have

$$\begin{aligned}\operatorname{div}\big(A_{\text{per}}(x)\,A_{\text{per}}(\sigma(x))\,v_{\text{per}}(\sigma(x))\big) &= \alpha\beta\ \operatorname{div}\big(v_{\text{per}}(\sigma(x))\big)\\ &= \alpha\beta\ \operatorname{div}\big(\nabla u_{\text{per}}(\sigma(x))\big) = 0\end{aligned} \tag{7}$$

and

$$\operatorname{curl}(A_{\text{per}}(\sigma(x))v_{\text{per}}(\sigma(x))) = \operatorname{div}(A_{\text{per}}(x)v_{\text{per}}(x)) = 0. \tag{8}$$

Remark 5. In dimension 2, the divergence and scalar curl are related by a rotation: for any vector field F and σ the rotation by $\pi/2$, we have $\operatorname{curl}(F \circ \sigma) = -\operatorname{div}(F)$. Since $A_{\text{per}}v_{\text{per}}$ is divergence-free by equation (4), its rotated counterpart is curl-free.

Using the fact that we work in dimension 2, Eq. (8) implies that the function

$$w_{\text{per}}(x) = A_{\text{per}}(\sigma(x))v_{\text{per}}(\sigma(x)) \tag{9}$$

can be written as

$$w_{\text{per}}(x) = \nabla h_{\text{per}}(x) + \langle w_{\text{per}}\rangle$$

using Helmholtz decomposition, where h_{per} is periodic.

Remark 6. In dimension 2, a curl-free vector field is necessarily a gradient. For periodic fields on Q, we have $F = \nabla h + \langle F\rangle$ for some periodic function h.

On the other hand, with w_{per} as in (9), (7) reads as

$$\operatorname{div}(A_{\text{per}} w_{\text{per}}) = 0.$$

Thus (5) holds with w_{per} replacing v_{per}. This gives

$$\begin{aligned} A^*(A_{\text{per}}(\sigma(x)) v_{\text{per}}(\sigma(x))) &= \langle A_{\text{per}}(A_{\text{per}}(\sigma(x)) v_{\text{per}}(\sigma(x)))\rangle \\ &= \alpha\beta \langle v_{\text{per}}(\sigma(x))\rangle. \end{aligned}$$

In addition, by using the same substitution and using invariance of the average under rotation,

$$A^* \langle A_{\text{per}}(\sigma(x)) v_{\text{per}}(\sigma(x))\rangle = A^* \langle A_{\text{per}} v_{\text{per}}\rangle = A^* A^* \langle v_{\text{per}}\rangle,$$

where we have used (5) again. Hence

$$A^* A^* \langle v_{\text{per}}\rangle = \alpha\beta \langle v_{\text{per}}(\sigma(x))\rangle = \alpha\beta\langle v_{\text{per}}\rangle,$$

since σ does not change the average.

We have thus proved that

$$(A^*)^2 \lambda = \alpha\beta\lambda, \qquad \forall \lambda \in \mathbb{R}^2.$$

This implies that

$$(A^*)^2 = \alpha\beta \, \mathrm{Id}.$$

Consequently, the polynomial $P(X) = X^2 - \alpha\beta$ satisfies $P(A^*) = 0$. Since the polynomial has two simple roots, A^* is diagonalizable. Recalling that the eigenvalues of A^* are positive, the only possible eigenvalue is $\sqrt{\alpha\beta}$. Thus

$$A^* = \sqrt{\alpha\beta}\, \mathrm{Id}.$$

And this concludes the proof. □

3.4 Random periodic checkerboard and laminated material

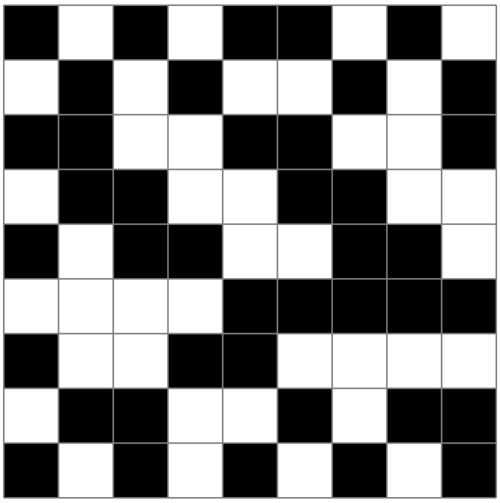

Figure 3.1: Random periodic checkerboard

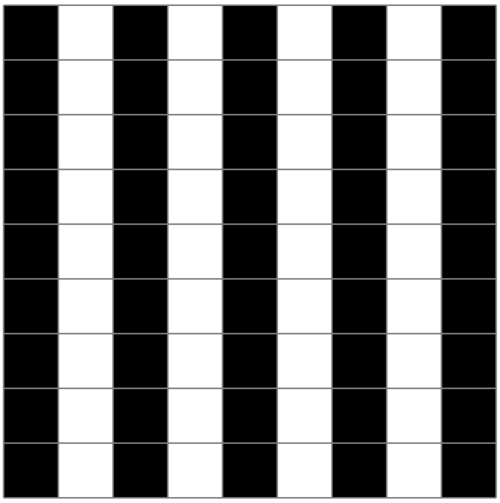

Figure 3.2: Vertical laminated material

3.5 Comments

It's interesting to study other structures aswell and how their homogenized matrices differ related to the geometry. The examples of a laminated material and a random checkerboard are studied thoroughly in the book of Blanc and Le Bris [3, pp.110-118]. In the case of a random checkerboard with the same coefficients randomly chosen independently each with probability $1/2$, we arrive at the same homogenized matrix. It is because even if the materials are different, the proportions of the phases are equal, and that makes the materials the same on average. The case of a laminated material, where instead of a checkerboard we have columns, gives us a different homogenized matrix:

$$\begin{pmatrix} \frac{1}{\langle \frac{1}{a_{\text{per}}} \rangle} & 0 \\ 0 & \langle a_{\text{per}} \rangle \end{pmatrix}$$

This has an intuitive explanation: the laminated material is the same as the checkerboard in one direction (the one where the blocks alternate), but not in the direction parallel to the columns, even if there is the same proportion of phases.

It is relevant to notice how important the geometry of the problem is and how generalizations are difficult. From the specific geometry of the problems arise different macroscopic behaviours, and some specific computations are available which make those problems solvable, like the rotation due to the symmetry of the checkerboard that we employed previously.

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